Designing a new adaptive multivariate control chart for simultaneously monitoring mean and variability of process under effects of multiple assignable causes

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Abstract

There have been some advances in multivariate control charts with the ability to monitor both the mean and variability of processes. However, due to the complexity of production processes, the assumption of single assignable causes is not close to the real-life conditions. As a novel contribution, this article proposes a new control chart for monitoring the mean and variability of a multivariate normal process which is under effects of multiple assignable causes. We develop a Markov chain model to compute the average run length and average time to signal (ATS) values. We also make it fully adaptive by varying all control chart parameters. The presented model involves complex non-linear models with a mix of continuous and discrete decision variables, and discontinuous, non-convex solution spaces. Therefore, one of the most suitable metaheuristic search approaches, Genetic algorithm is implemented. Numerical examples based on the Taguchi method are presented and sensitivity analyses are conducted to measure the performance of the proposed chart.

Keywords: Multivariate control chart, fully adaptive, Simultaneous monitoring, Markov chain, Taguchi, Genetic algorithm

1. Introduction

Traditionally, effective maintenance policies were implemented to increase the reliability of production systems and, subsequently, enhance the operational performance of the equipment (Taji et al. [1], Taji et al. [2]). However, focusing solely on reliability is not sufficient; it is equally important to consider the quality of the products by stabilizing the processes. So, in order to stabilize the processes and control the variability of the production process, process monitoring is applied to ensure that delivered products have satisfying quality. For this purpose, the most important tools are process control charts as statistical process control (SPC) tools. Researchers have been concentrating constantly on improving different forms of control charts in recent years. Univariate control charts are employed if the products have only one quality characteristic, while multivariate control charts are used in the case of more than one quality characteristic. One way of detecting small and moderate shifts in the process parameters (variability and mean) is by using adaptive control charts. This type of charts allows the charts parameters (sampling interval, sample size and control limits) to vary which is named variable parameters (VP). It is worth noting that some studies have only examined the process mean (Askari, M et al. [3], Ali Salmasnia et al. [4], Jafarian-Namin, S et al. [5]) and others have only monitored the process deviation (Saeed, N et al. [6]), while many recent studies have monitored both parameters simultaneously. Recently, Sabahno and Khoo [7] simultaneously monitored mean and variability of process under one assignable cause by an adaptive multivariate control chart. Taji et al. [8] presented an adaptive multivariate control chart that considered multi-assignable causes. The VP adaptive feature is also considered on multivariate control charts in many other studies such as: Faraz et al. [9] and Seif et al. [10] Aparisi et al.[11], Aparisi and Haro [12], Grigoryan and He [13], Sabahno et al. [14, 15], Taji et al. [16] and Lee and Khoo [17]. In all of the above studies either the process mean or process variability is monitored using adaptive control charts on multivariate production processes.

It has been proven that one way of improving the performance of the monitoring procedure and reducing the false alarm rates is simultaneous monitoring of the process parameters (mean and variability). Control charts that simultaneously monitor the parameters are divided into two categories single-chart and double-chart (one for each parameter) schemes. Some researchers who used two separate multivariate charts are mentioned in the following: Reynolds and Cho [18], Hawkins and Maboudou-Tchao [19], and Zhang and Chang [20]. Also, Khoo [21], Zhang, Li, and Wang [22], Wang, Yeh, and Li [23] and Sabahno et al. [24] used single multivariate charts. Concerning using adaptive features in simultaneous monitoring schemes, the following research can be mentioned Reynolds and Kim [25], Reynolds and Cho [26], and Sabahno, Castagliola, and Amiri [27, 28] and Sabahno et al. [24].

In case of dependence of estimators for mean and variability, the form of the coefficient of variation (CV) is used for measuring the variability, which is computed by dividing the standard deviation by the mean. Many researchers proposed Univariate control charts for monitoring the CV that we can refer to Castagliola et al. [29] and for adaptive ones to Castagliola et al. [30]. Yeong et al. [31] introduced the initial control chart for multivariate coefficient of variation (MCV). Nguyen et al. [32] and Khaw et al. [33] later enhanced their approach by incorporating adaptive features into their scheme.

Sabahno et al. [24] put forward a novel concurrent approach for monitoring the mean vector and covariance matrix of a process. They enhanced it with adaptive capabilities and assessed various performance metrics utilizing a Markov chain model. In a pioneering move in the field of

Statistical Process Control (SPC) control charts, they introduced a statistic to represent process variation. However, the distribution of this statistic becomes less appropriate for datasets with more than two quality characteristics. So, the mentioned issue was resolved by Sabahno et al. [7] by proposing a new statistic that has a deterministic distribution.

As our knowledge allows us, studies related to multivariate adaptive control charts that simultaneously monitor the mean and variability consider only one assignable cause (AC). The proposed multivariate adaptive control chart in this research simultaneously monitors the mean and variability and also considers multiple assignable causes for the first time in this field that make it so closer to the real life.

The paper is structured as follows: Section 2 presents problem statement and section 3 focuses on the development of VP adaptive features for the proposed scheme. Then, performance measures, derived using a Markov chain model, are discussed in Section 4. The employment of a GA to solve the model is explained in Section 5. Also, Section 6 presents the results of numerical analyses conducted to evaluate the proposed scheme. Finally, concluding remarks can be found in Section 7.

Notations:

m	Number of the assignable causes
r	Number of the quality characteristics
ω	Chart statistic
$\boldsymbol{\mu}_j$	Vector denoting the mean of the quality characteristics in state j ($j = 1, \ldots m$)
δ_j	Mahalanobis distance while the process is under the effect of AC j'th (j=1,…, m)
λ_j	The rate of the exponential distribution associated with assignable cause j (j=1,…, m)
$\phi(\, . \,)$	the standard normal cumulative distribution function
$H_r(\, . \,)$	chi-square cumulative distribution function
$F_{r.n-r.\delta}(\, . \,)$	the Fisher cumulative distribution function
MV	max-type statistic
UCL_1, UCL_2	The upper control limits of the control chart in the relaxed and tightened states respectively.
w	Warning limit
h_1, h_2	Sampling intervals in the relaxed and tightened states respectively.
n_1, n_2	Sampling size of the control chart in the relaxed and tightened states respectively.
RP	Relaxed Parameters
TP	Tightened Parameters
α	Type I error

β	type II error
ECT	The long run expected cost per time unit
EC	The expected cost of a transition epoch
ET	The expected duration of a transition epoch
T_j	Time to search and remove assignable cause j $(j = 1,2,\ldots,m)$
c	Fixed sampling cost
b	Variable sampling cost
R_j	Operational cost of the process per time unit in state j ($j = 0,1,\ldots,m$)
L_j	The cost of removing the effect of AC j'th ($j = 1,2,\ldots,m$)
L_0	The cost of investigation of a false alarm
$ENOF$	expected number of false alarms per time unit
ARL_0	Average run lengths in the in-control states
ARL_1	Average run lengths in the out-of-control states

2. Problem Statement

Taji et al. [8] developed a novel scheme to monitor the mean of normal multivariate processes. They applied one of the widely used control charts, Chi-square control chart. However, they assumed that assignable causes do not affect the variability. To make the model closer to the real life, we relax this assumption by proposing one single control chart that monitors mean and variation of the process simultaneously. Also, Sabahno and Khoo [7] proposed a multivariate adaptive control chart to simultaneously monitor the process parameters. However, they considered only one assignable cause that is far from real life. We assumed that more than one AC can affect the process.

We consider a production process which has *r* critical quality characteristics (QC) so that the QCs form a multivariate normal distribution. In the in-control state, the mean vector of QCs is $\boldsymbol{\mu}_0 = (\mu_1,\ \mu_2,\ \cdots,\ \mu_r)$ and covariance matrix is $\boldsymbol{\Sigma}_0$. It is assumed that the process is under the effect of *m* assignable causes (AC) and AC j'th changes mean vector to $\boldsymbol{\mu}_j$. The following statistic is computed:

$$\omega = n\left(\bar{\boldsymbol{X}} - \boldsymbol{\mu}_0\right)' \boldsymbol{\Sigma}_0^{-1} \left(\bar{\boldsymbol{X}} - \boldsymbol{\mu}_0\right) \qquad (1)$$

In state 0, i.e., in-control state, ω has a chi-square distribution with r degrees of freedom. In state j ($j = 1,2,\ldots,m$), ω follows a non-central chi-square distribution with r degrees of freedom and non-centrality parameter $n\delta_j$.

Where $\delta_j = n\left(\boldsymbol{\mu_j} - \boldsymbol{\mu}_0\right)' \boldsymbol{\Sigma}_j^{-1}(\boldsymbol{\mu_j} - \boldsymbol{\mu}_0)$, It is noteworthy that δ_j is affected by covariance matrix as well. In this article we assume that $\boldsymbol{\Sigma}_j = \rho_j\, \boldsymbol{\Sigma}_0$. So we have new δ_j as following:

$$\delta_j = n\left(\boldsymbol{\mu_j} - \boldsymbol{\mu}_0\right)' \left(\rho_j\, \boldsymbol{\Sigma}_0\right)^{-1} (\boldsymbol{\mu_j} - \boldsymbol{\mu}_0) \qquad (2)$$

Also, the time until the occurrence of *AC i'th* is assumed to follow exponential distribution with rate λ_i, so that $\lambda = \sum_{i=1}^{m} \lambda_i$. The choice of the exponential distribution for modeling the time to manifestation of out-of-control causes is primarily due to its consistency with process behavior. In many industrial and quality control contexts, the occurrence of assignable causes can often be modeled as a random process with a constant average rate. The exponential distribution fits well in such scenarios where the time between occurrences is exponentially distributed. Additionally, the exponential distribution has a simple mathematical form, which makes it analytically tractable and easy to work with when developing control charts. This simplicity also facilitates the estimation of parameters and the derivation of control limits.

Letting $T^2 = n\bar{\boldsymbol{X}}^T \boldsymbol{\Sigma}^{-1} \bar{\boldsymbol{X}}$ gives:

$$\Upsilon = \frac{T^2}{n-1} . \frac{n-r}{r} : F_{r,n-r,\delta_0}(x) \tag{3}$$

Sabahno and et al. [7] has shown that letting $\tilde{\gamma} = (\bar{X}^T \boldsymbol{\Sigma}^{-1} \bar{\boldsymbol{X}})^{-\frac{1}{2}}$ and substituting $n\tilde{\gamma}^{-2}$ for T^2 in Eq 3 gives,

$$\Upsilon = \frac{n\tilde{\gamma}^{-2}}{n-1} . \frac{n-r}{r} : F_{r,n-r,\delta_0}(x) . \tag{4}$$

Υ can be used for monitoring the variability even in case of dependency between the process mean and variability.

