Study of Darcy-Brinkman gravity modulated Thermal Bio-Convection Under Internal Heating Effect in a Casson Fluid Saturated Porous Medium

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Abstract

In the present study, microorganisms whose stimulus is influenced by gravity and viscosity is taken for consideration in Casson fluid saturated porous medium. The behaviour of microorganisms in Casson fluid due to gravitational modulation and internal heat generated in the system is studied analytically. Darcy-Brinkman model is implemented to analyze thermo bioconvection in horizontal flow of the fluid. Stationary mode of convection produces both linear and nonlinear stability. The threshold Rayleigh number, which addresses the initiation of bioconvection, is determined by linear stability analysis. This comprises parameters for which marginal stability curves are plotted to understand the stability and onset of convection of the system against wavenumber k_c. The equation for Nusselt number is also used to describe nonlinear stability and to graphically illustrate how various parameters affect heat transfer in the system. In heat transfer analysis, the Ginzburg-Landau equation which characterizes the modulation amplitude is significant. Here, the behaviour of gyrotactic microorganisms in Casson fluid due to various parameters under the influence of gravity modulation and internal heat is picturised. From this study, we find that the Casson parameter decreases the critical Rayleigh number leads to the advancement of onset of convection and reduces heat transfer in the system.

Keywords:

Casson fluid, Biothermal convection, Gyrotactic microorganisms, Internal heating, Porous medium, Gravity modulation.

1. Introduction

Many various fields find substantial practical applications in the study of fluid flow in a porous material. In particular, viscoplastic fluids are crucial in the phenomena of convection. We consider Casson fluid saturated porous material for our investigation. Also we are interested to discuss the interdisciplinary aspect by including the behaviour of microorganisms under the influence of various parameters in a Casson fluid saturated porous media. This study of thermal bioconvection is the major part of the current problem. There are several practical uses for these viscoplastic fluids. Casson, also known as the biviscous Bingham fluid, is one such well-known fluid. Based on yield stress P_y, this non-Newtonian fluid has two distinct natures: viscous fluid and rigid solid. At comparatively high shear stresses, it exhibits pseudo-plastic behavior. The primary objective of the study is to analyse the stability of the system due to the presence of microorganisms in a non-Newtonian Casson fluid.

E.C. Bingham [1] was the pioneer to introduce thorough mathematical model for Casson fluid based on its yield stress in early twentieth century. Pharmaceutical industries, coal processing industries, clay products and paint manufacturing industries and many such similar industries require Casson fluid. For the Casson fluid, Swathi et al. [2] conducted research on unstable twodimensional flow over stretched surfaces. Pramanik [3] investigated the Casson fluid's boundary layer flow. Heat transfer beneath an increasingly extending surface was involved in this. Imran et al. [4] studied hydromagnetic effect produced by extending wedge submerged in Casson fluid in steady two-dimensional flow under boundary layer theory. Exact solutions for the unstable Casson fluid flow in a porous media with an embedded infinite vertical oscillating plate and an applied MHD were discovered by Abid et al. [5]. The temperature-dependent viscosity and thermal conductivity of a steady incompressible Casson fluid laminar flow were examined by Animasaun et al. [6]. A research using both linear and nonlinear analysis was conducted on the Casson fluid by Keerthana et al. [7]. Saleem et al. [8] studied thermal radiation effect on non-Newtonian Carreau fluid with one parameter lie scaling. Then Tufail et al. [9]-[10] made an attempt to study joule heating and viscous dissipation on Casson fluid with two parameter lie convection and also a theoretical analysis of chemically reacting mixed convective Casson fluid flow. An intriguing addition to the study is the incorporation of a porous medium in conjunction with gravity modulation. While, Ali et al. [11] studied on bio-viscosity using Keller box technique. Majeed et al. [12] made an extensive study on numerical simulations of energy storage performance.

Later, Saleem et al. [13] and Tufail et al. [14] studied on MHD Casson fluid using Two-parameter Lie scaling approach and Maxwell fluid through shrinking sheet with the Lie group approach. On the other side, Maya et al. [15] impacted on the MHD mixed convective lid-driven cavity with wavy bottom surface. Rafaie et al. [16] and Ali et al. [17] made AI-based predictive approach using CFD-ANN and spacing effects on flows around two square cylinders respectively. With the same Lie group analysis, Saleem et al. [18] studied heat and mass transfer by non-Newtonian fluid in an irregular channel. Then, CNT-water and human blood based Casson nanofluid flow was studied by Alkasasbeh et al. [19]. On the same line Saleem et al. [20] made a study on Thermal effects of ternary Casson nanofluid flow over a stretching sheet. Also Saleem et al. [21] made a comparative study on Williamson and Casson fluid on exponential stretching sheet. Consequently, Iqbal et al. [22] studied Darcy Casson fluid flow in a vertical channel: a Lie group approach.

Scholars and scientists have recently developed an interest in a novel field of study related to bioconvection in a porous media. The study of microorganisms moving through a fluid due to convection is known as bioconvection, and it is particularly pertinent to oil production technology. Thus, theoretical investigation of the connection between bioconvection and natural convection is needed. When self-propelled microorganisms with a density greater than that of the surrounding fluid medium exist, convective patterns are created. This phenomenon is known as bioconvection. Platt [23] coined the term bioconvection and studied the shifting polygonal patterns in Tetrahymena dense cultures. Rayleigh-Taylor instability has been used by Plesset and Winet [24] to discuss bioconvection. Pedley et al. [25] introduced the theoretical bioconvective model for gyrotactic bacteria. Following that, Hill et al. [26] examined the biothermal convection patterns of an increasing number of bacteria in a finite depth layer.

Studying this process requires an understanding of the motion and significance of bioconvection in its interaction. Thermal convection in porous media has been the subject of numerous computational simulations developed by researchers to date. For gravitactic microorganism bioconvection, Chandrashekhar [27], Drazin and Reid [28], Vafai [29], and Childress et al. [30] are the first to model a comprehensive theory. Several academics have also developed mathematical models based on the idea of heat convection in porous and fluid media.

Darcy-Brinkman equations are commonly used method for examining flow in high porosity porous media. Under gravity modulation, Zhao et al. [31] investigated the same. Here, the chaotic nature of the fluid is highlighted. In this regard, numerous problems arose with the DarcyBrinkman model-based study of bioconvection in high porosity media. A comprehensive study on the temperature-induced instability of fluid layers in porous media was carried out by Ingham and Pop [32]. Nield and Bejan [33] discussed internal free convection in connection to porous media. A comprehensive investigation of heat transfer rates related to fluid flow in rotating porous media was provided by Vadasz [34]. The thermal instability of the system is the main focus of the works mentioned above. Similar studies on free or natural convection in fluid-saturated porous media were carried out, taking into account a number of variables like gravitational modulation, rotational modulation, and magnetic field, among others.

A gyrotactic microorganism is an organism that moves in a certain direction in response to viscous and gravitational forces. We shall focus on this idea in our research. The mathematical model of bioconvection theory describes the motile pattern formed by the microorganisms suspended in the fluid. Pedley et al. [35] created the linear stability theory to examine the stability of biothermal convection brought on by gravitactic and gyrotactic microorganisms in a shallow layer of a simple fluid.

An important investigation on the kinetics of biological processes in porous media was carried out by Avramenko, Nield, and Kuznetsov. If the permeability is below the threshold/critical value, bioconvection does not occur. On the other hand, if the permeability is higher than the threshold or critical value, bioconvection may happen. To explore the marginal stability of the bioconvective system, the threshold Rayleigh numbers are found for a range of physical parameters such as cell eccentricity, Peclet number, and gyrotactic number. Their investigation focused on the effects of bioconvection in both horizontal and vertical flow on gyrotactic bacteria floating in a porous media.

In their study, Hwang and Pedley [36] investigated the effects of uniform shear on instability caused by biothermal convection in a shallow media containing moving gyrotactic microorganisms. The results of the study indicate that bioconvective instability in a diluted solution can be caused by three distinct mechanisms: gravitational effect, cell gyrotaxis, and negative cross-diffusion flow. At sufficiently high velocities, shear acts as a stabilizing force, much like Rayleigh-Bénard convection (RBC). Sharma and Kumar [37] then used analytical and numerical techniques to investigate the effect of vertically vibrated high-frequency on the onset of biothermal convection in a diluted solution of gyrotactic bacteria. Their findings demonstrated that the bioconvective Peclet number and high frequency, low amplitude vertical vibration stabilize the system.

Zhao et al. [38] investigated biothermal convection in a medium with high porosity while accounting for a suspension of gyrotactic microorganisms, therefore expanding the use of this model. They did a stability study to look into how biothermal convection reacts to heat applied from below. Using the Brinkman-Darcy model, Kopp et al. [39] examined the effects of a vertically vibrating magnetic field in a porous media saturated by a nanofluid containing gyrotactic microorganisms and water as the base fluid. A larger concentration of gyrotactic microbes has been demonstrated in [39] to accelerate the onset of magnetic convection. Additionally, Kopp and Yanovsky [40] examined how the rotational effect and the Coriolis force interacted in a layer of porous material that was saturated with biothermal convection and gyrotactic microorganisms.

One of the biggest challenges in technical and engineering applications is controlling mass and heat transfer. Several methods of manipulating convective processes involve the application of modulation effects or external parametric effects to the system. Magnetic field modulation, rotational modulation, temperature modulation and gravity modulation are examples of frequently used modulation procedures. We use a convection control technique in this work that is based on the gravitational field modulation. Before explaining our choice of gravitational modulation, we give a brief synopsis of pertinent research that investigates the application of gravity modulation in various convective systems. Gravity modulation was initially used by Gresho and Sani [41] to increase the stability of a fluid layer heated from below. The authors suggest that this kind of gravity modulation might be accomplished by creating a vertical oscillation in a fluid layer in a continuous gravitational field. Malashetty and Begum [42] carried out more study on the effects of small-amplitude gravity modulation on the initiation of convection in fluid layers and fluid-saturated porous surfaces, as well as additional physical features for non-Newtonian fluids. Govender [43] conducted research on a porous layer subjected to gravity modulation under natural convection. In addition to considering synchronous and subharmonic solutions, the study included a linear stability analysis and a weak nonlinear analysis. It was shown that as the excitation frequency is raised, convection swiftly stabilizes until the transition point is achieved.

