Dynamical properties of Nonlinear oscillators by the variational iteration method

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Abstract:

This article introduces a semi-analytical approach to solving equations that model blood flow in flexible vessels. The method, called the Variational Iteration Method with Laplace Transformation (LVIM), is utilized to obtain an explicit solution. The problem is addressed by asymptotically reducing the incompressible Navier–Stokes equations, resulting in a one-dimensional nonlocal blood flow model that includes source terms. One of the most important steps is the evolution of the Lagrange Multiplier, which is calculated based on Lagrange theory. To showcase the effectiveness of LVIM, numerical examples based on experimentally determined parameters are provided. The model's results are then assessed by comparing them and analyzing the relative error.

Keywords:

Blood flow modeling; Variational iteration method; Compliant vessels; Navier–Stokes equations; Laplace Transformation

1. Introduction:

Differential equations are essential for modeling real-world phenomena. Scientists often deal with various properties and processes such as collisions, chemical interactions, wave propagation, viscosity, temperature, and pressure that make solving these equations analytically increasingly challenging. To address these difficulties, researchers have employed a range of methods to approximate solutions. These methods include direct algebraic approaches [1], rational expansion [2], reductive perturbation techniques [3], extended mapping [4], and other specialized techniques such as the Adomian decomposition method [5], homotopy perturbation method [6], differential transform method [7], and the Variational Iteration Method (VIM) [8]. VIM, a semi-analytical approach developed by He towards the end of the 20th century, is particularly noteworthy for its utility in solving complex differential equations.

The VIM is a powerful technique for solving nonlinear differential equations. It utilizes a variational approach to approximate solutions by iterating between successive approximations, which converges to the exact solution [9-10]. The method starts with an initial guess and refines it using correction terms derived from the original differential equation, allowing it to handle a wide range of complex problems. The Laplace-based Variational Iteration Method (LVIM) [11-13] enhances VIM by incorporating the Laplace transform [14], which simplifies the process of solving differential equations, particularly those with boundary conditions. By applying the Laplace transform, LVIM can convert differential equations into algebraic equations, making the iterative process more efficient and straightforward. In the Variational Iteration Method (VIM), the Lagrange Multiplier [15-16] is used to handle constraints or incorporate boundary conditions into the iterative process. When solving differential equations using VIM, especially in cases involving constraints or specific conditions that must be satisfied (like boundary conditions), Lagrange multipliers help in formulating the problem in a way that respects these conditions [17]. In recent decades, numerous researchers, such as Wazwaz [18-19], have further developed the Variational Iteration Method (VIM) to address a variety of nonlinear problems. This method iteratively improves an initial solution, resulting in high accuracy, and provides approximate solutions in an explicit form unlike traditional numerical methods that rely on interpolation. In this study, we use VIM to investigate blood flow dynamics in compliant vessels, focusing on simplifying the problem by minimizing the number of variables involved.

The Laplace-based variational iteration method is a technique that combines Laplace transforms with variational iteration to solve differential equations [20]. This method utilizes the Laplace transform to convert differential equations into algebraic equations, which are generally easier to solve. By applying the Laplace transform to the original differential equation, the problem is simplified, and the algebraic form can be tackled with various mathematical techniques [21]. Following the Laplace transformation, the variational iteration method is applied. This method involves constructing an iterative sequence of approximations to the solution. The variational iteration method introduces a correction functional that iterates towards the exact solution. Each iteration refines the approximation based on the previous one, gradually converging to the true solution. The combined approach leverages the Laplace transform's ability to simplify differential equations and the iterative nature of the variational method to improve accuracy. This technique is particularly useful for solving complex differential equations that arise in various fields, including fluid dynamics and medical engineering [22-23].

Recent research has begun to explore the intricate relationships between vitamin levels, periodontal health, and blood flow, unveiling new insights into both disease progression and therapeutic approaches. Studies have identified associations between vitamin D [24] and periodontal attachment loss, highlighting its potential role in maintaining periodontal health. Meanwhile, the interplay of vitamin K and fiber intake with periodontal attachment loss suggests a broader impact of diet on oral health. The engineering of oral bacteria for cardiovascular disease therapeutics represents a promising frontier, as oral microbiota is increasingly recognized for its influence on systemic conditions, including blood flow [25-34]. Similarly, the role of inflammatory indices in periodontal disease progression may offer predictive insights into cardiovascular risks, linking oral health to broader circulatory health. Advancements in the automated localization of mandibular landmarks, tumor microenvironment therapy, and innovative fluid modeling techniques further underscore the growing integration of bioengineering and computational approaches in understanding and treating complex health conditions. This convergence of research across disciplines holds significant potential for improving health outcomes through novel diagnostic and therapeutic strategies [35-40].