Following transformations are used to monitor the mean and variability simultaneously:

$$M = \phi^{-1}\left[H_{r,\delta}(\omega)\right] \tag{5}$$

$$CV = \phi^{-1}\left[F_{r,n-r,\delta}(\Upsilon)\right] \tag{6}$$

the following max-type statistic is used:

$$MV = \max\left\{|M|,|CV|\right\} \tag{7}$$

We propose some general equations so that by replacing control limits in different cases, probabilities can be calculated (See Appendix A).

3. Adaptive chi-square control chart and computations of type I and II errors

Control/warning limits (UCL/w), sampling interval (h) and sample size (n) are parameters of a control chart. Varying these parameters according to the current sample makes adaptive control chart. The parameters are divided into two categories: relaxed parameters (RP) and tightened parameters (TP).

UCL_1, h_1, n_1 are relaxed parameters and UCL_2, h_2, n_2 are considered as TP so that $n_2 \geq n_1$, $h_2 \leq h_1$ and $UCL_2 \leq UCL_1$. It has been proved by Park & Reynolds [34] that considering two warning limits has no major effect on the cost reduction and just increases the complexity of the chart. So, we considered only one warning limit in this paper.

Three regions of the proposed control chart include: central region $[0,w]$, warning region $[w,UCL_s]; s=1,2$ and out-of-control region $[UCL_s,\infty]; s=1,2$. At each sampling point, the following decisions must be made: (1) whether to search for an AC or not, (2) the parameters of the next sampling epoch. Depending on where the statistic is placed, the following activities are carried out.

If the statistic is within the control limit, the process does not stop and RPs are used for the next sampling point. If the statistic is within the warning limit, the process continues its operation but in the next point, TPs are applied. Finally, if the chart statistic falls in the out-of-control region, the process is stopped to investigate possible AC and then RP are the next sampling epoch parameters.

Accordingly, type I error is computed as in the following:

$$1-\alpha_s = P\left(MV \le UCL_s\right) = P\left(\max\left\{|M|,|CV|\right\} \le UCL_s\right) \tag{8}$$

$$= P\left(-UCL_s \le M \le UCL_s\right) P\left(-UCL_s \le CV \le UCL_s\right)$$

$$= \left(\phi\left(UCL_s\right)-\phi\left(-UCL_s\right)\right)\left(\phi\left(UCL_s\right)-\phi\left(-UCL_s\right)\right) = \left(\phi\left(UCL_s\right)-\phi\left(-UCL_s\right)\right)^2$$

$$;\ \forall s = 1,2$$

Then,

$$\alpha_s = 1-\left(\phi\left(UCL_s\right)-\phi\left(-UCL_s\right)\right)^2 \tag{9}$$

If AC j'th affects the process, the probability that the statistic is within the control limit (type II error) is equal to:

$$\beta_s = P\left(MV \le W \mid \delta = \delta_j\right) = P\left(\max\left\{|M|,|CV|\right\} \le W\right) \tag{10}$$

$$= P\left(-W \le M \le W\right) P\left(-W \le CV \le W\right)$$

$$= (H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(W\right)\right)\times\left(\frac{1}{\rho_j}\right)\right) - H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-W\right)\right)\times\left(\frac{1}{\rho_j}\right)\right))$$

$$\times(F_{r,n_s-r,\delta_j}\left(F_{r,n_s-r,\delta_0}{}^{-1}\left(\phi\left(cv_2\right)\right)\right)-F_{r,n_s-r,\delta_j}\left(F_{r,n_s-r,\delta_0}{}^{-1}\left(\phi\left(cv_1\right)\right)\right))\quad s=1,2;\; j=1,2,\ldots,m$$

4. Markov Chain model

Transition among states is calculated using a proper Markov chain. At each sampling point *t*, the pair $\left(z_t, a_t\right)$ represents the state of the Markov chain. If the process is under control $z_t = 0$, and in cases where AC i'th affects the process $Z_t = i(i = 12\ ...m)$ Also, $a_t = 0$ in the event that $0 < \omega_t < w$. Additionally, if the chart statistic falls in the warning region, (i.e., $w < \omega_t < UCL_s;\ s = 1,2)$ $a_t = 1$. Finally, $a_t = 2$ if the chart statistic falls in the out-of-control region that we call it action zone (i.e.. $UCL_s < \omega_t; s = 1,2)$. Accordingly, a $3\left(m+1\right)\times 3\left(m+1\right)$ matrix with the following structure in Figure 1 can define the transition of the states of the Markov chain.

Formulas of Transition probabilities

Suppose the process starts from the in-control state. For a period with length $h_s\left(s=1,2\right)$, the probability of the process going out of control is equal to $1-e^{-\lambda h_s}$. Therefore, the probability of occurrence of AC j'th during this period is equal to the following formula:

$$q_j\left(h_s\right)=\frac{\lambda_j}{\lambda}\left(1-e^{-\lambda h_s}\right);\; j=1,2,\ldots,m; s=1,2 \tag{11}$$

Elements of the Markov matrix (P_{ik}^{jl}) are defined as in the following:

$$P_{ik}^{jl} = P[a_t = l, Z_t = j \mid a_{t-1} = k, Z_{t-1} = i] \tag{12}$$

Matrix $\mathbf{P}$ arrays are categorized and calculated according to the formulas provided in Appendix B.

By solving the following linear equations, the steady state probabilities of the Markov matrix can be computed.

$$\pi_{lj} = \sum_{k=0}^{m}\sum_{i=0}^{2}\pi_{k,i}\, p_{k,i}^{l,j};\ \sum_{l=0}^{m}\sum_{j=0}^{2}\pi_{l,j} = 1 \tag{13}$$

The long run expected cost per time unit, ECT , is equal to the expected cost of a transition epoch, EC , over the expected duration of a transition epoch, ET .

$$ECT = \frac{EC}{ET} \tag{14}$$

The weighted average of the expected costs of all states forms the average cost of one transition step, and also the weighted average duration of all states produces the average duration of one transition step. Therefore, EC and ET give the following equations:

$$ET = h_1\sum_{i=0}^{m}\pi_{i0} + \sum_{i=0}^{m}\pi_{i2}\left(h_1 + T_i\right) + h_2\sum_{i=0}^{m}\pi_{i1} \tag{15}$$

$$EC = c + bn_1\sum_{i=0}^{m}(\pi_{i0} + \pi_{i2}) + bn_2\sum_{i=0}^{m}\pi_{i1} + \sum_{i=0}^{m}\pi_{i2}L_i + \sum_{i=0}^{m}\left(\pi_{i0}\zeta_i\left(h_1\right) + \pi_{i1}\zeta_i\left(h_2\right) + \pi_{i2}\zeta_0\left(h_1\right)\right) \tag{16}$$

In the last term of relation 16, $\zeta_i\left(h_s\right); s = 1,2$ denotes the expected operational costs of the process given that the process is under the effect of *AC 'i'th* at the start of an interval with length h_s . It is noteworthy that if the process is under the effect of *AC 'i'th* at the start of an interval, the process will not be affected by any other ACs and also, until the end of the process, AC will not be removed. Then, the following equation is derived to compute $\zeta_i(h_s)$.

$$\zeta_j(h_s) = R_j h_s; s = 1, 2; \ j = 1, 2, \ldots, m \tag{17}$$

To compute $\zeta_0(h_s)$, the following equation is derived:

$$\zeta_0(h_s) = R_0 h_s e^{-\lambda h_s} + \sum_{j=1}^{m} \int_0^{h_s} \frac{\lambda_j}{\lambda} \lambda e^{-\lambda t} \left(R_0 t + \zeta_j(h_s - t) \right) dt; s = 1,2 \tag{18}$$

The first term of Equation 18 computes the expected operational cost in the in-control state given that the process remains in the in-control state until the end of the interval. If the process is in-control at the beginning of the interval, the process may not remain in the in-control state until the end of the interval and assignable cause may occur during the interval. In this case, the second term of Eq 18 corresponds to the case that the process shifts from the in-control state to state j $(j = 1,2,\ldots,m)$ at time point *t* and remains in this state until the end of the interval.

Using the steady state probabilities of the Markov chain, the following equation can be employed to compute the probability of Type I error:

$$\alpha = \frac{\pi_{02}}{\pi_{00} + \pi_{01} + \pi_{02}} \tag{19}$$

The equation for computing expected number of false alarms per time unit (*ENOF*) is as follows:

$$ENOF = \frac{\pi_{02}}{ET} \tag{20}$$

For the power of the control chart in detecting the out-of-control states, it was proved by (Nenes et al., 2015) that the following equation holds:

$$1-\beta=\sum_{i\neq 0}\left(\frac{\pi_{i2}}{1-\left[\pi_{00}+\pi_{01}+\pi_{02}\right]}\right) \tag{21}$$

Average run lengths in the in-control and out-of-control states can be computed according to the probability of Type I and type II errors as in the following:

$$ARL_0=\frac{1}{\alpha} \tag{22}$$

$$ARL_1=\frac{1}{1-\beta} \tag{23}$$

Since time is the most important element in process monitoring schemes, Average Time to Signal (ATS) is by far the most important performance measure that should not be ignored. It can be calculated as bellow:

$$ATS=ET*ARL_1 \tag{24}$$

5. Solution technique

Due to the simultaneous existence of continuous and discrete variables in presented model, the problem solving space is non-convex (Salmasnia et al. [35]). For example, the sample size is a discrete variable, but the control limits are continuous. Also, in some mathematical relations, the decision variables are in integral limits, and some decision variables are in the cumulative density function of non-central chi-square distribution. Therefore, due to the complexity of the functions, it is almost impossible to use exact classical optimization methods (Panagiotido and Tagaras [36], Linderman et al. [37]).

Meta-heuristic methods are highly efficient in solving problems with real dimensions and complex mathematical models, because these methods reach acceptable solutions in a short period of time. In this article, in order to solve the mathematical models, the genetic algorithm (GA) is used, which is one of the most widely used meta-heuristic algorithms used in the field of study. The GA technique is most suitable, being simpler than exact algorithms and successfully applied previously (Shojaee M et al. [38]).

Using Matlab 2021 Optimization Toolbox, the results of solving four presented examples using the complete counting method and GA algorithm, as well as their solution times, are shown in Tables 1 and 2.

As can be seen from the Table 1 and Table 2, in the first three examples, the genetic algorithm obtains almost the same answers as the complete counting method, with the difference that it has a much lower solution time. For the fourth example, with the increase of the solution space, the complete counting method is not able to find the answer even in a time equal to 50,000 seconds, but the genetic algorithm reaches the answer in a short time. Therefore, by observing the results, we can conclude that the accuracy and ability of the genetic algorithm in reaching optimal solutions in relatively simple problems is acceptable. So, it is expected that this algorithm will have similar performance in the complex models that are proposed in the upcoming sections.