Under gravitational modulation, Kiran [44] investigated the nonlinear thermal instability in a viscoelastic, nanofluid-saturated porous medium. Under the same effect Kiran [45] made an analysis on weekly nonlinear statbility in a fluid layer bounded by rigid boundaries. In a porous media, Bhadauria et al. [46] investigated how gravitational modulation and internal heating affect oscillatory convection. Additionally, Hopf and Pitchfork bifurcations were investigated.

The stability of thermo-bioconvection into an anisotropic porous fluid layer saturated with Jeffrey liquid-a substance produced by gravitactic bacteria-was recently studied by Arpan et al. [47]. Their findings show that the system becomes unstable as the number of bioconvection peclets and the concentration of bacteria rise. For low concentrations of microbes, increasing the thermal anisotropy strength results in the system stabilizing and the convective cell size growing. Kopp and Yanovsky [48] have reported the effect of g -jitter on weakly nonlinear biothermal convection in a porous medium. They investigated how changes in gravity affect bio-porous convection, in which the bio-convective cell is solely modeled as being under the influence of vertical vibrations. The impact of internal heating on bio-Darcy convection was recently described by Akhila et al. [49]. Further, along with the gravity modulation, double diffusive biothermal convection in porous media was investigated by Akhila et al. [50] to study weekly nonlinear stability analysis. Chandan et al. [51] extended to Casson fluid under the same internal heating and gravity modulation. This was quite previously studied by Akhila et al. [52] in Casson fluid only under gravity modulation. It is found that internally heated layers even govern bioconvection in the presence of gravity modulations. Their findings indicate that when viscous and gravitational torques combine, microorganisms can swim in the direction of downwelling fluid (M.A. Bees [53]). On the other hand, there is no record of what happens when they cross the boundaries. Whether or not the convection of the swimming microorganisms could be impacted by the unequal heating at the boundary. The present work aims at providing an explanation of bioconvection in the presence of plate modulation. The literature makes it abundantly evident that there is limited number of studies have been found on the introduction of gravitactic microbes to a rotating porous layer that is responsive to gravity modulation. As a result, the above literature review motivated us to learn more about the interactions between gravitaxis, gyrotaxis, and thermal convection in a Casson fluid-saturated porous medium under the influence of internal heating and gravity modulation. This article's major objective is to investigate the behavior of weakly nonlinear thermal bioconvection in a porous media that is saturated with gyrotactic microorganism-containing Casson fluid.The study focuses on the effects of variation in the gravitational field modulation and internal heating effect using the Ginzburg-Landau (GL) model.

This work can be used to relevant fields in the biotechnological, medical, geoscientific, and environmental sciences. It is based on Brinkman and Darcy gravity modulated thermal bioconvection model for gyrotactic microorganisms in porous media saturated by Casson fluid in the presence of an internal heat source. This is an interdisciplinary field with a broad application in biomathematics.

2. Problem Statement and Formulation

The physical interpretation of the problem is as shown in Fig 1. Infinite horizontal parallel plates between which Casson fluid saturated porous medium of thickness h is considered. The walls of the horizontal plates are completely thermal conducting and perfectly impermeable. As shown in the geometry, there exist a temperature difference between lower and upper plates whose temperatures are given by $T_0 + \Delta T$ and T_0 respectively. Gyrotactic microorganisms present in the porous medium layer experience gravity modulation and undergo thermal bioconvection. The gravity modulation is given by $eg_0\left(1+\varepsilon^2 \delta Cos(\omega_g t)\right)$ that is acting vertically downwards parallel to z-axis. Here, e is the unit vector along z direction, g_0 is the magnitude of the mean gravitational field vector, amplitude and frequency of the gravity modulation are denoted by δ and ω_g, respectively, whereas ε is a minor dimensionless parameter. Here, we assume the pores of the porous medium are sufficiently large enough to accommodate gyrotactic microorganisms and allow them to swim. The Darcy-Brinkman model for the considered geometry are governing the problem as follows: [40], [48], [49]

$$\nabla . V_D = 0 \tag{1}$$

$$\frac{\rho_0}{\varepsilon}\frac{(\partial V_D)}{\partial t} = -\nabla P + \mu_C (1+\tfrac{1}{\chi})\nabla^2 V_D - (1-\beta(T-T_0))eg(t)\rho_0 - \frac{\mu}{K}V_D - (\delta\rho)\Upsilon n.e.g(t) \tag{2}$$

$$(\rho c)_m \frac{\partial T}{\partial t} + (\rho c)_f V_D . \nabla T - k_m \nabla^2 T = Q(T - T_0) \tag{3}$$

$$\frac{\partial n}{\partial t} = -\nabla .(nV_D + nW_c \hat{I} - \nabla n D_m) \tag{4}$$

$$g(t) = (1+\varepsilon^2 \delta Cos(\omega_g t))g_0 \tag{5}$$

Here, $\frac{V_D}{\in} = V$, where $\in$ is the porosity of porous media, connects the fluid velocity V and the Darcy velocity $V_D = (u, v, w)$. The viscosity of the Casson fluid is denoted by μ and μ_c consists of the Brinkman effective viscosity and dynamic viscosity. The porous media permeable nature is indicated by K, while the fluid's density at the reference temperature T_0 is indicated by ρ_0. Further, P, β and g are pressure, thermal expansion coefficient and gravitational acceleration respectively. A unit vector in the direction of z-axis is denoted by $e = (1, 0, 0)$. The fluid heat capacity is $(\rho c)_f$, whereas its effective heat capacity is $(\rho c)_m$ with effective thermal conductivity given by k_m. Furthermore, the difference in density between microorganisms and a base fluid is represented by $\delta\rho = \rho_m - \rho_f$. The concentration of microorganisms in the fluid is n. Microbes have a diffusivity of D_m and an average volume of Υ. Lastly, the average swimming velocity of an individual microorganism is $W_c I(0)$, (W_c is constant). The direction of motion of microorganisms with temperature are represented by the unit vector $I(t)$. The stress tensor for Casson fluid is modelled as [7]

$$\tau_{ik} = \begin{cases} (\mu_B + \frac{P_y}{\sqrt{2\Gamma}})e_{ik}, \Gamma > \Gamma_c \\ (\mu_B + \frac{P_y}{\sqrt{2\Gamma}})e_{ik}, \Gamma < \Gamma_c \end{cases} \tag{6}$$

where $e_{ik} = \frac{1}{2}(\frac{\partial u}{\partial x_k} + \frac{\partial w}{\partial x_i})$ and μ_B is dynamic viscosity. Since the Casson fluid is viscoplastic, the rate of deformation Γ of a non-Newtonian fluid (NNF) model surpasses its threshold value Γ_c. This is referred to as yield stress and is represented by P_y. The Casson parameter denoted by χ is defined as [7]

$$\chi = \frac{\mu_B \sqrt{2\Gamma}}{P_y} \tag{7}$$

The boundary conditions for the above governing equations are given as follows: [40], [48], [49]

$$w = 0, \quad T = T_0 + \Delta T, \quad J \cdot e = 0 \quad \text{at} \quad z = 0 \tag{8}$$

$$w = 0, \quad T = T_0, \quad J \cdot e = 0 \quad \text{at } z = h \tag{9}$$

where, $J = n\frac{V_D}{\in} + nW_c \hat{I} - D_m \nabla$ is the flux of the microorganisms through the system. The non-dimensional parameters that are to be included in analysing the problem is as follows: [40], [48], [49]

$$(x^*, y^*, z^*)h = (x, y, z)\,,\ V_D^* = V_D \frac{h}{\alpha_m}\,,\ t^* = \frac{(t\alpha_m)}{(h^2\sigma)}\,,\ \ T^* = \frac{(T - T_u)}{(T_d - T_u)}$$

$$P^* = \frac{PK}{\mu\alpha_m}\,,\ \sigma = \frac{(\rho c)_m}{(\rho c)_f}\,,\ \ n^* = n\Upsilon\,,\ \ \omega_g^* = \omega_g \frac{h^2\sigma}{\alpha_m} \tag{10}$$

where, $\alpha_m = \dfrac{k_m}{(\rho c)_f}$ is thermal diffusivity coefficient. On non-dimensionalising Eq. (1) to Eq. (4) using (8) and on removing the asterisks, we get the system of non-dimensional equations that follows:

$$\nabla \cdot V_D = 0 \tag{11}$$

$$\frac{1}{\Upsilon_a}\frac{\partial V_D}{\partial t} = -\nabla P + D_a(1+\tfrac{1}{\chi})\nabla^2 V_D - V_D - ef_m \frac{R_b}{L_b} n + ef_m RaT \tag{12}$$

$$\frac{\partial T}{\partial t} + (V_D \cdot \nabla)T = \nabla^2 T + R_i T \tag{13}$$

$$\frac{1}{\sigma}\frac{\partial n}{\partial t} = -\nabla(nV_D + \frac{Pe}{L_b} n\hat{I}(t) - \frac{1}{L_b}\nabla n) \tag{14}$$

where, $f_m = 1 + \varepsilon^2 \delta Cos(\omega_g t)$. The corresponding non-dimensional boundary conditions are given by [40], [48], [49]

$$T = 1,\ \ w = 0\,,\ nPe = \frac{dn}{dz}\,,\ \ \text{at}\ \ z = 0\,, \tag{15}$$

$$T = 0,\ \ \ w = 0\,,\ nPe = \frac{dn}{dz}\,,\ \ \text{at}\ \ z = 1. \tag{16}$$

In Eqs. (11)-(14), we introduce the non-dimensional parameters such as, $R_i = \dfrac{Qh^2}{k_m}$ is internal Rayleigh number, $\Upsilon_a = \dfrac{\in(\rho c)_m \mu_C}{\rho_0 k_m D_a} = \dfrac{\in \sigma \Pr}{D_a}$ is the modified Vadasz number, $D_a = \dfrac{\mu_C K}{\mu h^2}$ is the Darcy number, $\Pr = \dfrac{\mu}{\alpha_m \rho_0}$ is the Prandtl number, $R_b = \dfrac{g(\delta\rho)hK}{\mu D_m}$ is the Rayleigh-Darcy number for bioconvection, $L_b = \dfrac{\alpha_m}{D_m}$ is the Lewis number for bioconvection, $Ra = \dfrac{\rho_0 ghK\beta\Delta T}{\mu\alpha_m}$ is the Darcy-Rayleigh number, and $Pe = \dfrac{W_C h}{D_m}$ is the Peclet number for bioconvection.

