Limit cycles and Integrability of a class of 3-dimension chaotic systems

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Abstract

We focus on a chaotic differential system in 3-dimension, including an absolute term and a line of equilibrium points. Which describes as $\dot{x}=y, \dot{y}=-ax+yz, \dot{z}=b|y|-cxy-x^2$. This system has an implementation by using electronic components. The first purpose of this paper is to provide sufficient conditions for the existence of a limit cycle bifurcating from the zero-Hopf equilibrium point located at the origin of the coordinates. The second aim is to study the integrability of each differential system, one defined in half-space $y \geq 0$ and the other in half-space $y < 0$. We prove that these two systems have no polynomial, rational, or Darboux first integrals for any value of a, b and c. Furthermore, we provide a formal series and an analytic first integral of these systems. We also find Darboux polynomials and exponential factors.

Keywords: Chaos system, Limit cycle, Zero-Hopf bifurcation, Darboux integrability, First integral.

1 Introduction and the main results

Recent decades have seen a significant increase in the study of piecewise smooth differential systems, mainly because this type of system offers more realistic models in many applications, such as those involving switched circuit modeling, some mechanical problems, and control theory. For more details, one can see [1, 2, 3, 4, 5]. In [6], the authors listed eight chaotic systems. One of them, which has a line of equilibrium points, is given in the following:

$$\dot{x}=y, \quad \dot{y}=-ax+yz, \quad \dot{z}=b|z|-cxy-x^2, \tag{1}$$

where a, b and c are real parameters. System (1) has a line equilibrium points, namely $E=(0,0,z_0)$, where $z_0 \in \mathbf{R}$. Studying these kinds of systems is important because: this system belongs to a family of dynamical systems that have a "hidden attractor." There is evidence that hidden attractor plays a crucial role in theoretical and engineering fields. In recent years, chaotic systems with hidden attractors have garnered significant interest and made substantial progress [7, 8, 9]. The complexity of chaotic systems can be used to study a variety of engineering issues, such as image encryption, secure communication, control, and synchronisation. When working with differential equations, it is important to determine whether a differential system exhibits chaos [10, 11]. It’s worth mentioning that the absolute-value function is a possible nonlinear option for constructing chaotic systems with hidden attractors [8, 9, 12]. The authors in [13] have investigated the dynamics of a system (1) and observed its chaotic attractors and multistability based on initial conditions.

The limit cycles play a significant role in the dynamical systems when they exist. The limit cycles of a piecewise differential system are a highly challenging problem to analyze. The authors in [14, 15] studied the limit cycles of the piecewise differential systems, linear or nonlinear, using the first integrals. The authors in [4] studied the limit cycles of the piecewise differential systems. The authors established the Melnikov function method and the averaging method for finding limit cycles of piecewise smooth near-integrable systems. While, the authors in [16] studied the limit cycles bifurcating from a zero-Hopf equilibrium point using the averaging theory for Lipschitz differential systems.

Our first aim is to extend the dynamical features of system (1) by showing that it can exhibit a zero-Hopf equilibrium point for appropriate parameter values for which one limit cycle can bifurcate from the origin. Although much research has been done on limit cycles of smooth systems, we employ a version of the averaging method for non-smooth differential systems to examine limit cycles that bifurcate from zero-Hopf point. The second objective of this paper is to study the integrability of system (1). It is worth noting that, there is currently no sophisticated method for handling the integrability of non-smooth vector fields, especially when the system is defined on non-compact manifolds. Moreover, the classical method of integrability cannot be directly used for the non-smooth system (1). The absolute value term $|y|$ is regarded in system (1) as a piecewise function, which is defined as:

$$|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}. \tag{2}$$

The non-smooth differential system (1) is formed by the following two smooth differential systems: Considering $y \geq 0$, then system (1) becomes

$$\dot{x} = y, \quad \dot{y} = -ax + yz, \quad \dot{z} = by - cxy - x^2. \tag{3}$$

Also considering $y < 0$, then system (1) becomes

$$\dot{x} = y, \quad \dot{y} = -ax + yz, \quad \dot{z} = -by - cxy - x^2. \tag{4}$$

Consequently, the classical method can be used to study integrability of systems (3) and (4). More precisely, we use the Darboux theory of integrability to report the existence or non-existence of Darboux polynomials, exponential factors, Darboux first integrals, polynomials, and rational first integrals. Moreover, we provide the existence of the formal and analytic first integral of systems (3) and (4).

The following is the main result, which relates to limit cycles.

Theorem 1. *Consider the differential system* (1) *with* $b = \varepsilon\beta$, *and* $\varepsilon > 0$ *sufficiently small. System* (1) *has one unstable limit cycle* $\left(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)\right)$ *which bifurcates from a non-isolated zero-Hopf equilibrium point located at the origin, if* $a, \beta > 0$.

The proof of Theorem 1 is given in Section 3, which uses the averaging theory; this method is described in the appendix (Theorem 7). The next results, which are related to the integrability of systems (3) and (4) are summarized in the following theorems.

Theorem 2. *The following statements hold;*

1. *Systems* (3) *and* (4) *have no polynomial first integrals.*
2. *Systems* (3) *and* (4) *have a unique irreducible Darboux polynomial represented as* y, *with cofactor* z *if and only if* $a = 0$.

3. Systems (3) *and* (4) *have no rational first integrals.*

4. Systems (3) *and* (4) *have a single exponential factor* e^x *with cofactor* y.

5. Systems (3) *and* (4) *have no first integrals of the Darboux type.*

Theorem 3. *System* (3) *has a formal first integral in the neighborhood of equilibrium points* $E=(0,0,z_0)$ *for following cases:*

1. $z_0 > 2\sqrt{a}$, *where* $a>0$.

2. $a<0$ *and* $z_0 \in \mathbb{R} \setminus \{0\}$.

Moreover, system (3) *has an analytic first integral in the neighborhood of equilibrium points* $E=(0,0,z_0)$ *for* $z_0 \in (-2\sqrt{a}, 2\sqrt{a}) \setminus \{0\}$ *and* $a>0$.

The proof of Theorems 2 and 3 are given in Section 4. It is important to note that the proof of Theorem 2 for system (4) is similar to the one of system (3). So, we only consider the proofs on system (3). It is worth noting that, the statement of Theorem 2 and 3 are also true for system (1) when $b=0$, and its proof is clearly alike.