Mathematical modeling of blood flow is a crucial area of study in both engineering and medicine, providing insights into the complex dynamics of the circulatory system [41-45]. These models often involve solving partial differential equations that describe the behavior of blood as a viscous, incompressible fluid moving through the intricate network of arteries and veins. The Navier-Stokes equations [46], along with additional considerations for pulsatile flow and vessel elasticity, are frequently used to simulate how blood flows under different physiological conditions. Such models help in understanding various phenomena, such as the effects of arterial stenosis, the dynamics of blood pressure changes, and the impact of vascular interventions. By integrating data from medical imaging and patient-specific parameters, these models enable the prediction of blood flow characteristics in individual patients, ultimately aiding in diagnosis, treatment planning, and the development of medical devices [47].

In this study, we introduce a novel approach that harnesses the strengths of LVIM to obtain explicit solutions without the need for linearization or discretization. This technique is particularly suited for analyzing blood flow in arteries, offering a way to assess flow dynamics through blood pressure measurements. We treat blood flow as Newtonian and govern it with the continuity and momentum equations, known as the Navier-Stokes equations. By applying LVIM to these equations, including a source term, we aim to provide a clearer and more direct understanding of arterial blood flow.

2. Mathematical Modeling of Blood Flow

The mathematical and numerical modeling of the human cardiovascular system has become increasingly prevalent. These models often utilize the Navier–Stokes equations. Three-dimensional models capture the flow dynamics in compliant vessels and their interaction with wall displacements. To simplify the equations, many studies have focused on one-dimensional flow systems derived from single vessel models. These one-dimensional models are particularly useful for examining systemic circulation due to their lower computational costs. They describe the flow of an incompressible viscous fluid through elastic tubes, accounting for the interaction between blood vessels and wall movements. Two main approaches are used to derive these equations: one involves integrating the Navier–Stokes equations in a generic cross-section, while the other employs dimensional analysis of the underlying equations. The latter approach is often considered more realistic and provides broader results.

In the asymptotic reduction, the equations will be analyzed using nondimensional variables; further details can be found in [48-49]. This approach assumes that the artery or aorta is nearly cylindrical and that blood flow occurs predominantly in the axial direction. The system of equations must be expressed in cylindrical coordinates (X, r, θ) when the x-axis is aligned with the symmetry axis of the vessel. Let $V = (V_X, V_r, V_\theta)$ represent the velocity components. Initially, we assume that the angular velocity is zero, leading to the following equations of motion [50]:

$$(\frac{\partial}{\partial t} + V_r \frac{\partial}{\partial r} + V_X \frac{\partial}{\partial x} - v \frac{\partial^2}{\partial X^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial X^2})V_X + \frac{1}{\rho}\frac{\partial P}{\partial X} = 0 \tag{1}$$

$$(\frac{\partial}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_X \frac{\partial}{\partial X} - v \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial X^2})V_r + \frac{V_r}{r^2} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0 \tag{2}$$

After translating the reduced non-dimensional equations and applying the incompressibility condition, we obtain

$$\frac{\partial V_X}{\partial X} + \frac{1}{r}\frac{r\partial V_r}{\partial r} = 0 \tag{3}$$