Each above mentioned parameter has their significance in heat transfer phenomena of the system.

2.1 Basic State (Conduction State):

Since the fundamental state preceeding the onset of convection is known as the conduction state. Thus, we presume the fundamental (conduction) state is totally time independent and is as below by [40], [48], [49]

$$P = P_b(z)\,, \quad T = T_b(z)\,, \; V_D = V_b = 0\,, \quad n = n_b(z)\,. \tag{17}$$

In order to obtain the stable profiles of microbe concentration $n_b(z)$, temperature $T_b(z)$, and pressure variation $P_b(z)$ in the conduction state, the following equations must be solved:

$$\frac{d^2T_b}{dz^2} + R_i T_b(z) = 0 \tag{18}$$

$$\frac{dn_b}{dz} = n_b(z)Pe \tag{19}$$

$$\frac{dP_b}{dz} = -\frac{R_b}{L_b} n_b(z) + RaT_b(z) \tag{20}$$

The temperature distribution $T_b(z)$ is obtained by integrating Eq. (18) and taking into account the boundary conditions in Eqs. (15)-(16). The result is as shown below:

$$T_b(z) = \frac{\sin(\sqrt{R_i}(1-z))}{\sin(\sqrt{R_i})} \tag{21}$$

Moreover, $n_b(z)$ has a solution that we obtain as: [48],

$$\frac{n_b(z)}{n_b(0)} = e^{(zPe)} \tag{22}$$

where, $n_b(0)$ is the density of the Casson fluid at the bottom of the layer. The $n_b(0)$ (constant) is found as

$$n_b(0) = \frac{(<n> Pe)}{e^{Pe}-1}\,, \quad <n> = \int_0^1 n_b(z)dz \tag{23}$$

Since, Peclet number Pe is thought to be extremely small, $n_b(z) \approx n_0$, indicating that $n_b(z)$ is roughly constant in the layer, according to Eq. (22). This case is considered to simplify the convective analysis.

Given the assumption that $P = P_0$ at $z = 1$, the pressure distribution can be determined in the conduction state as follows:

$$P_0+\frac{R_b}{L_b Pe}n_b(0)(e^{Pe}-e^{zPe})+\frac{Ra}{\sqrt{R_i}Sin(\sqrt{R_i})}(Cos(\sqrt{R_i}(1-z))-1)=P_b(z) \quad (24)$$

2.2 Perturbed state:

Perturbation is the small disturbance arises in the fluid layer on heating from below. Velocity, pressure, temperature and concentration of microorganisms and pressure of the perturbed state is stated as follows:

$$n=n'+n_b(z),\quad T=T'+T_b(z),\quad V_D=V',\quad P=P'+P_b(z),\quad \hat{I}(t)=e+m'(t) \quad (25)$$

The following equation can be used to determine the perturbation of the unit vector, which represents the direction in which the microorganisms are swimming, when taking gravity modulation into account. ([40], [48], [49])

$$m'(t)=B_0(1+\varepsilon^2\delta Cos(\omega_g t))\zeta i-B_0(1+\varepsilon^2\delta Cos(\omega_g t))\xi j+0\cdot e \quad (26)$$

i and j here represent the respective unit vectors pointing in the x and y directions. $B_0=(\frac{\mu\alpha_\perp}{\rho_0 g_0 d})(\frac{\alpha_m}{h^2})$ represents the dimensionless parameter, in the absence of modulation, that indicates the reorientation of microorganisms owing to the action of a gravitational moment in relation to viscous resistance.

In Eq. (26) the parameters ζ and ξ in the x and y components of vector m' are

$$\zeta=-(1-\alpha_0)\frac{\partial w'}{\partial x}+(1+\alpha_0)\frac{\partial u'}{\partial z}\quad \text{and}\quad \xi=(1-\alpha_0)\frac{\partial w'}{\partial y}-(1+\alpha_0)\frac{\partial v'}{\partial z}, \quad (27)$$

Here, α_0 is the cell eccentricity defined as $\alpha_0=(r_M^2-r_m^2)/(r_M^2+r_m^2)$, where the semi-major and semi-minor axes of the spherical cell are denoted by the numbers r_M and r_m, respectively. The following are the equations for the variables V', T' and n' if we replace (25) into Eqs. (11)-(14):

$$\nabla\cdot V'=0 \quad (28)$$

$$\frac{1}{\Upsilon_a}\frac{\partial V'}{\partial t}=-\nabla P'+D_a(1+\tfrac{1}{\chi})\nabla^2 V'-V'-ef_m\frac{R_b}{L_b}n'+ef_m RaT' \quad (29)$$

$$\frac{\partial T'}{\partial t}+w'\frac{dT_b}{dz}+(V'\cdot\nabla)T'=\nabla^2 T'+R_i T' \quad (30)$$

$$\frac{1}{\sigma}\frac{\partial n'}{\partial t}t=-\nabla(n'V')-w'\frac{dn_b}{dz}-\frac{Pe}{L_b}\frac{\partial n'}{\partial z}-\frac{1}{L_b}\nabla^2 n'+PeG_0 n_b(1-\varepsilon^2\delta Cos(\omega_g t))\Lambda \quad (31)$$

Where, $\Lambda=(1+\alpha_0)\frac{d^2 w'}{dz^2}+(1-\alpha_0)(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2})$

The nondimensional orientation parameter in the absence of modulation is $G_0 = D_m B_0 / h^2$. For the 2D flow model, we introduce ψ as the stream function to determine the velocities as follows:

$$u' = \frac{\partial \psi}{\partial z} \quad , \qquad w' = \frac{\partial \psi}{\partial x} \tag{32}$$

The dimensionless governing equations that follow (after asterisks are removed) are obtained by transferring Eq. (32) into Eqs. (29)-(31), applying the outcomes for the fundamental state, and eliminating the pressure term.

$$\left(\tfrac{1}{\Upsilon_a}\frac{\partial}{\partial t} + 1 - D_a\left(1+\tfrac{1}{\chi}\right)\nabla^2\right)\nabla^2\psi = f_m \frac{R_b}{L_b}\frac{\partial n'}{\partial x} - f_m Ra \frac{\partial T'}{\partial x} \tag{33}$$

$$\frac{\partial \psi}{\partial x} - (\nabla^2 + R_i)T' = -\partial T' / \partial t + \frac{\partial(\psi, T')}{\partial(x,z)} \tag{34}$$

$$PeG_0(2-f_m)n_0\alpha\frac{\partial \psi}{\partial x} + \frac{Pe}{L_b}\frac{\partial n'}{\partial z} - \frac{1}{L_b}\nabla^2 n' = -\frac{1}{\sigma}\frac{\partial n'}{\partial t} + \frac{\partial(\psi, n')}{\partial(x,z)} \tag{35}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad \alpha = \nabla^2 + \alpha_0\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)$$

We rescale the system, concentrating on the stationary mode of thermal bioconvection, by establishing $\tau = \varepsilon^2 t$ with a little time variation. The nonlinear system of equations (33)-(35) can be expressed in matrix form as shown below:

$$\begin{pmatrix} \nabla^2 - D_a(1+\tfrac{1}{\chi})\nabla^4 & f_m Ra \frac{\partial}{\partial x} & -f_m \frac{R_b}{L_b}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -(\nabla^2 + R_i) & 0 \\ PeG_0(2-f_m)n_0\alpha\frac{\partial}{\partial x} & 0 & \frac{Pe}{L_b}\frac{\partial}{\partial z} - \frac{1}{L_b}\nabla^2 \end{pmatrix} \begin{pmatrix} \psi \\ T \\ n \end{pmatrix} = \begin{pmatrix} -\frac{\varepsilon^2}{\Upsilon_a}\frac{\partial}{\partial \tau}\nabla^2\psi \\ -\varepsilon^2\frac{\partial T}{\partial \tau} + \left(\frac{\partial(\psi,T)}{\partial(x,z)}\right) \\ -\frac{\varepsilon^2}{\sigma}\frac{\partial n}{\partial \tau} + \left(\frac{\partial(\psi,n)}{\partial(x,z)}\right) \end{pmatrix} \tag{36}$$

To solve the system shown in Eq. (36), impermeable boundary conditions need to be considered as follows:

$$\psi = \nabla^2\psi = T = n = 0 \;\text{ at }\; z = 0 \;\text{ and }\; z = 1 \tag{37}$$

3. Weakly nonlinear instability analysis

We employ the subsequent asymptotic expansions to Eq. (36) in order to investigate the stationary instability:

$$Ra = Ra_c + \varepsilon^2 Ra_2 + \varepsilon^4 Ra_4 + \ldots, \tag{38}$$

$$\psi = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \ldots, \tag{39}$$

$$n = \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \ldots, \tag{40}$$

$$T = \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + ..., \tag{41}$$

The critical Rayleigh number, Ra_c, denotes the temperature at which convection begins in the absence of gravity modulation.