2 Integrability for smooth system

In this section, we present some results on the Darboux theory of integrability taht we will use throughout this investigation. The researchers [17, 18, 19, 20] have contributed into great details about the Darboux theory of this type of system.
The vector field corresponding to system (3) is

$$\chi = y\frac{\partial}{\partial x} + (-ax+yz)\frac{\partial}{\partial y} + (by - cxy - x^2)\frac{\partial}{\partial z}. \tag{5}$$

We called $f \in \mathbb{C}[x,y,z]$ is a Darboux polynomial of the vector field χ if there exists a polynomial $K \in \mathbb{C}[x,y,z]$ (which is the cofactor of f) such that $\chi f = Kf$, where $\mathbb{C}$ denotes the ring of all complex polynomials in the variables x,y,z. Here, the cofactor is a polynomial of degree at most 1. If $f(x,y,z)$ is a Darboux polynomial of system (3), then $f(x,y,z)=0$ is an invariant algebraic surface in $\mathbb{C}^3$, i.e. if an orbit has a point on the surface $f(x,y,z)=0$, the entire orbit is contained on it.

A Darboux polynomial $f(x,y,z) \in \mathbb{C}[x,y,z]$ with a zero cofactor is defined as a polynomial first integral of system (3). When f is a rational function that is not a polynomial, we say that f is a rational first integral. If f is a formal series, then f is called a formal first integral. Moreover, if f is an analytic function, we say that f is an analytic first integral.

For coprime polynomials g and h in $\mathbb{C}[x,y,z]$, the non-constant function $E=e^{(g/h)}$ is said to be an exponential factor of system (3) if $\chi E = LE$, where the polynomial L is a cofactor of E with a maximum degree of one, see [21, 22].

If a first integral f of system (3) is of the following form,

$$f_1^{\lambda_1} f_2^{\lambda_2} \ldots f_p^{\lambda_p} E_1^{\mu_1} E_2^{\mu_2} \ldots E_q^{\mu_q}, \tag{6}$$

it is called a Darboux type or a Darboux first integral, where f_j are Darboux polynomials, E_k are exponential factors, $\lambda_j, \mu_k \in \mathbb{C}$ for $j=1,..,p,\ k=1,\ldots,q$. The following theorem gives the condition for enough number of Darboux polynomials, and exponential factors need to have a Darboux first integral.

Theorem 4. *Assume that system* (3) *admits* p *invariant algebraic surfaces* $f_j = 0$ *with cofactors* K_i *for* $i = 1,\ldots,p$ *and* q *exponential factors* $E_j = e^{(g_j/h_j)}$ *with cofactors* L_j *for* $j = 1,\ldots,q$ *. Then there exist* λ_i *and* μ_j *in* $\mathbb{C}$ *, which are not all zero, such that*

$$\sum_{i=1}^{p}\lambda_i K_i + \sum_{j=1}^{q}\mu_j L_j = 0, \tag{7}$$

if and only if system (3) *has Darboux first integrl* H *of the form* (6).

The following results are related to the existence of Darboux polynomials. For proof, see [23, 24].

Lemma 1. *For system* (3), *the existence of a rational first integral indicates the presence of either a polynomial first integral or two Darboux polynomials with a non-zero cofactor.*

Proposition 1. *Both of the following statements hold.*

1. *If* $E = e^{(g/h)}$ *is an exponential factor for the polynomial system* (3) *and* h *is not a constant polynomial, then* $h = 0$ *is an invariant algebraic surface.*
2. *Eventually,* e^{g} *can be an exponential factor, coming from the multiplicity of the infinity.*

To prove Theorem 2, we use the weight-homogeneous polynomials, which have been used extensively in several standard systems [19, 25, 26]. A polynomial $h(x)$ with $x \in \mathbb{R}^3$ is said to be weight homogeneous if there exist $r = (r_1,\ldots,r_n) \in \mathbb{N}^n$ and $m \in \mathbb{N}$ such that $h(\mu^{r_1}x_1,\ldots,\mu^{r_n}x_n) = \mu^m h(x)$ for all $\mu > 0$. r is called the weight exponent and m is called the weight degree.

We use the following two result from [27, 28] to prove Theorem 3. Firstly, we consider an analytic differential system

$$\dot{x} = f(x), \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{8}$$

where $f(x)$ is a vector-valued function satisfying $f(0) = 0$. We denote by $A = \partial f/\partial x$ the Jacobian matrix of system (8) at $x = 0$.

Theorem 5. *Assume that the eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *of the Jacobian matrix A satisfy* $\lambda_1 = 0$ *and* $k_2\lambda_2 + k_3\lambda_3 \neq 0$ *for any* $k_2, k_3 \in \mathbb{Z}^+ \cup \{0\}$ *with* $k_2 + k_3 \geq 1$ *. Then the system* (8) *has a formal first integral in the neighborhood of* $x = 0$ *if and only if the equilibrium point* $x = 0$ *is not isolated. In particular, if the equilibrium point* $x = 0$ *is isolated, then system (8) has no analytic first integral in a neighborhood of* $x = 0$ *.*

Theorem 6. *For the local analytic differential system* (8), *assume that* $\lambda_1 = 0$ *, and* λ_2, λ_3 *either all have positive real parts or all have negative real parts. Then the system* (8) *has an analytic first integral in a neighborhood of* $x = 0$ *if and only if the equilibrium point* $x = 0$ *is not isolated.*

3 Proof of Theorem 1

Proof of Theorem 1. In system (1), consider $b = \varepsilon\beta$, and $\varepsilon \geq 0$. If $\varepsilon = 0$, then the Jacobian matrix of system (1) at the equilibrium point $E = (0, 0, z_0)$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ -a & z_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Setting $z_0 = 0$, then the above matrix has one zero eigenvalue and two eigenvalues $\pm\sqrt{-a}$. Assuming that $a > 0$, then system (1) has a non-isolated zero-Hopf equilibrium localizing at the origin of the coordinates with eigenvalues $\lambda_1 = 0,\ \lambda_{2,3} = \pm i\sqrt{a}$.