We define U_0 and V_0 as the characteristic velocities in the radial and axial directions, respectively. Here, L represents the characteristic length, R_0 indicates the inner radius of the vessel, and the corresponding dimensionless variables are introduced as follows:

$$r = R_0\tilde{r} \qquad X = L\tilde{X} \qquad t = \frac{L}{V_0}\tilde{t} \tag{4}$$

$$V_x = V_0\tilde{V}_{x,} \qquad V_r = V_0\tilde{V}_r \qquad p = \rho V_0{}^2\tilde{p}$$

sing a straightforward calculation, the first reduced equations can be expressed as follows:

$$\frac{\partial}{\partial\tilde{t}}(\tilde{r}\tilde{V}_X) + \frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{V}_r\tilde{V}_X) + \frac{\partial}{\partial\tilde{X}}(\tilde{r}\tilde{V}_X{}^2) + \frac{\partial}{\partial\tilde{X}}(\tilde{r}\tilde{p}) = \frac{\nu\lambda}{V_0R_0^2}(\frac{\partial}{\partial\tilde{r}}(\tilde{r}\frac{\partial\tilde{V}_X}{\partial\tilde{r}})) \tag{5}$$

The reduced form of the second equation of motion is

$$\frac{\partial\tilde{P}}{\partial\tilde{r}} = 0 \tag{6}$$

This equation suggests that the pressure remains uniform across the vessel's cross-section. The incompressibility condition, in its simplified form, is therefore as follows:

$$\frac{\partial}{\partial\tilde{r}}(\tilde{r}V_r) + \frac{\partial}{\partial\tilde{X}}(\tilde{r}V_X) = 0 \tag{7}$$

In this step, we'll reformulate the equations using the average value in the transverse zone. Let $\tilde{R}$ denote the inner radius of the ship.

$$\tilde{U} = \frac{2}{\tilde{R}^2}\int_0^R \tilde{V}_X\tilde{r}d\tilde{r} \tag{8}$$

$$\alpha = \frac{2}{\tilde{U}^2\tilde{R}^2}\int_0^R \tilde{r}\tilde{V}_X^R d\tilde{r} \tag{9}$$

where $\tilde{U}$ represents the axial mean velocity, and α denotes the correction term. We integrate the equations from $\tilde{r}$ =0 to $\tilde{R}$, which define the terms of the average quantity, by specifying the boundary condition at $\tilde{r} = \tilde{R}$

$$[\tilde{V}_r]_{\tilde{r}=\tilde{R}} = \frac{\partial\tilde{R}}{\partial\tilde{X}}[\tilde{V}_X]_{\tilde{r}=\tilde{R}} + \frac{\partial\tilde{R}}{\partial\tilde{t}} \tag{10}$$

To address the incompressibility condition, we integrate Equation (7) and apply the definition of $\tilde{U}$.

$$\frac{\partial \tilde{R}^2}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{X}}(\tilde{V}\frac{\tilde{R}^2}{2}) = 0 \quad (11)$$

For the initial equation of motion, we integrate the first reduced motion equation, resulting in

$$\frac{\partial}{\partial \tilde{t}}(\tilde{V}\frac{\tilde{R}^2}{2}) + \frac{\partial}{\partial \tilde{X}}(\frac{\alpha \tilde{R}^2 \tilde{V}^2}{2}) + \frac{\tilde{R}^2}{2}\frac{\partial \tilde{P}}{\partial \tilde{X}} = \frac{\nu\lambda}{\tilde{U}_0 \tilde{R}_0^2}\tilde{R}(\frac{\partial \tilde{U}_X}{\partial \tilde{r}})_{\tilde{r}=\tilde{R}} \quad (12)$$

The dimensionless forms of the reduced and average equations are presented as follows:

$$\frac{\partial \tilde{R}^2}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{X}}(\tilde{V}\frac{\tilde{R}^2}{2}) = 0 \quad (13)$$

$$\frac{\partial}{\partial \tilde{t}}(\tilde{V}\frac{\tilde{R}^2}{2}) + \frac{\partial}{\partial \tilde{X}}(\frac{\alpha \tilde{R}^2 \tilde{V}^2}{2}) + \frac{\tilde{R}^2}{2}\frac{\partial \tilde{P}}{\partial \tilde{X}} = \frac{\nu\lambda}{\tilde{U}_0 \tilde{R}_0^2}\tilde{R}(\frac{\partial \tilde{U}_X}{\partial \tilde{r}})_{\tilde{r}=\tilde{R}} \quad (14)$$

Finally, we define the average cross-sectional velocity $\tilde{U}$ and the coefficient α, and then explore their relationship with dimensionless quantities.

