

New frequency predictions for a simple pendulum: Application of Harmonic Balance and Akbari-Ganji methods

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Abstract

In the analysis of nonlinear dynamical systems, developing an accurate understanding of simple mechanical models—such as the pendulum—is of fundamental importance in both engineering and physics. Although the simple pendulum is often introduced in its linearized form for small oscillations, its true behavior becomes highly nonlinear at larger amplitudes. The nonlinear pendulum, therefore, serves as a classical yet powerful example for exploring the rich dynamics that emerge in real-world systems where linear approximations fail. In this study, the non-linear dynamic analysis of a simple pendulum is revisited. Two new formulas for the period and frequency are proposed based on the Harmonic Balance Method and the Akbari-Ganji Method. Furthermore, to obtain more accurate results, improvements are made to the formulas of the harmonic balance method and the Akbari-Ganji method. These improvements provide more reliable outcomes, especially in systems requiring high accuracy. Two of the most prominent formulas in the literature are derived using the Akbari-Ganji Method. As a result of this, the frequencies obtained by the present method and the other methods are compared. The obtained results emphasize the accuracy and efficiency of the proposed approaches. Consequently, this study encourages the use of alternative methods in the analysis of non-linear dynamic systems.

Keywords: Simple pendulum, Non-linear equation, Harmonic Balance Method, Akbari-Ganji Method.

1. Introduction

Dynamic analysis of a simple pendulum is a practically important problem that has been extensively investigated in the literature. The large-amplitude oscillations of the pendulum have also attracted considerable attention, leading to the proposal of several useful formulas for estimating the period and frequency. Since an exact analytical solution is not attainable, most studies have focused on developing better approximations. Cumber [1] examined various forced pendulum systems, including horizontally, vertically, radially, and angularly forced pendulums, as well as the angularly forced double pendulum. He concluded that simple pendulums are highly interesting and worthy of detailed analysis. Wang et al. [2] analyzed the dynamics of a simple pendulum subjected to variable damping and solved the governing equations using numerical methods. Ganley [3] suggested a formula for the period for very large amplitudes. Parwani [4] also estimated the period of the non-linear pendulum by comparing the modified equation of motion with the perturbative solution and gave a new useful result. Fulcher and Davis [5] investigated the pendulum's motion experimentally and theoretically. They solved the equation using the perturbation method of Kryloff and Bogoliuboff [6] to find the two lowest order corrections to the amplitudes and the period. Kidd and Fogg [7] suggested an approximate formula for the period. Beléndez et al. [8] investigated the non-linear oscillation of a simple pendulum and presented the formulas not only for the period, but also for the angular displacement and amplitude. Some researchers [9-11] revisited the non-linear oscillation of the simple pendulum and obtained the frequency and period using the Homotopy Perturbation Method (HPM). Butikov [12] studied the non-linear motion of the pendulum for extremely large amplitudes. New handy-used formulas based on concrete theories are still welcome since the analytical solution cannot be obtained.

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Compared to other semi-analytical methods such as the Differential Transform Method (DTM), the Adomian Decomposition Method (ADM), and the Homotopy Perturbation Method, the Akbari-Ganji Method is much easier to use [13]. According to this method, the solution depends only on the use of the initial conditions and the governing equation itself and its derivatives. The most advantageous feature of the AGM is that it transforms the problem into a system of algebraic equations [14]. Due to its simplicity and versatility, AGM has found wide application in various engineering problems [15–24]. In the present paper, we revisit the non-linear motion of the simple pendulum. Two new formulas are proposed. For this purpose, the harmonic balance method and the Akbari-Ganji method [25] developed for non-linear differential equations are utilized.

2. Analysis

The equation of motion for the simple pendulum is simply shown to be in the form

$$\ddot{\theta} + \omega_0^2 \sin \theta = 0, \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0, \quad \omega_0 = \sqrt{\frac{g}{l}}. \quad (1)$$

Here, l is the length of the pendulum, θ is the angle measured from the vertical, and g is the acceleration due to gravity. For small displacements, Eq.1 takes the form

$$\ddot{\theta} + (\omega_0^2) \theta = 0 \quad (2)$$

whose solution is given by

$$\theta(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t). \quad (3)$$

Here, ω_0 is the frequency, and A and B are the constants to be determined from the initial conditions.

