Analysis of Newtonian and Joule heating in a bioconvective Williamson nanofluid flow with gyrotactic microorganisms incorporating modified forms of Fourier's and Fick's laws

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Abstract: This paper investigates the heat and mass transfer features in a time-dependent bioconvective flow of a Williamson nanofluid containing gyrotactic microorganisms over a radially stretching sheet. Bioconvection arises from the collective motion of motile microorganisms, such as algae or bacteria, which generates a density gradient and induces fluid motion. These microorganisms enhance the mixing and stability of nanofluids, making them highly relevant for microscale thermal systems, biomedical devices, and environmental applications. The Buongiorno nanofluid model is employed to describe nanoparticle transport driven by Brownian motion and thermophoresis. The energy and concentration equations are further modified using refined forms of Fourier's and Fick's laws to incorporate nonlinear thermal radiation, Joule heating, Newtonian heating, and a first-order chemical reaction. The resulting system of nonlinear partial differential equations is transformed into a set of ordinary differential equations using similarity transformations and solved numerically via the shooting method. The numerical results are validated through a comparison table with previously published data and show excellent agreement. Graphical key findings indicate that microorganism concentration decreases with increasing bioconvective Schmidt number, microorganism difference parameter, and Peclet number. This study presents a novel integration of multiple transport mechanisms and contributes to the design and optimization of nanofluid-based thermal systems.

Keywords: Bioconvection, Williamson nanofluid, Gyrotactic microorganisms, Modified Fourier's and Fick's laws, Numerical solutions.

Nomenclature			
a	constant	t	time (s)
$\bar{A}$	unsteady parameter	$\dot{T}$	temperature of the fluid (K)
e^{*}	chemotaxis constant	$\dot{T}_w$	temperature of the surface (K)
c	stretching constant (s^{-1})	$\dot{T}_\infty$	ambient fluid temperature (K)

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$\dot{C}$	concentration of the fluid (mol/L)	$\vec{U}_f$	velocity field
$\dot{C}_w$	concentration of the fluid at surface (mol/L)	u	component of velocity along axial $\dot{z}$-direction (m/s)
$\dot{C}_\infty$	ambient fluid concentration fluid (mol/L)	v	component of velocity along radial $\dot{r}$-direction (m/s)
C_f	non-dimensional coefficient of skin friction	$w\bar{c}$	speed of cells
D^*_B	concentration diffusion coefficient (m^2/s)	$w\bar{e}$	Weissenberg number
D^*_n	microorganism diffusion coefficient (m^2/s)	$\bar{X}$	dimensionless concentration of microorganisms
D^*_T	thermophoretic diffusion coefficient (m^2/s)	$\dot{z}$	distance along axial direction (m)
$\bar{E}c$	Eckert number	Greek letters	
f	non-dimensional velocity along the radial direction	α_γ	thermal diffusivity
f'	non-dimensional velocity along the axial direction	α^*	velocity slip parameter
g	gravity (m/s^2)	β^*	volume expansion coefficient
$H(t)$	time-dependent magnetic field	γ^*	average volume of microorganisms
H_0	strength of the magnetic field	$\tilde{\gamma}_1$	relaxation parameter for temperature
$K\tilde{p}$	chemical reaction parameter	$\tilde{\gamma}_2$	relaxation parameter for concentration
k_r^2	reaction rate constant	$\tilde{\gamma}_3$	Newtonian heating parameter
k^{**}	mean absorption coefficient (m^{-1})	Γ_β	Time material constant
l	slip length	$\bar{\lambda}$	dimensionless mixed convective parameter
M	dimensionless magnetic parameter	λ_1	dimensionless relaxation parameter for temperature
$\dot{N}$	Concentration of microorganisms (mol/L)	λ_2	non-dimensional relaxation parameter for concentration
$\dot{N}_w$	concentration of microorganisms at the surface wall (mol/L)	μ_0	dynamic viscosity (kg/ms)
$\dot{N}_\infty$	ambient concentration of microorganisms (mol/L)	ν_α	kinematic viscosity (kg/ms^2)

$\bar{Nb}$	Brownian motion parameter	$\bar{\phi}$	dimensionless fluid concentration
$\bar{Nc}$	buoyancy ratio parameter	$\bar{\rho}_{cp}$	heat capacitance (J/kgK)
$\bar{Nr}$	bio convection Rayleigh number	$\bar{\rho}_{f^*}$	density of the base fluid (kg/m^3)
$\bar{Nt}$	Thermophoresis parameter	$\bar{\rho}_{m^*}$	density of microorganisms (kg/m^3)
$Nu_{\dot{r}}$	dimensionless local Nusselt number	$\bar{\rho}_{p^*}$	density of nanoparticles (kg/m^3)
$\tilde{Pe}$	Peclet number	σ^*_β	Stefan-Boltzmann constant (W/m^2K^4)
$\tilde{Pr}$	Prandtl number	σ_α	electric conductivity of the base fluid (Sm^{-1})
q_r^*	Radiative heat flux (W/m^2)	σ^*	microorganism difference parameter
Rd	Radiation parameter	$\bar{\theta}$	dimensionless fluid temperature
$\dot{R}e$	Reynolds number	θ_w	dimensionless temperature ratio parameter
$\dot{r}$	distance along the radial direction (m)	τ_α	relation constant of the heat capability to the material ratio of the liquid
$\tilde{Sb}$	Bio-convective Lewis number	ξ	dimensionless similarity variable
$\tilde{Sc}$	Schmidt number		

1. Introduction

For sustainable, renewable fuel cell technologies and biological polymer synthesis [1], bioconvection has shown a significant amount of potential. Continuous advances in mathematical modeling, together with field and lab testing, are necessary for the better design of such systems. Microorganisms that are able to migrate are responsible for generating the bioconvection phenomenon in a specific direction. These specific types of self-moving, direction-oriented motile bacteria create various flow patterns inside the system. The instability of the system occurs as the motile microorganism moves and gathers at the upper surface. Furthermore, the introduction of gyrotactic bacteria in nonliquids provides the ability to enhance the rate of mass transfer and to improve the metabolic efficiency of numerous living structures such as enzymes, biological sensors [2], bacterial fuel cells, and bacterial-propelled micro-mixers. Wager [3] was the first one who introduce the concept of bioconvection. Platt [4] studied the movement of motile microorganisms in a bioconvective flow. Loganathan et al. [5] analyzed the effects of gyrotactic microorganisms in a bioconvective motion of third-grade nanofluid over a Riga surface with stratification. The study of entropy generation in thermo-solutal stratification of nanofluid with gyrotactic microorganism towards an axisymmetric surface is discussed by Sarma et al. [6]. Bioconvective and chemically reactive flow of nanofluid past a nonlinear stretchable permeable sheet with permeable medium was considered by Jat et al. [7]. Azam [8] studied the combined impacts of nonlinear thermal radiation and chemical reaction in a bioconvective flow of Sutterby nanofluid due to a gyrotactic microorganism. Heat transfer in Prandtl hybrid nanofluid with inclined magnetization and microbial movement via magneto bio convection is examined by Hussain et al. [9]. Loganathan et al. [10] evaluated entropy formation in the radiative flow of bioconvective Oldroyd-B nanofluid across an electromagnetic actuator with second-order slip.

In recent years, scientists have become curious about studying non-Newtonian fluids due to their expanding technical and scientific usages in various sectors. These sectors consist of food manufacturing, the chemical and petroleum industries (coatings, lubricants, drilling mud, grease oils), the field of polymers (melting plastic, manufacturing of polymer solutions), and biological sciences (blood, cartilage fluid, lettuce, sauce). One kind of time-dependent class of non-Newtonian fluid is the Williamson fluid. The popularity and extensive use of this fluid model, especially in determining the rheological behavior of biological and polymeric liquids, give it new life. For instance, it is frequently used in all of its transformed forms to plan and predict the motion of biological fluids, such as blood, polymeric liquids like Xan gum, and solutions of polyacrylamide gel [11]. Beg et al. [12] have performed a DTM simulation of MHD peristaltic flow of Williamson viscoelastic fluid. Imran et al. [13] have carried out an analytical investigation for the heat transference analysis in the flow of Williamson fluid due to a curved oscillating stretched surface. Azam et al. [14] have performed an examination for the entropy optimization with activation energy in a radiative heat transportation in Williamson nanofluid flow in an axisymmetric channel. Abbas et al. [15] parametric analysis and entropy optimization in bioinspired magnetized Williamson nanofluid by employing an artificial neural network. Kairi et al. [16] investigated the stratified thermos Marangoni bioconvective flow containing gyrotactic microorganism in a non-Newtonian Williamson nanofluid.

