# The state-of-the-art methodologies for reliability analysis of imperfect repair

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**Abstract**

The research analyzes a Markovian machine repair problem (MRP) with a controllable strategy and imperfect repair. Imperfect repair means the repairer services failed machines, but the service may not be successful. The chosen controllable threshold-based strategy manages the admission of failed units to prevent significant expected waiting times. Repairers stop admitting new failed units until the queue size decreases. Unadmitted failed units may undergo repair elsewhere, incurring additional costs. The number of failed units and expected service-gained units are crucial in the control policy. Using the Laplace transform method, the study derives the transient-state distribution, establishes performance measures, and calculates the system's reliability function and mean time to the first failure. Extensive numerical experiments and sensitivity analyses provide a comprehensive understanding of the system.

*Keywords.* Imperfect repair; $F$-policy; Start-up time; Warm standby; Reliability; Sensitivity analysis.

## 1 Introduction

In today's world, machines and devices are deeply intertwined with daily life, making it nearly impossible to imagine human existence without them. Machines have become an integral part of human life, seamlessly integrating into society and playing a crucial role in meeting the growing demands for products and services. Fault-tolerant systems (FTSs) are especially important in ensuring uninterrupted operations within socio-techno-economic constraints. These systems find wide applications in various industries like textiles, automobiles, and Fast-moving consumer goods. Machine interference occurs when there's a mismatch between the units and the repairer. This research addresses the novel concept of imperfect repair in FTSs with finite active/standby units. It focuses on strategic control to improve maintenance and enhance redundancy for increased utility and reliability. In machining systems, unexpected unit failures due to wear and tear lead to increased costs, delays, and operational inefficiencies. Implementing a preventive maintenance strategy with standby units can help promptly replace failed units, improving system reliability despite additional costs.

Previous research has focused on managing failed units in systems, often through threshold-based policies. These policies control the arrival of failed units to minimize expected downtime. For instance, under the commonly studied $F$-policy, failed units are not allowed to enter the system when the number of waiting units reaches capacity. They are only permitted to enter after the queue size drops to a specified level.

Service systems have often been studied assuming consistently successful service provision. However, real-world situations may involve instances of unsuccessful service attempts before achieving success. For example, the COVID-19 pandemic has forced academic institutions to shift to untested online teaching methods, leading to network glitches hindering effective knowledge transfer. Cloud computing has emerged as a significant enhancer of educational efficiency, dynamically allocating computing and storage resources for teaching materials and addressing network errors during online sessions. The term "unreliable service" describes this interplay of unsuccessful and successful service instances, where customers experiencing unsuccessful service rejoin the queue until they receive successful service.

This paper comprehensively explores the novel service regime of imperfect repair in FTSs, aiming to bridge gaps in existing literature by introducing controlled arrival processes for failed units. Its objectives are to formulate a stochastic model of the machining system considering imperfect repair and controlled arrival, propose a computationally efficient numerical scheme for calculating transient-state probabilities, and establish the system's reliability and queueing characteristics. The study also addresses sensitivity and relative sensitivity analysis, offering valuable insights for decision-makers. Additionally, it discusses potential applications of the FTS in various domains. The methodology involves a systematic approach, including an extensive literature survey, the introduction of a novel model, the formulation of a stochastic model, and the development of numerical schemes and exploration of system characteristics using mathematical theories. The paper is organized into sections covering literature review, model description, Chapman-Kolmogorov differential-difference equations, transient-state probabilities derivation, system characteristics, cost function formulation, sensitivity analysis, identification of standard models, numerical results, and conclusion.

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## 2 Literature Review

The exploration of machine interference has been a prominent focus in the literature, with foundational reviews (*cf.* Stecke and Aronson [1], Valdez-Flores and Feldman [2], Haque and Armstrong [3]). The articles on machine interference problems (MIPs) highlight significant applications across various sectors, including telecommunications (Kryvinska [4]), cloud computing (Luo, Meng, Qiu et al. [5]), computer networking (Bunday, Bokhari, and Khorram [6]), artificial intelligence ( Liu, Yang, Zio et al. [7]), ambulance fleet management (Firooze, Rafiee, and Zenouzzadeh [8]), and more. Zagia, Motamedi-Sedeh, and Ostadi [9] introduced a hybrid model optimizing facility maintenance scheduling for efficiency and cost savings.