We will use the averaging theory to estimate the limit cycle, which bifurcates from the origin. Before this, we must formulate system (1) into normal form (42) in the appendix. We start by rescaling the variables (x, y, z) to $(\varepsilon X, \varepsilon Y, \varepsilon Z)$, which gives us

$$\dot{X} = Y, \qquad \dot{Y} = \varepsilon YZ - aX, \qquad \dot{Z} = \varepsilon\beta |Y| - \varepsilon cXY - \varepsilon X^2. \tag{9}$$

Now, using the change of variables $(X, Y, Z) \to (u, v, w)$ by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

The following differential system is obtained

$$\dot{u} = -\sqrt{a}v, \qquad \dot{v} = \sqrt{a}u + \varepsilon vw, \qquad \dot{w} = \varepsilon(\sqrt{a}\beta |v| + c\sqrt{a}uv - u^2). \tag{10}$$

Also using the cylindrical coordinates (r, θ, w) defined as $u = r\cos\theta$ and $v = r\sin\theta$, then system (10) becomes

$$\begin{aligned} \dot{r} &= \varepsilon wrsin^2\theta, \\ \dot{\theta} &= \sqrt{a} + \varepsilon w\cos\theta\sin\theta, \\ \dot{w} &= \varepsilon(\sqrt{a}\beta |rsin\theta| + r^2cos\theta(c\sqrt{a}sin\theta - cos\theta)). \end{aligned} \tag{11}$$

We utilize θ as a new independent variable. System (11) then becomes the following system which has been expand up to ε order,

$$\frac{dr}{d\theta} = \varepsilon F_1(r, \theta, w) + O(\varepsilon^2), \qquad \frac{dw}{d\theta} = \varepsilon F_2(r, \theta, w) + O(\varepsilon^2), \tag{12}$$

where

$$F_1(r, \theta, w) = \frac{1}{\sqrt{a}} rwsin^2\theta,$$

$$F_2(r, \theta, w) = \beta |r||sin\theta| + cr^2cos\theta sin\theta - (r^2cos^2\theta)/\sqrt{a}.$$

This gives the averaged function

$$\begin{aligned} F_{10}(r, w) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(r, \theta, w)d\theta = \frac{rw}{2\sqrt{a}}, \\ F_{20}(r, w) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(r, \theta, w)d\theta = \frac{4\beta\sqrt{a}|r| - \pi r^2}{2\pi\sqrt{a}}. \end{aligned} \tag{13}$$

The system $F_{10}(r, w)$=$F_{20}(r, w)$=0 has a unique solution (r^*, w^*) with $r^* > 0$ for $\beta > 0$, namely

$$(r^*, w^*) = \left(\frac{4\beta\sqrt{a}}{\pi}, 0\right).$$

By Theorem 7 in Appendix each solution (r^*, w^*) of the averaged function $(F_{10}(r, w), F_{20}(r, w))$ whose determinant of the Jacobian matrix is

$$\det\left(\frac{\partial F_0(r, w)}{(\partial r, \partial w)}\right)_{(r^*, w^*)} = -\frac{4\beta^2}{\pi^2} \neq 0, \qquad \text{where} \quad \beta \neq 0,$$

provides a limit cycle $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ of system (12), for $\varepsilon \neq 0$ sufficiently small. Which satisfying $(r(\theta,\varepsilon), w(\theta,\varepsilon)) \to (r^*, w^*) + O(\varepsilon)$. This limit cycle writes in system (11) as $\left(r(t,\varepsilon), \theta(t,\varepsilon), w(t,\varepsilon)\right) = \left(r^*, \theta, w^*\right) + O(\varepsilon)$. The limit cycle for system (10) takes form $\left(u(t,\varepsilon), v(t,\varepsilon), w(t,\varepsilon)\right) = \left(r^*\cos\theta, r^*\sin\theta, w^*\right) + O(\varepsilon)$. Passing to limit cycle to system (9) we have $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon)) = (r^*\cos\theta, r^*\sin\theta, w^*) + O(\varepsilon)$. Going back to system (1), the limit cycle $\left(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)\right) = \varepsilon(r^*\cos\theta, r^*\sin\theta, w^*) + O(\varepsilon^2)$ satisfies

$$(x(0,\varepsilon), y(0,\varepsilon), z(0,\varepsilon)) \to \varepsilon(r^*, 0, w^*) + O(\varepsilon^2) \to \varepsilon((4\beta\sqrt{a})/\pi, 0, 0) + O(\varepsilon^2).$$

When $\varepsilon \to 0$, this limit cycle tends to the equilibrium point located at the origin of the coordinates. Since, the eigenvalues of the matrix $\left(\dfrac{\partial F_0(r,w)}{(\partial r, \partial w)}\right)$ evaluated at $\left(4\beta\sqrt{a}/\pi, 0\right)$ are $\mp 2\beta/\pi$, by Theorem 8 in Appendix, it follows that the limit cycle defining by $\left(4\beta\sqrt{a}/\pi, 0\right)$ is unstable. ∎

Here, we will exhibit an example showing that one unstable limit cycle is born at the origin when $\varepsilon \to 0$ of Theorem 1. we consider $a = 2.5$, $c = 9$ and $b = \varepsilon\beta$, where $\beta = 2$ and $\varepsilon = 0.0001$. We obtain one positive solution for $r^* > 0$, with the initial condition $(0.0004026336968, 0, 0)$ shown in Figure 1.

4 Proof of Theorems 2 and 3

Proof of Therem 2(1). Let $H = H(x,y,z)$ be a polynomial first integral of system (3). Then, it satisfies

$$y\frac{\partial H}{\partial x} + (-ax + yz)\frac{\partial H}{\partial y} + (by - cxy - x^2)\frac{\partial H}{\partial z} = 0. \tag{14}$$

We can write H in the form $H(x,y,z) = \sum_{i=0}^{n} H_i(x,y) z^i$, where each H_i is a polynomial in the variables x, y. Now, computing the coefficient in (14) of z^{n+1}, we obtain $y\,\partial/\partial y\, H_n(x,y) = 0$. That is $H_n(x,y) = F_n(x)$, where $F_n(x)$ is a polynomial of the variable x. Computing also the coefficient in (14) of z^n , we obtain $y\,\partial/\partial y\, H_{n-1}(x,y) + y\, d/dx\, F_n(x) = 0$. That is $H_{n-1}(x,y) = -d/dx\, F_n(x) + F_{n-1}(x)$, where $F_{n-1}(x)$ is a polynomial of the variable x. Computing the coefficient in (14) of z^{n-1}, we get

$$y\frac{\partial}{\partial y} H_{n-2}(x,y) + ax\frac{\mathrm{d}}{\mathrm{d}x} F_n(x) - y^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} F_n(x) + y\frac{\mathrm{d}}{\mathrm{d}x} F_{n-1}(x) + (b - cx)nyF_n(x) - nx^2 F_n(x) = 0.$$

Solving the above equation with respect to $H_{n-2}(x,y)$, we obtain

$$H_{n-2}(x,y) = \frac{1}{2} y^2 \frac{d^2 F_n(x)}{dx^2} - y\frac{dF_{n-1}(x)}{dx} - (b - cx)nyF_n(x) + (nx^2 F_n(x) - ax\frac{dF_n(x)}{dx})\ln(y)) + F_{n-2}(x).$$

Since H_{n-2} is a polynomial, then we must have $nx^2 F_n - ax\, d/dx\, F_n = 0$. This gives $F_n(x) = C_n e^{\frac{nx^2}{2a}}$, where C_n is a constant. Since F_n is a polynomial, this implies that $n = 0$. Therefore, for $n \geq 1$, then $H_n = 0$. Let $H = H_0(x,y)$, we substitute it in (14), and computing the coefficient in (14) of z^i for $i = 0,1$. We obtain for $i = 1$: $y\frac{\partial H_0}{\partial y}=0$, that is $H_0(x,y)=F_0(x)$, where $F_0(x)$ is a polynomial of the variable x.

for $i = 0$: $y\frac{\partial}{\partial y}F_0(x)=0$, that is $F_0(x)=\mathrm{C}$, where C is a constant.