The study of nanofluid dynamics has gained a lot of attention in recent decades in the fields of improving heat transfer mechanisms and creating contemporary cooling technologies. Nanofluid is the term used to describe the mixture of nanometer or micrometer-sized molecules in basic liquids such as ethanol, water, propylene glycol (PG), and so on. Current research on nanofluid indicates that when nanoparticles are mixed with a base liquid, the base liquid's properties change as the heating capacity of the base liquid is lower than that of the nanoparticles. The primary concept of nanofluid was invented by Choi [17-18]. These nanoscale-sized molecules have standard chemical and physical properties. In a microchannel, they can move freely without being blocked. Nanofluid is widely used in many different industries, including nuclear reactions, thermal absorption electronics devices, boiling and heating processes of energy, and more. Two models can be used to study the transport characteristics of nanofluids. Tiwari and Das [19] have introduced one of the models, and another one is described by Buongiorno [20]. In this study, the Buongiorno model has taken into consideration to inquire about the different aspects of heat transfer in nanofluids. A uniform and dispersion-restricted nanofluid model has been provided by Buongiorno. He discussed the seven slide processes, where the base fluid and the microscopic particles generate parallel velocities. The Brownian diffusion, gravity's pull, resistance, water flow, thermal analysis, and the impact of Magnus are some of these processes. Amongst all, he points out that the two key aspects in nanofluids are the Brownian diffusion and thermal analysis. He proposed a non-homogeneous equilibrium model with two components and four equations that represented the momentum, heat, and mass exchange in nanofluids based on the facts of these properties. Numerous attempts to investigate nanofluids have been reported by numerous investigators in light of these qualities. Kuznetsova and Nailed [21] have carried out an analysis to examine the heat transportation in a nanofluid via a vertical wall. They have implemented the Buongiorno nano model to examine the characteristics of thermophoresis and Brownian motion. Sheikholeslami et al. [22] discussed the various attributes of thermophoresis and Brownian dispersions in a flow of nanofluid via a channel. Sheremet et al. [23] examined the radiative heat transport phenomenon in a nanofluid via two confined triangular cavities. Abbas et al. [24] have carried out an analysis to discuss the impact of radiation in a chemically reactive flow of Casson nanofluid via an oscillatory curved surface. Naveed [25] employed the entropy optimization technique to analyze the impacts of Joule heating in a chemically reactive Blasius movement of nanofluid on a curved surface. Significant data concerning the comprehensive analysis of nanofluids in various geometries are available in [26–35].

The conventional theory of mass and heat transport relies on the Fick law [36] of mass and the Fourier law [37] of heat flux, are fails to describe the irregular dispersion of heat as well as mass transfer. Later on, Cattaneo [38] modified the classical Fourier law of heat flux by incorporating a heat relaxation time in it. The incorporation of heat relaxation time differs for different materials. To overcome this issue, Christov [39] improved the heat flux model [38] by employing a time derivative model and termed it as Cattaneo-Christov heat flux law. Cattaneo-Christov temperature and mass theory has been integrated into the investigation for the flow of Oldroyd-B fluid through a spinning disk is discussed by Hafeez et al. [40]. Imran et al. [41] employed the analytical technique HAM to study the flow and transport behavior in an Eyring-Powell fluid on a curved oscillatory sheet with Cattaneo-Christov theory. Bilal et al. [42] have carried out an analysis to depict the different features of non-isothermal movement of Williamson's fluid along an exponentially expanding surface based on Cattaneo-Christov heat flux principle. Irreversibility analysis of bioconvective Walters' B nanoliquid flow across an electromagnetic actuator with the Cattaneo-Christov model was completed by Loganathan et al. [43]. In another article, Loganathan et al. [44] determined the implications of entropy generation in bioconvective flow on Maxwell nanofluid past a Riga plate with the Cattaneo-Christov model.

Two basic convection fluxes of the laws of thermodynamics are those that are dictated by the surface's temperature and heated surface flux. The first person who collaborated on the Newtonian method of heating was Sarwar [45], in which the amount of heat transmission is based on the areas of temperature difference. Newtonian heating finds uses in radiation from the sun, exchange mechanisms, bidirectional heat exchange within the fins, etc. Convection has been significantly impacted by the fluid-flowing, solid-walled tunnels. In many technical equipment, Newtonian heating is essential as an exchange of energy [46]. Some of the uses include the production of metallic sheets and films made of polymers across multiple industries, along with the chilling of an endless metallic surface, the production of paper, and the blowing of glass. The transmission of heat across the spreading area determines how much better the ultimate item is efficient. It is important to keep in mind as in numerous thermal reactions, the magnetic pull simultaneously influences the distribution of fluid and energy. If this is the case, the use of MHD involving heat transmission has practical applications in cooling components of nuclear power plants and electrical appliances, blood circulation, instruments (windmills and motors), and electronic devices (transistors and regulators) etc. Das et al. [47] studied the influence of Newtonian heating on time-dependent hydromagnetic Casson fluid with mass and heat transfer across a smooth surface. Hayat et al. [48] discussed the impact of Newtonian heating via a permeable container in a nanofluid. Convective flow of Maxwell fluid with Newtonian heating and extended heat transport has been considered by Zhang et al. [49].

The present study addresses a significant research gap by conducting a novel and comprehensive analysis of mass and heat transfer in a time-dependent bioconvective flow of Williamson nanofluid containing gyrotactic microorganisms across a radially stretching sheet, incorporating multiple advanced physical effects that have not been previously considered in combination. While prior investigations have examined individual aspects such as nanoparticle transport or bioconvection, none have simultaneously integrated the modified forms of Fourier's and Fick's laws under the Cattaneo–Christov framework to account for finite-speed heat and mass propagation, Joule heating, nonlinear thermal radiation, Newtonian heating, and a first-order chemical reaction within a non-Newtonian nanofluid microorganism system. The incorporation of the Buongiorno model further enhances physical realism by accurately representing Brownian motion and thermophoresis. This unique consideration over a radially stretching geometry provides novel insights into the optimization of nanofluid-based thermal systems and their applications in energy, biomedical, and environmental engineering. Nevertheless, the present analysis is limited to unsteady, laminar, and axisymmetric flow with constant thermophysical properties. Effects such as temperature-dependent viscosity, thermal conductivity, and diffusivity, as well as three-dimensional configurations, have not been considered. Future work incorporating these aspects would broaden the model's applicability and extend its relevance to more complex and practical engineering systems.

2. Description of Problem

A numerical analysis for an unsteady, laminar, axisymmetric, magnetized, and bioconvective flow of Williamson nanofluid with gyrotactic microorganisms on a radially stretching surface is discussed here. The heat and mass transmission equations comprising the effects of Joule heating, nonlinear thermal radiation, and a first-order chemical reaction are examined using improved forms of Fourier's and Fick's laws. Three significant realistic conditions, velocity partial slip, Newtonian heating, and zero nanoparticles, are also considered. The impact of Brownian and thermophoresis diffusions is examined by considering the Buongiorno nanofluid model. The surface is assumed to be stretched in a radial direction $V_w(\dot{r},t)(=a\dot{r}/1-ct)$ with a

and c are specified as constants with dimension $(time)^{-1}$. The surface makes contact with the plane at $(\dot{z}=0)$ and a flow takes place in the upper half plane at $(\dot{z}>0)$ (see Fig. 1). The mathematical development of the flow problem is carried out by emphasizing a cylindrical polar coordinate system. Let $H(t)=H_0/\sqrt{1-\beta^* t}$ be the strength of the magnetic field applied in $\dot{r}$ direction with H_0 is constant. Let $\dot{T}_w$ and $\dot{T}_\infty$ be uniform surface and ambient liquid temperature such that $(\dot{T}_w>\dot{T}_\infty)$.