3.1 Stability analysis in lowest order system

At the first order, the system simplifies to a linear model with negligible nonlinear effects. The following is an expression for the system:

$$\begin{pmatrix} \nabla^2 - D_a(1+\frac{1}{\chi})\nabla^4 & Ra_c \frac{\partial}{\partial x} & -\frac{R_b}{L_b}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -(\nabla^2 + R_i) & 0 \\ PeG_0 n_0 \alpha \frac{\partial}{\partial x} & 0 & \frac{Pe}{L_b}\frac{\partial}{\partial z} - \frac{1}{L_b}\nabla^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ T_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{42}$$

Subject to the boundary constraints provided by Eq. (37), the first order system's solution is as follows:

$$\psi_1 = A(\tau)\sin(k_c x)\sin(\pi z)$$

$$T_1 = -\frac{(A(\tau)k_c)}{a^2 - R_i} Cos(k_c x)Sin(\pi z), \quad a^2 = k_c^2 + \pi^2$$

$$n_1 = \frac{k_c}{a^2} PeG_0 n_0 L_b ((1-\alpha_0)k_c^2 + (1+\alpha_0)\pi^2) A(\tau)Cos(k_c x)Sin(\pi z) \tag{43}$$

The following is the expression for the critical (threshold) Rayleigh number Ra_c:

$$Ra_c = \frac{a^2(1 + D_a(1+\frac{1}{\chi})a^2)(a^2 - R_i)}{k_c^2} - PeG_0 n_0 R_b ((1-\alpha_0)k_c^2 + (1+\alpha_0)\pi^2)\left(1 - \frac{R_i}{a^2}\right) \tag{44}$$

Wave number k_c is obtained by minimizing Ra_c with respect to k_c^2. This is used to determine the onset of convection. It is possible to differentiate Ra_c with respect to k_c^2 and then set the derivative to zero. By resolving this equation, we may get the associated wave number for the convection's beginning.

But regular bioconvection, which is induced by the movement of bacteria, occurs when the system is not internally heated. In this instance, the bioconvection-governing parameter is the bioconvective Rayleigh number R_b.

3.2 Stability analysis in second order system

The Jacobian terms in the right-hand side of Eq. (36) introduce the nonlinear effects at this order and explain how fluid velocity, temperature, and microbiological diffusivity interact. The system of the resulting equations in second order can be represented alternatively as follows:

$$\begin{pmatrix} \nabla^2 - D_a(1+\frac{1}{\chi})\nabla^4 & Ra_c \frac{\partial}{\partial x} & -\frac{R_b}{L_b}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -(\nabla^2 + R_i) & 0 \\ PeG_0 n_0 \alpha \frac{\partial}{\partial x} & 0 & \frac{Pe}{L_b}\frac{\partial}{\partial z} - \frac{1}{L_b}\nabla^2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ T_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix} \tag{45}$$

where,

$$N_{21} = 0, \qquad N_{22} = \left(\frac{\partial(\psi_1, T_1)}{\partial(x,z)} \right), \quad N_{23} = \left(\frac{\partial(\psi_1, n_1)}{\partial(x,z)} \right) \tag{46}$$

The first-order solutions given in Eqs. (43) can be used to generate the second-order solutions. The subsequent expressions represent the second-order solutions that take boundary conditions (37) into account.

$$\psi_2 = 0 \tag{47}$$

$$T_2 = \frac{-k_c^2 \pi}{2(a^2 - R_i)(4\pi^2 - R_i)} A^2(\tau) Sin(2\pi z) \tag{48}$$

$$n_2 = \frac{k_c^2 L_b \Pi}{8a^2 \pi} A^2(\tau) Sin(2\pi z) \tag{49}$$

where, $\Pi = PeG_0 n_0 L_b((1-\alpha_0)k_c^2 + (1+\alpha_0)\pi^2)$.

To evaluate the heat transfer using the stationary convective mode and the Nusselt number $Nu(\tau)$, we make use of the following expression.

$$Nu(\tau) = 1 + \frac{\left[[\frac{k_c}{2\pi} \int_0^{\frac{2\pi}{k_c}} \frac{\partial T_2}{\partial z} dx] \right]_{(z=0)}}{\left[[\frac{k_c}{2\pi} \int_0^{\frac{2\pi}{k_c}} \frac{\partial T_b}{\partial z} dx] \right]_{(z=0)}} \tag{50}$$

$$Nu(\tau) = 1 + \frac{k_c^2 \pi^2 Sin\sqrt{R_i}}{(a^2 - R_i)(4\pi^2 - R_i)\sqrt{R_i} Cos\sqrt{R_i}} A^2(\tau) \tag{51}$$

It is possible to compute the heat transfer quotient $Nu(\tau)$ after obtaining the expression for the amplitude $A(\tau)$. The asymptotic expansion established in Eqs. (38)-(41) demonstrates that the gravity modulation is important only at the third order in terms of ε. For gravity modulation on heat transfer, the mean Nusselt number Nu is defined across a suitable interval $(0, 2\pi)$.

$$Nu = \frac{1}{2\pi} \int_0^{2\pi} Nu(\tau) d\tau \tag{52}$$

Mean Nusselt number Nu is quantitatively assessed in relation to many parameters, and Fig. 6 presents an equivalent graphical analysis.

3.3 Stability analysis of the system in its third order

To study the stability analysis at this order, we get a matrix as below:

$$\begin{pmatrix} \nabla^2 - D_a(1+\frac{1}{\chi})\nabla^4 & Ra_c \frac{\partial}{\partial x} & -\frac{R_b}{L_b}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -(\nabla^2 + R_i) & 0 \\ PeG_0 n_0 \alpha \frac{\partial}{\partial x} & 0 & \frac{Pe}{L_b}\frac{\partial}{\partial z} - \frac{1}{L_b}\nabla^2 \end{pmatrix} \begin{pmatrix} \psi_3 \\ T_3 \\ n_3 \end{pmatrix} = \begin{pmatrix} N_{31} \\ N_{32} \\ N_{33} \end{pmatrix} \quad (53)$$

where,

$$N_{31} = \left(\frac{a^2}{\Upsilon_a}\frac{\partial A(\tau)}{\partial \tau} - Ra_c \frac{k_c^2 A(\tau)}{(a^2 - R_i)} \delta Cos(\Omega\tau) + Ra_2 \frac{k_c^2 A(\tau)}{(a^2 - R_i)} + \frac{R_b}{L_b}\frac{k_c^2 A(\tau)}{a^2} \Pi \delta Cos(\Omega\tau) \right) Sin(k_c x) Sin(\pi z) \quad (54)$$

$$N_{32} = \left(\frac{k_c}{a^2}\frac{\partial A(\tau)}{\partial \tau} - \frac{k_c^3 A^3(\tau)\pi^2}{(4\pi^2 - R_i)(a^2 - R_i)} Cos(2\pi z) \right) Cos(k_c x) Sin(\pi z) \quad (55)$$

$$N_{33} = \left(\frac{k_c \Pi}{\sigma a^2}\frac{\partial A(\tau)}{\partial \tau} + \frac{k_c \Pi}{L_b} \delta Cos(\Omega\tau) A(\tau) - \frac{k_c^3 A^3(\tau)}{4a^2} \Pi L_b Cos(2\pi z) \right) Cos(k_c x) Sin(\pi z) \quad (56)$$

The solvability condition given below can be used to extract the Ginzburg-Landau equation of amplitude for the stationary mode of convection with time-periodic coefficients from the third ordered solution.

$$\int_{z=0}^{z=1} \int_{x=0}^{x=\frac{2\pi}{k_c}} \left[\psi_1^* N_{31} + T_1^* N_{32} + n_1^* N_{33} \right] dxdz = 0 \quad (57)$$

where, ψ_1^*, T_1^* and n_1^* are obtained by taking the adjoint of Eq.(43). On solving Eq. (58), we obtain ampitude equation which is given by

$$B_1 \frac{\partial A}{\partial \tau} - B_2(\tau) A + B_3 A^3 = 0 \quad (58)$$

where,

$$B_1 = \frac{a^2}{\Upsilon_a} - Ra_c \frac{k_c^2}{(a^2 - R_i)^2} + \frac{R_b}{\sigma}\frac{k_c^2}{a^4}\Pi \quad (59)$$

$$B_2(\tau) = \frac{k_c^2}{(a^2 - R_i)} Ra_c \left(\frac{Ra_2}{Ra_c} - \delta Cos(\Omega\tau) \right) + 2\frac{k_c^2)}{a^2} \frac{R_b}{L_b} \Pi \delta Cos(\Omega\tau) \tag{60}$$

$$B_3 = k_c^4 \left(\frac{Ra_c \pi^2}{(a^2 - R_i)^2 (4\pi^2 - R_i)} + \frac{R_b L_b \Pi}{4a^4} \right) \tag{61}$$

There is no autonomy in the amplitude equation given by Eq. (58). The numerical solution has been obtained using the built-in Mathematica function NDSolve. The problem is solved by using the beginning condition $A(0) = A_0$, where A_0 is the chosen initial condition for the amplitude of convection. We assume $Ra_2 \approx Ra_c$ since we are interested in the nonlinearity approaching the critical stage of convection. Thus, in this theory of weakly nonlinear convective instability, a modest expansion parameter ε^2 is the relatively small deviation of the Rayleigh number Ra from its critical value Ra_c, as shown by

$$\varepsilon^2 = \frac{Ra - Ra_c}{Ra_c} << 1$$

As a result, the analytical solution for the unmodulated case of the aforementioned Eq. (58) is as follows:

$$A_u(\tau) = \frac{A_0}{\sqrt{\frac{B_3}{B_2} A_0^2 + \left(1 - \frac{B_3}{B_2} A_0^2\right) exp\left(-\frac{2\tau B_2}{B_1}\right)}} \tag{62}$$

where the unmodulated amplitude of convection is represented by $A_u(\tau)$. B_1 and B_3 are as given in Eqs. (59) and (61), whereas $B_2 = \frac{k_c^2}{(a^2 - R_i)} Ra_2$.