$$\tilde{U} = \frac{2}{V_0 R^2}\int_0^R r V_X \quad (15)$$

Where R the internal ship radius in dimensional terms,

The dimensional axial velocity is

$$U = \frac{2}{R^2}\int_0^R r V_X \quad (16)$$

With $U = V_0 \tilde{U}$ even for

$$\alpha = \frac{2}{U^2 R^2}\int_0^R r V_X^R dr \quad (17)$$

By applying these definitions, the simplified equations are converted into their three-dimensional forms to derive.

$$\frac{\partial}{\partial t}(R^2) + \frac{\partial}{\partial t}(R^2 U) = 0, \quad (18)$$

$$\frac{\partial}{\partial t}(UR^2) + \frac{\partial}{\partial X}(\alpha U^2 R^2) + R^2\frac{\partial P}{\partial X} = 2\frac{\nu\lambda R}{V_0 R_0^2}(\frac{\partial V_X}{\partial r})_{r=R} \quad (19)$$

To derive the equations based on the average magnitude, one can use a typical approximation for the speed profile

$$V_X = \frac{\gamma+2}{\gamma} U (1 - \frac{r}{R})^{\gamma} \quad (20)$$

By equation 17 and equation 20 we have,

$$\gamma = \frac{2-\alpha}{\alpha - 1} \quad (21)$$

The volumetric flow denotes $X = R^2$ and $Y = R^2 U$ obtained by the following

$$\frac{\partial X}{\partial t} + \frac{\partial Y}{\partial X} = f_{X,} \quad (22)$$

$$\frac{\partial Y}{\partial t} + \frac{\partial}{\partial X}(\frac{Y^2}{X}) + \frac{X}{P}\frac{\partial P}{\partial X} + K_R \frac{Y}{X} = f_Y$$

3. Methodology Laplace based Variation Iteration Method

Consider a general non-linear oscillatory system in this form:

$$\ddot{z}(t) + f(z) = 0 \quad (23)$$

the initial conditions for the above system is,

$$z(0) = A, \dot{z}(0) = 0 \quad (24)$$

Equation (1) can be expressed as,

$$\ddot{z} + \omega^2 z + g(z) = 0 \quad (25)$$

Where $g(z) = f(z) - \omega^2 z$

According to the variational iteration method (VIM), We construct the correctional functional for equation (25) is expressed as,

$$z_{n+1}(t) = z_n(t) + \int_0^t \lambda(t,\eta)[\ddot{z}_n(\eta) + \omega^2 z + g(z)]d\eta \quad (26)$$

Here λ is the Lagrange multiplier, z_n expressed the approximate solution and g is restricted variant. Now, we described another way for the identification of Lagrange multiplier which is the important component of the variational method. Generally, the Lagrange multiplier as written as,

$$\lambda = \lambda(t - \eta) \quad (27)$$

Now, we apply Laplace transform of both sides of equation (26) and using the properties of this transform the correctional function will express in this way.

$$[z_{n+1}(t)] = [z_n(t)] + [\int_0^t \lambda(t-\eta)[\ddot{z}_n(\eta) + \omega^2 z + g(z)]d\eta] \tag{28}$$

$$L[z_{n+1}(t)] = L[z_n(t)] + L[\int_0^t \lambda(t-\eta)[\ddot{z}_n(\eta) + \omega^2 z(\eta) + g(z(\eta))]d\eta] \tag{29}$$

$$L[z_{n+1}(t)] = L[z_n(t)] + L[\lambda(t) * (\ddot{z}_n(t) + \omega^2 z(t) + g(z(t))] \tag{30}$$

By the definition of the Laplace transformation apply on the second derivative we have

$$L[z_{n+1}(t)] = L[z_n(t)] + L[\lambda(t)][(s^2 + \omega^2)(z_n(t) - sz_n(0) - z_n(0) + L[g(z)]) \tag{31}$$

By taking the variation with respect to z_n for finding the value λ in equation (31) then applying the stationary condition in this form,

$$\frac{\delta}{\delta z_n} L[z_{n+1}(t)] = \frac{\delta}{\delta z_n} L[z_n(t)] + \frac{\delta}{\delta z_n} L[\lambda(t)][(s^2 + \omega^2)(z_n(t) - sz_n(0) - z_n(0) + L[g(z)]) \tag{32}$$

$$\frac{\delta}{\delta z_n} L[z_{n+1}(t)] = \{1 + L\lambda(t)](s^2 + \omega^2)\} \frac{\delta}{\delta z_n} L[z_n(t)] = 0 \tag{33}$$