2.1. Harmonic Balance Method

To solve Eq.1, we will use the harmonic balance method. First, we expand $\sin(\theta)$ into the Taylor series :

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \frac{\theta^7}{5040} + \frac{\theta^9}{362880}. \quad (4)$$

We take the first three terms and insert Eq.4 into Eq.1:

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} \right) = 0. \quad (5)$$

Let us now assume a solution

$$\theta = A \cos \omega t, \quad (6)$$

where A and ω are the amplitude and the frequency, respectively. Inserting this term and its derivatives into Eq.1 gives

$$-A\omega^2 \cos \omega t + A\omega_0^2 \cos \omega t - \frac{\omega_0^2}{6} A^3 \cos^3 \omega t + \frac{\omega_0^2}{120} A^5 \cos^5 \omega t = 0. \quad (7)$$

Using the equalities

$$\begin{aligned} \cos^3 \omega t &= \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t, \\ \cos^5 \omega t &= \frac{10}{16} \cos \omega t + \frac{5}{16} \cos 3\omega t + \frac{1}{16} \cos 5\omega t, \end{aligned} \quad (8)$$

we find

$$\begin{aligned} -A\omega^2 \cos \omega t + A\omega_0^2 \cos \omega t - \frac{\omega_0^2}{6} A^3 \left(\frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \right) \\ + \frac{\omega_0^2}{120} A^5 \left(\frac{10}{16} \cos \omega t + \frac{5}{16} \cos 3\omega t + \frac{1}{16} \cos 5\omega t \right) = 0. \end{aligned} \quad (9)$$

Equating now the coefficients of like terms yields

$$\omega = \omega_0 \sqrt{1 - \frac{A^2}{8} + \frac{A^4}{192}} \quad (10)$$

Eq.10 gives $\omega = 2.6637$ rad/s for $\theta(0) = \pi/2$ and $\omega = 2.92005$ rad/s for $\theta(0) = \pi/3$. This formula is identical to the one obtained by Belendez and colleagues for small values of x using the Homotopy Perturbation Method [9]. They propose the formula $\omega = \omega_0 \sqrt{(2J_1(A))/(A)}$ for the same problem. For small A 's, expanding Bessel function into the series and taking the first three terms, we obtain the same as Eq.4 and this leads to Eq.10. Since the interval of interest in the present problem is $0 \leq \theta \leq \pi/2$, we can infer that the results of HPM will also be valid for small angles as well. To improve the correctness of the results, we take the first four terms in Eq. 4 and find the frequency again:

$$\omega = \omega_0 \sqrt{1 - \frac{A^2}{8} + \frac{A^4}{192} - \frac{A^6}{9216}}. \quad (11)$$

Eq.11 gives $\omega = 2.6607$ rad/s for $\theta(0) = \pi/2$ and $\omega = 2.9198$ rad/s for $\theta(0) = \pi/3$. A comparison of the results of Eq.11 with the results of the Runge-Kutta method and some of the other techniques available in the literature is given in Table 1.

2.2. Akbari-Ganji Method

To obtain more correct values, one could include additional terms in Eq. 4 or assume a more general solution of the form $\theta = A_1 \cos \omega t + A_2 \cos 3\omega t \dots$, where ω , A_1 , A_2 are constants to be determined. However, this procedure requires tedious computations. Instead, we wish to use a relatively simpler

and more effective Akbari-Ganji method (AGM). It will be demonstrated shortly that several previously obtained solution forms can also be derived using this method. According to the Akbari-Ganji Method, the process begins by assuming a trial solution that includes a set of unknown coefficients. The initial conditions are then applied to this assumed form, resulting in a system of algebraic equations. To fully determine all unknowns, the original differential equation, along with its necessary derivatives, is used to generate as many independent equations as there are unknowns. The main differential equation and its derivatives are used to reach the number of algebraic equations that equal the number of unknowns. Since there is no damping in the system, we choose the solution of Eq.1 in the form

$$\theta(t) = K \sin(\omega t + \varphi) \quad (12)$$

where K , ω and φ are constants to be determined. Let us assume that the initial conditions are in the form $\theta(0) = A$, $\dot{\theta}(0) = B$. Appealing to Eq.12 yields

$$A = K \sin(\varphi), B = K \omega \cos(\varphi). \quad (13)$$

We need an extra equation. To this end, we differentiate Eq.12 and replace $t = 0$ in the equation as follows:

$$\begin{aligned} \dot{\theta} &= K \omega \cos(\omega t + \varphi) \rightarrow \dot{\theta}(0) = K \omega \cos(\varphi), \\ \ddot{\theta} &= -K \omega^2 \sin(\omega t + \varphi) \rightarrow \ddot{\theta}(0) = -K \omega^2 \sin(\varphi). \end{aligned} \quad (14)$$