4. Observations and Discussion

The Ginzberg-Landau equation for amplitude in stationary mode of biothermal convection along with gravity modulation in Casson fluid saturated porous medium has been derived by the use of perturbation approach in the inquiry. For given initial values, a numerical solution to the GL problem is found. The obtained amplitude is used to calculate heat transfer parameter called Nusselt number Nu which describes heat transmission rate in the system. For the corresponding wave number k_c, the linear theory yields the critical or threshold Rayleigh number Ra_c, which provides a clear picture of the commencement of convection. In the current investigation, we encounter a number of characteristics that affect the system's heat transmission and commencement of convection. The transition point we observe in the curves represents critical Rayleigh number. In Fig. 2, the relationship between Ra_c and k_c is plotted for a range of parameter values. We can see how the Darcy number affects the beginning of convection in Fig. 2(a). The critical Rayleigh number increases in proportion to the growing value of D_a. As a result, convection takes longer to start, stabilizing the system for a longer period of time. Whereas, in Figs. 2(b,c,d and e) we observe the common nature. We detect decreasing values in the critical

Rayleigh number with increasing values of the Casson parameter (χ), internal Rayleigh number R_i, cell eccentricity (α_0), and modified bioconvection Rayleigh number Ra_B. In other words, higher values of the aforementioned parameters cause convection to begin earlier and cause the system to become unstable. Therefore, in addition to gravity modulation and the internal heating impact, all the study's characteristics determine when biothermal convection begins in a porous medium saturated with Casson fluid. For our study, the range of χ is taken to be between 0.1 and 0.3 . In Fig. 2(e), we observe that the threshold Rayleigh number decreases with increasing χ. The system becomes unstable as a result of the early development of convection. This viscoelastic fluid plays some significant role in the commencement of bioconvection which leads to various prominent applications in the current scenario. Oil drilling is one such important application where microorganism motility comes into action. The presence of Casson fluid may help in better improvement in convection.

Table 1 presents a novel observation and it displays the critical Rayleigh number Ra_c and the associated wavenumber k_c for all available parameters. These values are in good agreement with the graphical values.

The next set of graphs are exhibited in Fig.3. These graphs explain the heat transfer rates with respect to timescale τ in the system due to variation in the existing various parameters. Heat transfer in the system is governed by Nusselt number Nu given in Eq. (51). Modified Vadasz number Υ_a is varied between 0.5 and 1.5 in equal interval to notice the variation in the graph. This parameter increases the heat transfer rate for small timescale. As time increases, there is least variation in heat transfer rates in the system which is seen in Fig. 3(a). Whereas in Fig. 3(b), the effect of variation of modified bioconvection Rayleigh number $(Ra_B = PeG_0 n_0 R_b)$ is exhibited. The values for Ra_B is taken as $(0,3,6)$ for the investigation of heat transfer. As the value of Ra_B increases, there is a decrease in Nusselt number throughout the timescale τ. Hence there is a decrease in heat transfer in the system. This depends on gyrotactic number G_0 and concentration of microorganism n_0. Cell eccentricity α_0 is one such important parameter which describes the shape of the microorganisms which has an impact on bioconvection. The value α_0 =o indicates the spheroidal shape of microorganism which helps in easy convection leads to increase in heat transfer. As the value of α_0 increases, there exist irregularity in the shape of microorganisms. This leads to disturbance in convective quality. Hence, there is a decrease in the value of Nusselt number which is clearly shown in Fig. 3(c). Some significant fact can be observed in Fig. 3(d) which has the variation in the value of Ω. Irrespective of the value of Ω, the maximum and minimum value of Nusselt number remains uniform throughout the timescale. Only the frequency of increase and decrease in heat transfer varies. Fig. 3(e) showcases the variation of δ i.e., variation of amplitude in the system. The value of δ is taken between 0.1 and 0.3 . As we can notice in the graph, for smaller δ values, there exist smaller values of Nusselt number. As the value of δ increases, there is an increase in heat transfer in the system. Fig. 3(f) shows the impact of variation in internal Rayleigh number R_i on Nusselt number Nu. Internal Rayleigh number is varied between 0.2 and 0.6 in equal intervals. There is a natural increase in temperature due to increase in R_i due to which there is an increase in heat transer in the system. The results are in good agreement with [48] and [49]. Finally, Fig. 3(g) is an important part of the current problem.

The effect of presence of Casson parameter χ on heat transfer in biothermal convection is shown in this plot. Increase in the value of χ in ten's and hundred's, there is a slight variation in heat transfer in the system. As χ is found as a reciprocal term in governing equations, we can conclude that increase in χ leads to small decrease in heat transfer in the system.

Figs. 4 and 5 show a set of remarkable graphs that represent streamlines and isotherms, respectively. When the Rayleigh number rises over the critical Rayleigh number in the Rayleigh-Benard experimental setup, convection cells gradually form as the bottom layer heats up. Isothermal lines and streamlines serve as visual representations of such cells. We discuss these lines for different periods. Streamlines travel in two directions: one in a clockwise direction and the other in an anticlockwise way. Temperature variations with respect to the variable time scale τ give rise to isotherms. All of the aforementioned graphs describe the effects of parameters, gravity modulation, and internal heat source.

The following series of graphs, displayed in Fig. 6, examine the mean Nusselt number in relation to various parameters. Nu against Ω is plotted for different modified Vadasz numbers in Fig. 6(a). The mean Nusselt number decreases with an increase in Υ_a. Figs 6(b,c) and 6(e) show the same behavior, but in different ways. For altering Ra_B, respectively, Nu is plotted against α_0 and R_b in these graphs. However, Fig. 6(d) shows a different nature from the ones mentioned before. Plotting the mean Nusselt number against R_i in this case shows that it increases as D_a increases. The effect of Casson parameter χ on mean Nusselt number against internal Rayleigh number R_i is observed in Fig. 6(e). Increasing values of χ decreases mean Nusselt number which is the significant nature of Casson parameter.

The comparison of the analytical and numerical Nusselt number values derived for an unmodulated case of the amplitude equation is shown in Fig. 7. The coincidence of the heat transfer data shown here indicates a fair agreement between the numerical and analytical validations.

5. Conclusions

In our study, Darcy-Brinkman model is employed on Casson fluid under the effect of internal heating and gravity modulation in porous medium to study thermal bioconvection of gyrotactic microorganism. In the first stage, we developed a theory of linear stability on stationary mode of bioconvection to derive critical Rayleigh number which is used to analyse the onset of bioconvection by plotting marginal stability curves. In the second stage, nonlinear instability analysis is done to study heat transfer analysis in the system. The numerous parameters of the problem are visually displayed to demonstrate how they affect the system's heat transport and marginal stability. We apply perturbation theory to analyze the data, paying particular attention to the tiny parameter ε, which denotes the divergence from the threshold Rayleigh number. We have drawn several inferences from the results of our numerical study. These results shed light on the relationship between thermal bioconvection in porous medium saturated with Casson fluid and gravity modulation as well as internal heating effects. The following significant findings are derived from our findings:

1. The parameter Darcy number (D_a) leads to stabilize the system as we find delayed onset of convection.
2. The parameters namely, modified bioconvection Rayleigh number (Ra_B), cell eccentricity (α_0), internal Rayleigh number (R_i) and Casson parameter (χ) helps in destabilizing the system as advancement in onset of convection can be noticed.
3. As modified Vadasz number (Υ_a) is raised for a short interval of time, an increase in heat transfer is observed.
4. Because of internal heating effects and gravity modulation in Casson fluid-saturated porous media, an increase in the (Ra_B) number reduces heat transfer.
5. The spherical form of the microbe aids in increasing the efficiency of heat transmission mechanism. The more asymmetrical the shape, the less heat is transferred by the system.
6. An increase in the modulation frequency Ω has no effect on heat transport. The rate of heat transmission only rises with frequency.
7. Increasing the modulation amplitude δ improves heat transmission.
8. One important feature of the internal Rayleigh number R_i is that as R_i increases, heat transmission is improved.
9. The Casson parameter χ diminishes heat transfer on its increasing values due to its reciprocal behavior.
10. Convective cells are displayed in streamlines in a clockwise and counterclockwise pattern.
11. Convection increases with time scale with respect to temperature, as shown by isotherms.
12. Numerical and analytical results for Nusselt number are in good agreement with each other.

Our study sheds light on the relationship between stationary thermal bioconvection in a porous media saturated with non-Newtonian Casson fluid and internal heating and gravitational modulation. By analyzing the impacts of different factors on Υ_a, Ra_B, α_0, Ω, δ, R_i and χ, we can comprehend the convection process better. Acquiring this understanding will assist in efficiently managing and regulating the system's heat transfer. By modifying the above listed parameters, heat transfer in a system with a non-Newtonian fluid (NNF) in a porous media containing gyrotactic microorganisms can be enhanced or controlled. This knowledge could lead to the development of convection control methods, more efficient heat transfer systems, and improved thermal management across a range of applications. Casson fluid is often used to model blood flow in the human body. In the polymer industry, Casson fluid models are used to study the flow of polymer melts and solutions. This helps in optimizing the manufacturing processes of plastic products. Casson fluid models are applied to study the flow of food products like chocolate, ketchup, and honey. These fluids exhibit shear-thinning behavior, which is essential for processing and packaging.

Nomenclature

h thickness of the porous layer

T_d temperature at lower boundaries

T_u temperature at upper boundaries

g_0 Mean gravity

$\vec{g}$ acceleration due to gravity

e unit vector in the direction of z – axis

δ amplitude of gravity modulation

ω_g frequency of gravity modulation

Ω non-dimensional frequency of gravity modulation

ε small dimensionless parameter

V_D Darcy velocity

$\in$ porosity of the porous medium

T Temperature

T_0 reference temperature

ρ_0 density of the fluid at reference temperature

K permeability of the porous medium

P pressure

β thermal expansion coefficient

μ_c Brinkman effective viscosity and dynamic viscosity

μ fluid viscosity

$(\rho c)_f$ heat capacity of the fluid

χ Casson parameter

$(\rho c)_m$ effective heat capacity

k_m effective thermal conductivity

n concentration of microorganisms

$\delta\rho$ density difference

Υ average microorganism volume

D_m diffusivity of microorganisms

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Conflict of interest

The authors state that there is no conflict of interest.