From the conceptualization of standby units (Taylor and Jackson [10]), the literature has been extensively enriched (*cf.* Yun [11], Cho and Parlar [12]). The fault-tolerant systems with multi-active and standby units were analyzed for availability with imperfect coverage (Ke, Su, Wang et al. [13]), reliability in a fuzzy environment (Shekhar, Jain, and Bhatia [14]), reliability in a probabilistic environment (Shekhar, Jain, Raina et al. [15]), reliability with switching failure and reboot delay (Shekhar, Kumar, and Varshney [16]), reliability characterizing the temperature deviation procedure via a two-stage Wiener process (Ma, Liu, Yang et al. [17]), and availability and mean time to failure (MTTF) in a fuzzy environment (Devanda, Shekhar, and Kaswan [18]).

The total cost of a mathematical model was scrutinized for coordinating production scheduling and Condition-Based Maintenance (CBM) planning in a manufacturing system with a single machine (Sharifi and Taghipour [19]) and parallel-machine (Sharifi, Ghaleb, and Taghipour [20]) experiencing multiple failures and discrete stages of deterioration. Maintenance planning and production scheduling were jointly optimized in intelligent manufacturing systems (*cf.* Ghaleb, Taghipour, Sharifi et al. [21], Ghaleb, Zolfagharinia, and Taghipour [22], Ghaleb, Taghipour, and Zolfagharinia [23]), addressing factors such as new job arrivals, due date changes, stochastic deterioration-based failures, minimal repairs, and CBM. Recent research explores applications for multiunit systems, including studies on bipropellant rocket engines with electric pump-fed systems (Bai, Xu, Li et al. [24]) and hydrogen-air-steam mixture gas behavior under steam condensation (Liu, Sun, Bian et al. [25]). Structural reliability and design analysis for complex systems have been conducted (*cf.* She, Wang, Peng et al. [26], Qi, Yu, Meng et al. [27]), alongside the development of an iterative threshold algorithm of Log-Sum Regularization for sparse problems (Zhou, Liu, Zhang et al. [28]).

After the inception of the $F$-policy (Gupta [29]), subsequent research extensively explored the concept of controllable arrival in MRPs for reliability characteristics analysis. Steady-state results were derived for a single removable and unreliable server using the matrix analytical method (Wang and Yang [30]), while transient results were investigated for a retrial system with working breakdowns and randomized setup time using the Laplace transform technique (Yen, Wang, and Wu [31]).

The concept of unreliable service was introduced in a Markovian queue with a single server, both without (Patterson and Korzeniowski [32]) and with (Patterson and Korzeniowski [33]) working vacations. Recognizing its practicality, unreliable service as imperfect repair in MRPs with standby provisioning, a strategic threshold-based $F$-policy, and vacation interruption was examined (Shekhar, Varshney, and Kumar [34]), highlighting its importance in reducing power consumption and preventing thermal trip errors through discouragement and feedback strategies (Shekhar, Gupta, Kumar et al. [35]). The reliability of multi-unit systems with standby provisioning was explored, considering failures, degradation, random delays, and probabilistic imperfections (Shekhar, Devanda, and Kaswan [36]).

Although FTSs have been widely studied, the incorporation of imperfect repair mechanisms, where failed units are subject to controlled arrival processes, remains underexplored in the literature. Addressing this gap, our research introduces a more realistic model that:

- Pioneers the incorporation of imperfect repair mechanisms, presenting a novel service regime beyond traditional fault-tolerance approaches.
- Explores controlled arrival processes for failed units, offering in-depth understanding of system dynamics.
- Extends the understanding of service scenarios to include unsuccessful attempts before achieving success, relevant in contexts like online teaching during the COVID-19 pandemic.

These findings collectively advance the understanding of system reliability and performance in real-world scenarios.

## 3 Model Description

In this section, we outline a fault-tolerant model, integrating active and standby units, repair processes, and controlled failed unit arrivals. It incorporates parameters like failure rates, repair rates, and strategic thresholds, rooted in reliability and queueing theories. Key assumptions include focusing on reliability within a service regime involving imperfect repair.

- The proposed MRP focuses on reliability within a service regime involving imperfect repair by a single repairer, where the arrival of failed units is controlled by the $F$-policy.

- The system comprises $M$ identical active units and $S$ warm standby units for enhanced reliability and availability, with standby units promptly replacing failed active units.
- During normal operation, all $M$ active units operate simultaneously, and the system continues in short mode until there are at least $m; 1 \leq m \leq M-1$ active units operational.
- The preventive strategy regulates failed units, allowing an excess up to $K = M + S - m + 1$.

Future research may extend to incorporate switching failures and significant switching delays. Our model comprises three processes.

**Failure process**: The failure process involves independent exponential failures for active and standby units, with degradation upon exhausting standby units.