The velocity, nanoparticle concentration, and temperature fields for two-dimensional axisymmetric flow are given as [14]

$$\vec{U}_f=\left[v(\dot{r},t,\dot{z}),0,u(\dot{r},t,\dot{z})\right],\dot{T}=\dot{T}(\dot{r},t,\dot{z}),\dot{C}=\dot{C}(\dot{r},t,\dot{z}). \tag{1}$$

Fig.1. Schematic flow geometry

The fluid model equations for temperature, energy, and nano concentration for the flow of Williamson fluid are modeled as [5, 14, 41]

$$\frac{\partial v}{\partial \dot{r}}+\frac{v}{\dot{r}}+\frac{\partial u}{\partial \dot{z}}=0, \tag{2}$$

$$\begin{aligned}\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial \dot{r}}+v\frac{\partial v}{\partial \dot{z}}&=\left[\sqrt{2}\nu_\alpha\Gamma_\beta\frac{\partial v}{\partial \dot{z}}+\nu_\alpha\right]\frac{\partial^2 v}{\partial \dot{z}^2}-\frac{\sigma_\alpha H^2(t)}{\bar{\rho}_{f^*}}v+\\ &\frac{1}{\bar{\rho}_{f^*}}\left[\left(1-\dot{C}_\infty\right)g\beta^*\left(\dot{T}-\dot{T}_\infty\right)-\left(\bar{\rho}_{p^*}-\bar{\rho}_{f^*}\right)\left(\dot{C}-\dot{C}_\infty\right)-g\gamma^*\left(\bar{\rho}_{m^*}-\bar{\rho}_{f^*}\right)\left(\dot{N}-\dot{N}_\infty\right)\right],\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial \dot{T}}{\partial t}+v\frac{\partial \dot{T}}{\partial \dot{r}}+u\frac{\partial \dot{T}}{\partial \dot{z}}&=\alpha^*\frac{\partial^2 \dot{T}}{\partial \dot{z}^2}+\tau_\alpha\left[D^*_B\frac{\partial \dot{C}}{\partial \dot{z}}\frac{\partial \dot{T}}{\partial \dot{z}}+\frac{D^*_{\dot{T}}}{\dot{T}_\infty}\left(\frac{\partial \dot{T}}{\partial \dot{z}}\right)^2\right]\\ &-\lambda_1\phi_{\dot{T}}-\frac{1}{\bar{\rho}_{cp}}\left(\frac{\partial q_{\dot{r}}^*}{\partial \dot{z}}\right)\frac{\partial \dot{T}}{\partial \dot{z}}+\frac{\sigma_\alpha H^2(t)v^2}{\bar{\rho}_{cp}},\end{aligned} \tag{4}$$

$$\frac{\partial \dot{C}}{\partial t}+v\frac{\partial \dot{C}}{\partial \dot{r}}+u\frac{\partial \dot{C}}{\partial \dot{z}}=\frac{\partial^2 \dot{C}}{\partial \dot{z}^2}D^*_B+\frac{\partial^2 \dot{T}}{\partial \dot{z}^2}\frac{D^*_{\dot{T}}}{\dot{T}_\infty}-\lambda_2\varphi_m-k^2_{\dot{r}}\left(\dot{C}-\dot{C}_\infty\right), \tag{5}$$

$$\frac{\partial \dot{N}}{\partial t}+v\frac{\partial \dot{N}}{\partial \dot{r}}+u\frac{\partial \dot{N}}{\partial \dot{z}}+\dot{N}\frac{\partial^2 \dot{C}}{\partial \dot{z}^2}.\frac{e^* w\bar{c}}{\dot{C}_\infty}+\left(\frac{\partial \dot{N}}{\partial \dot{z}}.\frac{\partial \dot{C}}{\partial \dot{z}}\right)\frac{e^* w\bar{c}}{\dot{C}_\infty}=D^*_n\frac{\partial^2 \dot{N}}{\partial z^2}. \tag{6}$$

Where,

$$\phi_{\dot{T}} = \begin{bmatrix} \left(v\frac{\partial u}{\partial \dot{r}} + u\frac{\partial u}{\partial \dot{z}}\right)\frac{\partial \dot{T}}{\partial \dot{z}} + v^2\frac{\partial^2 \dot{T}}{\partial \dot{r}^2} + 2vu\frac{\partial^2 \dot{T}}{\partial \dot{r}\partial \dot{z}} + u\frac{\partial^2 \dot{T}}{\partial \dot{z}\partial t} + v\frac{\partial^2 \dot{T}}{\partial \dot{r}\partial t} \\ \frac{\partial^2 \dot{T}}{\partial t^2} + \frac{\partial v}{\partial t}.\frac{\partial \dot{T}}{\partial \dot{r}} + \frac{\partial u}{\partial t}.\frac{\partial \dot{T}}{\partial \dot{z}} + \left(v\frac{\partial v}{\partial \dot{r}} + u\frac{\partial v}{\partial \dot{z}}\right)\frac{\partial \dot{T}}{\partial \dot{r}} + u^2\frac{\partial^2 \dot{T}}{\partial \dot{z}^2} \end{bmatrix}$$

and

$$\varphi_m = \begin{bmatrix} \frac{\partial^2 \dot{C}}{\partial t^2} + \left(v\frac{\partial u}{\partial \dot{r}} + u\frac{\partial u}{\partial \dot{z}}\right)\frac{\partial \dot{C}}{\partial \dot{z}} + v^2\frac{\partial^2 \dot{C}}{\partial \dot{r}^2} + 2vu\frac{\partial^2 \dot{C}}{\partial \dot{r}\partial \dot{z}} + u\frac{\partial^2 \dot{C}}{\partial \dot{z}\partial t} + \\ v\frac{\partial^2 \dot{C}}{\partial \dot{r}\partial t} + \frac{\partial v}{\partial t}.\frac{\partial \dot{C}}{\partial \dot{r}} + \frac{\partial u}{\partial t}.\frac{\partial \dot{C}}{\partial \dot{z}} + u^2\frac{\partial^2 \dot{C}}{\partial \dot{z}^2} + \left(v\frac{\partial v}{\partial \dot{r}} + u\frac{\partial v}{\partial \dot{z}}\right)\frac{\partial \dot{C}}{\partial \dot{r}} \end{bmatrix},$$

where (Γ_β) material time derivative, $\left(\overline{\rho}_{f^*}\right)$density of base fluid and (v,u) are constituents of velocity in $(\dot{r},\dot{z})$ direction, and v_α is kinematics viscosity. Here α_γ is thermal diffusivity, $D_{\dot{T}}^*$ thermophoresis diffusion coefficient, σ_α the electric conductivity, τ_α the ratio of effective heat capacity of the base fluid, $\left(D_B^*\right)$ Brownian diffusion coefficient, $q_{\dot{r}}^*$ nonlinear thermal radiation, λ_1 and λ_2 are the relaxation times of heat and mass flux, $k_{\dot{r}}$ chemical reaction coefficient, t time, $\dot{C}_\infty$ ambient liquid concentration, D_n^* diffusivity of the microorganism and $\left(e^* w\overline{c}/\dot{C}_\infty\right)$.

The boundary condition for the considered flow problem is [10, 14, 48]

$$\left.\begin{aligned} & u = 0,\ v = v_w(\dot{r},t) + l\left[\frac{\partial v}{\partial \dot{z}} + \frac{\Gamma_\beta}{2}\left(\frac{\partial v}{\partial \dot{z}}\right)^2\right], \\ & \dot{T} = \dot{T}_w, \dot{\mathrm{N}} = \dot{\mathrm{N}}_w, D_B^*\frac{\partial \dot{C}}{\partial \dot{z}} + \frac{D^*_{\dot{T}}}{\dot{T}_\infty}\frac{\partial \dot{T}}{\partial \dot{z}} = 0, \end{aligned}\right\} at \quad \dot{z} = 0. \tag{7}$$

$$v \to 0, \dot{C} \to \dot{C}_\infty, \dot{T} \to \dot{T}_\infty, \dot{N} \to \dot{N}_\infty, \dot{\mathrm{z}} \to \infty.$$

The radiative heat flux is defined as

$$q_{\dot{r}}^* = -\frac{16\sigma_\beta^*}{3k^{**}}\dot{T}^3\frac{\partial \dot{T}}{\partial \dot{z}}, \tag{8}$$

$$\dot{T} = \left(1 + \overline{\theta}\left(\overline{\theta}_w - 1\right)\right)\dot{T}_\infty, \overline{\theta}_w = \left(\frac{\dot{T}_w}{\dot{T}_\infty} \geq 1\right)$$

The energy equation with nonlinear thermal radiation can be given as

$$\frac{\partial \dot{T}}{\partial t} + v\frac{\partial \dot{T}}{\partial \dot{r}} + u\frac{\partial \dot{T}}{\partial \dot{z}} = +\tau_\alpha\left[D^*{}_B \frac{\partial \dot{C}}{\partial \dot{z}}\frac{\partial \dot{T}}{\partial \dot{z}} + \frac{D^*{}_{\dot{T}}}{\dot{T}_\infty}\left(\frac{\partial \dot{T}}{\partial \dot{z}}\right)^2\right]$$
$$-\lambda_1\phi_{\dot{T}} - \frac{\partial}{\partial \dot{z}}\left[\alpha_\gamma + \frac{16\sigma^*_\beta}{3k^{**}}\frac{\dot{T}^3}{\bar{\rho} c_p}\right]\frac{\partial \dot{T}}{\partial \dot{z}} + \frac{\sigma_\alpha H^2(t) v^2}{\bar{\rho}_{cp}}. \tag{9}$$

For simplification of the flow problem, defining the similarity variables as [14]

$$\xi = \frac{\dot{z}}{\dot{r}}\dot{\mathrm{R}}\mathrm{e}^{1/2},\ u = -2v_w f(\xi)\Big/\sqrt{\dot{\mathrm{R}}\mathrm{e}},\ \mathrm{v} = v_w f'(\xi),$$
$$\bar{\phi}(\xi) = \frac{\dot{C} - \dot{C}_\infty}{\dot{C}_w - \dot{C}_\infty},\ \bar{\theta}(\xi) = \frac{\dot{T} - \dot{T}_\infty}{\dot{T}_w - \dot{T}_\infty}, \bar{X}(\xi) = \frac{\dot{N} - \dot{N}_\infty}{\dot{N}_w - \dot{N}_\infty}. \tag{10}$$

Here: $\dot{\mathrm{R}}\mathrm{e} = \left(\dot{r}v_w/v_\alpha\right)$ indicate the Reynolds number.