Data availability statement

Data sets generated during the current study are available from the corresponding author on reasonable request.

Authors' contributions

PMB - Conceptualization, resources, supervision, validation, writing-review and editing.
APA - Investigation, methodology, resources, software, writing original draft.
AJC - Conceptualization, supervision, validation, formal analysis.

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Figure 1: Physical interpretation of the problem

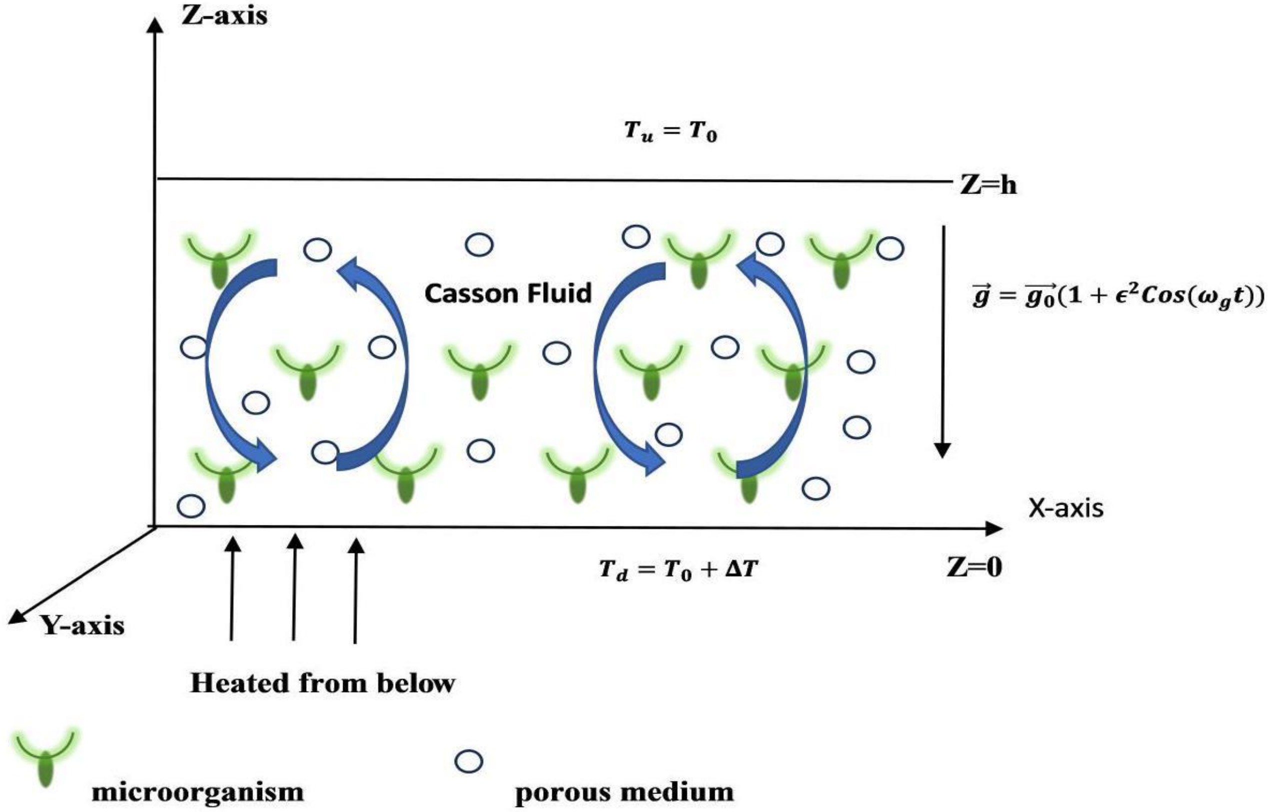


Table I: Numerical analysis of critical Rayleigh number Ra_c with respect to respective wavenumbers k_c for variable parameters:

Fixed Parameters	Variable Parameter	Ra_c	k_c
$\mathrm{Ra}_B = 15, \alpha_0 = 0.8$ $\chi = 1, \mathrm{R}_i = 0.2$	$Da =$ 0.3	154.012	2.34041
	$Da =$ 0.5	414.12	2.29041
	$Da =$ 0.8	808.673	2.26195
$Da = 0.5, \alpha_0 = 0.4,$ $\chi = 1, \mathrm{R}_i = 0.2$	$\mathrm{Ra}_B =$ 0.0	692.299	2.24928
	$\mathrm{Ra}_B =$ 15	440.365	2.37519
	$\mathrm{Ra}_B =$ 25	269.106	2.47756
$Da = 0.8, \mathrm{Ra}_B = 15$ $\chi = 1, \mathrm{R}_i = 0.2$	$\alpha_0 =$ 1.0	789.447	2.2385
	$\alpha_0 =$ 2.0	714.081	2.13535
	$\alpha_0 =$ 3.0	632.832	2.05069
$Da = 0.3, \mathrm{Ra}_B = 15$ $\alpha_0 = 0.8, \chi = 1$	$\mathrm{R}_i =$ 1.0	145.861	2.32738
	$\mathrm{R}_i =$ 5.0	104.387	2.23292
	$\mathrm{R}_i =$ 8.0	71.5258	2.08295
$Da = 0.3, \mathrm{Ra}_B = 15$ $\alpha_0 = 0.8, \mathrm{R}_i = 8$	$\chi =$ 0.1	803.156	1.6875
	$\chi =$ 0.3	400.495	1.7532
	$\chi =$ 0.6	128.175	1.95338

Figure 2: Stationary Rayleigh number Ra_c against wave number k_c

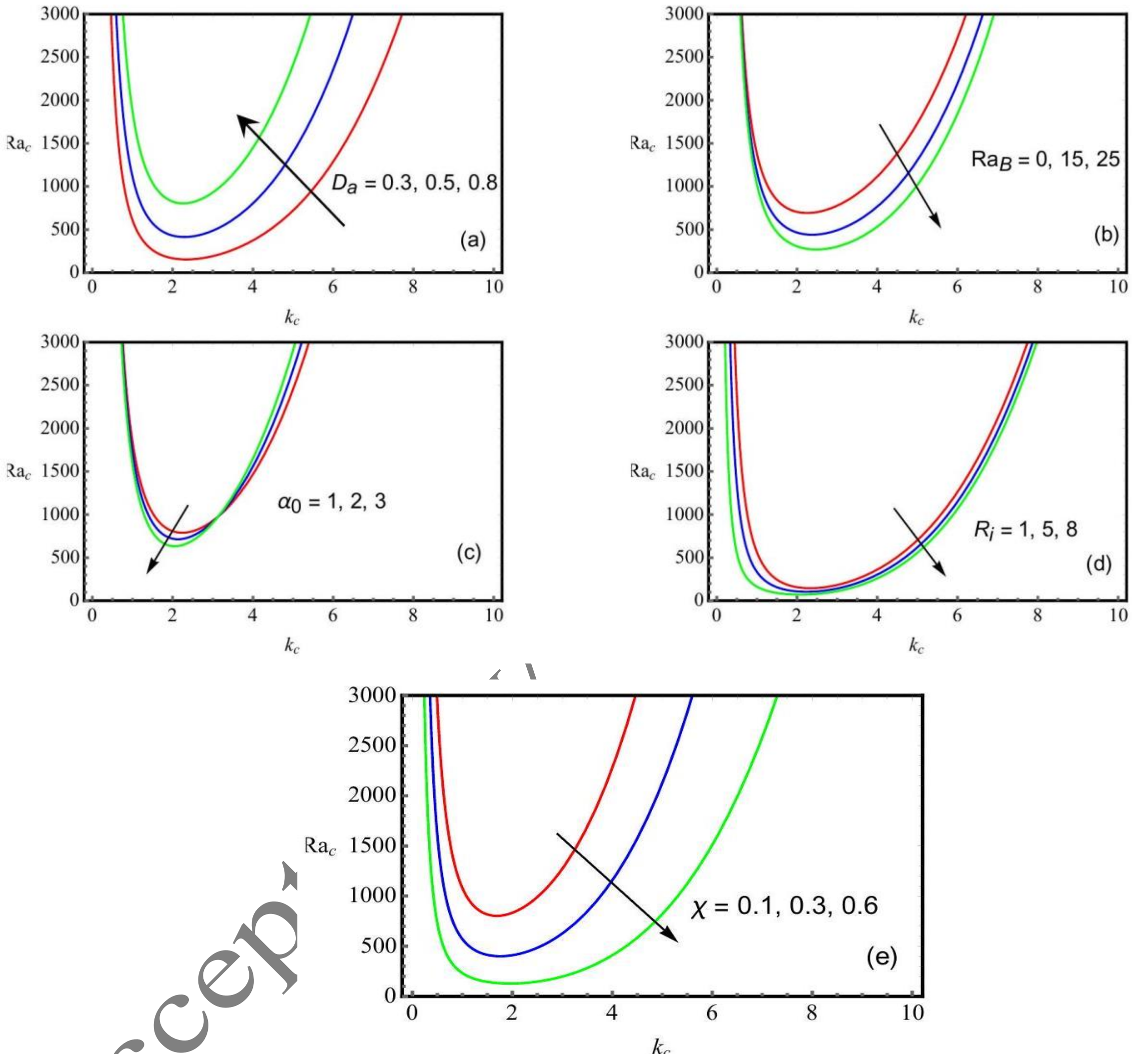


Figure 3: Graphical representation of the Nusselt number Nu versus time τ with respect to various parameters.