Inserting the terms in $\theta(0)$, $\dot{\theta}(0)$, $\ddot{\theta}(0)$ into Eq.1, we have

$$\ddot{\theta}(0) + \omega_0^2 \sin(\theta(0)) = 0 \rightarrow -K \omega^2 \sin(\varphi) + \omega_0^2 \sin(K \sin(\varphi)) = 0. \quad (15)$$

Solving the equations in Eqs. 13 and 15 simultaneously, we get

$$\omega = \omega_0 \sqrt{\frac{\sin(A)}{A}}, \quad \varphi = \arctan\left(\frac{A}{B} \omega\right), \quad K = \sqrt{A^2 + \left(\frac{B^2 A}{\omega_0^2 \sin(A)}\right)}. \quad (16)$$

Thus, the angle $\theta(t)$ is obtained in the form

$$\theta(t) = \sqrt{A^2 + \left(\frac{B^2 A}{\omega_0^2 \sin(A)}\right)} \sin\left(\omega_0 \sqrt{\frac{\sin(A)}{A}} t + \arctan\left(\frac{A}{B} \omega\right)\right). \quad (17)$$

It is important to note that the present method incorporates the initial angular velocity into its formulation. As an illustrative example, we consider the initial conditions $\theta(0) = \pi/6$, $\dot{\theta}(0) = 0$, $l=1$, and plot the angular displacement θ as a function of time t . Figure 1a presents a comparison between the results obtained using the proposed method and those obtained via the Runge-Kutta method. As shown, the two curves nearly coincide, indicating good agreement. In Fig.1b, the values $\theta(0) = A = \pi/3$, $\dot{\theta}(0) = 0$ are taken. It is observed that as the initial angular displacement $\theta(0)$ increases, the discrepancy between the results of the present method and the Runge-Kutta solution

becomes more noticeable. When the results of the present method are compared with those obtained by other methods, it can be said that the present formula in Eq. 16 yields accurate results between $0 < \theta < 30^\circ$. Table 1 summarizes the frequencies calculated by the present and various previously proposed methods for initial angles in the range $0 < \theta < \pi/2$. Taking the Runge-Kutta method as the reference solution, the present method provides acceptable frequency estimates for $0 < \theta < 30^\circ$. After the angles $\theta > 30^\circ$, the proposed method tends to underestimate the frequency compared to other methods. To improve the accuracy of Eq. 16 in the range $30 < \theta < 90^\circ$ a corrective transformation is introduced: $\theta = k\theta$, where k is a constant. Taking the derivatives and inserting them into Eq.1 gives

$$\ddot{\theta} + \frac{\omega_0^2}{k} \sin(k\theta) = 0. \quad (18)$$

We assume a solution

$$\theta(t) = b \sin(\omega t + \varphi) \quad (19)$$

from which we can write

$$\dot{\theta}(t) = b\omega \cos(\omega t + \varphi), \quad \ddot{\theta}(t) = -b\omega^2 \sin(\omega t + \varphi). \quad (20)$$

Here, b is a constant to be determined. Inserting these expressions into Eq.18 and evaluating at $t = 0$, we obtain

$$\omega = \omega_0 \sqrt{\frac{\sin(kA)}{(kA)}}, \quad \varphi = \arctan\left(\frac{A}{B} \omega\right), \quad b = \sqrt{A^2 + \left(\frac{B^2}{\omega^2}\right)}. \quad (21)$$

The new form of the frequency expression ω is more general, incorporating the correction factor k , which must be determined to ensure consistency between the results obtained using the Akbari-Ganji Method and those from the Runge-Kutta Method. As shown in Table 1, for an initial angle of $\theta = \pi/2$, the frequency obtained via the Runge-Kutta Method is $\omega = 2.6536$ rad/s. Substituting this value into Eq. 21 yields $k = 0.8679$, which closely approximates $\sqrt{3}/2$, a value also noted by Parwani in his analysis. This result implies that the constant $\sqrt{3}/2$ in Parwani's formula corresponds to the specific case of an initial angle $A = \pi/2$. However, upon closer examination of the Runge-Kutta results, it becomes evident that the value of k is not fixed but varies with the initial angle A . This dependency is clearly illustrated in Table 2, where different values of A correspond to distinct values of k . However, it does not show enormous changes between $0 < \theta < 90^\circ$. Thus, while Eq.16 can be used without k between $0 < \theta < 30^\circ$, Eq.21 can also be used in the entire interval using the values of k 's from Table 2 for each initial angle. Figure 2 presents the angular displacement θ as a function of time t , with each curve computed using a different correction coefficient k . The results obtained via the improved Akbari-Ganji Method are compared with those from the Runge-Kutta Method for three different initial angles. The close agreement between the two approaches demonstrates the effectiveness of incorporating the correction factor k into the analytical formulation.