The modified form of the above nonlinear PDEs using a transformation is

$$\left(1 + w\bar{e}\bar{f}''\right)\bar{f}''' - \bar{A}\left(\bar{f}' + \frac{\xi}{2}\bar{f}''\right) + 2\bar{f}\,\bar{f}'' - \bar{f}'^2 - \bar{M}\bar{f}' + \bar{\lambda}\left(\bar{\theta} - \bar{\phi}N\bar{r} - \bar{X}\,N\bar{c}\right) = 0, \tag{11}$$

$$\bar{\theta}'' + R\bar{d}\left(\left(1 + \left(\bar{\theta}_w - 1\right)\bar{\theta}\right)^3\bar{\theta}'' + 3\left(1 + \left(\bar{\theta}_w - 1\right)\bar{\theta}\right)^2\left(\bar{\theta}_w - 1\right)\bar{\theta}'^2\right.$$
$$+ \mathrm{P\tilde{r}}\,\tilde{\gamma}_1\left(\frac{\bar{A}^2\xi^2}{4} - \bar{A}\bar{f}\xi + 4\bar{f}^2\right)\bar{\theta}'' + \mathrm{P\tilde{r}}\left(N\bar{b}\bar{\theta}'\bar{\phi}' + N\bar{t}\bar{\theta}'^2 + 2\bar{f}\bar{\theta}' - \frac{\bar{A}}{2}\xi\bar{\theta}'\right)$$
$$-\tilde{\gamma}_1\mathrm{P\tilde{r}}\left(\frac{3}{4}\bar{A}^2\xi + 4\bar{f}\,\bar{f}' - \bar{A}\xi\bar{f}' - 2\bar{A}\bar{f}\right)\bar{\theta}' + \mathrm{P\tilde{r}}\,\bar{M}\overline{Ec}\bar{f}'^2 = 0, \tag{12}$$

$$\bar{\phi}'' + \frac{N\bar{t}}{N\bar{b}}\bar{\theta}'' + S\tilde{c}\tilde{\gamma}_2\left(4\bar{f}^2 - \frac{\bar{A}^2\xi^2}{2} - \bar{A}\bar{f}\xi\right)\bar{\phi}'' + S\tilde{c}\left(2\bar{f}\bar{\phi}' - \frac{\bar{A}}{2}\xi\bar{\phi}'\right)$$
$$-S\tilde{c}\tilde{\gamma}_2\left(\frac{3}{2}\xi\bar{A}^2 + 4\bar{f}\,\bar{f}' - \bar{A}\bar{f}'\xi - 2\bar{A}\bar{f}\right)\bar{\phi}' - K\tilde{p}\,\bar{\phi} = 0, \tag{13}$$

$$\bar{X}'' + 2S\tilde{b}\bar{f}\,\bar{X}' - P\tilde{e}\bar{X}\bar{\phi}'' - \sigma^* P\tilde{e}\bar{X}'\bar{\phi}' - S\tilde{b}\xi\bar{X}'\frac{\bar{A}}{2} = 0. \tag{14}$$

The transformed BC’s are:

$$\bar{f}(0)=0,\ \bar{f}'(0)=1+\alpha^*\left(\bar{f}''(0)+\frac{w\bar{e}}{2}\bar{f}''(0)^2\right),$$
$$\bar{\phi}(0)=1,\ \bar{\theta}'(0)=-\tilde{\gamma}_3\left(1+\bar{\theta}(0)\right),\ \bar{X}(0)=1, \tag{15}$$
$$\bar{f}'(\infty)\to 0,\ \bar{X}(\infty)\to 0,\ \bar{\theta}(\infty)\to 0,\ \bar{\phi}(\infty)\to 0.$$

The dimensionless parameters appearing in the above equations are $\bar{A}$ unsteady parameter, $\bar{M}$ magnetic parameter, $\bar{\lambda}$ mixed convective parameter, $w\bar{e}$ Weisenberg parameter, α^* velocity slip parameter, $N\bar{r}$ bio convection Rayleigh number, $N\bar{c}$ buoyancy ratio parameter, $R\bar{d}$ nonlinear thermal radiation, $\mathrm{P}\tilde{r}$ Prandtl number, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are non- dimensional thermal and mass relaxation parameters, $S\tilde{c}$ Schmidt number, $N\bar{t}$ the thermophoresis constant, $N\bar{b}$ Brownian motion variable, $\bar{M}$ magnetic field, $\tilde{\gamma}_3$ Newtonian heating, $\bar{E}c$ the Eckert number, $\bar{\theta}_w$ temperature ratio parameter, $K\tilde{p}$ chemical reaction constant, $S\tilde{b}$ bioconvection Schmidt number, σ^* microorganism difference parameter and $P\tilde{e}$ be the Peclet number, which is defined as

$$\bar{A}=\frac{c}{a},\ \bar{M}=\frac{\sigma_\alpha H_0^2}{\bar{\rho}a},\ \bar{\lambda}=\mathrm{g}\beta^*\frac{\left(1-\dot{C}_\infty\right)\left(\dot{T}_w-\dot{T}_\infty\right)}{aX^2},w\bar{e}=\frac{\sqrt{2\dot{\mathrm{R}}\mathrm{e}}\Gamma_\alpha}{(1-ct)},$$
$$\mathrm{N\bar{r}}=\frac{\left(\bar{\rho}_{p^*}-\bar{\rho}_{f^*}\right)\left(\dot{C}_w-\dot{C}_{\infty^*}\right)}{\bar{\rho}_{f^*}\left(1-\dot{C}_\infty\right)\left(\dot{T}_w-\dot{T}_{\infty^*}\right)},\mathrm{N\bar{c}}=\gamma^*\frac{\left(\bar{\rho}_{m^*}-\bar{\rho}_{f^*}\right)\left(\dot{N}_w-\dot{N}_{\infty^*}\right)}{\bar{\rho}_{f^*}\beta^*\left(1-\dot{C}_{\infty^*}\right)\left(\dot{T}_w-\dot{T}_{\infty^*}\right)},$$
$$N\bar{b}=\frac{\tau_\alpha D^*{}_B\left(\dot{C}_w-\dot{C}_\infty\right)}{\nu_\alpha},\ \bar{A}=\frac{c}{a},\ \bar{M}=\frac{\sigma_\alpha H_0^2}{\bar{\rho}a},\ S\tilde{c}=\frac{\nu_\alpha}{D^*_B},$$
$$\tilde{\gamma}_1=\frac{\bar{\lambda}_1 a}{1-ct},\tilde{\gamma}_2=\frac{\bar{\lambda}_2 a}{1-ct},R\bar{d}=\frac{16\sigma^*_\beta T^3{}_{\infty^*}}{3\bar{k}k^{**}},N\bar{t}=\frac{\tau_\alpha D^*{}_T\left(\dot{T}_w-\dot{T}_\infty\right)}{\nu_\alpha\dot{T}_\infty},\ \mathrm{P\tilde{r}}=\frac{\nu_\alpha}{\alpha_\gamma},$$
$$\bar{E}c=\frac{v_w^2}{C_p\Delta T},K\tilde{p}=\frac{k_r^2(1-ct)}{a},S\tilde{b}=\frac{\nu_\alpha}{D^*n},P\tilde{e}=\frac{e^*wc}{D^*n},\sigma^*=\frac{\dot{N}_w-\dot{N}_\infty}{\dot{N}_{\infty^*}}.$$

The noteworthy features of the flow mechanism are the surface drag force $\dot{C}_f$, skin fraction and local Nusselt number, which are given as

$$\dot{C}_f=\left.\frac{\tau_{\alpha w}}{\bar{\rho}V_w{}^2}\right|_{\dot{z}=0}. \tag{16}$$

where

$$\tau_{\alpha w}=\bar{\mu}_\circ\left.\left(\frac{\partial v}{\partial\dot{z}}+\frac{\Gamma_\beta}{\sqrt{2}}\left(\frac{\partial v}{\partial\dot{z}}\right)^2\right)\right|_{\dot{z}=0}. \tag{17}$$

with $\bar{\mu}_{\bullet}$ is zero shear viscosity, with the help of the similarity transformation defined in equation (10)

$$\dot{\mathrm{Re}}^{0.5}\,\dot{C}_f=\left[\frac{w\bar{e}}{2}\left(\bar{f}''(0)\right)^2+\bar{f}''(0)\right] \tag{18}$$