6. Numerical Examples

In this section, the behavior of the chart is shown for different values of parameters (See Table3). Small (0,0.1), moderate (0.5,1) and large (3) mean shifts are considered separately, where $m=3$ possible assignable causes occur, $L_0=50, \lambda=0.1,\ c=1, R_0=0, \rho_0=1, b=0$. A robust parameter design is done using Taguchi Method to study the effects of the parameters with fewer number of cases (See Table 4). Therefore, based on Taguchi Method, 25 cases are investigated.

The results of this analysis which are presented in Table 5 show that:

When either the mean or covariance matrix shifts from their in-control values, the chart signals. After a slight decrease in the chart's ARL/ATS, we have an increasing value up to about $\delta = 0.5$ and then performance of the chart gets better and signals faster (See Figure 2 and Figure 3). Also, the ARL and ATS trends are very close in most cases.

We also ran the analyses for the cases of $r = 2,\ \rho \in \{0.25,\ 0.5,\ 1,\ 1.5,\ 3\}$ and mean shifts $\in \{0,\ 0.1,\ 0.5,\ 1,\ 3\}$, while other assumptions are $m = 3$ possible assignable causes occur, $L_0 = 50,\ \lambda = 0.1,\ c = 1,\ R_0 = 0, \rho_0 = 1,\ b = 0\ R = 150, L = 100$. The results are shown in the Table 6.

7. Discussion

The adaptive nature of the multivariate control chart allows for real-time adjustments based on current process data. This leads to more accurate detection of shifts and variability in process parameters, enabling more effective responses to deviations. By monitoring both the mean and variability, managers can obtain a more comprehensive view of the process performance, reducing the likelihood of overlooking critical changes that could affect product quality.

It can be seen from figure 4-8, ECT decreases with the increase in the size of the shifts. The only exception is where the variation shift is large. As an example, in the case of $\rho = 1.5$ when $\delta = 0$ the ECT is 137.3, while for $\delta = 1$ ECT becomes 98.1. The logical explanation for the aforementioned conclusions is that an increase of δ means that identifying assignable causes by the chart is easier and it subsequently decreases the value of ECT. Also, for downward variation shift and the same mean shifts, ECT decreases as ρ grows. For instance, for the case of $\rho = 0.25$ and $\delta = 3$, ECT is equal to 134.62, While for the case of $\rho = 1$ and $\delta = 3$, ECT becomes 94.4.

When variation shift is downward (i.e. $\rho \in \{0.25,\ 0.5,\ 1)$, the performance of the chart for small mean shifts is almost similar in all cases and ATS is declining. For example, for all cases where δ is equal to one, ATS is equal to 1.36 for $\rho = 0.25$, ATS is equal to 1.35 for ρ=0.5 and for $\rho = 1$ is equal to 1.12. The reason is that it is easier for the chart to detect changes in cases of bigger sigma.

For the case of $\rho = 0.25$, ATS first peaks as δ increases from 1.21 to 2.29 and then decreases to 0.69 but ECT is still declining. For the case where variation is under control, as δ increases to 3, the performance of the chart drops and ATS reaches 1.45.

We also examined the effect of changes in r on ATS. As it is clear from Figure 9, increasing the number of qualitative characteristics does not have an increasing effect and pattern.

8. Conclusion

This article proposed a new multivariate adaptive control chart for simultaneously monitoring mean and variability of the multivariate normal process which was under the effects of multiple assignable causes. Genetic algorithm was performed as the presented model was non-convex. The performance measures of the control chart, i.e., ATS and ARL, was calculated using a developed Markov chain model. We designed and conducted numerical analyses based on a special method called Taguchi method and simple method. Taguchi analysis was performed in order to study the effects of parameters on the performance of the proposed chart. For future work, we suggest to extend the proposed model through integrating with other fields such as production and inventory planning.

Disclosure Statement

The author reports there are no competing interests to declare.

Appendix A. General equations for calculating probabilities

In case of out of control ($\delta = \delta_j$) we have:

$$P\left(m_1 \le M \le m_2\right) = P\left(m_1 \le \phi^{-1}\left[H_r(\omega)\right] \le m_2\right) = P\left(\phi(m_1) \le \left[H_r(\omega)\right] \le \phi(m_2)\right) \tag{A.1}$$

$$= P\left(H_r^{-1}\phi(m_1) \le \omega \le H_r^{-1}\phi(m_2)\right)$$

$$= H_{r,\delta_j}\left(H_r^{-1}\left(\phi(m_2)\right) \times \left(\frac{1}{\rho_j}\right)\right) - H_{r,\delta_j}\left(H_r^{-1}\left(\phi(m_1)\right) \times \left(\frac{1}{\rho_j}\right)\right)$$

Similarly, for CV:

$$P\left(cv_1 \le CV \le cv_2\right) = P\left(cv_1 \le \phi^{-1}\left[F_{r,n-r,\delta}(\Upsilon)\right] \le cv_2\right) \tag{A.2}$$

$$= P\left(\phi(cv_1) \le \left[F_{r,n-r,\delta}(\Upsilon)\right] \le \phi(cv_2)\right) = P\left(F_{r,n-r,\delta_0}{}^{-1}\phi(cv_1) \le \Upsilon \le F_{r,n-r,\delta_0}{}^{-1}\phi(cv_2)\right)$$

$$= F_{r,n-r,\delta_j}\left(F_{r,n-r,\delta_0}{}^{-1}\left(\phi(cv_2)\right)\right) - F_{r,n-r,\delta_j}\left(F_{r,n-r,\delta_0}{}^{-1}\left(\phi(cv_1)\right)\right)$$

In case of in-control state we have ($\delta = \delta_0$ *and* $\rho_j = 1$ *for all* j):

$$P\left(m_1 \le M \le m_2\right) = P\left(m_1 \le \phi^{-1}\left[H_r(\omega)\right] \le m_2\right) = P\left(\phi(m_1) \le \left[H_r(\omega)\right] \le \phi(m_2)\right) \tag{A.3}$$

$$= P\left(H_r^{-1}\phi(m_1) \le \omega \le H_r^{-1}\phi(m_2)\right) = H(H_r^{-1}\phi(m_2) - H(H_r^{-1}\phi(m_1) = \phi(m_2) - \phi(m_1)$$

Similarly, for CV, the following equation is derived

$$P\ (cv_1 \le CV \le cv_2) = P\left(cv_1 \le \phi^{-1}\left[F_{r,n-r,\delta}(\Upsilon)\right] \le cv_2\right) \tag{A.4}$$

$$= P\left(\phi(cv_1) \le \left[F_{r,n-r,\delta}(\Upsilon)\right] \le \phi(cv_2)\right)$$

$$= P\left(F_{r,n-r,\delta}(\Upsilon)^{-1}\phi(cv_1) \le \Upsilon \le F_{r,n-r,\delta}(\Upsilon)^{-1}\phi(cv_2)\right)$$

$$= F_{r,n-r,\delta}(F_{r,n-r,\delta}(\Upsilon)^{-1}\phi(cv_2) - F_{r,n-r,\delta}(F_{r,n-r,\delta}(\Upsilon)^{-1}\phi(cvt_1) = \phi(cv_2) - \phi(cv_1)$$

Appendix B. Arrays of the transition probability matrix P

For cases that AC j’th affects the process at the beginning of the interval, and the chart statistic is within the central or warning scope, no signal is issued by the chart. So, the AC is not removed. The transition probability for these cases can be calculated from the equation B.1-22.

For cases that chart statistic is within the central scope at the beginning and the end of the interval:

$$P_{j0}^{j0} = P\left[a_t = 0, Z_t = j | a_{t-1} = 0, Z_{t-1} = j\right] = P\left(MV \le W | \delta = \delta_j \ and \ s = 1\right) \tag{B.1}$$

$$= P\left(\max\left\{|M|, |CV|\right\} \le W\right) = P\left(-W \le M \le W\right) P\left(-W \le CV \le W\right)$$

$$= (H_{r,\delta_j}\left(H_r^{-1}\left(\phi(W)\right) \times \left(\frac{1}{\rho_j}\right)\right) - H_{r,\delta_j}\left(H_r^{-1}\left(\phi(-W)\right) \times \left(1/\rho_j\right)\right))$$

$$\times(F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(W\right)\right)\right)-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(-W\right)\right)\right)) \quad j=1,2,\ldots,m$$

For cases that chart statistic is within the warning scope at the beginning of the interval and statistic is within the central scope at the end of the interval, still no signal is issued (AC j'th is not removed):

$$P_{j1}^{j0}=P\left[a_t=0,Z_t=j|a_{t-1}=1,Z_{t-1}=j\right]=P\left(MV\le W|\delta=\delta_j \text{ and } s=2\right) \quad \text{(B.2)}$$

$$=P\left(\max\left\{|M|,|CV|\right\}\le W\right)=P\left(-W\le M\le W\right)P\left(-W\le CV\le W\right)$$

$$=(H_{r,\delta_j}\left(H_r{}^{-1}\left(\phi\left(W\right)\right)\times\left(\frac{1}{\rho_j}\right)\right)-H_{r,\delta_j}\left(H_r{}^{-1}\left(\phi\left(-W\right)\right)\times\left(1/\rho_j\right)\right))$$

$$\times(F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(W\right)\right)\right)-F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(-W\right)\right)\right)) \quad j=1,2,\ldots,m$$

For cases that chart statistic is within the central scope at the beginning of the interval and statistic is within the warning scope at the end of the interval, still no signal is issued (AC j'th is not removed):

$$P_{j0}^{j1}=P\left[a_t=1,Z_t=j|a_{t-1}=0,Z_{t-1}=j\right]=P\left(W\le MV\le UCL_1|\delta=\delta_j \quad s=1\right) \quad \text{(B.3)}$$

$$=P\left(W\le\max\left\{|M|,|CV|\right\}\le UCL_1\right)=P\left(-UCL_1\le M\le UCL_1\right)P\left(-UCL_1\le CV\le UCL_1\right)$$

$$-P\left(-W\le M\le W\right)P\left(-W\le CV\le W\right)$$

$$=((H_{r,\delta_j}\left(H_r{}^{-1}\left(\phi\left(UCL_1\right)\right)\times\left(\frac{1}{\rho_j}\right)\right)-H_{r,\delta_j}\left(H_r{}^{-1}\left(\phi\left(-UCL_1\right)\right)\times\left(1/\rho_j\right)\right)$$

$$\times F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(UCL_1\right)\right)\right)-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(-UCL_1\right)\right)\right))$$

$$-((H_{r,\delta_j}(H_r{}^{-1}\left(\phi\left(W\right)\right)\times\left(1/\rho_j\right)-H_{r,\delta_j}\left(H_r{}^{-1}\left(\phi\left(-W\right)\right)\times\left(1/\rho_j\right)\right)$$

$$\times F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(W\right)\right)\right)-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(-W\right)\right)\right))\quad j=1,2,\ldots,m$$