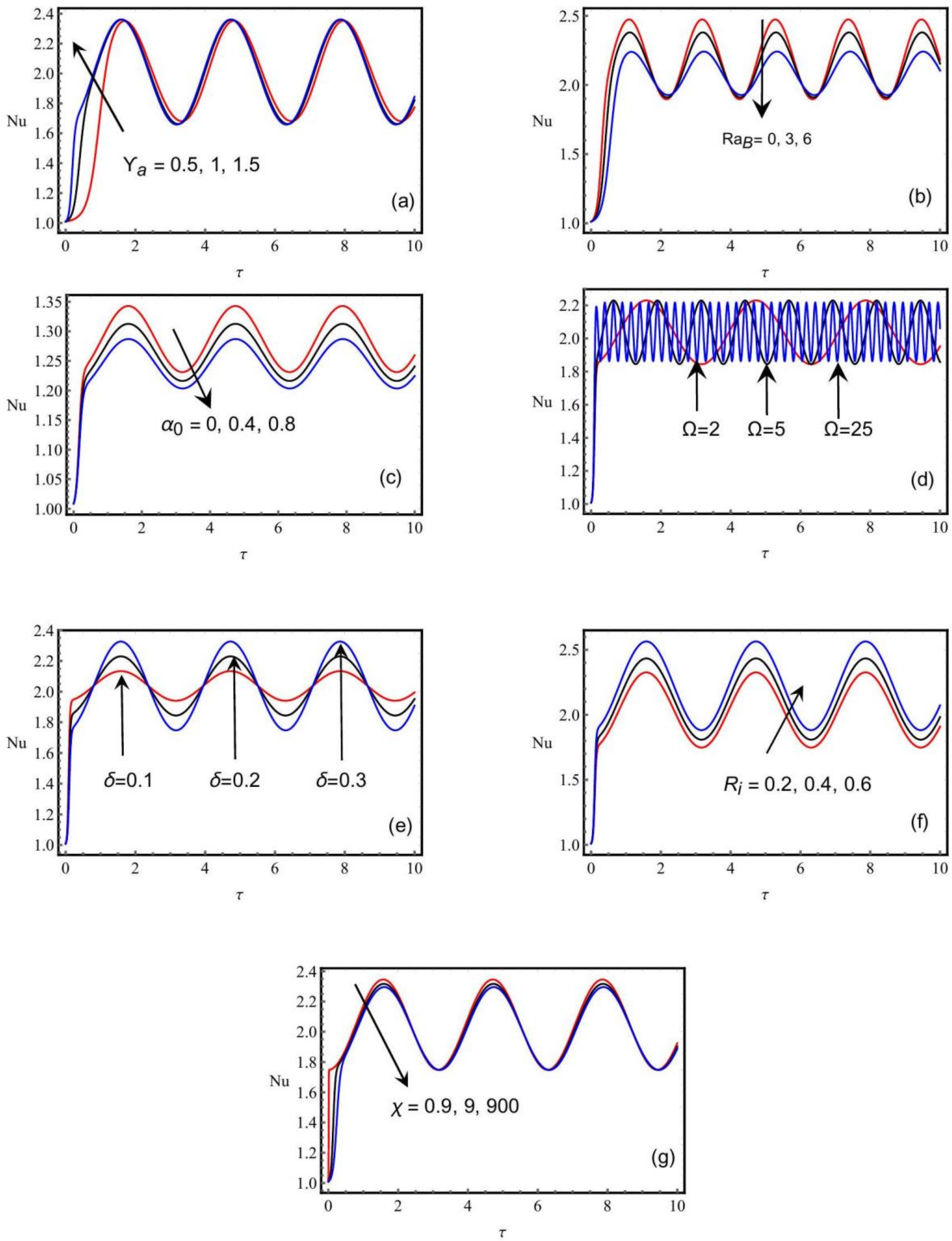
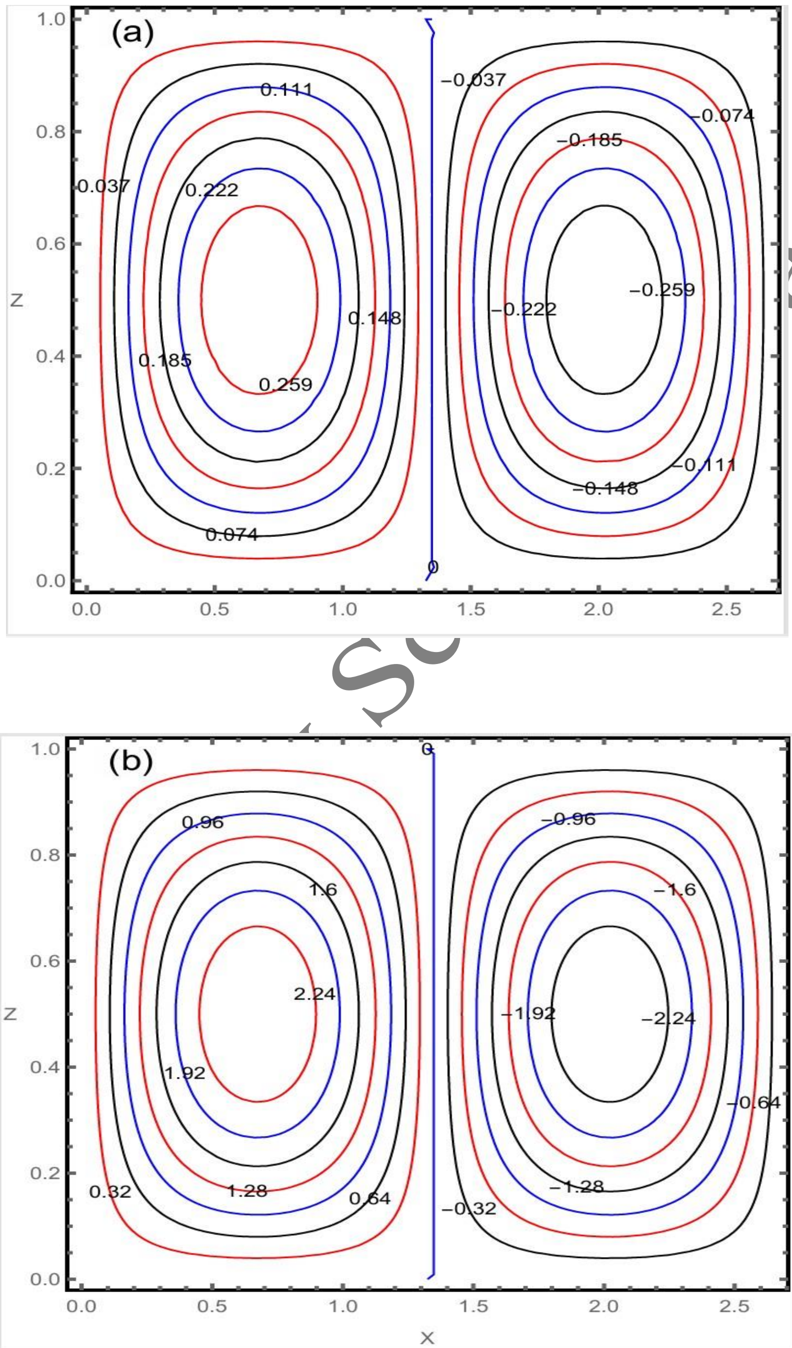