3. Discussion and Conclusions

In this study, two new analytical formulas have been developed for estimating the period and frequency of a simple pendulum using the Harmonic Balance Method (HBM) and the Akbari-Ganji Method (AGM). A key distinction between the two approaches lies in the treatment of initial conditions: while the HBM assumes a zero initial angular velocity $\dot{\theta}(0) = 0$, the AGM formulation incorporates the initial angular velocity directly into the analysis. The accuracy of the proposed formulas has been validated through comparisons with numerical results obtained using the Runge-Kutta method, as well as with several other approximate analytical methods. The Harmonic Balance Method was found to yield satisfactory results for small amplitude oscillations. However, Eq. 16, derived initially via AGM, showed limitations at higher amplitudes and required refinement for improved accuracy. To address this, an enhanced version of Eq. 16 was proposed using a correction coefficient k based on the Akbari-Ganji framework. This improved formula coincides with Parwani's result when the initial angle is $A = \pi/2$, yielding $k = \sqrt{3}/2$. Further analysis revealed that k is not a universal constant but rather depends on the initial angular displacement A , as documented in Table 2. This angle-dependent behavior provides a more flexible and accurate model for predicting the pendulum's frequency across a broader range of initial conditions. As an alternative formulation, the solution form $\theta(t) = b \cos(\omega t + \phi)$ was also investigated. When applying AGM to this form, the resulting frequency expression $\omega = \omega_0 \sqrt{\cos(\theta)}$ did not yield acceptable accuracy. However, modifying this to $\omega = \omega_0 \sqrt{\cos(k\theta)}$ and calibrating k using the known result $\theta = \pi/2$ for $\omega = 2.6536$ rad/s, led to $k = 0.4903$. This value is nearly identical to Kidd and Fogg's well-known approximation $\omega = \omega_0 \sqrt{\cos(\theta/2)}$, highlighting the AGM's capacity to reproduce other established formulas through appropriate transformations. For students in physics and mechanics, these two proposed formulas can serve as useful alternatives for analyzing non-linear pendulum motion. Moreover, the ability of the Akbari-Ganji Method to accommodate non-zero initial angular velocity offers a distinct advantage over many traditional approaches. Several formulas derived in previous studies were shown to be recoverable using AGM, emphasizing its generality and utility. Looking ahead, the Akbari-Ganji Method shows strong potential for broader applications, including damped pendulums and multi-degree-of-freedom systems such as the double pendulum. It is also well-suited for tackling other systems governed by non-linear differential equations. However, one limitation of the method lies in its scalability—solving the resulting system of algebraic equations becomes increasingly complex as the number of variables and equations grows, and may become intractable in some cases.

4. Author Declarations

The authors have no conflicts to disclose.

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Table 1: Comparison of angular frequency values ω calculated using different methods. Results are presented for various initial angles θ , including values obtained using the Akbari-Ganji Method (AGM), the corrected AGM with the coefficient k , the Runge-Kutta Method, the Harmonic Balance Method (HBM), Homotopy Perturbation Method (HPM) and other formulas reported in the literature.