- Each of the active and standby units experiences independent failures, and the time-to-failure for each active and standby unit follows an exponential distribution with mean time to failure $\frac{1}{\lambda}$ and $\frac{1}{\nu}$ (where $0 < \nu < \lambda$), respectively.
- Upon switching to the on-the-go state on the active unit's failure, the standby unit inherits the same failure and working characteristics as those of an active unit.
- In the event that all available standby units are exhausted, the time to failure for each active unit is degraded with a mean time to failure of $\frac{1}{\lambda_d}$ (where $0 < \lambda < \lambda_d$).

**Repair process:** The repair process assumes immediate repair without delay with inspection for perfect and imperfect repair.

- When a unit becomes futile, immediate repair is essential without any delay. If the repairer is available, the failed unit undergoes instant repair; otherwise, it waits in the queue.
- The queue discipline of this repairable system is $FCFS$ (first-come, first-served).
- The time-to-repair follows an exponential distribution with a mean time of $\frac{1}{\mu}$.
- After repair, the unit undergoes inspection. Generally, perfect repair is assumed, but in practice, it may be imperfect.
- The inter-time-to-inspect for both perfect and imperfect repair follows an exponential distribution with mean rates $\beta_1$ and $\beta_2$, respectively.
- The unit with imperfect repair rejoins the queue until it undergoes perfect repair.
- The fixed unit is considered as good as a new active or standby unit and is returned to the pool of active units or standby units when the system is operating in short or normal mode, respectively.

**Controlled process:** A controlled process prevents additional failed units from joining the queue mitigating prolonged waiting times.

- When the number of failed units reaches the system capacity $K$, the $F$-policy prevents additional failed units from joining the queue until the system reverts to normal mode. This strategy aims to mitigate prolonged waiting times and is crucial for the system's efficiency. Excluded failed units may undergo repair at an external facility, incurring additional costs.
- Upon resuming, the system experiences a random setup time following an exponential distribution with a parameter of $\gamma$. The setup rate determines how quickly failed units, initially excluded due to capacity constraints, can rejoin the repair queue after the system transitions back to normal mode. It is a crucial parameter that influences the controlled admission of failed units based on the system's capacity strategy, affecting the speed at which the system allows previously excluded failed units to rejoin the repair process and mitigate prolonged waiting times.

The failure, degraded failure, perfect repair, imperfect repair, setup, etc., of each unit are independent events. The model is illustrated in the transition diagram in Figure 1. Blue nodes represent the state with a specific number of failed units, and the red node labeled $F$ represents the system failure state. Transition arrows indicate state transitions with marked rates. Each row of blue nodes represents a $j$-th system state, with each node in the row denoting the state of the system $(j, n)$, where $n$ is the number of failed units or the failure state $F$.

Initially, all active and standby units are operational, with no failed units in the system at $t = 0$. If there are $n$ failed units in the system at time $t$, the state-dependent effective failure rate of units is expressed as:

$$\lambda_n = \begin{cases} M\lambda + (S-n)\nu; & \text{if } n = 0,1,2,3,...,S-1 \\ (M+S-n)\lambda_d; & \text{if } n = S, S+1, ..., K-1, K \\ 0; & \text{otherwise} \end{cases}$$

For mathematical modeling, the following notations are defined. The state of the failed units $J(t)$ in the system at time $t$ is defined as follows:

$$J(t) \equiv \begin{cases} 0; \text{ The newly failed unit cannot be allowed to ingress into the system for repair, while the existing failed unit is promptly} \\ \quad \text{repaired and reinstated back into the system.} \\ 1; \text{ Access to the system to repair the newly failed unit has been prohibited, and the repairer is currently occupied.} \\ 2; \text{ The newly failed unit is granted access to the system for repair, while the repairer is engaged.} \\ 3; \text{ The newly failed unit is allowed to ingress into the system for repair, and the existing failed unit immediately after} \\ \quad \text{the repair for inspection.} \end{cases}$$

and

$$[N(t)] \equiv \text{Number of failed units in the system at the time } t$$

$$[F(t)] \equiv \text{The state when system has been failed at the time } t.$$

With the above definition of the system states, the system states form a continuous-time Markov chain $(J(t), N(t)) \cup F(t)$; $t \geq 0$ in the state space, which can be represented as:

$$\begin{aligned} \Psi = &\{(j,n) \mid j=0; n=1,2,...,K-2,K-1\} \cup \{(j,n) \mid j=1; n=0,1,2,...,K-1\} \\ &\cup\{(j,n) \mid j=2; n=0,1,2,...,K-2\} \cup \{(j,n) \mid j=3; n=1,2,...,K-2\} \cup F \end{aligned}$$

The probabilities of the different states in the system at any time $t$ are defined as follows:

$P_{j,n}(t) \equiv$ Probability that at time $t$ there are $n$ failed units in the system and the system is in the state $j$, where $(j,n) \in \Psi$.