Figure 4: Streamlines : (a) $\tau = 0.0$, (b) $\tau = 0.1$, (c) $\tau = 0.4$



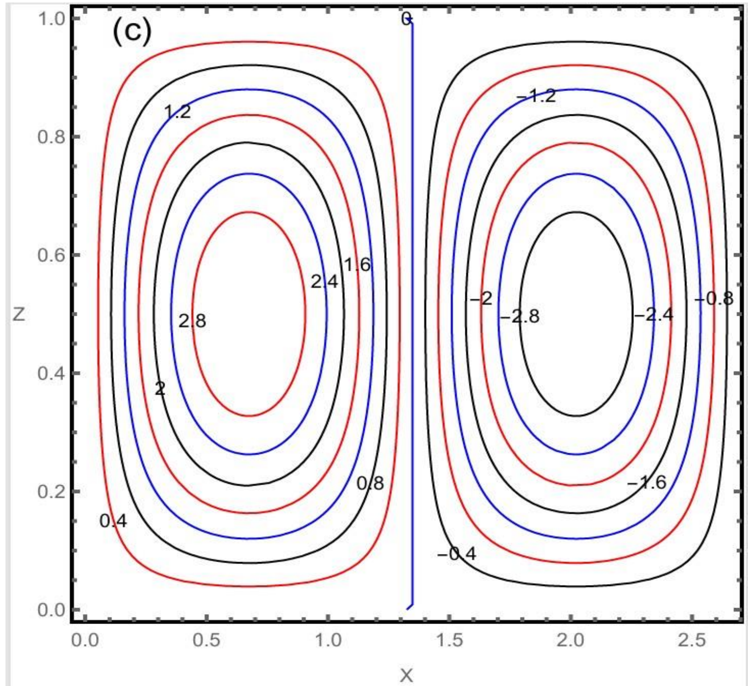
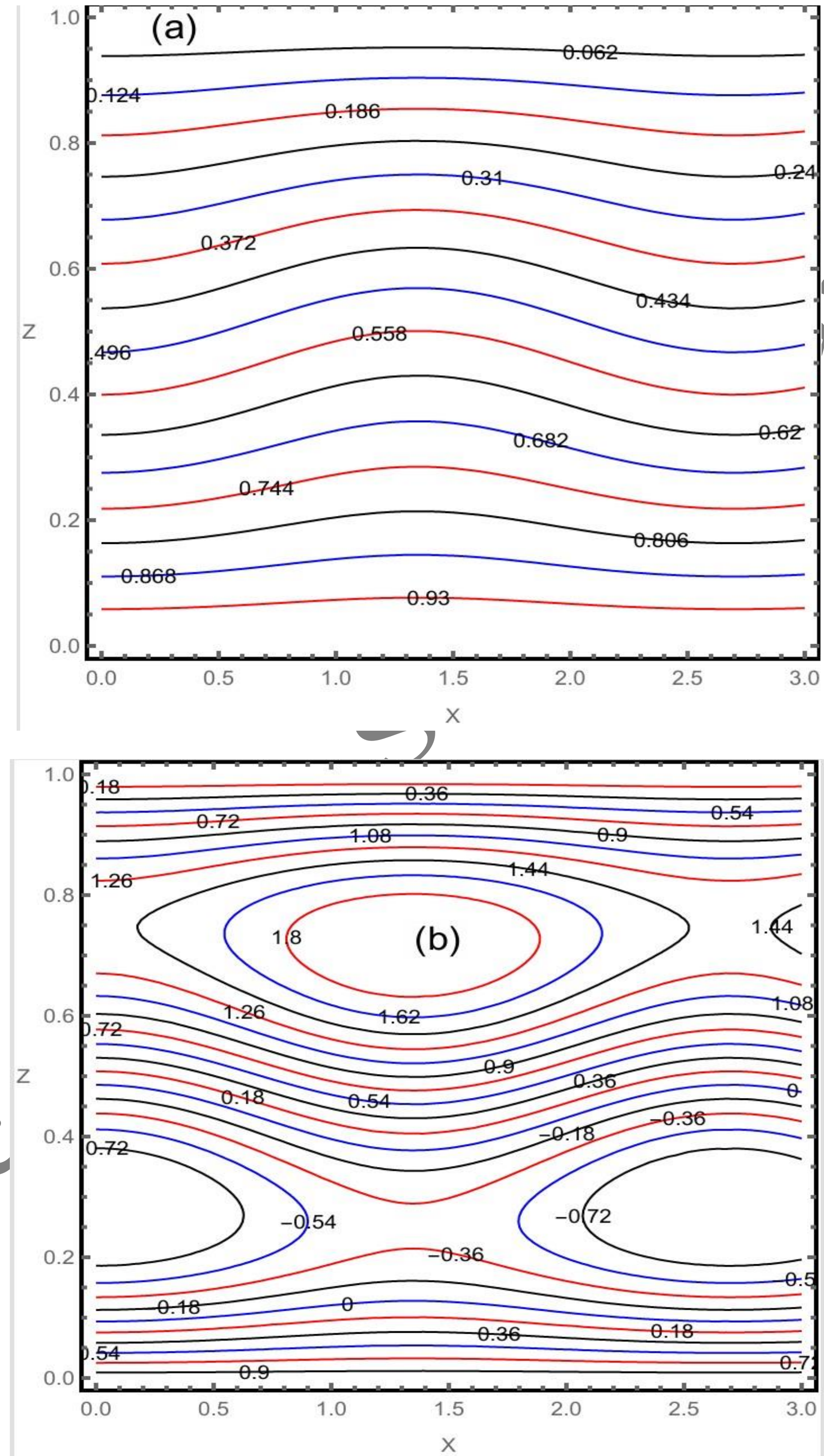

(c)
1.0
0.8
0.6
0.4
0.2
0.0
z
0.0
0.5
1.0
1.5
2.0
2.5
x
0
1.2
1.6
2.4
2.8
2
0.8
0.4
−1.2
−2
−2.8
−2.4
−0.8
−1.6
−0.4

Figure 5: Isotherm images at : (a) $\tau = 0.0$, (b) $\tau = 0.1$, (c) $\tau = 0.4$



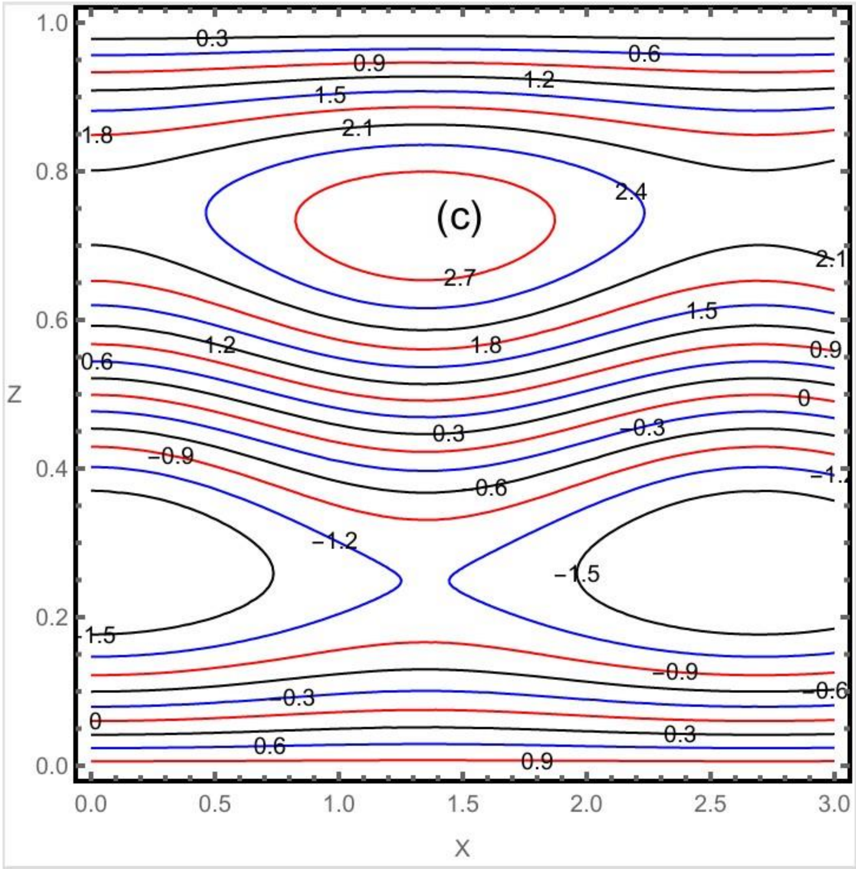

(c)
z
x
0.0
0.2
0.4
0.6
0.8
1.0
0.5
1.5
2.0
2.5
3.0

Figure 6: The graphical representation of the mean (average) Nusselt number Nu with respect to various parameters.

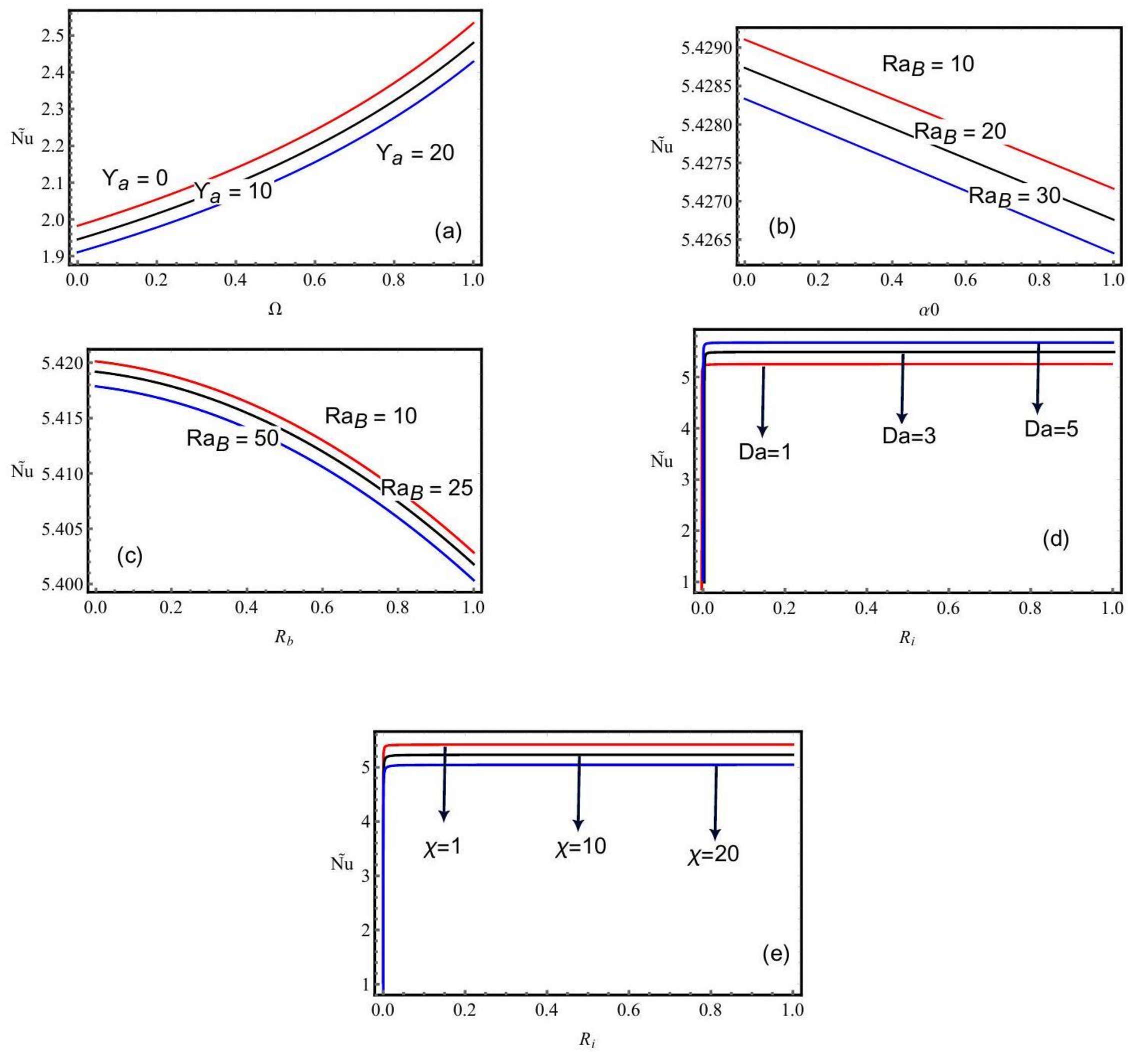


Figure 7: Analysis between analytical and numerical results of Nusselt number.