For cases that chart statistic is within the warning scope at the beginning of the interval and statistic is within the warning scope again at the end of the interval, still no signal is issued (AC j'th is not removed):

$$P_{j1}^{j1}=P\left[a_t=1,Z_t=j|a_{t-1}=1.Z_{t-1}=j\right]=P\left(W\le MV\le UCL_2|\delta=\delta_j\quad s=2\right) \tag{B.4}$$

$$=P\left(W\le\max\left\{|M|.|CV|\right\}\le UCL_2\right)=P\left(-UCL_2\le M\le UCL_2\right)P\left(-UCL_2\le CV\le UCL_2\right)$$

$$-P\left(-W\le M\le W\right)P\left(-W\le CV\le W\right)$$

$$=((H_{r.\delta_j}(H_r{}^{-1}\left(\phi\left(UCL_2\right)\right)\times\left(1/\rho_j\right)-H_{r.\delta_j}\left(H_r{}^{-1}\left(\phi\left(-UCL_2\right)\right)\times\left(1/\rho_j\right)\right)$$

$$\times F_{r.n_2-r.\delta_j}\left(F_{r.n_2-r.\delta_0}{}^{-1}\left(\phi\left(UCL_2\right)\right)\right)-F_{r.n_2-r.\delta_j}\left(F_{r.n_2-r.\delta_0}{}^{-1}\left(\phi\left(-UCL_2\right)\right)\right))$$

$$-((H_{r.\delta_j}(H_r{}^{-1}\left(\phi\left(W\right)\right)\times\left(1/\rho_j\right)-H_{r.\delta_j}\left(H_r{}^{-1}\left(\phi\left(-W\right)\right)\times\left(1/\rho_j\right)\right)$$

$$\times F_{r.n_2-r.\delta_j}\left(F_{r.n_2-r.\delta_0}{}^{-1}\left(\phi\left(W\right)\right)\right)-F_{r.n_2-r.\delta_j}\left(F_{r.n_2-r.\delta_0}{}^{-1}\left(\phi\left(-W\right)\right)\right))\quad j=1.2.\ldots.m$$

For cases that chart statistic is within the central scope at the beginning of the interval and statistic is within the action scope at the end of the interval, AC j'th is not removed:

$$P_{j0}^{j2}=P\left[a_t=2,Z_t=j|a_{t-1}=0,Z_{t-1}=j\right]=P\left(UCL_1\le MV\le inf\,|\delta=\delta_j\ \ s=1\right) \tag{B.5}$$

$$=1-P\left(0\le MV\le UCL_1|\delta=\delta_j\right)=1-P\left(\max\left\{|M|,|CV|\right\}\le UCL_1\right)$$

$$=1-P\left(-UCL_1\le M\le UCL_1\right)P\left(-UCL_1\le CV\le UCL_1\right)$$

$$=1-[(H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(UCL_1\right)\right)\times\left(\frac{1}{\rho_j}\right)\right)-H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-UCL_1\right)\right)\times\left(1/\rho_j\right)\right))$$

$$\times(F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}^{-1}\left(\phi\left(UCL_1\right)\right)\right)-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}^{-1}\left(\phi\left(-UCL_1\right)\right)\right))] \quad j=1,2,\ldots,m$$

For cases that chart statistic is within the warning scope at the beginning of the interval and statistic is within the action scope at the end of the interval:

$$P_{j1}^{j2}=P\left[a_t=2,Z_t=j|a_{t-1}=1,Z_{t-1}=j\right]=P\left(UCL_2\le MV\le inf|\delta=\delta_j \quad s=2\right) \tag{B.6}$$

$$=1-P\left(0\le MV\le UCL_2|\delta=\delta_j\right)=1-P\left(\max\left\{|M|,|CV|\right\}\le UCL_2\right)$$

$$=1-P\left(-UCL_2\le M\le UCL_2\right)P\left(-UCL_2\le CV\le UCL_2\right)$$

$$=1-[(H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(UCL_2\right)\right)\times\left(\frac{1}{\rho_j}\right)\right)-H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-UCL_2\right)\right)\times\left(1/\rho_j\right)\right))$$

$$\times(F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}^{-1}\left(\phi\left(UCL_2\right)\right)\right)-F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}^{-1}\left(\phi\left(-UCL_2\right)\right)\right))]$$

$$j=1,2,\ldots,m$$

For cases that no AC affects the process at the beginning of the interval and then an AC affects the process, the transition probability corresponding to these scenarios can be computed from the equations B.7-B.12.

If chart statistic is within the central scope at the beginning and the end of the interval:

$$P_{00}^{j0}=P\left[a_t=0,Z_t=j|a_{t-1}=0,Z_{t-1}=0\right]=q_j\left(h_1\right)\times P\left(MV\le W|\delta=\delta_j \quad s=1\right) \tag{B.7}$$

$$= q_j\left(h_1\right)\times[(H_{r,\delta_j}\left(H_r^{-1}\left(\phi(W)\right)\times\left(\frac{1}{\rho_j}\right)\right)-H_{r,\delta_j}\left(H_r^{-1}\left(\phi(-W)\right)\times\left(1/\rho_j\right)\right))$$

$$\times(F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi(W)\right)\right)-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi(-W)\right)\right))]$$

If chart statistic is within the warning scope at the beginning and within the central zone at the end of the interval:

$$P_{01}^{j0} = P\left[a_t = 0, Z_t = j | a_{t-1} = 1, Z_{t-1} = 0\right] = q_j\left(h_2\right)\times P\left(MV \le W | \delta = \delta_j \;\; s = 2\right) \tag{B.8}$$

$$= q_j\left(h_2\right)\times\left[(H_{r,\delta_j}\left(H_r^{-1}\left(\phi(W)\right)\times\left(\frac{1}{\rho_j}\right)\right)-H_{r,\delta_j}\left(H_r^{-1}\left(\phi(-W)\right)\times\left(1/\rho_j\right)\right)\right)$$

$$\times(F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi(W)\right)\right)-F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi(-W)\right)\right))]$$

If chart statistic is within the central scope at the beginning and within the warning zone at the end of the interval:

$$P_{00}^{j1} = P\left[a_t = 1, Z_t = j | a_{t-1} = 0, Z_{t-1} = 0\right] \tag{B.9}$$

$$= q_j\left(h_1\right)\times P\left(W \le MV \le UCL_1 | \delta = \delta_j \;\;\; s = 1\right)$$

$$= q_j\left(h_1\right)\times[((H_{r,\delta_j}\left(H_r^{-1}\left(\phi(UCL_1)\right)\times\left(\frac{1}{\rho_j}\right)\right)-H_{r,\delta_j}\left(H_r^{-1}\left(\phi(-UCL_1)\right)\times\left(1/\rho_j\right)\right)$$

$$\times F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi(UCL_1)\right)\right)-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi(-UCL_1)\right)\right))$$

$$-((H_{r,\delta_j}(H_r^{-1}\left(\phi(W)\right)\times\left(\frac{1}{\rho_j}\right)-H_{r,\delta_j}\left(H_r^{-1}\left(\phi(-W)\right)\times\left(\frac{1}{\rho_j}\right)\right)$$

$$\times F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi(W)\right)\right)-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi(-W)\right)\right))\,]$$

If chart statistic is within the warning scope at the beginning and the end of the interval:

$$P_{01}^{j1} = P\left[a_t = 1, Z_t = j | a_{t-1} = 1, Z_{t-1} = 0\right] = q_j\left(h_2\right) \times P\left(W \le MV \le UCL_2 | \delta = \delta_j \quad s = 2\right) \qquad \text{(B.10)}$$

$$= q_j\left(h_2\right) \times [((H_{r,\delta_j}(H_r^{-1}\left(\phi\left(UCL_2\right)\right) \times \left(1/\rho_j\right)$$

$$-H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-UCL_2\right)\right) \times \left(\frac{1}{\rho_j}\right)\right) \times F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(UCL_2\right)\right)\right)$$

$$-F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(-UCL_2\right)\right)\right)) - ((H_{r,\delta_j}(H_r^{-1}\left(\phi\left(W\right)\right) \times \left(1/\rho_j\right)$$

$$-H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-W\right)\right) \times \left(1/\rho_j\right)\right)$$

$$\times F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(W\right)\right)\right) - F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(-W\right)\right)\right))]$$

If chart statistic is within the central scope at the beginning and within the action zone at the end of the interval:

$$P_{00}^{j2} = P\left[a_t = 2, Z_t = j | a_{t-1} = 0, Z_{t-1} = 0\right] = q_j\left(h_1\right) \times P\left(UCL_1 \le MV \le inf | \delta = \delta_j \quad s = 1\right) \qquad \text{(B.11)}$$

$$= q_j\left(h_1\right) \times [1 - [(H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(UCL_1\right)\right) \times \left(\frac{1}{\rho_j}\right)\right)$$

$$-H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-UCL_1\right)\right) \times \left(1/\rho_j\right)\right)) \times (F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(UCL_1\right)\right)\right)$$

$$-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(-UCL_1\right)\right)\right))]]$$

If chart statistic is within the warning scope at the beginning and within the action zone at the end of the interval:

$$P_{01}^{j2} = P\left[a_t = 2, Z_t = j | a_{t-1} = 1, Z_{t-1} = 0\right] = q_j\left(h_2\right) \times P\left(UCL_2 \le MV \le inf | \delta = \delta_j \quad s = 2\right) \qquad \text{(B.12)}$$

$$= q_j\left(h_2\right) \times [1 - [(H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(UCL_2\right)\right) \times \left(\frac{1}{\rho_j}\right)\right)$$

$$-H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-UCL_2\right)\right) \times \left(1/\rho_j\right)\right)) \times (F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(UCL_2\right)\right)\right)$$

$$-F_{r,n_2-r,\delta_j}\left(F_{r,n_2-r,\delta_0}{}^{-1}\left(\phi\left(-UCL_2\right)\right)\right))]]$$

Consider the cases that an AC is discovered and removed at the beginning of the interval and another AC occurs by the end of the interval. The transition probability related to these cases can be computed from the equations B.13-B.15.