θ [rad]	ω [rad/s]								
	AGM	AGM (Corrected)	Runge -Kutta	HBM	HPM [9]	Parwani [4]	Molina [26]	Kidd and Fogg [7]	Lima and Arun [27]
$\pi/18$	3.1241	3.1262	3.1262	3.1261	3.1261	3.1261	3.1261	3.1261	3.1261
$\pi/12$	3.1142	3.1191	3.1191	3.1187	3.1187	3.1187	3.1187	3.1187	3.1187
$\pi/9$	3.1003	3.1088	3.1088	3.1083	3.1083	3.1083	3.1082	3.1082	3.1082
$5\pi/36$	3.0825	3.0954	3.0954	3.0949	3.0949	3.0949	3.0948	3.0947	3.0948
$\pi/6$	3.0607	3.0796	3.0796	3.0786	3.0786	3.0785	3.0784	3.0783	3.0784
$7\pi/36$	3.035	3.0602	3.0602	3.0593	3.0593	3.0592	3.059	3.0588	3.059
$2\pi/9$	3.0054	3.038	3.038	3.0372	3.0372	3.037	3.0366	3.0362	3.0367
$\pi/4$	2.9719	3.0125	3.0125	3.0121	3.0121	3.0118	3.0112	3.0105	3.0113
$5\pi/18$	2.9345	2.9851	2.9851	2.9842	2.9842	2.9837	2.9827	2.9818	2.983
$11\pi/36$	2.8933	2.9547	2.9547	2.9534	2.9534	2.9527	2.9513	2.9498	2.9516
$\pi/3$	2.8483	2.9215	2.9215	2.9198	2.9198	2.9188	2.9167	2.9147	2.9173
$13\pi/36$	2.7995	2.8851	2.8851	2.8834	2.8834	2.8821	2.8792	2.8764	2.8799
$7\pi/18$	2.7469	2.8443	2.8443	2.8443	2.8443	2.8424	2.8385	2.8348	2.8395
$15\pi/36$	2.6905	2.8015	2.8015	2.8024	2.8024	2.8	2.7947	2.7898	2.796
$4\pi/9$	2.6304	2.7554	2.7554	2.7578	2.7579	2.7547	2.7478	2.7413	2.7495
$17\pi/36$	2.5666	2.706	2.706	2.7106	2.7107	2.7065	2.6976	2.6894	2.6998
$\pi/2$	2.499	2.6536	2.6536	2.6607	2.6608	2.6556	2.6442	2.6338	2.647

Table 2: Computed values of k for various values of θ , illustrating the relationship between these parameters.

θ and k values									
θ	$\pi/18$	$\pi/12$	$\pi/9$	$5\pi/36$	$\pi/6$	$7\pi/36$	$2\pi/9$	$\pi/4$	$5\pi/18$
k	0.8595	0.8522	0.8558	0.8596	0.8571	0.8599	0.8611	0.8633	0.8619
θ	$11\pi/36$	$\pi/3$	$13\pi/36$	$7\pi/18$	$15\pi/36$	$4\pi/9$	$17\pi/36$	$\pi/2$	
k	0.8612	0.8606	0.8606	0.8632	0.8640	0.8651	0.8666	0.8679	

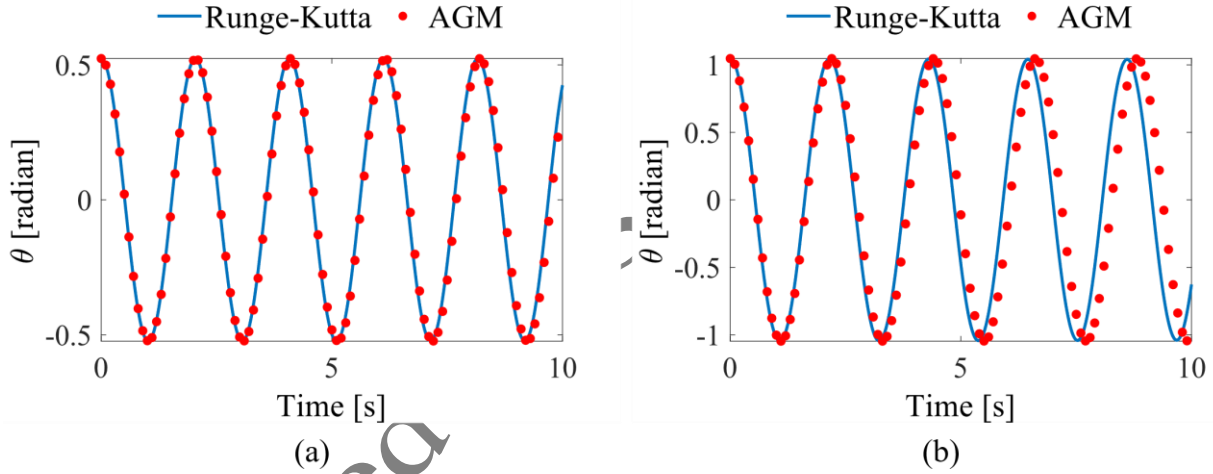


Figure 1: Comparison of numerical solutions for the angular displacement θ as a function of time t , obtained using the Runge-Kutta Method (solid line) and the Akbari-Ganji Method (dotted line). (a) $\theta(0) = A = \pi/6$, $\dot{\theta}(0) = 0$. (b) $\theta(0) = A = \pi/3$, $\dot{\theta}(0) = 0$.