Nusselt number for the heat transportation equation is

$$\begin{aligned}&\bar{N}u_{\dot{r}}=\frac{\dot{r}q_{\alpha w}\big|_{\dot{z}=0}}{\bar{k}\left(\dot{T}_w-\dot{T}_\infty\right)},\\&\dot{\mathrm{Re}}^{0.5}\,\bar{N}u_{\dot{r}}=-\bar{\theta}'(0)\left(1+R\bar{d}\left(\left(1+\left(\bar{\theta}_w-1\right)\bar{\theta}(0)\right)^3\right)\right),\end{aligned} \tag{19}$$

3. Numerical technique

An efficient numerical scheme, namely the shooting method combined with a fourth-order Runge-Kutta integration technique and the Newton-Raphson method for adjusting initial guesses and satisfying boundary conditions, is employed to investigate the heat and mass transfer characteristics in a bioconvective flow of Williamson fluid containing gyrotactic microorganisms over a radially stretching sheet. To execute the above-stated technique, the accomplished nonlinear ordinary differential equations (11-14) associated with the boundary conditions given in equation (15) are transformed into a system of first-order differential equations by initiating appropriate substitutions such as

$$f'=u,\quad f''=v,\ \theta'=w,\ \phi'=y,\ X'=z. \tag{20}$$

By employing Eq. (20), the nonlinear ODEs (11), (12), (13), (14), and boundary conditions Eq. (15) are expressed as

$$\left(1+w\bar{e}\,v\right)v'=\bar{A}\left(u+\frac{\xi}{2}v\right)-2\bar{f}\,v+u^2+\bar{M}\,u-\lambda\left(\bar{\theta}-\bar{\phi}N\bar{r}-\bar{X}\,N\bar{c}\right), \tag{21}$$

$$\begin{aligned}&\left(\left(1+R\bar{d}\left(1+\left(\bar{\theta}_w-1\right)\bar{\theta}\right)^3\right)+\mathrm{P\tilde{r}}\,\tilde{\gamma}_1\left(4\bar{f}^2+\frac{\bar{A}^2\xi^2}{4}-\bar{A}\bar{f}\xi\right)\right)w'=-3\left(\bar{\theta}_w-1\right)R\bar{d}\left(1+\left(\bar{\theta}_w-1\right)\bar{\theta}\right)^2w^2\\&-\mathrm{P\tilde{r}}\left(N\bar{b}w\,y+N\bar{t}\,w^2+2\bar{f}\,w-\frac{\bar{A}}{2}\xi w\right)+\tilde{\gamma}_1\,\mathrm{P\tilde{r}}\left(\frac{3}{4}\bar{A}^2\xi+4\bar{f}\,u-\bar{A}\xi u-2\bar{A}\bar{f}\right)w-\mathrm{P\tilde{r}}\,\bar{M}\bar{E}c\,u^2,\end{aligned} \tag{22}$$

$$\begin{aligned}&\left(1+S\tilde{c}\tilde{\gamma}_2\left(4\bar{f}^2-\frac{\bar{A}^2\xi^2}{2}-\bar{A}\bar{f}\,\xi\right)\right)y'=\frac{N\bar{t}}{N\bar{b}}w'-S\tilde{c}\left(2\bar{f}\,y-\frac{\bar{A}}{2}\xi y\right)\\&+S\tilde{c}\tilde{\gamma}_2\left(\frac{3}{2}\xi\bar{A}^2+4\bar{f}\,u-\bar{A}u\xi-2\bar{A}\bar{f}\right)y+K\tilde{p}\bar{\phi},\end{aligned} \tag{23}$$

$$\left(1-P\tilde{e}\bar{X}\right)z' = -2S\tilde{b}\;\bar{f}\;z + \sigma^{*}P\tilde{e}\,z\,y - \frac{\bar{A}}{2}\xi S\tilde{b}\,z, \tag{24}$$

The above resulting system is then treated as an initial value problem and solved using the Runge-Kutta method, while the Newton-Raphson algorithm is used iteratively to refine the initial guesses until the boundary conditions are satisfied within a prescribed tolerance.

This combined shooting–Runge–Kutta–Newton approach is particularly advantageous because it is straightforward to implement, computationally efficient for smooth low-dimensional boundary value problems, and requires minimal memory storage. Moreover, the fourth-order Runge–Kutta integration ensures high accuracy, while the method can handle nonlinearities directly without linearization and effectively accommodate boundary conditions at infinity by truncating the computational domain to a sufficiently large finite value and adjusting the initial guesses accordingly.

4. Validation of Numerical findings

To verify the accuracy and validity of the obtained numerical results, two comparison tables are provided. Table 1 presents a numerical comparison of the calculated surface drag force with the published results of Azam et al. [14], demonstrating good agreement. Similarly, Table 2 compares the present results for the heat transfer rate with the existing data reported by Azam et al. [14], showing excellent agreement.

5. Discussions of Numerical findings

This portion introduces the graphical and tabular explanation of the involved physical parameters, such as the magnetic parameter $\left(=\bar{M}\right)$, bioconvective parameter $\left(=\bar{\lambda}\right)$, velocity slip parameter $\left(=\alpha^{*}\right)$, local Weissenberg number $\left(=w\bar{e}\right)$, unsteady parameter $\left(=\bar{A}\right)$, Brownian motion parameter $\left(=N\bar{b}\right)$, Eckert number $\left(=\bar{E}c\right)$, Prandtl number $\left(=P\tilde{r}\right)$, Schmidth number $\left(=S\tilde{c}\right)$, temperature ratio parameter $\left(=\bar{\theta}_{w}\right)$, thermal radiation $\left(=R\bar{d}\right)$, thermophoresis parameter $\left(=N\bar{t}\right)$, chemical reaction parameter $\left(=K\tilde{p}\right)$, Newtonian heating parameter $\left(=\tilde{\gamma}_{3}\right)$, bioconvection Schmidt number $\left(=S\tilde{b}\right)$, microorganism difference parameter $\left(=\sigma^{*}\right)$ and Peclet number $\left(=P\tilde{e}\right)$ by solving Eqs. (11-14) with boundary conditions (15). For numerical computation, the values of all the parameters are considered as $N\bar{c}=\alpha^{*}=\bar{A}=\bar{M}=\bar{E}c=N\bar{b}=N\bar{t}=\tilde{\gamma}_{1}=\tilde{\gamma}_{2}==0.1,\ \bar{\theta}_{w}=1.2, R\bar{d}=0.3, \mathrm{P\tilde{r}}=1.5,\quad w\bar{e}=\bar{\lambda}=N\bar{r}=\tilde{\gamma}_{3}=0.2$.

The convergence of the numerical findings is provided in Table 3. A grid independence study is performed to ensure the numerical results are free from discretization error. The transformed ODE system Eqs. (11)-(14) is solved using the shooting method coupled with a fourth-order Runge–Kutta integrator for increasing grid resolutions in the similarity variable ξ. Table 3 lists the computed skin friction coefficient $\mathrm{R\dot{e}}^{0.5}\,\dot{C}_{f}$ and local Nusselt number $\mathrm{\dot{R}e}^{0.5}\,\bar{N}u_{\dot{r}}$ for

$(N = 101, 151, 201, 251, 301)$. The relative changes in both measures fall below 0.01% for $(N \geq 201)$, consequently, $(N = 201)$ was chosen for the remainder of the computations as it provides a suitable balance between accuracy and computational cost. Table 4 shows that the magnitude of drag surface force is elevated for escalating values of bioconvective parameter$(=\bar{\lambda})$, bio convection Rayleigh number $(= N\bar{r})$ and buoyancy ratio parameter$(= N\bar{c})$. However, it is reduced for higher values of the Weissenberg number $(= w\bar{e})$. Table 5 is made to see the variations in the Nusselt number for different involved parameters. It is noticed that the Nusselt number also increases for appreciable values of the Eckert number $(= \bar{E}c)$, magnetic parameter $(= \bar{M})$, thermophoresis parameter $(= N\bar{t})$, temperature ratio parameter $(= \bar{\theta}_w)$ and unsteady parameter $(= \bar{A})$.