$$P_{i2}^{j0} = P\left[a_t = 0, Z_t = j | a_{t-1} = 2, Z_{t-1} = i\right] = q_j\left(h_1\right) \times P\left(0 \le MV \le W | \delta = \delta_j \quad s = 1\right) \qquad \text{(B.13)}$$

$$= q_j\left(h_1\right) \times [(H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(W\right)\right) \times \left(\frac{1}{\rho_j}\right)\right) - H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-W\right)\right) \times \left(1/\rho_j\right)\right))$$

$$\times (F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(W\right)\right)\right) - F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(-W\right)\right)\right))]$$

$$j = 1,2,\ldots,m; i = 0,1,\ldots,m$$

If chart statistic is within the warning zone at the end of the interval:

$$P_{i2}^{j1} = P\left[a_t = 1, Z_t = j | a_{t-1} = 2, Z_{t-1} = i\right] = q_j\left(h_1\right) \times P\left(W \le MV \le UCL_1 | \delta = \delta_j \;\; s = 1\right) \qquad \text{(B.14)}$$

$$= q_j\left(h_1\right) \times [((H_{r,\delta_j}(H_r^{-1}\left(\phi\left(UCL_1\right)\right) \times \left(1/\rho_j\right) - H_{r,\delta_j}(H_r^{-1}\left(\phi\left(-UCL_1\right)\right)$$

$$\times \left(1/\rho_j\right)) \times F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(UCL_1\right)\right)\right)$$

$$-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(-UCL_1\right)\right)\right)) - ((H_{r,\delta_j}(H_r^{-1}\left(\phi\left(W\right)\right) \times \left(1/\rho_j\right)$$

$$-H_{r,\delta_j}\left(H_r^{-1}\left(\phi\left(-W\right)\right) \times \left(1/\rho_j\right)\right) \times F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(W\right)\right)\right)$$

$$-F_{r,n_1-r,\delta_j}\left(F_{r,n_1-r,\delta_0}{}^{-1}\left(\phi\left(-W\right)\right)\right)]$$

$$j=1,2,\ldots,m; i=0,1,\ldots,m$$

If chart statistic is within the action scope at the end of the interval:

$$P_{i2}^{j2}=P\left[a_t=2,Z_t=j|a_{t-1}=2,Z_{t-1}=i\right]=q_j\left(h_1\right)\times P\left(UCL_1\leq MV\leq inf|\delta=\delta_j\ \ s=1\right) \quad \text{(B.15)}$$

$$=q_j\left(h_1\right)\times[1-[(H_{r,\delta_{j,1}}\left(H_r{}^{-1}\left(\phi\left(UCL_1\right)\right)\times\left(\frac{1}{\rho_j}\right)\right)$$

$$-H_{r,\delta_{j,1}}\left(H_r{}^{-1}\left(\phi\left(-UCL_1\right)\right)\times\left(1/\rho_j\right)\right))\times(F_{r,n_1-r,\delta_{j,1}}\left(F_{r,n_1-r,\delta_{0,1}}{}^{-1}\left(\phi\left(UCL_1\right)\right)\right)$$

$$-F_{r,n_1-r,\delta_{j,1}}\left(F_{r,n_1-r,\delta_{0,1}}{}^{-1}\left(\phi\left(-UCL_1\right)\right)\right))]] \quad j=1,2,\ldots,m; i=0,1,\ldots,m$$

When the process is not affected by any AC at the beginning of the sampling interval and no AC is expected to occur until the end of the interval, transition probabilities are obtained from the relations B.16-B.24.

$$P_{00}^{00}=P\left[a_t=0,Z_t=0|a_{t-1}=0,Z_{t-1}=0\right] \quad \text{(B.16)}$$

$$=e^{-\lambda h_1}\times P\left(0\leq MV\leq W|\delta=\delta_0\ \ s=1\right)=e^{-\lambda h_1}\times P\left(\max\left\{|M|,|CV|\right\}\leq W\right)$$

$$=e^{-\lambda h_1}\times P\left(-W\leq M\leq W\right)P\left(-W\leq CV\leq W\right)$$

$$=e^{-\lambda h_1}\times\left[\left(\phi\left(W\right)-\phi\left(-W\right)\right)\times\left(\phi\left(W\right)-\phi\left(-W\right)\right)\right]=e^{-\lambda h_1}\times\left(\phi\left(W\right)-\phi\left(-W\right)\right)^2$$

If chart statistic is within the warning scope at the beginning and within the central zone at the end of the interval:

(B.17)

$$P_{01}^{00} = P\left[a_t = 0, Z_t = 0 | a_{t-1} = 1, Z_{t-1} = 0\right] = e^{-\lambda h_2} \times P\left(0 \le MV \le W | \delta = \delta_0 \;\; s = 2\right)$$

$$= e^{-\lambda h_2} \times \left(\phi(W) - \phi(-W)\right)^2$$

If chart statistic is within the action scope at the beginning and within the central zone at the end of the interval:

$$P_{02}^{00} = P\left[a_t = 0, Z_t = 0 | a_{t-1} = 2 Z_{t-1} = 0\right] = e^{-\lambda h_1} \times P\left(0 \le MV \le W | \delta = \delta_0 \;\; s = 1\right) \tag{B.18}$$

$$= e^{-\lambda h_1} \times \left(\phi(W) - \phi(-W)\right)^2$$

If chart statistic is within the central scope at the beginning and within the warning zone at the end of the interval:

$$P_{00}^{01} = P\left[a_t = 1, Z_t = 0 | a_{t-1} = 0, Z_{t-1} = 0\right] = e^{-\lambda h_1} \times P\left(W \le MV \le UCL_1 | \delta = \delta_0 \;\; s = 1\right) \tag{B.19}$$

$$= e^{-\lambda h_1} \times P\left(W \le \max\left\{|M|, |CV|\right\} \le UCL_1\right)$$

$$= e^{-\lambda h_1} \times [P\left(-UCL_1 \le M \le UCL_1\right) P\left(-UCL_1 \le CV \le UCL_1\right)$$

$$-P\left(-W \le M \le W\right) P\left(-W \le CV \le W\right)$$

$$= e^{-\lambda h_1} \times \left[\left(\phi(UCL_1) - \phi(-UCL_1)\right)^2 - \left(\phi(W) - \phi(-W)\right)^2\right]$$

If chart statistic is within the warning scope at the beginning and the end of the interval:

$$P_{01}^{01} = P\left[a_t = 1, Z_t = 0 | a_{t-1} = 1, Z_{t-1} = 0\right] = e^{-\lambda h_2} \times P\left(W \le MV \le UCL_2 | \delta = \delta_0 \;\;\; s = 2\right) = \tag{B.20}$$

$$= e^{-\lambda h_2} \times \left[\left(\phi(UCL_2) - \phi(-UCL_2)\right)^2 - \left(\phi(W) - \phi(-W)\right)^2\right]$$

If chart statistic is within the action scope at the beginning and within the warning zone at the end of the interval:

$$P_{02}^{01} = P\left[a_t = 1, Z_t = 0 | a_{t-1} = 2, Z_{t-1} = 0\right] = e^{-\lambda h_1} \times P\left(W \le MV \le UCL_1 | \delta = \delta_0 \;\; s = 1\right) \tag{B.21}$$

$$= e^{-\lambda h_1} \times \left[\left(\phi\left(UCL_1\right) - \phi\left(-UCL_1\right)\right)^2 - \left(\phi\left(W\right) - \phi\left(-W\right)\right)^2\right]$$

If chart statistic is within the central scope at the beginning and within the action zone at the end of the interval:

$$P_{00}^{02} = P\left[a_t = 2, Z_t = 0 | a_{t-1} = 0, Z_{t-1} = 0\right] = e^{-\lambda h_1} \times P\left(UCL_1 \le MV \le inf | \delta = \delta_0 \;\; s = 1\right) \tag{B.22}$$

$$= e^{-\lambda h_1} \times \left[1 - P\left(0 \le MV \le UCL_1 | \delta = \delta_{0,1}\right)\right] = e^{-\lambda h_1} \times \left[1 - \left(\left(\phi\left(UCL_1\right) - \phi\left(-UCL_1\right)\right)^2\right)\right]$$

If chart statistic is within the warning scope at the beginning and within the action zone at the end of the interval:

$$P_{01}^{02} = P\left[a_t = 2, Z_t = 0 | a_{t-1} = 1, Z_{t-1} = 0\right] = e^{-\lambda h_2} \times P\left(UCL_2 \le MV \le inf | \delta = \delta_0 \;\; s = 2\right) \tag{B.23}$$

$$= e^{-\lambda h_2} \times \left[1 - P\left(0 \le MV \le UCL_2 | \delta = \delta_{0,2}\right)\right] = e^{-\lambda h_2} \times \left[1 - \left(\left(\phi\left(UCL_2\right) - \phi\left(-UCL_2\right)\right)^2\right)\right]$$

If chart statistic is within the action scope at the beginning and the end of the interval:

$$P_{02}^{02} = P\left[a_t = 2, Z_t = 0 | a_{t-1} = 2, Z_{t-1} = 0\right] = e^{-\lambda h_1} \times P\left(UCL_1 \le MV \le inf | \delta = \delta_0 \;\; s = 1\right) \tag{B.24}$$

$$= e^{-\lambda h_1} \times \left[1 - P\left(0 \le MV \le UCL_1 | \delta = \delta_{0,1}\right)\right] = e^{-\lambda h_1} \times \left[1 - \left(\left(\phi\left(UCL_1\right) - \phi\left(-UCL_1\right)\right)^2\right)\right]$$

Finally, when the process is affected by AC i'th at the beginning of the sampling interval and chart statistic is within the out-of-control scope, the signal is issued by the chart. So, the AC is removed and the process returns to the in-control state. Transition probabilities are obtained from the following relations.

$$P_{i2}^{00} = P\left[a_t = 0, Z_t = 0 | a_{t-1} = 2, Z_{t-1} = i\right] = \left(1 - \sum_{i=1}^{m} q_i\left(h_1\right)\right) \times P\left(0 \le MV \le w | \delta = \delta_0 \;\; s = 1\right) \tag{B.25}$$

$$= \left(1 - \sum_{i=1}^{m} q_i\left(h_1\right)\right) \times \left(\phi\left(w\right) - \phi\left(-w\right)\right)^2 \; i = 1,2,\ldots,m$$