Hence, $H(x,y,z)=C$. So, system (3) has no polynomial first integrals. ■

Now, we will start the proof of Theorem 2 (2). We recall that a Darboux polynomial of the system (3) is a non-constant polynomial $f \in \mathbb{C}[x,y,z]$ such that

$$y\frac{\partial f}{\partial x}+(-ax+yz)\frac{\partial f}{\partial y}+(by-cxy-x^2)\frac{\partial f}{\partial z}=Kf\ . \tag{15}$$

For some polynomial $K=k_0+k_1x+k_2y+k_3z$, where $k_i \in \mathbb{C}$ for $i=0,1,2,3$. Firstly, we want to show that $k_0=k_1=k_2=0$.

Lemma 2. $k_2=0$.

Proof. We write f as the form $f(x,y,z)=\sum_{i=0}^{n} f_i(x,z)y^i$, where each f_i is a polynomial of the variables x,z. Computing the coefficient in (15) of term y^{n+1}. We have

$$\frac{\partial}{\partial x}f_n(x,z)+(b-cx)\frac{\partial}{\partial z}f_n(x,z)-k_2f_n(x,z)=0.$$

Solving the above equation, we obtain $f_n(x,z)=G_n(1/2\,cx^2-bx+z)e^{k_2x}$, where G_n is an arbitrary polynomial of the variable x and z. Since f_n is a polynomial then must be $G_n=0$ or $k_2=0$. Now, suppose that $G_n=0$ and $k_2\neq 0$. Consequently, we obtain $f_n=0$ for $n\geq 1$. Thus $f=f_0(x,z)$. We substitute it in (15) and compute the terms of y, we obtain

$$\frac{\partial}{\partial x}f_0(x,z)+(b-cx)\frac{\partial}{\partial z}f_0(x,z)-k_2f_0(x,z)=0.$$

Then, $f_0(x,z)=G_0\left(1/2\,cx^2-bx+z\right)e^{k_2x}$ for an arbitrary function G_0 of the variables x and z. If $k_2\neq 0$, we get a contradiction with the fact that f_0 must be non-constant polynomial. Hence, $k_2=0$. This complete the proof of the lemma. ■

Lemma 3. $k_0=k_1=0$ and $k_3\in\mathbb{N}^+$.

Proof. From Lemma 2, we can consider $k_2=0$. We write $f=\sum_{i=0}^{n} f_i(x,y)z^i$, where each f_i is a polynomial of the variables x,y. Computing the coefficient in (15) of the terms z^{n+1}, we obtain

$$y\frac{\partial}{\partial y}f_n(x,y)-k_3f_n(x,y)=0.$$

Solving the above equation, we obtain $f_n(x,y)=F_n(x)y^{k_3}$. Since $f_n(x,y)$ is a polynomial, it is possible $k_3\in\mathbb{N}^+\cup\{0\}$. Computing also the terms of z^n in (15), we obtain

$$y\frac{\partial f_{n-1}(x,y)}{\partial y}-k_3f_{n-1}(x,y)+y^{k_3+1}\frac{dF_n(x)}{dx}-(k_0+k_1x)F_n(x)y^{k_3}-ak_3xy^{k_3-1}F_n(x)=0.$$

Solving the above equation with respect to f_{n-1}, we obtain

$$f_{n-1}(x,y)=\left((k_0+k_1x)\ln(y)F_n(x)-y\frac{dF_n(x)}{dx}-\frac{ak_3xF_n(x)}{y}+F_{n-1}(x)\right)y^{k_3},$$

where $F_{n-1}(x)$ is a polynomial of x. Since f_{n-1} is a polynomial, then $F_n(x)=0$ or $k_0=k_1=0$ and $k_3\in\mathbb{Z}^+$. Now, suppose that $F_n(x)=0$ and $k_0\neq 0, k_1\neq 0$. This gives that $f_n=0$ for $n\geq 1$, this implies that $f=f_0(x,y)$. Then from (15) and computing the terms of z^i for $i=1,0$, we have

$$y\frac{\partial}{\partial y}f_0(x,y)-k_3f_0(x,y)=0, \tag{16}$$

$$y\frac{\partial}{\partial x}f_0(x,y)-ax\frac{\partial}{\partial y}f_0(x,y)-(k_0+k_1x)f_0(x,y)=0. \tag{17}$$

Solving equation (16), we obtain $f_0(x,y)=F_0(x)y^{k_3}$. By substituting f_0 in equation (17), we obtain

$$y^{k_3+1}\frac{d}{dx}F_0(x)-ak_3xy^{k_3-1}-(k_0+k_1x)F_0(x)=0.$$

That is $F_0(x)=Ce^{\frac{ak_3x^2+k_1x^2y+2k_0xy}{2y^2}}$, where C is an arbitrary constant. This contradict with the fact that f_0 is a polynomial of the variable x and y. This implies that $k_0=k_1=0$ and $k_3\in\mathbb{Z}^+$. This completes the proof of the lemma. ■

Next, we demonstrate the proof of Theorem 2(2).

Proof of Theorem 2(2). From Lemmas 2 and 3, we can write $K=k_3z$,where $k_3\in\mathbb{Z}^+$. Now, We choose the change of variables $x=X, y=Y, z=\lambda^{-1}Z, t=\lambda T$. Then, system (3) becomes

$$\dot{X}=\lambda Y,\quad \dot{Y}=-a\lambda X+YZ,\quad \dot{Z}=b\lambda^2Y-c\lambda^2XY-\lambda^2X^2, \tag{18}$$

where the dot represents derivative with respect to the variable T. Set $F(X,Y,Z)=\lambda^n f(X,Y,\lambda^{-1}Z)$ and $K(X,Y,Z)=\lambda K(X,Y,\lambda^{-1}Z)=k_3Z$, where n denotes the highest weight degree in the weight homogeneous components of f in the variables (x,y,z) with weight degree $(0,0,1)$.