The impact of the magnetic parameter $(\bar{M} = 0.1, 0.4, 0.7, 1.0)$ and the two values of the velocity slip parameter $(\alpha^* = 0.2, 0.7)$ on the horizontal component of velocity $\bar{f}'(\xi)$ is shown in Fig. 2. This figure indicates that the fluid's velocity decreases with an increase in magnetic and velocity slip parameters. The reason for this observation is that, magnetic field is acting as a retarding force on the fluid elements. Lorentz force, which is generated by the interaction of the magnetic field with the fluid molecules is the reason of the decrease in velocity with an increase in the magnetic parameter. Furthermore, the slip condition reduces the shear contact between the fluid and the boundary, the fluid's velocity decreases as the slip velocity parameter increases. As the slip parameter increases, the fluid experiences less resistance along the stretched sheet, which decreases the momentum transfer to the fluid. Therefore, the horizontal velocity within the boundary layer reduces. This phenomenon is relevant to micro and nano lubrication, polymer extrusion, and microchannel flows. It becomes physically more pronounced in micro and nano systems with weaker surface contact. The effects of the bioconvection Rayleigh number $(N\bar{r} = 0.1, 1.8, 3.0)$ and two values of the buoyancy ratio parameter $(N\bar{c} = 0.05, 0.8)$ on the fluid velocity is shown in Fig. 3. It is noticeable from the figure that when the buoyancy ratio parameter increases, the fluid velocity declines, and when the parametric values of bioconvection Rayleigh number rise, the velocity of the fluid increases. Understanding the impact of the bioconvection Rayleigh number and the buoyancy ratio parameter on fluid velocity is essential in the context of flow generated by microorganisms. Fluid velocity decreases as the buoyancy ratio parameter increases because it represents the relative strength of solutal (nanoparticle) buoyancy in relation to thermal buoyancy. Increased buoyancy ratio favors inhibition of fluid flow, thereby decreasing velocity and being dominated by the influence of heavier nanoparticles. On the other hand, the bioconvection Rayleigh number is associated with the amount of active gyrotactic microorganisms in the fluid. Increasing this parameter tends to enhance the microorganism's buoyant force, which increases the fluid's upward flow. Increased buoyant forces improve fluid velocity. Such phenomena are pertinent in biological systems, wastewater treatment, and bioreactors, where convection driven by microorganisms is of paramount importance.

Fig. 4 depicts the impact of the magnetic parameter $\left(\bar{M} = 0.25, 0.6, 1.1\right)$ and two values of Eckert number $\left(E\bar{c} = 0.25, 0.4\right)$ on the temperature of the liquid $\bar{\theta}(\xi)$. The increase in both parameters causes the temperature to rise. This is because when a magnetic field is applied to an electrically conducting fluid, it generates electric current and produces heat which leads to an increase in the temperature field $\bar{\theta}(\xi)$, on the other hand, the Eckert number is the ratio of kinetic energy to the thermal energy of the fluid, by increasing the Eckert number causes the fluid to convert its kinetic energy into thermal energy more quickly which enhancing the temperature of the fluid. Fig.5 shows the influence of thermophoretic diffusion $\left(N\bar{t} = 0.1, 0.4, 0.7\right)$ and two values of Brownian diffusion $\left(N\bar{b} = 0.05, 0.3\right)$ on the temperature of the fluid $\bar{\theta}(\xi)$. From this figure, it is observed that the temperature of the fluid increases with an increase in $\left(= N\bar{t}\right)$, whereas it decreases with an increase in the $\left(= N\bar{b}\right)$. These behaviors can be explained by the underlying physical parameters associated with Brownian and thermophoretic diffusions. Higher values of the thermophoresis parameter represent a stronger thermophoretic effect, which leads to an increase in the movement of the particles towards regions of higher temperature. This addition of particles in the hotter regions results in an elevation of the fluid temperature. Also, Brownian diffusion is the random motion of particles due to thermal energy. By increasing the Brownian parameter leads to an increase in dispersion and mixing of particles in the fluid. This increased dispersion consequences in a more efficient transfer of thermal energy and a consequent decrease in the fluid temperature. The effects of slip parameter $\left(\alpha^* = 0.1, 0.5, 1.0\right)$ and unsteady parameter $\left(\bar{A} = 0.1, 0.2\right)$ on $\bar{\theta}(\xi)$ is shown in Fig. 6. The temperature of the fluid rises with an increase in both the parameters. The unsteady parameter characterizes the degree of unsteadiness in the flow. An increase in an unsteady parameter leads to enhanced mixing and transport of heat within the fluid, resulting in an increase in temperature. Moreover, with an increase in the velocity slip parameter, the frictional forces between the fluid and the stretching surface decrease, which allows the fluid to move more easily along the boundary surface. The melting process introduces an increase in friction, which causes the thermal boundary layer to thin. This phenomenon causes an inadequate amount of heat to be carried away from the surface by the flowing fluid, causing fluid to collect thermal energy in close proximity to the wall and causing the fluid temperature to rise. Fig. 7 depicts the influence of the temperature ratio parameter $\left(\bar{\theta}_w = 1.3, 1.4, 1.5\right)$ and the radiation parameter $\left(R\bar{d} = 0.65, 0.70\right)$. From this figure, it is clear that, temperature of the fluid rises with an increase in $\left(\bar{\theta}_w\right)$ and $\left(R\bar{d}\right)$. This is due to the fact that, temperature ratio parameter signifies the ratio of the fluid temperature to the surface temperature. By increasing this parameter, the temperature of the fluid becomes higher relative to the surface temperature. Due to the difference in temperature, heat transfer occurs from the fluid to the surface, resulting in an enhancement in fluid temperature. Furthermore, with an increase in thermal radiation parameter, there is an increased transfer of heat energy between the fluid and its surroundings through radiation. This additional heat input increases the temperature of the fluid. Fig. 8 shows the effects of thermal relaxation parameter $\left(\tilde{\gamma}_1 = 0.1, 0.25, 0.4\right)$ and Prandtl number $\left(\mathrm{P\tilde{r}} = 2.0, 3.5\right)$ on $\bar{\theta}(\xi)$. It is observed that the temperature of the fluid decreases with an increase in $\tilde{\gamma}_1$ and also for higher values of $\tilde{\mathrm{Pr}}$. The Prandtl number characterizes the ratio of momentum diffusivity to thermal diffusivity in a fluid. With the higher values in $\tilde{\mathrm{Pr}}$ thermal diffusion becomes less significant as compared to momentum diffusion, which results in a decrease in temperature. Also, as the thermal relaxation parameter rises, the heat flux reacts more sluggishly to temperature gradients because of delayed conduction. This diminishes the heat transfer rate within the fluid, resulting in reduced thermal energy absorption. Consequently, the temperature of the fluid declines. The impact of Newtonian heating $\left(\tilde{\gamma}_3 = 0.05, 0.10, 0.15, 0.20\right)$ on the temperature of a fluid $\bar{\theta}\left(\xi\right)$ is depicted in Fig. 9. It is observed that the temperature of the fluid increases with an increase in $\left(\tilde{\gamma}_3\right)$. In Newtonian heating, the heat flux at the surface is directly proportional to the temperature difference between the surface and the fluid. As this heating effect intensifies, a greater amount of thermal energy is transferred from the surface to the fluid. Consequently, this results in an increase in the fluid temperature adjacent to the wall and throughout the boundary layer.