If chart statistic is within the warning zone at the end of the interval:

$$P_{i2}^{01} = P\left[a_t = 1, Z_t = 1 | a_{t-1} = 2, Z_{t-1} = i\right] \tag{B.26}$$

$$= \left(1 - \sum_{i=1}^{m} q_i\left(h_1\right)\right) \times P\left(w \le MV \le UCL_1 | \delta = \delta_0 \;\; s = 1\right)$$

$$= \left(1 - \sum_{i=1}^{m} q_i\left(h_1\right)\right) \times \left[\left(\phi\left(UCL_1\right) - \phi\left(-UCL_1\right)\right)^2 - \left(\phi\left(W\right) - \phi\left(-W\right)\right)^2\right]$$

$$i = 1,2,\ldots,m$$

If chart statistic is within the action zone at the end of the interval:

$$P_{i2}^{02} = P\left[a_t = 2, Z_t = 2 | a_{t-1} = 2, Z_{t-1} = i\right] \tag{B.27}$$

$$= \left(1 - \sum_{i=1}^{m} q_i\left(h_1\right)\right) \times P\left(UCL_1 \le MV \le inf | \delta = \delta_0 \;\; s = 1\right)$$

$$= \left(1 - \sum_{i=1}^{m} q_i\left(h_1\right)\right) \times \left[1 - \left(\left(\phi\left(UCL_1\right) - \phi\left(-UCL_1\right)\right)^2\right)\right]$$

$$i = 1,2,\ldots,m$$

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$\rho_1 = \rho,\ \rho_2 = 1.5\rho, \rho_2 = 2\rho$

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Table 6: ATS values for cases with $r = 2$

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	Z_t			0			1		.		$m-1$			m	
	a_t		0	1	2	0	1	2	.	0	1	2	0	1	2
	Z_{t-1}	a_{t-1}							.						
		0	P_{00}^{00}	P_{00}^{01}	P_{00}^{02}	P_{00}^{10}	P_{00}^{11}	P_{00}^{12}	.	$P_{00}^{m-1,0}$	$P_{00}^{m-1,1}$	$P_{00}^{m-1,2}$	$P_{00}^{m,0}$	$P_{00}^{m,1}$	$P_{00}^{m,2}$
	0	1	P_{01}^{00}	P_{01}^{01}	P_{01}^{02}	P_{01}^{10}	P_{01}^{11}	P_{01}^{12}	.	$P_{01}^{m-1,0}$	$P_{01}^{m-1,1}$	$P_{01}^{m-1,2}$	$P_{01}^{m,0}$	$P_{01}^{m,1}$	$P_{01}^{m,2}$
		2	P_{02}^{00}	P_{02}^{01}	P_{02}^{02}	P_{02}^{10}	P_{02}^{11}	P_{02}^{12}	.	$P_{02}^{m-1,0}$	$P_{02}^{m-1,1}$	$P_{02}^{m-1,2}$	$P_{02}^{m,0}$	$P_{02}^{m,1}$	$P_{02}^{m,2}$
		0	P_{10}^{00}	P_{10}^{01}	P_{10}^{02}	P_{10}^{10}	P_{10}^{11} .	P_{10}^{12}	.	$P_{10}^{m-1,0}$	$P_{10}^{m-1,1}$	$P_{10}^{m-1,2}$		$P_{10}^{m,1}$	$P_{10}^{m,2}$
	1	1	P_{11}^{00}	P_{11}^{01}	P_{11}^{02}	P_{11}^{10}	P_{11}^{11}	P_{11}^{12}	.	$P_{11}^{m-1,0}$	$P_{11}^{m-1,1}$	$P_{11}^{m-1,2}$	$P_{11}^{m,0}$	$P_{11}^{m,1}$	$P_{11}^{m,2}$
		2	P_{12}^{00}	P_{12}^{01}	P_{12}^{02}	P_{12}^{10}	P_{12}^{11}	P_{12}^{12}	.	$P_{12}^{m-1,0}$	$P_{12}^{m-1,1}$	$P_{12}^{m-1,2}$	$P_{12}^{m,0}$	$P_{12}^{m,1}$	$P_{12}^{m,2}$
$P =$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
		0	$P_{m-1,0}^{00}$	$P_{m-1,0}^{01}$	$P_{m-1,0}^{02}$	$P_{m-1,0}^{10}$	$P_{m-1,0}^{11}$	$P_{m-1,0}^{12}$	.	$P_{m-1,0}^{m-1,0}$	$P_{m-1,0}^{m-1,1}$	$P_{m-1,0}^{m-1,2}$	$P_{m-1,0}^{m,0}$	$P_{m-1,0}^{m,1}$	$P_{m-1,0}^{m,2}$
	$m-1$	1	$P_{m,1,1}^{00}$	$P_{m,1,1}^{01}$	$P_{m,1,1}^{02}$	$P_{m,1,1}^{10}$	$P_{m,1,1}^{11}$	$P_{m,1,1}^{12}$	.	$P_{m,1,1}^{m-1,0}$	$P_{m,1,1}^{m-1,1}$	$P_{m,1,1}^{m-1,2}$	$P_{m,1,1}^{m,0}$	$P_{m,1,1}^{m,1}$	$P_{m,1,1}^{m,2}$
		2	$P_{m-1,2}^{00}$	$P_{m-1,2}^{01}$	$P_{m-1,2}^{02}$	$P_{m-1,2}^{10}$	$P_{m-1,2}^{11}$	$P_{m-1,2}^{12}$	.	$P_{m-1,2}^{m-1,0}$	$P_{m-1,2}^{m-1,1}$	$P_{m-1,2}^{m-1,2}$	$P_{m-1,2}^{m,0}$	$P_{m-1,2}^{m,1}$	$P_{m-1,2}^{m,2}$
		0	$P_{m,0}^{00}$	$P_{m,0}^{01}$	$P_{m,0}^{02}$	$P_{m,0}^{10}$	$P_{m,0}^{11}$	$P_{m,0}^{12}$	.	$P_{m,0}^{m-1,0}$	$P_{m,0}^{m-1,1}$	$P_{m,0}^{m-1,2}$	$P_{m,0}^{m,0}$	$P_{m,0}^{m,1}$	$P_{m,0}^{m,2}$
	m	1	$P_{m,1}^{00}$	$P_{m,1}^{01}$	$P_{m,1}^{02}$	$P_{m,1}^{10}$	$P_{m,1}^{11}$	$P_{m,1}^{12}$	.	$P_{m,1}^{m-1,0}$	$P_{m,1}^{m-1,1}$	$P_{m,1}^{m-1,2}$	$P_{m,1}^{m,0}$	$P_{m,1}^{m,1}$	$P_{m,1}^{m,2}$
		2	$P_{m,2}^{00}$	$P_{m,2}^{01}$	$P_{m,2}^{02}$	$P_{m,2}^{10}$	$P_{m,2}^{11}$	$P_{m,2}^{12}$	.	$P_{m,2}^{m-1,0}$	$P_{m,2}^{m-1,1}$	$P_{m,2}^{m-1,2}$	$P_{m,2}^{m,0}$	$P_{m,2}^{m,1}$	$P_{m,2}^{m,2}$