$P_F(t) \equiv$ Probability that at time $t$ the system is in the failed state.

## 4 The governing equation

We have developed forward Chapman-Kolmogorov differential-difference equations and initial conditions for the threshold-based failed unit arrival controlled strategy with imperfect repair and an exponential setup time, as detailed in the transition diagram of Figure 1. These equations balance inflow and outflow rates to describe state probabilities and governing parameters. To solve the system of first-order, first-degree differential equations with initial conditions for transient-state probabilities, we use the Laplace transform of state probabilities and their derivatives, resulting in the following system of linear equations.

$$u\ddot{P}_{0,n}(u) = -(\gamma + \beta_1 + \beta_2)\ddot{P}_{0,n}(u) + \mu\ddot{P}_{1,n}(u) \ ; \ 1 \leq n \leq S \tag{1}$$
$$u\ddot{P}_{0,n}(u) = -(\beta_1 + \beta_2)\ddot{P}_{0,n}(u) + \mu\ddot{P}_{1,n}(u) \ ; \ S+1 \leq n \leq K-1 \tag{2}$$
$$u\ddot{P}_{1,0}(u) = -\gamma\ddot{P}_{1,0}(u) + \beta_1\ddot{P}_{0,1}(u) \tag{3}$$
$$u\ddot{P}_{1,n}(u) = -(\mu+\gamma)\ddot{P}_{1,n}(u) + \beta_2\ddot{P}_{0,n}(u) + \beta_1\ddot{P}_{0,n+1}(u) \ ; \ 1 \leq n \leq S \tag{4}$$
$$u\ddot{P}_{1,n}(u) = -\mu\ddot{P}_{1,n}(u) + \beta_2\ddot{P}_{0,n}(u) + \beta_1\ddot{P}_{0,n+1}(u) \ ; \ S+1 \leq n \leq K-2 \tag{5}$$
$$u\ddot{P}_{1,K-1}(u) = -(\mu + \lambda_{K-1})\ddot{P}_{1,K-1}(u) + \beta_2\ddot{P}_{0,K-1}(u) + \lambda_{K-2}\ddot{P}_{2,K-2}(u) + \lambda_{K-2}\ddot{P}_{3,K-2}(u) \tag{6}$$
$$u\ddot{P}_{2,0}(u) - 1 = -\lambda_0\ddot{P}_{2,0}(u) + \gamma\ddot{P}_{1,0}(u) + \beta_1\ddot{P}_{3,1}(u) \tag{7}$$
$$u\ddot{P}_{2,n}(u) = -(\lambda_n + \mu)\ddot{P}_{2,n}(u) + \gamma\ddot{P}_{1,n}(u) + \lambda_{n-1}\ddot{P}_{2,n-1}(u) + \beta_2\ddot{P}_{3,n}(u) + \beta_1\ddot{P}_{3,n+1}(u) \ ; \ 1 \leq n \leq S \tag{8}$$
$$u\ddot{P}_{2,n}(u) = -(\lambda_n + \mu)\ddot{P}_{2,n}(u) + \lambda_{n-1}\ddot{P}_{2,n-1}(u) + \beta_2\ddot{P}_{3,n}(u) + \beta_1\ddot{P}_{3,n+1}(u) \ ; \ S+1 \leq n \leq K-3 \tag{9}$$
$$u\ddot{P}_{2,K-2}(u) = -(\lambda_{K-2} + \mu)\ddot{P}_{2,K-2}(u) + \lambda_{K-3}\ddot{P}_{2,K-3}(u) + \beta_2\ddot{P}_{3,K-2}(u) \tag{10}$$
$$u\ddot{P}_{3,1}(u) = -(\lambda_1 + \beta_1 + \beta_2)\ddot{P}_{3,1}(u) + \gamma\ddot{P}_{0,1}(u) + \mu\ddot{P}_{2,1}(u) \tag{11}$$
$$u\ddot{P}_{3,n}(u) = -(\lambda_n + \beta_1 + \beta_2)\ddot{P}_{3,n}(u) + \gamma\ddot{P}_{0,n}(u) + \mu\ddot{P}_{2,n}(u) + \lambda_{n-1}\ddot{P}_{3,n-1}(u) \ ; \ 2 \leq n \leq S \tag{12}$$
$$u\ddot{P}_{3,n}(u) = -(\lambda_n + \beta_1 + \beta_2)\ddot{P}_{3,n}(u) + \mu\ddot{P}_{2,n}(u) + \lambda_{n-1}\ddot{P}_{3,n-1}(u) \ ; \ S+1 \leq n \leq K-2 \tag{13}$$
$$u\ddot{P}_F(u) = \lambda_{K-1}\ddot{P}_{1,K-1}(u) \tag{14}$$