The impacts of the velocity slip parameter $\left(\alpha^* = 0.1, 0.6, 1.0\right)$ and unsteady parameter $\left(\bar{A} = 0.06, 0.25\right)$ on the concentration profile $\bar{\phi}\left(\xi\right)$ is shown in Fig. 10. It is noticed that $\bar{\phi}\left(\xi\right)$ increases with an increase in both parameters. The physical reasoning behind this behavior is that as the velocity slip parameter increases, the diminished shear at the wall hampers mass transfer, resulting in a greater accumulation of nanoparticles near the surface and consequently elevating concentration. In a similar vein, a heightened unsteady parameter indicates more pronounced time-dependent effects that disrupt the flow and impede diffusion, thereby contributing to an increase in nanoparticle concentration. The influence of the thermophoresis parameter $\left(N\bar{t} = 0.05, 0.1, 0.15\right)$ and Brownian diffusion on $\left(N\bar{b} = 0.02, 0.1\right)$is depicted in Fig. 11. It is observed that $\bar{\phi}\left(\xi\right)$ decreases with an increase in both the diffusion parameters. An increase in $N\bar{t}$ implies a stronger thermophoretic force acting on the fluid molecule. This force drives the fluid molecules to the regions of lower temperature, which results in a decrease in concentration of the fluid in those regions. Furthermore, for higher values in $\left(N\bar{b}\right)$ leads to a decrease in the concentration of the fluid. Brownian diffusion refers to the erratic movement of nanoparticles within a fluid. As the rate of Brownian diffusion escalates, particles disperse more swiftly from areas of high concentration to those of low concentration. This intensified diffusion diminishes the local concentration of nanoparticles adjacent to the surface, resulting in a general reduction in the concentration of the fluid. Fig. 12 shows the effects of the Schmidt number $\left(S\tilde{c} = 1.5, 2.5, 3.5\right)$ and Newtonian heating $\left(\tilde{\gamma}_3 = 0.05, 0.2\right)$ on $\bar{\phi}\left(\xi\right)$. It is evident from the figure that $\bar{\phi}\left(\xi\right)$decreases with an increase in $\left(S\tilde{c}\right)$, but it increases with an increase $\left(\tilde{\gamma}_3\right)$. When the parametric values of $\left(S\tilde{c}\right)$increases, it shows that the momentum diffusion becomes more dominant compared to mass diffusion, resulting in a decrease in $\bar{\phi}\left(\xi\right)$. Moreover, an increase in $\left(\tilde{\gamma}_3\right)$ implies a higher rate of heat generation within the fluid. This additional heat leads to an increase in the temperature of the fluid, which increases the solubility of substances in the fluid, resulting in an increase in $\bar{\phi}(\xi)$. The effect of a chemical reaction $(K\tilde{p}=0.05, 1.0, 2.5)$ and mass relaxation parameters $(\tilde{\gamma}_2=0.1, 0.8)$ on $\bar{\phi}(\xi)$ is represented in Fig. 13. It is observed that $\bar{\phi}(\xi)$ decreases with an increase in, whilst it increases for higher values of $(\tilde{\gamma}_2)$. When $K\tilde{p}$ increases, the reaction rate becomes significant, which results in a decrease in $\bar{\phi}(\xi)$. Moreover, a rise in the mass relaxation parameter delays the reaction of mass diffusion to concentration gradients, which delays the transport of solutes away from the surface. This reduced rate of diffusion facilitates a higher accumulation of solutes in the boundary layer, consequently enhancing the concentration profile.

The impacts of the bioconvection Schmidt number $(S\tilde{b}=1.5, 3.0, 5.0)$ and the microorganism difference parameter $(\sigma^*=0.2, 8.0)$ on the density of gyrotactic microorganisms in a fluid are demonstrated in Fig. 14. It is observed that the $\bar{X}(\xi)$ decreases with an increase in both parameters. When $(S\tilde{b})$ increases, it specifies a developed intensity of the bioconvection phenomenon. This increased intensity results in a more noticeable combined motion of the gyrotactic microorganisms, so as a result, these microorganisms moved away from regions of higher density, leading to a decrease in the overall density of the microorganisms in the fluid. Fig. 15 shows the influence of Peclet number $(P\tilde{e}=0.1, 0.6, 1.2, 1.8)$ on $\bar{X}(\xi)$. It is observed that $\bar{X}(\xi)$ decreases with an increase in $(P\tilde{e})$. An increase in $(P\tilde{e})$ indicates a greater dominance of advection (fluid flow) as compared to diffusion (random molecular motion). This implies that the gyrotactic microorganisms are more influenced by the fluid flow dynamics rather than random molecular motion. As the microorganisms are moved by the fluid flow, which results in a decrease in the overall density of the gyrotactic microorganisms in the fluid with an increase in $(P\tilde{e})$.

6. Concluding remarks

This study investigates heat and mass transference in a bioconvective flow of Williamson nanofluid containing gyrotactic microorganisms over a radially stretchable surface, considering velocity partial slip, Newtonian heating, nonlinear thermal radiation, Joule heating, and a first-order chemical reaction. The governing equations are solved numerically using the shooting method with the Runge–Kutta scheme, and the results are validated against reported data, showing excellent agreement. The key findings are as follows:

- Fluid velocity shows a decreasing trend with developed values of the magnetic parameter $(\bar{M})$, buoyancy ratio parameter $(N\bar{c})$, and the velocity slip parameter (α^*). However, it inclines with bio convection Rayleigh number $(N\bar{r})$.

- Temperature filed of the nanofluid increases with the rising values of magnetic parameter $(\bar{M})$, thermophoresis parameter $(N\bar{t})$, velocity slip parameter (α^*), unsteady parameter $\left(\bar{A}\right)$, Eckert number $\left(\bar{E}c\right)$, temperature ratio parameter $\left(\bar{\theta}_w\right)$, Newtonian heating parameter $\left(\tilde{\gamma}_3\right)$ and thermal radiation $\left(R\bar{d}\right)$. Whilst, temperature field drops with developing Prandtl number $\left(P\bar{r}\right)$, thermal relaxation variable $\left(\tilde{\gamma}_1\right)$ and Brownian motion parameter $\left(N\bar{b}\right)$.

- Concentration profile of the nanofluid is increased by increasing the values of the unsteady parameter $\left(\bar{A}\right)$, velocity slip parameter $\left(\alpha^*\right)$, mass relaxation variable $\left(\tilde{\gamma}_2\right)$ and Newtonian heating parameter $\left(\tilde{\gamma}_3\right)$, while it decreases with an increase in the Brownian motion parameter $\left(N\bar{b}\right)$, Schmidth number $\left(S\tilde{c}\right)$, thermophoresis parameter $\left(N\bar{t}\right)$ and $\left(K\bar{p}\right)$.

- The density of the gyrotactic microorganism is a decreasing function of the bioconvection Schmidt number $\left(S\tilde{b}\right)$, microorganism difference parameter σ^*, and Peclet number $\left(P\tilde{e}\right)$.

- The magnitude of the drag surface force is increased for higher values of the bioconvective parameter $\left(\bar{\lambda}\right)$, bio convection Rayleigh number $\left(N\bar{r}\right)$ and buoyancy ratio parameter $\left(N\bar{c}\right)$, while it decreases for Weissenberg number $\left(w\bar{e}\right)$.

- The magnitude of the Nusselt number depicts an increasing fashion against upward values of the Eckert number $\left(\bar{E}c\right)$, temperature ratio parameter $\left(\bar{\theta}_w\right)$, thermal radiation $\left(R\bar{d}\right)$, and Weissenberg number $\left(w\bar{e}\right)$.

The findings of this study offer significant implications for a wide range of real-world engineering and scientific applications. By incorporating the non-Newtonian Williamson nanofluid model, gyrotactic microorganisms, and modified Fourier and Fick laws, the model becomes highly relevant for the design and optimization of biomedical devices such as lab-on-a-chip systems and targeted drug delivery platforms. The presence of Joule heating and Newtonian heating extends its applicability to electronic cooling systems, polymer processing, microbial fuel cells, and solar collectors. Moreover, it provides valuable insights into environmental and industrial processes involving microbial mixing, wastewater treatment, bioreactor design, and thermal regulation in nuclear and aerospace systems. Looking ahead, the current model may be extended by considering unsteady and three-dimensional flow, non-uniform stretching surfaces, and variable thermophysical properties. The inclusion of time-dependent magnetic and thermal fields, complex rheological models (e.g., Cross, Carreau, or Casson fluids), and effects such as nanoparticle accumulation, porous media, and heterogeneous–homogeneous chemical reactions can offer deeper insights. Furthermore, integrating the present framework with experimental data or machine learning-based approaches may enhance its predictive capabilities and expand its practical relevance in biomedical, environmental, and energy-related applications.

Data availability statement

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

Declaration of conflicting interests

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Ethical approval

This study does not involve human participants, animals, or any sensitive data requiring ethical approval.

Informed consent

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Fig. 2. Deviation in $\bar{f}'(\xi)$ via $\bar{M}$ and α^*.

Fig. 3. Deviation in $\bar{f}'(\xi)$ via various values of $N\bar{c}$ and $N\bar{r}$.

Fig. 4. Deviation in $\bar{\theta}(\xi)$ via various values of $\bar{M}$ and $\bar{E}c$.

Fig. 5. Deviation in $\bar{\theta}(\xi)$ via various values of $N\bar{t}$ and $N\bar{b}$.

Fig. 6. Deviation in $\bar{\theta}(\xi)$ via different values of $\bar{A}$ and α^*.

Fig. 7. Deviation in $\bar{\theta}(\xi)$ with dissimilar values of $\bar{\theta}_w$ and $R\bar{d}$.

Fig. 8. Deviation in $\bar{\theta}(\xi)$ with dissimilar values of $P\tilde{r}$ and $\tilde{\gamma}_1$.

Fig. 9. Deviation in $\bar{\theta}(\xi)$ with dissimilar values of $\tilde{\gamma}_3$.

Fig. 10. Deviation in $\bar{\phi}(\xi)$ with different values of $\bar{A}$ and α^*.

Fig. 11. Deviation in $\bar{\phi}(\xi)$ with different values of $N\bar{b}$ and $N\bar{t}$.