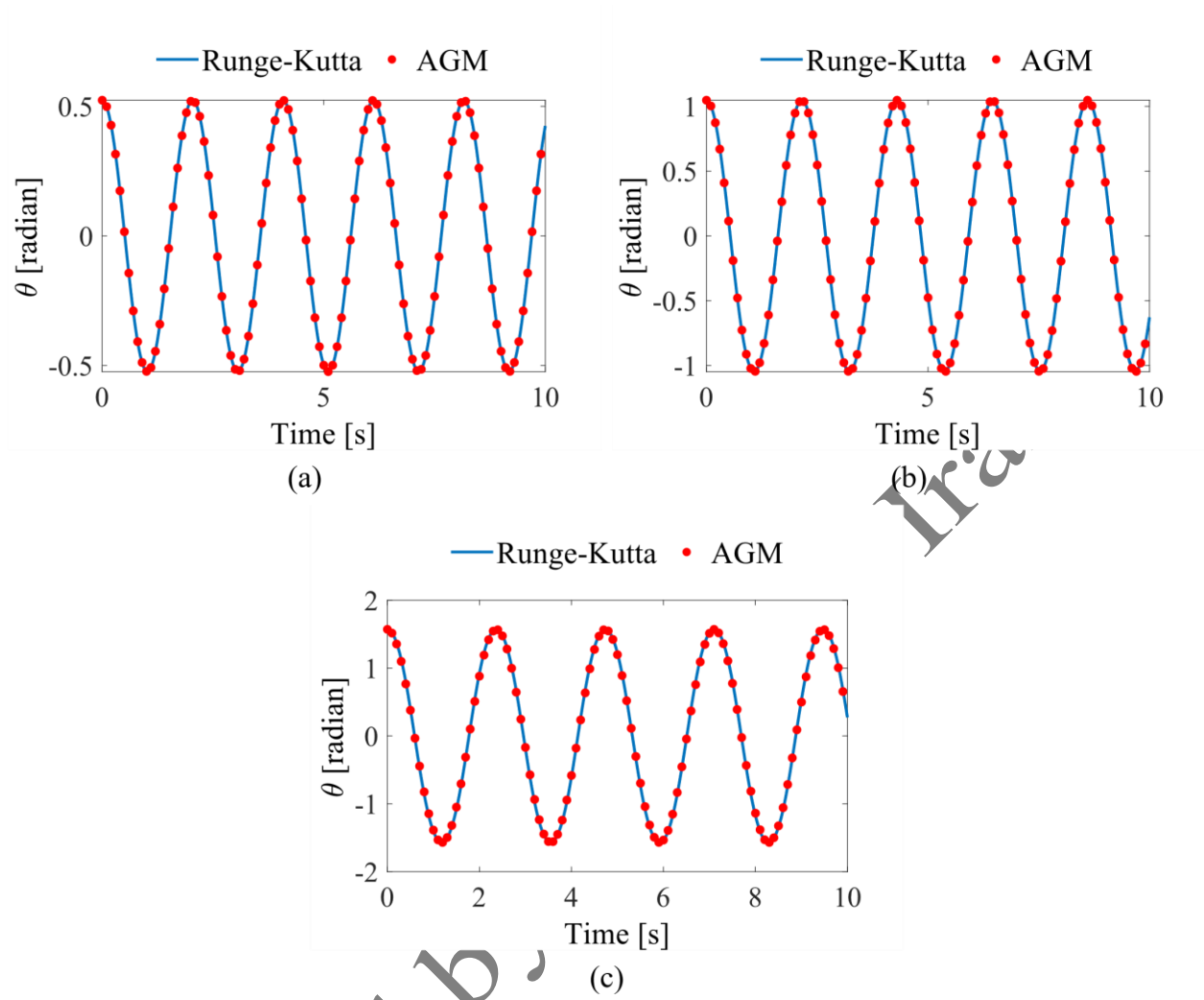


Figure 2: Comparison of numerical solutions for the angular displacement θ as a function of time t , obtained using the Runge-Kutta Method (solid lines) and the Akbari-Ganji Method with correction coefficient k (dotted lines). (a) $\theta(0) = A = \pi / 6$ and $k = 0.8571$; (b) $\theta(0) = A = \pi / 3$ and $k = 0.8606$; (c) $\theta(0) = A = \pi / 2$ and $k = 0.8679$.

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