Assume that $F(\lambda,X,Y,Z)=\sum_{i=0}^{n}\lambda^iF_{n-i}(X,Y,Z)$, where F_i is a weight homogeneous polynomial of the variables X,Y and Z with weight degree j for $j=0,1,...,n$. From the definition of Darboux polynomial, we have

$$\begin{aligned}&\lambda Y\sum_{i=0}^{n}\lambda^i\frac{\partial F_{n-i}}{\partial X}+(-a\lambda X+YZ)\sum_{i=0}^{n}\lambda^i\frac{\partial F_{n-i}}{\partial Y}\\&+(b\lambda^2Y-c\lambda^2XY-\lambda^2X^2)\sum_{i=0}^{n}\lambda^i\frac{\partial F_{n-i}}{\partial Z}=(k_3Z)\sum_{i=0}^{n}\lambda^iF_{n-i}.\end{aligned} \tag{19}$$

Computing the terms with λ^0 in (19), we get $YZ\,\partial/\partial Y\,F_n(X,Y,Z)-k_3ZF_n(X,Y,Z)=0$. That is $F_n(X,Y,Z)=H_n(X,Z)Y^{k_3}$, where $H_n(X,Z)$ is a polynomial of the variable X and Z. Since F_n is a homogeneous polynomial of weight degree n, we can assume that $H_n=W_n(X)Z^n$ with W_n is an arbitrary polynomial of the variable X.

Computing the coefficient of λ in (19), we obtain

$$YZ\frac{\partial F_{n-1}(X,Y,Z)}{\partial Y}-k_3ZF_{n-1}(X,Y,Z)+Z^nY^{k_3+1}\frac{dW_n(X)}{dX}-ak_3XZ^nY^{k_3-1}W_n(X)=0.$$

Solving the above equation, we obtain

$$F_{n-1}(X,Y,Z)=-Z^{n-1}\left(ak_3XW_n(X)Y^{k_3-1}+Y^{k_3+1}\frac{dW_n(X)}{dX}\right)+H_{n-1}(X,Z)Y^{k_3}.$$

Since F_{n-1} is a homogeneous polynomial of weight degree $n-1$, we can assume that $H_{n-1}=W_{n-1}(X)Z^{n-1}$, where W_{n-1} is an arbitrary polynomial of the variable X. Computing the coefficient of λ^2 in (19), we obtain

$$\begin{aligned}&YZ\frac{\partial F_{n-2}(X,Y,Z)}{\partial Y}-k_3ZF_{n-2}(X,Y,Z)+(k_3-1)a^2k_3W_n(X)X^2Z^{n-1}Y^{k_3-2}\\&+\left((nb-nXc)W_n(X)+\left(\frac{dW_{n-1}(X)}{dX}-ak_3XW_{n-1}(X)\right)\right)Z^{n-1}Y^{k_3+1}\\&+\left((-nX^2-ak_3)W_n(X)+\left(aX\frac{dW_n(X)}{dX}-Y^2\frac{d^2W_n(X)}{dX^2}\right)\right)Z^{n-1}Y^{k_3}=0.\end{aligned}$$

That is

$$\begin{aligned}F_{n-2}(X,Y,Z)=&H_{n-2}(X,Z)Y^{k_3}+Z^{n-2}Y^{k_3}\ln(Y)\left((ak_3+nX^2)W_n(X)-aX\frac{dW_n(X)}{dX}\right)\\&+Z^{n-2}\left(Y^{k_3-2}\left(\frac{1}{2}(k_3-1)k_3a^2X^2W_n(X)\right)-ak_3XY^{k_3-1}W_{n-1}(X)\right)n\\&+Z^{n-2}Y^{k_3+1}\left(n(cX-b)W_n(X)+\frac{1}{2}Y\frac{d^2W_n(X)}{dX^2}-\frac{dW_{n-1}(X)}{dX}\right),\end{aligned}\tag{20}$$

where $H_{n-2}(X,Z)$ is a polynomial of the variables X and Z. For $F_{n-2}(X,Y,Z)$ to be a polynomial, we consider two cases;

Case 1. if $n=a=0$ and $W_n(X)\neq 0$, then equation (20) becomes

$$F_{n-2}(X,Y,Z)=\frac{Y^{k_3+2}}{2Z^2}\left(\frac{d^2W_n(X)}{dX^2}-\frac{dW_{n-1}(X)}{dX}\right)+H_{n-2}(X,Z)Y^{k_3}.$$

Which is impossible, because F_{n-2} is a polynomial.

Case 2. If $W_n(X)=0$, we can infer from equation (20) that

$$F_{n-2}(X,Y,Z)=-Z^{n-2}\left(ak_3XY^{k_3-1}W_{n-1}(X)+Y^{k_3+1}\frac{dW_{n-1}(X)}{dX}\right)+H_{n-2}(X,Z)Y^{k_3}.$$

Since F_{n-2} is a polynomial of weight degree $n-2$, we can assume that $H_{n-2}(X,Z)=W_{n-2}(X)Z^{n-2}$. Similarly to the above process. By computing the coefficient of λ^i for $i=3\ldots n-1$ in (19), we can derive $W_{n-1}(X)=\ldots=W_3=0$ and

$$F_{n-3}(X,Y,Z)=-Z^{n-3}\left(ak_3XY^{k_3-1}W_{n-2}(X)+Y^{k_3+1}\frac{dW_{n-2}(X)}{dX}\right)+H_{n-3}(X,Z)Y^{k_3},$$

$$\vdots$$

$$F_1(X,Y,Z)=-Z\left(ak_3XY^{k_3-1}W_2(X)+Y^{k_3+1}\frac{dW_2(X)}{dX}\right)+H_1(X,Z)Y^{k_3}$$

Since F_j is a polynomial of weight degree j, we can assume that $H_j(X,Z)=W_j(X)Z^j$, for

$j=n-3,\ldots,1$. Now, computing the coefficient of λ^n in (19), we obtain

$$YZ\frac{\partial F_0(X,Y,Z)}{\partial Y}-k_3ZF_0(X,Y,Z)+ak_3Z\left(a(k_3-1)X^2Y^{k_3-2}W_2(X)-XY^{k_3-1}W_1(X)\right)$$
$$+ZY^{k_3}\left((ak_3-2X^2)W_2(X)+aX\frac{dW_2(X)}{dX}+2(b-cX)YW_2(X)+Y\frac{dW_1(X)}{dX}-Y^2\frac{d^2W_2(X)}{dX^2}\right)=0.$$