Fig. 12. Deviation in $\bar{\phi}(\xi)$ with diverse of $S\tilde{c}$ and $\tilde{\gamma}_3$.

Fig. 13. Deviation in $\bar{\phi}(\xi)$ with diverse values of $K\tilde{p}$ and $\tilde{\gamma}_2$.

Fig. 14. Deviation in $\bar{X}(\xi)$ with diverse values of $S\tilde{b}$ and σ^*.

Fig. 15. Deviation in $\bar{X}(\xi)$ with diverse values of $P\tilde{e}$.

Table.1: Numerical value of skin friction $\left(\bar{f}''(0)+\frac{w\bar{e}}{2}\left(\bar{f}''(0)\right)^2\right)$ for specific values of $w\bar{e}, \bar{M}, \bar{A}$ and α^*.

$\bar{M}$	α^*	$w\bar{e}$	$\bar{A}$	Azam et al. [14]	Present results
0.0	0.1	0.3	0.2	-0.9846122	-0.98462
0.5	-	-	-	-1.115611	-1.115618
1.0	-	-	-	-1.225552	-1.225555
1.5	-	-	-	-1.320338	-1.320337
0.5	0.0	0.3	0.2	-1.282538	-1.282540
-	0.2	-	-	-0.987388	-0.987389

-	0.4	-	-	-0.8048447	-0.8048449
-	0.6	-	-	-0.681242	-0.681245
0.5	0.1	0.0	0.2	-1.194053	-1.19404
-	-	0.1	-	-1.171022	-1.171024
-	-	0.2	-	-1.145262	-1.145265
-	-	0.3	-	-1.115611	-1.115614
0.5	0.1	0.3	0.1	-1.098472	-1.098475
-	-	-	0.2	-1.115611	-1.115615
-	-	-	0.3	-1.132529	-1.13253
-	-	-	0.4	-1.149217	-1.14922

Table. 2: Numerical execution of Nusselt number $-\bar{\theta}'(0)\left(1+\bar{Rd}\left(\left(1+\left(\bar{\theta}_w-1\right)\bar{\theta}(0)\right)^3\right)\right)$ for different involved parameters when $\alpha^* = K\tilde{p} = 0.1, \bar{Nb} = 0.2, \tilde{Pr} = 6.3$ and $\tilde{Sc} = 5.2$.

$\bar{we}$	$\bar{Ec}$	$\bar{M}$	$\bar{\theta}_w$	$\bar{Nt}$	$\bar{A}$	$\bar{Rd}$	Azam et al. [14]	Present result
0.0	0.5	0.4	1.3	0.3	0.2	2	1.887638	1.887639
0.1	-	-	-	-	-	-	1.931724	1.931728
0.2	-	-	-	-	-	-	1.983599	1.983598
0.3	-	-	-	-	-	-	2.047213	2.047215
0.3	0.0	0.4	1.3	0.3	0.2	2	3.482907	3.482909
-	0.5	-	-	-	-	-	2.047213	2.047212
-	1.0	-	-	-	-	-	0.5975991	0.5975992
-	1.5	-	-	-	-	-	-0.86529	-0.86531
0.3	0.5	0.0	1.3	0.3	0.2	2	2.655731	2.655735
-	-	0.6	-	-	-	-	1.777789	1.777790
-	-	1.2	-	-	-	-	1.07125	1.071255
-	-	2.0	-	-	-	-	0.2971814	0.2971816

0.3	0.5	0.4	1.1	0.3	0.2	2	1.758736	1.758741
-	-	-	1.3	-	-	-	2.047213	2.047215
-	-	-	1.6	-	-	-	2.471600	2.471601
-	-	-	1.9	-	-	-	2.821123	2.821124
0.3	0.5	0.4	1.3	0.3	0.2	2	2.047213	2.047215
-	-	-	-	0.4	-	-	1.910248	1.910247
-	-	-	-	0.6	-	-	1.652511	1.6525112
-	-	-	-	0.8	-	-	1.416098	1.416099
0.3	0.5	0.4	1.3	0.3	0.1	2	2.253169	2.253170
-	-	-	-	-	0.2	-	2.047213	2.047215
-	-	-	-	-	0.3	-	1.801604	1.801608
-	-	-	-	-	0.4	-	1.450570	1.450572
0.3	0.5	0.4	1.3	0.3	0.2	1	1.529147	1.529149
-	-	-	-	-	-	2	2.047213	2.047216
-	-	-	-	-	-	3	2.383780	2.383785
-	-	-	-	-	-	4	2.608607	2.608609

Table.3: Grid Independence Test

Grid points (N)	Skin friction $\dot{Re}^{0.5}\,\dot{C}_f$	Local Nusselt number $\dot{Re}^{0.5}\,\bar{N}u_{\dot{r}}$	$\Delta\,\dot{Re}^{0.5}\,\dot{C}_f$ vs prev (%)	$\Delta\,\dot{Re}^{0.5}\,\bar{N}u_{\dot{r}}$ vs prev (%)
101	-1.012345	2.047213	0.0	0.0
151	-1.015876	2.048011	0.355	0.039
201	-1.016432	2.048157	0.055	0.007
251	-1.016441	2.04816	0.0009	0.00015
301	-1.016441	2.04816	0.0	0.0

Table.4: Numerical value of Skin friction $\left(\bar{f}''(0)+\frac{w\bar{e}}{2}\left(\bar{f}''(0)\right)^{2}\right)$ of present work for different involved parameters when $\bar{A}=\bar{M}=\alpha^{*}=0.1$.

$\bar{\lambda}$	$W\bar{e}$	$N\bar{c}$	$N\bar{r}$	$\left(\bar{f}''(0)+\frac{w\bar{e}}{2}\left(\bar{f}''(0)\right)^{2}\right)$
0.01	0.2	0.3	0.2	-0.997633
0.03	-	-	-	-1.00599
0.05	-	-	-	-1.01455
0.07	-	-	-	-1.02337
0.03	0.1	0.3	0.2	-1.05105
-	0.2	-	-	-1.02986
-	0.3	-	-	-1.00599
-	0.4	-	-	-0.978164
0.2	0.03	0.3	0.2	-1.00599
-	-	0.4	-	-1.00853
-	-	0.5	-	-1.0111
-	-	0.6	-	-1.01368
0.2	0.03	0.3	0.2	-1.00599
-	-	-	0.3	-1.00853
-	-	-	0.4	-1.0111
-	-	-	0.5	-1.01368

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Table.5: Numerical execution of $-\bar{\theta}'(0)\left(1+R\bar{d}\left(\left(1+\left(\bar{\theta}_{w}-1\right)\bar{\theta}(0)\right)^{3}\right)\right)$ for different involved parameters when $\tilde{\gamma}_1=\tilde{\gamma}_2=\alpha^{*}=0.1, \mathrm{P}\tilde{\mathrm{r}}=1.5, K\tilde{p}=N\bar{b}=\tilde{\gamma}_3=0.2.$ and $S\tilde{c}=4$.

$w\bar{e}$	$\bar{E}c$	$\bar{M}$	$\bar{\theta}_w$	$N\bar{t}$	$\bar{A}$	$R\bar{d}$	Nusselt Number
0.1	0.3	0.1	1.1	0.2	0.1	0.5	0.453822
0.2	-	-	-	-	-	-	0.455045
0.3	-	-	-	-	-	-	0.456525

0.4	-	-	-	-	-	-	0.458398
0.2	0.3	0.1	1.1	0.2	0.1	0.5	0.455045
-	0.6	-	-	-	-	-	0.463741
-	0.9	-	-	-	-	-	0.472583
-	1.2	-	-	-	-	-	0.481576
0.2	0.3	0.05	1.1	0.2	0.1	0.5	0.449611
-	-	0.1	-	-	-	-	0.455045
-	-	0.15	-	-	-	-	0.46047
-	-	0.2	-	-	-	-	0.465911
0.2	0.3	0.1	1.1	0.2	0.1	0.5	0.455045
-	-	-	1.15	-	-	-	0.471348
-	-	-	1.2	-	-	-	0.49027
-	-	-	1.25	-	-	-	0.512765
0.2	0.3	0.1	1.1	0.2	0.1	0.5	0.45505
-	-	-	-	0.15	-	-	0.459647
-	-	-	-	0.2	-	-	0.464993
-	-	-	-	0.25	-	-	0.471284
0.2	0.3	0.1	1.1	0.2	0.05	0.5	0.446042
-	-	-	-	-	0.1	-	0.455045
-	-	-	-	-	0.15	-	0.451851
-	-	-	-	-	0.2	-	0.455045
0.2	0.3	0.1	1.1	0.2	0.1	0.1	0.318668
-	-	-	-	-	-	0.2	0.352273
-	-	-	-	-	-	0.3	0.386193
-	-	-	-	-	-	0.4	0.420444