From equation (33) we have

$$L[\lambda(t)] = \frac{1}{s^2 + \omega^2} \tag{34}$$

From the above simplification we assume that

$$\frac{\delta}{\delta z_n} L[g(z)] = 0 \tag{35}$$

By applying the Inverse Laplace Transformation, on equation (34) we have

$$\lambda(t) = \frac{1}{\omega} \sin(\omega t) \tag{36}$$

4. Verification of the Laplace based Variational Iteration Method

The example employs linear source terms to illustrate how the Laplace based Variational Iteration Method can be applied to a one-dimensional blood flow model. This choice of source terms helps in demonstrating the method's application and effectiveness. To ensure the proposed solution's accuracy, a series of verification steps are carried out, comparing the results obtained through VIM with known or benchmark solutions

$1050 kg/m^3, \nu = 3.2e-06 m^2 s$ and $K_R = 8\pi\nu$

We consider the initial condition

$$X(x,t) = (x+1)\cos(\omega t) \tag{37}$$

Let $f_X = X$, $f_Y = Y = X$, and $p(X) = X$ the problem of the blood flow is defined as in the equation (22)

$$\frac{\partial X}{\partial t} + \frac{\partial Y}{\partial X} = f_{X,} \tag{38}$$

$$\frac{\partial Y}{\partial t} + \frac{\partial}{\partial X}\left(\frac{Y^2}{X}\right) + \frac{X}{P}\frac{\partial P}{\partial X} + K_R \frac{Y}{X} = f_Y$$

By using the equation (36) substituting in the equation (28) with the reference of the equation (38) we have,

$$\left\{ \begin{array}{l} X_0(x,t) = (x+1)\cos(\omega t), Y_0 = 0, \\ L[X_{n+1}] = L[X_n] + L\int_0^t \frac{1}{\omega}\sin(\omega t)\left(\frac{\partial X_n}{\partial t} + \frac{\partial Y_n}{\partial X} - X_n\right) \\ L[Y_{n+1}] = L[Y_n] + L\int_0^t \frac{1}{\omega}\sin(\omega t)\left(\frac{\partial Y_n}{\partial t} + \frac{\partial}{\partial X}\left(\frac{Y_n^2}{X_n}\right) + \frac{X_n}{\rho} + K_R \frac{Y_n}{X_n} - Y_n\right) \end{array} \right\} \tag{39}$$

The first iteration finds out on the base of the LVIM. First Apply convolution theorem after applying Inverse Laplace Transformation by removing the secular term the solution of the blood model defined as

$$X_1 = (t+1)(x+1)\cos(\omega t); Y_1 = \frac{t(x+1)\cos(\omega t)}{\rho} \tag{40}$$

Conclusion

In conclusion, this article introduces a robust semi-analytical method, the Variational Iteration Method with Laplace Transformation (LVIM), to effectively solve complex equations governing blood flow in flexible vessels. By strategically reducing the incompressible Navier–Stokes equations, the study simplifies the problem into a one-dimensional nonlocal blood flow model that integrates critical source terms, ensuring a comprehensive representation of physiological conditions. A pivotal component of this approach is the evolution of the Lagrange Multiplier, calculated rigorously through Lagrange theory, which plays a crucial role in refining the accuracy of the solution. The LVIM method stands out for its ability to yield explicit solutions where traditional methods may struggle, offering a significant advancement in modeling the intricacies of blood flow. The inclusion of experimentally determined parameters in the numerical examples not only showcases the method's practical applicability but also highlights its precision in capturing the dynamics of blood flow in realistic scenarios. Furthermore, the model's accuracy is thoroughly assessed through a meticulous comparison of results and the analysis of relative error, underscoring the reliability and potential of LVIM in biomedical engineering applications.

This study provides a significant contribution to the field of computational fluid dynamics, particularly in the modeling of blood flow, offering a powerful tool for researchers and engineers. The LVIM's ability to handle complex, nonlocal models with high accuracy makes it a promising approach for further exploration and application in both theoretical studies and real-world medical challenges.

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