Solving the above equation, we obtain

$$F_0(X,Y,Z)=H_0(X,Z)Y^{k_3}+\left(\frac{1}{2}Y^2\frac{d^2W_2(X)}{dX^2}-2(b-cX)YW_2(X)-Y\frac{dW_1(X)}{dX}\right)Y^{k_3}$$
$$+\ln(Y)\left(-aX\frac{dW_2(X)}{dX}-(ak_3-2X^2)W_2(X)\right)Y^{k_3}-ak_3\left(\frac{a(k_3-1)X^2W_2(X)}{2Y^2}-\frac{XW_1(X)}{Y}\right)Y^{k_3},$$

where $H_0(X,Z)$ is a polynomial of the variables X and Z. Since $F_0(X,Y,Z)$ is a polynomial, this required that $W_2(X)=0$, which implies that

$$F_1(X,Y,Z)=W_1(X)ZY^{k_3}, \tag{21}$$

$$F_0(X,Y,Z)=-\frac{dW_1(X)}{dX}Y^{k_3+1}-ak_3XW_1(X)Y^{k_3-1}+H_0(X,Z)Y^{k_3}. \tag{22}$$

Since F_0 is a polynomial of weight degree 0, we can assume that $H_0(X,Z)=W_0(X)$, which is a polynomial for X only. Lastly, computing the coefficient of λ^{n+1} in (19), we obtain

$$\left(\left(-ak_3-X^2\right)+(b-cX)Y\right)W_1(X)Y^{k_3}+\left(aX\frac{dW_1(X)}{dX}-Y^2\frac{d^2W_1(X)}{dX^2}+Y\frac{dW_0(X)}{dX}\right)Y^{k_3}$$
$$+\left(\frac{a^2k_3(k_3-1)X^2W_1(X)}{Y^2}-\frac{ak_3XW_0(X)}{Y}\right)Y^{k_3}=0. \tag{23}$$

The above equation is a polynomial of variables X and Y, computing the coefficients in (23) of terms $Y^{k_3+2},Y^{k_3+1},Y^{k_3}$ and Y^{k_3-1} respectively, we obtain

$$-\frac{d^2}{dX^2}W_1(X)=0, \tag{24}$$

$$\frac{d}{dX}W_0(X)+(b-cX)W_1(X)=0, \tag{25}$$

$$aX\frac{d}{dX}W_1(X)-(X^2+k_3)W_1(X)=0, \tag{26}$$

$$-ak_3XW_0(X)=0. \tag{27}$$

Solving the equation (24) for $W_1(X)$, we can write $W_1(X)=d_1X+d_0$, where d_1 and d_0 are arbitrary constant. Substituting $W_1(X)=d_1X+d_0$ in equation (26), we obtain

$$-d_1X^3-d_0X^2+ad_1(1-k_3)X-ad_0k_3=0. \tag{28}$$

One possibility that the above polynomial becomes zero is that $d_1=d_0=0$. This implies that $W_1(X)=0$, $F_1(X,Y,Z)=0$ and $F_0(X,Y,Z)=W_0(X)Y^{k_3}$. Substituting $W_1(X)=0$ in the equation (25) and solve it for $W_0(X)$, we obtain $W_0(X)=C_0$, where C_0 is a constant. This gives that from equation (27), $-aC_0k_3X=0$. Since $k_3>0$ and $C_0\neq 0$, then $a=0$, thus, $F_0(X,Z,Y)=C_0Y^{k_3}$.

To sum up, From $F(\lambda, X, Y, Z)=\sum_{i=0}^{n}\lambda^i F_{n-i}(X,Y,Z)$, we have $F=C_0Y^{k_3}$. This conclude that, $f=y^{k_3}$ is Darboux polynomial with cofactor $K = k_3 z$. This completes the proof of Theorem 2(2). ■

Proof of Theorem 2(3). The proof comes directly from Theorem 2 (1) and (2) with Lemma 1.

Proof of Theorem 2 (4). Let $E=e^{\frac{g}{h}}$ be an exponential factor of the system (3) with cofactor $L=l_0+l_1x+l_2y+l_3z$, where $g,h\in\mathbb{C}[x,y,z]$ with g and h are relatively prime and $l_i\in\mathbb{C}$ for $i=0,1,2,3$. To give the complete proof, we consider two cases;

Case (1): If $a\neq 0$, then from Theorem 2 (2) and Proposition 1, h is a constant (let say $h=1$). Thus, $E=e^g$, then we have

$$y\frac{\partial e^g}{\partial x}+(-ax+yz)\frac{\partial e^g}{\partial y}+(by-cxy-x^2)\frac{\partial e^g}{\partial z}=Le^g. \tag{29}$$

Simplifying

$$y\frac{\partial g}{\partial x}+(-ax+yz)\frac{\partial g}{\partial y}+(by-cxy-x^2)\frac{\partial g}{\partial z}=L, \tag{30}$$

Let g be $g(x,y,z)=\sum_{i=0}^{n} g_i(x,y)z^i$, where each g_i is a polynomial of the variables x and y. Firstly, we consider $n\geq 2$. Now, computing the coefficient of z^{n+1} in (30), we obtain $y\,\partial/\partial y\, g_n(x,y)=0$. That is $g_n(x,y)=G_n(x)$, where $G_n(x)$ is a polynomial of the variable x. Computing also the coefficient of z^n in (30), we obtain

$$y\frac{\partial}{\partial y}g_{n-1}(x,y)+y\frac{d}{dx}G_n(x)=0.$$

That is $g_{n-1}(x,y)=-y\frac{d}{dx}G_n(x)+G_{n-1}(x)$, where $G_{n-1}(x)$ is an arbitrary polynomial of the variable x. Next, computing the coefficient in (30) of z^{n-1}, we obtain

$$y\frac{\partial g_{n-2}(x,y)}{\partial y}+ax\frac{dG_n(x)}{dx}+y\frac{dG_{n-1}(x)}{dx}-y^2\frac{d^2G_n(x)}{dx^2}+n(b-cx)yG_n(x)-nx^2G_n(x)=0.$$