Figure 1: Transition probability matrix P

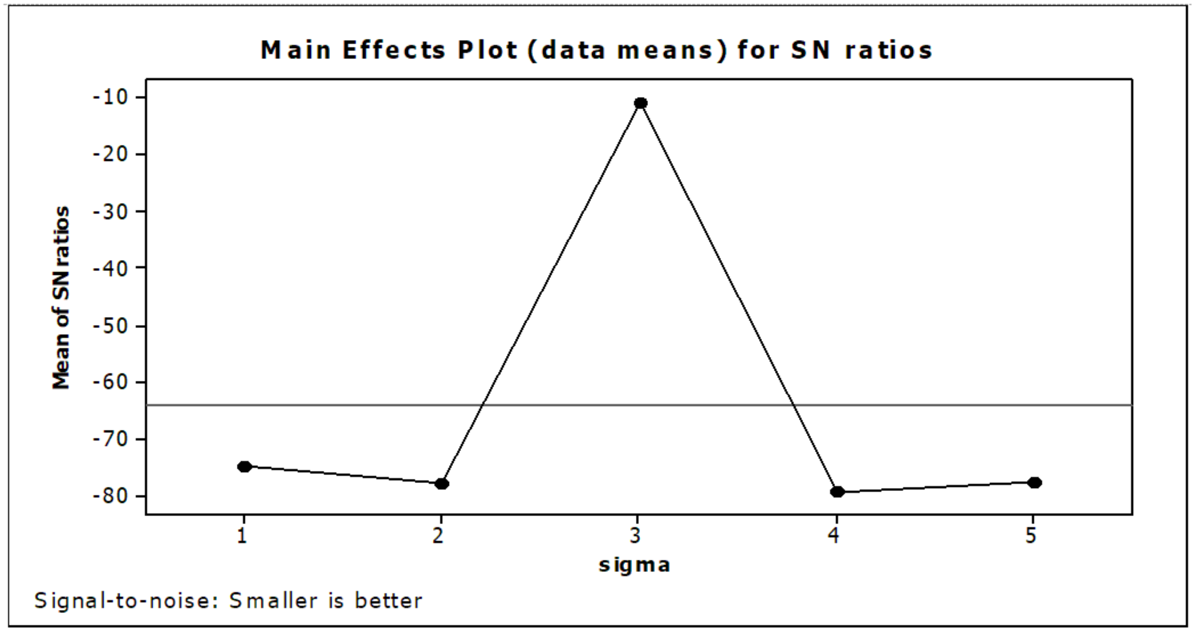

Figure 2: *ARL* versus mean vector shifts

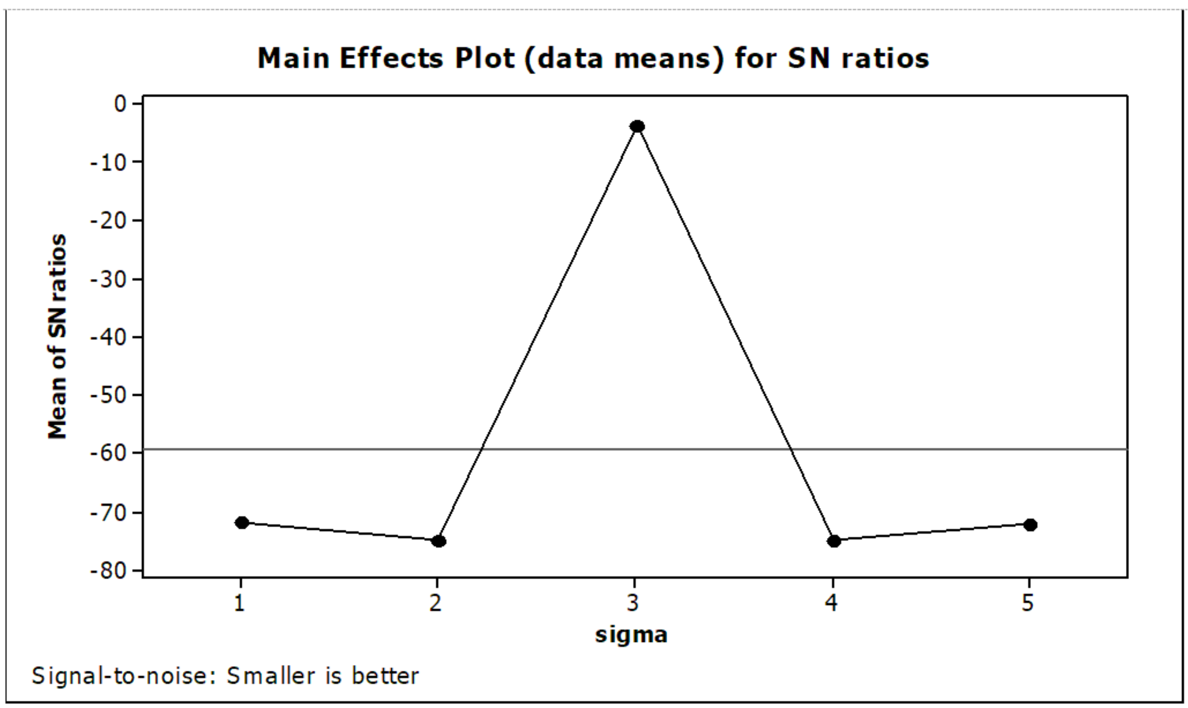

Figure 3: *ATS* versus mean vector shifts

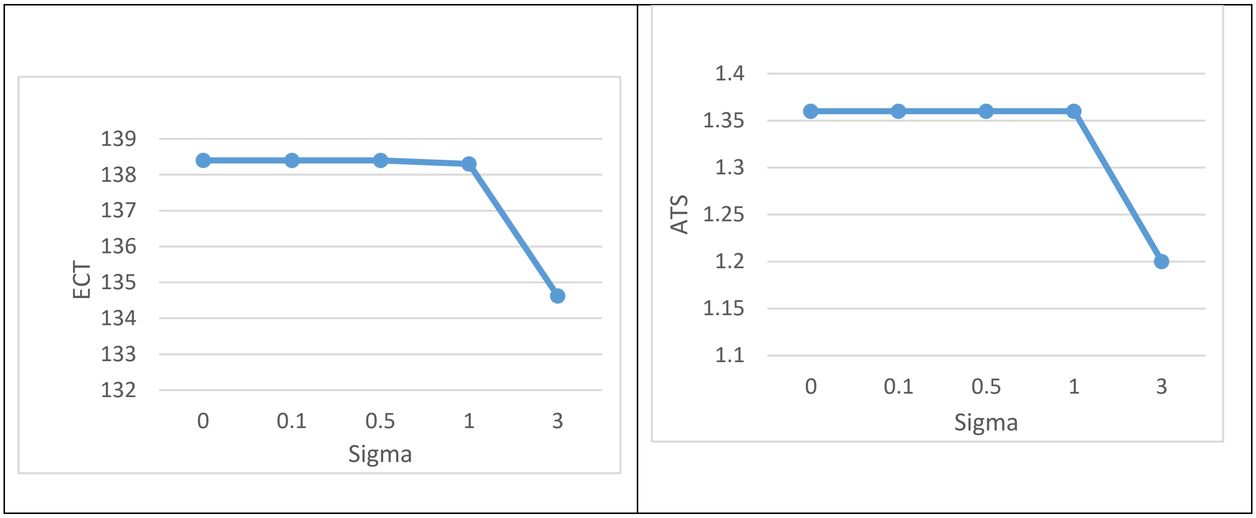

Figure 4: ATS and ECT values for cases with r= 2, ρ=0.25

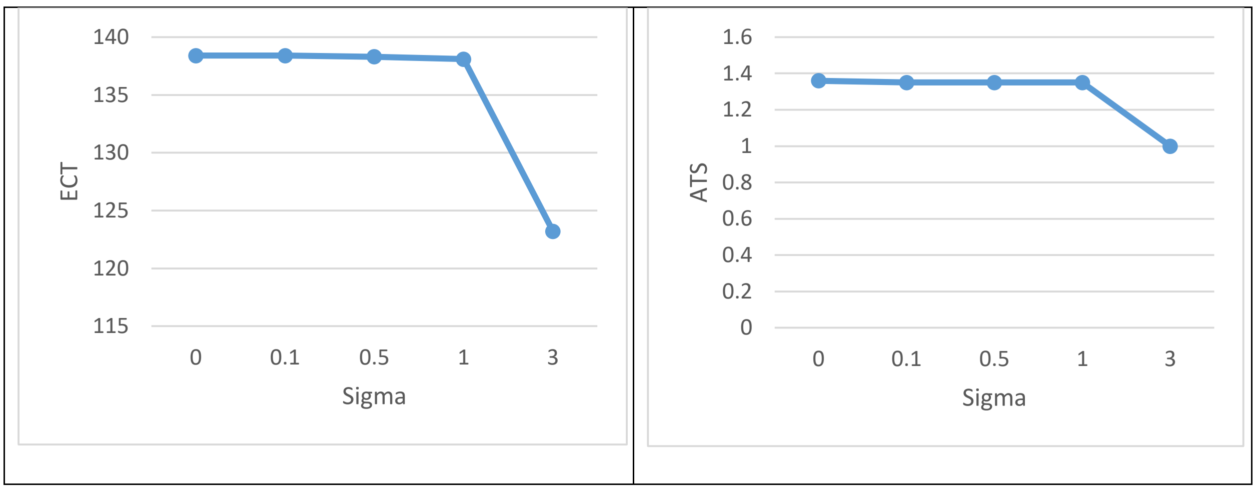

Figure 5: ATS and ECT values for cases with $r = 2$, ρ=0.5

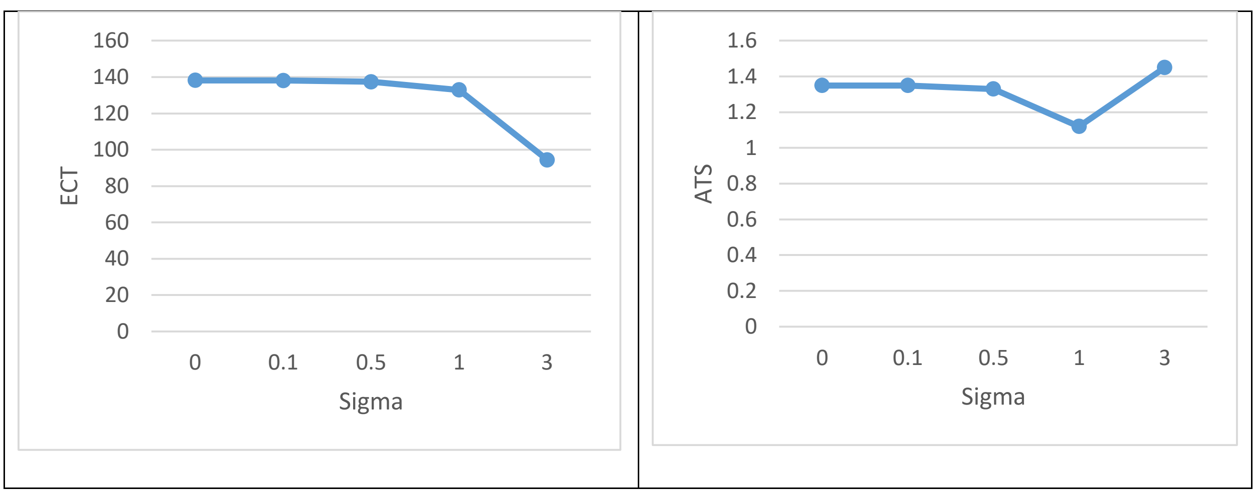

Figure 6: ATS and ECT values for cases with $r = 2$, ρ=1

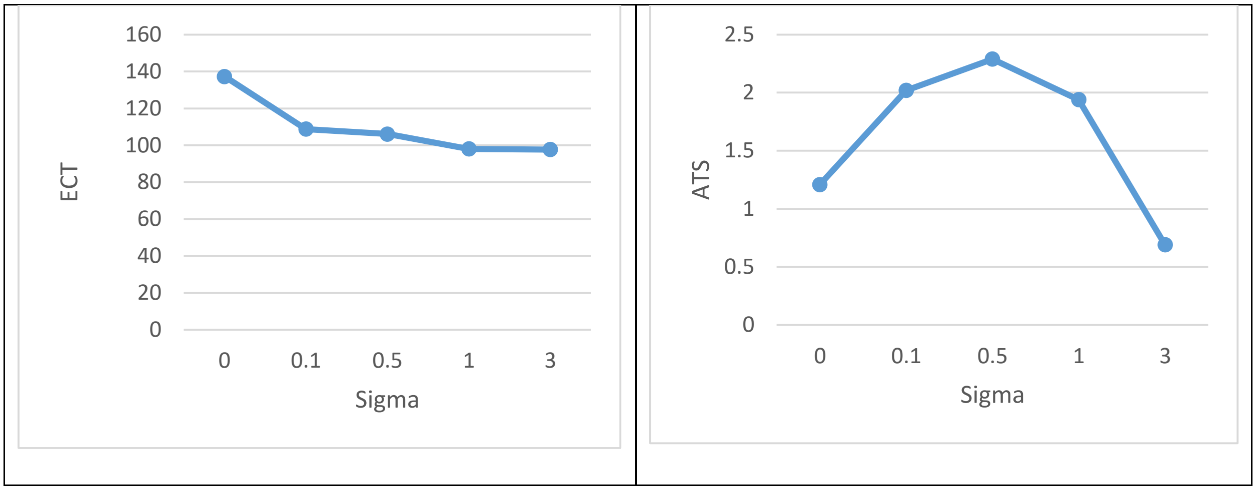

Figure 7: ATS and ECT values for cases with $r = 2$, ρ=1.5

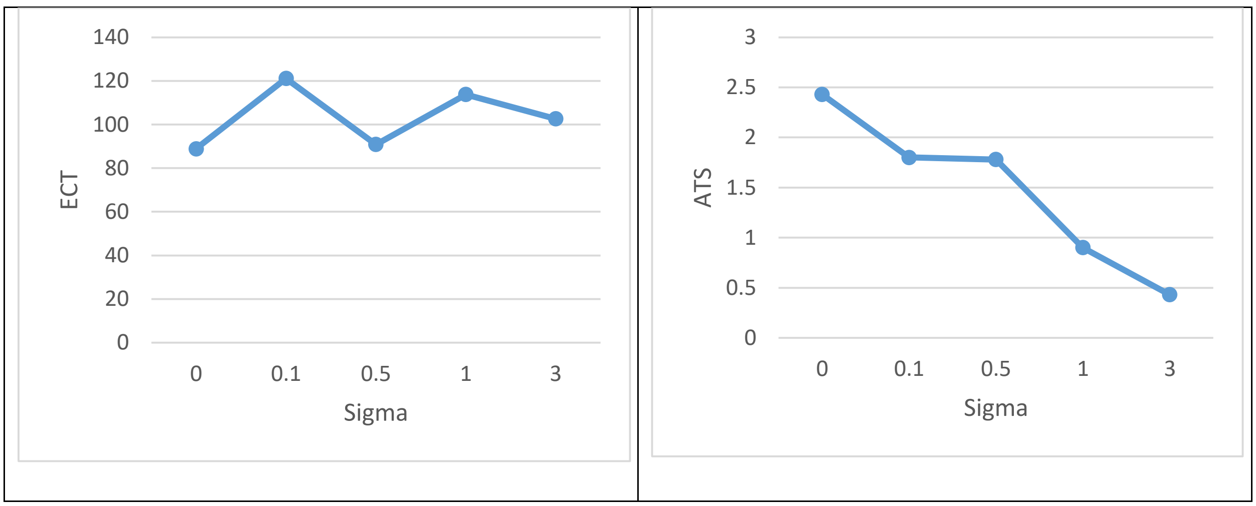

Figure 8: ATS and ECT values for cases with $r = 2$, ρ=3

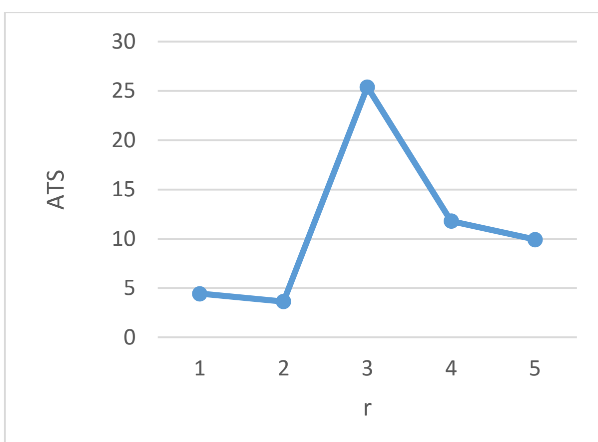

Figure 9: ATS values for cases with $r = 2, 3, 4, 5, 6$

Table 1: The results of GA and exact method

Decision variable Case	n		k		h		w	
	GA	Exact	GA	Exact	GA	Exact	GA	Exact
1	1	1	3.16	3.2	6.15	6.2	3.2	7.1
2	2	1	6.2	5.3	15.4	17.3	3.54	4.1
3	3	1	3.2	3.1	6.6	6.8	2.98	8.8
4	$n_1 = 4$ $n_2 = 6$	-	$k_1 = 4.6$ $k_2 = 8.43$	-	$h_1 = 6.66$ $h_2 = 1.97$	-	$w_1 = 0.65$ $w_2 = 0.33$	-

Table 2: Comparison of computational time and total cost of GA algorithm and exact method

Case	Total cost		Computational time (Second)		Solution space
	GA	Exact	GA	Exact	
1	56.98	56.98	121	17785	8×10^7
2	90.38	90.21	118	17462	8×10^7
3	227.5	227.48	122	18050	8×10^7
4	217.37	No answer	125	More than 50000	64×10^{14}

Table 3: The level of each parameter (for all levels $R_1 = R,\ R_2 = 1.5R,\ R_3 = 2R,$ $L_1 = L,\ L_2 = 1.5L,\ L_3 = 2L,\ \lambda_1 = \lambda,\ \lambda_2 = 1.5\lambda,\ \lambda_3 = 2\lambda,\ \delta_1 = \delta,\ \delta_2 = 1.5\delta,\ \delta_3 = 2\delta,$ $\rho_1 = \rho,\ \rho_2 = 1.5\rho, \rho_2 = 2\rho$

Level	r	R	L	δ	ρ
1	2	100	50	0	0.25
2	3	150	100	0.1	0.5
3	4	200	150	0.5	1
4	5	250	200	1	1.5
5	6	300	300	3	3

Table 4: Input parameters for the 25 cases

Case	r	R	L	δ	ρ	**Case**	r	R	L	δ	ρ
1	1	1	1	1	1	**14**	3	4	1	3	5
2	1	2	2	2	2	**15**	3	5	2	4	1
3	1	3	3	3	3	**16**	4	1	4	2	5
4	1	4	4	4	4	**17**	4	2	5	3	1
5	1	5	5	5	5	**18**	4	3	1	4	2
6	2	1	2	3	4	**19**	4	4	2	5	3
7	2	2	3	4	5	**20**	4	5	3	1	4
8	2	3	4	5	1	**21**	5	1	5	4	3
9	2	4	5	1	2	**22**	5	2	1	5	4
10	2	5	1	2	3	**23**	5	3	2	1	5
11	3	1	3	5	2	**24**	5	4	3	2	1
12	3	2	4	1	3	**25**	5	5	4	3	2
13	3	3	5	2	4						

Table 5: Design for proposed chi-control chart

Case	n_1	n_2	h_1	h_2	UCL_1	UCL_2	w	ECT	β	ATS	ARL_1
1	3.30	5.01	1.60	0.11	0.30	0.20	0.10	90.00	0.04	1.60	1.04
2	3.68	5.61	1.35	0.10	0.30	0.20	0.10	138.40	0.05	1.36	1.05
3	3.41	5.25	0.47	0.18	5.34	1.49	1.02	173.50	0.80	1.74	4.99
4	4.18	7.97	0.45	0.22	4.31	1.86	1.50	197.00	0.77	1.72	4.43