We can solve the above equation for $g_{n-2}(x,y)$, we obtain

$$g_{n-2}(x,y)=\frac{1}{2}y^2\frac{d^2G_n(x)}{dx^2}-y\frac{dG_{n-1}(x)}{dx}+(-b+cx)nyG_n(x)+\ln(y)\left(nx^2G_n(x)-ax\frac{dG_n(x)}{dx}\right)+G_{n-2}(x),$$

where $G_{n-2}(x)$ is a polynomial of the variable x. Since $g_{n-2}(x,y)$ is a polynomial and $a\neq 0$, it is required that $G_n(x)=0$. This implies that $g_n=0$ for $n\geq 2$. Hence, we have $g(x,y,z)=g_0(x,y)+g_1(x,y)z$. The equation (30) becomes

$$\left(yz^2-axz\right)\frac{\partial g_1}{\partial y}+yz\frac{\partial g_1}{\partial x}+\left(yz-ax\right)\frac{\partial g_0}{\partial y}+y\frac{\partial g_0}{\partial x}+(by-x^2-cxy)g_1=L. \tag{31}$$

Compute the coefficients in (31) of z^2, z^1 and z^0 respectively, we obtain the following differential equations

$$y\frac{\partial g_1(x,y)}{\partial y}=0, \tag{32}$$

$$-ax\frac{\partial g_1(x,y)}{\partial y}+y\frac{\partial g_0(x,y)}{\partial y}+y\frac{\partial g_1(x,y)}{\partial x}-l_3=0, \tag{33}$$

$$-ax\frac{\partial g_0(x,y)}{\partial y}+y\frac{\partial g_0(x,y)}{\partial x}+(by-x^2-cxy)g_1(x,y)-l_0-l_1x-l_2y=0. \tag{34}$$

Solve the equation (32) for $g_1(x,y)$, we obtain $g_1(x,y)=G_1(x)$, where $G_1(x)$ is a polynomial of the variable x. Substituting $g_1(x,y)=G_1(x)$ in (33), we obtain

$$g_0(x,y)=-y\frac{dG_1(x)}{dx}+l_3\ln(y)+G_0(x),$$

where $G_0(x)$ is a polynomial of the variable x. Since $g_0(x,y)$ is a polynomial, then $l_3=0$. Therfore, equation (34) can be simplified into

$$y\frac{dG_0(x)}{dx}+\left((b-cx)G_1(x)-l_2\right)y-y^2\frac{d^2G_1(x)}{dx^2}+\left(ax\frac{dG_1(x)}{dx}-x^2G_1(x)-l_0-l_1x\right)=0.$$

Computing the coefficients of y^i for $i=2,1,0$ respectively, we can derive the following:

$$-\frac{d^2G_1(x)}{dx^2}=0, \tag{35}$$

$$\frac{dG_0(x)}{dx}+(b-cx)G_1(x)-l_2=0, \tag{36}$$

$$ax\frac{dG_1(x)}{dx}-x^2G_1(x)-l_1x-l_0=0. \tag{37}$$

Solve the equation (35) for $G_1(x)$, we obtain $G_1(x)=a_1x+a_0$, where a_1, a_0 are arbitrary constant. Then, the equation (37) becomes $-a_1x^3+a_0x^2+(aa_1-l_1)x-l_0=0.$ Since $a\neq 0$, it is required that $a_1=a_0=l_1=l_0=0$. Now, solving equation (36) for G_0, we obtain $G_0(x)=l_2x+c_0$, where c_0 is constant. As a result, we have $g=g_0=l_2x+c_0$, that is $e^{l_2x+c_0}$ is an exponential factor with cofactor $L=l_2y$.

Case (2): When $a=0$, then according to Proposition 1 and Theorem 2 (2), the exponential factors of system (3) can be expressed as $E=e^{\frac{g}{y^m}}$ for some non-negative integer m, where $g\in\mathbb{C}[x,y,z]$, in which g and y^m are relatively prime. By definition of the exponential factor, directly, we have

$$y\frac{\partial g}{\partial x}+yz\frac{\partial g}{\partial y}+(by-cxy-x^2)\frac{\partial g}{\partial z}-mzg=Ly^m. \tag{38}$$

Now, we consider two cases;

Case I: For $m\geq 1$, the restriction of g to $y=0$ is denoted as g' is the polynomial, defined by $g(x,y,z)|_{y=0}=g'$. Then equation (38), becomes

$$-x^2\frac{\partial g'}{\partial z}-mzg'=0. \tag{39}$$

Let $g'(x,z)=\sum_{i=0}^{n} g_i'(x,z)$, where each g_i' is a homogeneous polynomial of degree i of the variables x and z. By computing the terms of degree $n+1$ from (39), we obtain

$$-x^2\frac{\partial g_n'(x,z)}{\partial z}-mzg_n'(x,z)=0. \tag{40}$$

Solving the above equation for g_n', we obtain

$$g_n'(x,z) = G_n(x)e^{-\frac{mz^2}{2x^2}}, \tag{41}$$

Since $g_n'(x,z)$ is a polynomial and $m \geq 1$, this gives that $G_n(x)=0$. Then, $g_i'(x,z)=0$ for each $i=0,\ldots,n$, thus $g'(x,z)=0$. This case can not be taken.

Case II: For $m=0$, we have $E=e^g$, where $g \in \square[x,y,z]$. Setting $a=0$, in Theorem 2 (4) Case (1), we obtain $e^{g(x,y,z)}=e^{l_2 x+c_0}$ with the cofactor $L=l_2 y$. This completes the proof of Theorem 2 (4). ■

Proof of Theorem 2 (5). According to Theorem 4, system (3) has a first integral of the Darboux type if and only if there are λ_i and μ_j, which are not all zero, satisfying equation (7). By Theorems 2 (2) and (4), we can consider the following cases;

1. If $a \neq 0$, then system (3) has no Darboux polynomials. By Theorem 2 (4), there is only one exponential factor e^x with cofactor $L=y$. Hence, the equation (7) becomes $\mu_1 y=0$. This gives that $\mu_1=0$.
2. If $a=0$, then system (3) has one Darboux polynomial $f=y$ with cofactor $K=z$. By Theorem 2 (4), there is only one exponential factor e^x with cofactor $L=y$. Hence, the equation (7) becomes $\lambda_1 z+\mu_1 y=0$. This gives that $\lambda_1=\mu_1=0$.

This completes the proof of Theorem 2 (5). ■

Proof of Theorem 3. System (3) has a line of equilibrium points formed by $E=(0,0,z_0)$, where $z_0 \in \square$. The characteristic polynomial of the Jacobian matrix at $E=(0,0,z_0)$ is given by $\lambda^3 - z_0\lambda^2 + a\lambda=0$. Hence, the eigenvalues are $\lambda_1=0, \lambda_{2,3}=1/2(z_0 \pm \sqrt{z_0^2-4a})$. We consider following cases:

1. If $a>0$, then $\lambda_2\lambda_3=a$ and $\lambda_2^2=\left(1/2(z_0+\sqrt{z_0^2-4a})\right)^2>0$, for $z_0^2 \geq 4a$. Hence,

   $k_2\lambda_2+k_3\lambda_3=\dfrac{1}{\lambda_2}(k_2\lambda_2^2+k_3\lambda_2\lambda_3) \neq 0$, for all $k_2,k_3 \in \square \cup \{0\}$ with $k_2+k_3>0$. By Theorem 5, system (3) has a formal first integral in a neighborhood of $E=(0,0,z_0)$ except the origin.