5	3.69	3.91	0.26	0.15	9.79	5.23	4.10	200.67	0.87	1.97	7.88
6	6.03	7.61	0.25	0.10	4.34	2.80	1.38	86.25	0.89	1.83	9.12
7	5.02	5.20	0.28	0.15	9.22	4.87	3.00	104.70	0.92	3.39	12.60
8	5.50	5.51	1.31	0.25	0.30	0.20	0.10	212.60	0.04	1.32	1.04
9	5.28	5.62	1.26	0.17	0.53	0.33	0.15	288.00	0.14	1.29	1.16
10	8.65	9.39	0.74	0.23	0.82	0.34	0.10	156.90	0.21	0.77	1.26
11	8.25	9.28	12.47	9.12	8.36	6.84	6.40	100.32	0.98	620.00	50.00
12	6.01	8.82	16.95	8.16	10.57	7.31	6.22	150.20	0.99	1600.00	100.00
13	5.15	7.80	0.36	0.13	6.81	2.44	1.66	180.20	0.94	5.49	17.20
14	5.66	7.00	0.58	0.24	2.66	2.31	0.37	99.70	0.38	0.50	1.62
15	5.04	5.33	0.83	0.10	0.30	0.20	0.10	187.40	0.04	0.83	1.04
16	9.70	11.97	14.16	0.13	9.03	7.89	6.76	100.32	0.97	471.60	33.30
17	7.22	8.43	3.86	0.11	0.38	0.28	0.10	233.70	0.05	3.86	1.05
18	7.06	7.09	1.02	0.10	0.33	0.21	0.10	128.20	0.06	1.02	1.06
19	8.20	10.57	0.52	0.21	2.29	2.05	0.85	133.40	0.42	0.62	1.73
20	6.62	8.15	0.50	0.10	2.07	1.92	0.79	196.90	0.64	0.78	2.78
21	9.62	12.12	17.15	2.72	10.66	7.54	6.19	100.23	0.98	857.00	50.00
22	8.64	10.32	0.57	0.33	2.40	2.24	0.11	79.80	0.40	0.63	1.67
23	7.04	7.77	0.60	0.30	2.56	2.43	0.10	118.22	0.34	0.52	1.52
24	8.19	9.25	0.85	0.34	0.98	0.82	0.10	200.40	0.21	0.85	1.27
25	7.13	9.94	0.23	0.11	4.53	1.32	1.05	237.90	0.85	1.28	7.00

Table 6: ATS values for cases with $r = 2$

δ	n_1	n_2	h_1	h_2	UCL_1	UCL_2	w	ECT	ATS
$\rho = 0.25$									
0	3	3	1.36	0.11	0.31	0.20	0.10	138.40	1.36
0.1	4	5	1.34	0.10	0.55	0.45	0.10	138.40	1.36
0.5	5	6	1.36	0.10	0.32	0.20	0.10	138.40	1.36
1	4	4	1.35	0.14	0.32	0.20	0.10	138.30	1.36
3	7	8	1.10	0.41	1.22	1.12	0.15	134.62	1.20
$\rho = 0.5$									
δ	n_1	n_2	h_1	h_2	UCL_1	UCL_2	w	ECT	ATS
0	3	3	1.36	0.10	0.30	0.20	0.10	138.40	1.36
0.1	4	5	1.33	0.10	0.52	0.29	0.11	138.40	1.35
0.5	6	6	1.34	0.17	0.32	0.22	0.10	138.30	1.35
1	5	6	1.35	0.37	0.30	0.20	0.10	138.10	1.35
3	9	11	0.59	0.25	1.87	1.70	1.03	123.20	1.00
$\rho = 1$									
δ	n_1	n_2	h_1	h_2	UCL_1	UCL_2	w	ECT	ATS
0	3	5	1.34	0.28	0.31	0.20	0.10	138.21	1.35
0.1	5	6	1.33	0.42	0.49	0.21	0.10	138.06	1.35
0.5	4	5	1.20	0.42	1.11	0.86	0.13	137.40	1.33
1	6	7	0.91	0.48	1.65	1.14	0.10	132.90	1.12

3	3	3	0.22	0.10	6.37	2.48	1.98	94.40	1.45

$\rho = 1.5$

δ	n_1	n_2	h_1	h_2	UCL_1	UCL_2	w	ECT	ATS
0	3	4	1.05	0.33	1.41	1.17	0.15	137.30	1.21
0.1	3	3	0.30	0.13	4.76	2.15	1.42	108.80	2.02
0.5	3	5	0.28	0.12	6.03	2.11	1.52	106.10	2.29
1	3	3	0.26	0.11	6.31	2.29	1.71	98.10	1.94
3	3	6	0.22	0.11	7.51	2.24	1.91	97.70	0.69

$\rho = 3$

δ	n_1	n_2	h_1	h_2	UCL_1	UCL_2	w	ECT	ATS
0	5	6	0.20	0.10	8.35	2.73	1.98	88.90	2.43
0.1	3	5	0.67	0.17	2.56	2.07	1.10	121.10	1.80
0.5	3	5	0.26	0.10	8.12	2.66	2.13	90.84	1.78
1	3	6	0.60	0.44	2.45	1.87	1.36	113.80	0.90
3	5	8	0.49	0.30	2.85	2.74	0.10	102.60	0.43