   On the other hand, if $z_0^2<4a$ i.e., $-2\sqrt{a}<z_0<2\sqrt{a}$, then $\lambda_{2,3}=1/2(z_0 \pm i\sqrt{4a-z_0^2})$. Hence, either all eigenvalues have positive real parts or all have negative real parts. By Theorem 6, system (3) has an analytic first integral in a neighborhood of the equilibrium points $E=(0,0,z_0)$ for $z_0 \in (-2\sqrt{a},2\sqrt{a})\setminus\{0\}$.
2. If $a<0$, then $k_2\lambda_2+k_3\lambda_3=(k_2-k_3)(\sqrt{z_0^2-4a})/2+(k_2+k_3)z_0/2 \neq 0$, for all non-negative integers k_2 and k_3, such that $k_2+k_3>0$ and $z_0 \neq 0$. Otherwise, if

   $(k_2-k_3)(\sqrt{z_0^2-4a})/2+(k_2+k_3)z_0/2=0$, then $\dfrac{k_3-k_2}{k_3+k_2}=\dfrac{z_0}{\sqrt{z_0^2-4a}}$.

   Which is impossible, because $z_0 \in \square\setminus\{0\}$ and $z_0^2-4a>0$, we can pick some value for z_0 in which the right side of the aforementioned equation becomes irrational. Consequently, by Theorem 5 and the equilibrium points $(0,0,z_0)$ being non-isolated, then system (3) has a formal first integral in a neighborhood of $(0,0,z_0)$ except the origin.

This conclude the proof of Theorem 3. ■

Conclusion

In our research, we delved into the study of limit cycles and the integrability of a non-smooth continuous chaotic system featuring a line equilibrium point with a "hidden attractor." We observed that the zero-Hopf bifurcation occurs at the origin of the coordinate, leading to the bifurcating of a single limit cycle from this equilibrium point. Moreover, we proved that system (1) has a unique irreducible invariant algebraic surface when the parameter a is zero. Subsequently, we proved that the system contains only one exponential factor. We also showed that the system has neither a polynomial first integral nor a rational first integral for any value of parameters. Additionally, we proved that the system is not Darboux integrable. Finally, we verified that the system has formal and analytic first integral in a neighborhood at the equilibrium point of the system where $a \neq 0$ and $a > 0$ respectively.

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Appendix: Averaging theory of first order for limit cycles

Now we'll go through the basic averaging theory for Lipschitz differential systems which we'll need to prove result of isolated limit cycle bifurcate from zero-Hopf point. The following theorem offers a first-order of the averaging theory for differential system which founded in [29, 30] and used in [16, 31, 32, 33]. For more information and the proof see previous references.

Theorem 7. Consider the differential equation

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x), \tag{42}$$

where $F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n, F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions and 2π -periodic in t, where $\Omega \in \mathbb{R}^n$ open. We set $F_{10} : \Omega \to \mathbb{R}^n$ and define

$$F_{10}(x) = \frac{1}{T}\int_0^T F_1(s,z)ds, \tag{43}$$

and assume that,

1. F_1 and F_2 are locally Lipschitz in x,
2. for $\alpha \in \Omega$ with $F_{10}(\alpha) = 0$, there exists a neighborhood V of α such that $F_{10}(z) \neq 0$ for all $z \in \bar{V} \setminus \{\alpha\}$ and $d_B(F_{10}, V, \alpha) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small, there exists an isolated $T-$periodic solution $x(t,\varepsilon)$ of system (42) such that $x(0,\varepsilon) \to \alpha$ as $\varepsilon \to 0$.

Where $d_B(F_{10}, V, \alpha)$ denotes the Brouwer degree of F_{10} in the neighborhood V of α. The improvement of Theorem 7 is considered in [4] as follows

Theorem 8. According to Theorem 7, for small ε, the condition $\det(DF_{10}(\alpha)) \neq 0$ guarantees the existence and uniqueness of a T-periodic solution $x(t,\varepsilon)$ of system (42) such that $x(0,\varepsilon) \to \alpha$ as $\varepsilon \to 0$. Moreover, the periodic solution $x(t,\varepsilon)$ is stable, if all of the eigenvalue of Jacobian matrix $DF_{10}(\alpha)$ have negative real portions. The periodic solution $x(t,\varepsilon)$ is unstable if any of the eigenvalues have positive real parts.

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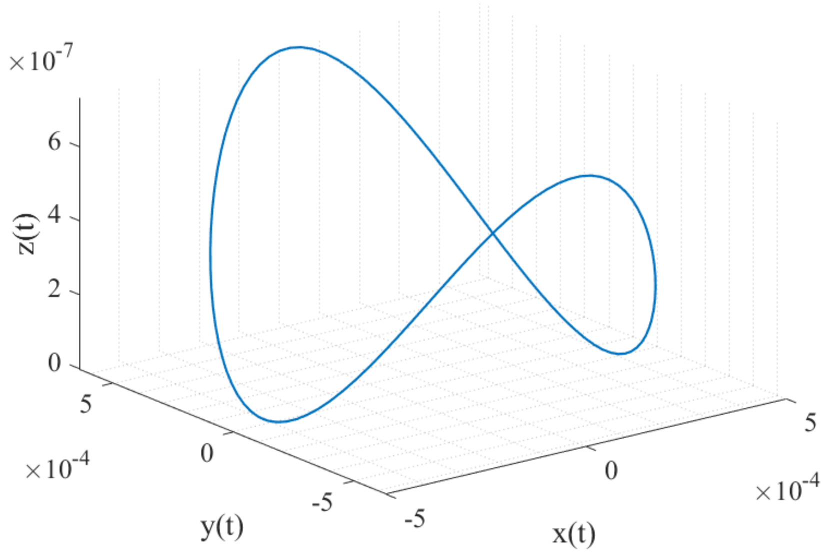
Figure 1: Limit cycle bifurcating from origin in system (1), for $a = 2.5$, $c = 9$, $b = \varepsilon\beta$, $\beta = 2$, $\varepsilon = 0.0001$ with initial condition: $(0.0004026336968, 0, 0)$.



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