We represent the transient-state probabilities subscript in a unary code for ease of the solution procedure.

$$
\begin{aligned}
&[P_{0,1}(t), P_{0,2}(t), ..., P_{0,K-1}(t)]^T \equiv [\pi_1(t), \pi_2(t), ..., \pi_{K-1}(t)]^T \\
&[P_{1,0}(t), P_{1,1}(t), ..., P_{1,K-1}(t)]^T \equiv [\pi_K(t), \pi_{K+1}(t), ..., \pi_{2K-1}(t)]^T \\
&[P_{2,0}(t), P_{2,1}(t), ..., P_{2,K-2}(t)]^T \equiv [\pi_{2K}(t), \pi_{2K+1}(t), ..., \pi_{3K-2}(t)]^T \\
&[P_{3,1}(t), P_{3,2}(t), ..., P_{3,K-2}(t)]^T \equiv [\pi_{3K-1}(t), \pi_{3K}(t), ..., \pi_{4K-4}(t)]^T \\
&P_F(t) \equiv \pi_{4K-3}(t)
\end{aligned}
$$

The Laplace transform probabilities relevant to the problem can be calculated using the following equation.

$$\ddot{\pi}_r(u) = L\{\pi_r(t)\};\ 1 \leq r \leq 4K-3$$

Delimitate the subsequent column vectors of order $4K-3$.

$$\ddot{\mathbf{\Xi}}(u) = [\ddot{\pi}_1(u), \ddot{\pi}_2(u), \ddot{\pi}_3(u), ..., \ddot{\pi}_{4K-4}(u), \ddot{\pi}_{4K-3}(u)]^T, \tag{15}$$

$$\mathbf{\Xi}(0) = [\pi_1(0), \pi_2(0), \pi_3(0), ..., \pi_{4K-4}(0), \pi_{4K-3}(0)]^T \tag{16}$$

We represent the system of linear equations (Equations 1-14) in matrix form using the aforementioned column vectors as follows.

$$\digamma(u)\ddot{\mathbf{\Xi}}(u) = \mathbf{\Xi}(0) \tag{17}$$

Here, $\digamma(u)$ is the coefficient square matrix of order $4K-3$. Applying Cramer's rule to the matrix equation 17, we explicitly express $\ddot{\pi}_r(u)$ as follows.

$$\ddot{\pi}_r(u) = \frac{|\digamma_r(u)|}{|\digamma(u)|};\ 1 \leq r \leq 4K-3 \tag{18}$$

Here, $\digamma_r(u)$ is a square matrix of order $4K-3$. It is derived from $\digamma(u)$ by replacing its $r^{th}$ column with the right-hand side column vector $\mathbf{\Xi}(0)$. To determine $\ddot{\pi}r(u)$ from Equation 18, we initially calculate the denominator $|\digamma(u)|$. Notably, $|\digamma(u)|$ becomes singular due to the balanced inflow and outflow rates inherent in its nature, resulting in $u=0$ as one latent root. Additionally, we identify $u=-\xi$ as another nonzero latent root of $|\digamma(u)|=0$.

$$\digamma(-\xi) = \mathbf{A} - \xi\mathbf{I} \tag{19}$$

where $\mathbf{A} = \digamma(0)$ and $\mathbf{I}$ is an identity matrix of order $4K-3$. The expression can be alternatively represented as

$$\digamma(-\xi)\ddot{\mathbf{\Xi}}(u) = (\mathbf{A} - \xi\mathbf{I})\ddot{\mathbf{\Xi}}(u) \tag{20}$$

Let $\xi_h(\neq 0)$, for $h = 1, 2, 3, \ldots, 4K-5, 4K-4$, denote $4K-4$ distinct latent roots of $|\mathbf{A} - \xi\mathbf{I}| = 0$, which may be real or complex numbers. Consider $\xi_1, \xi_2, \xi_3, \ldots, \xi_{n_1}$ as $n_1$ real latent roots, and $\xi_{n_1+1}, \bar{\xi}_{n_1+1}, \xi_{n_2+1}, \bar{\xi}_{n_2+1}, \ldots, \xi_{n_1+n_2}, \bar{\xi}_{n_1+n_2}$ as $2n_2$ complex latent roots, existing in conjugate pairs such that $n_1 + 2n_2 = 4K-4$. Therefore,

$$|\digamma(u)| = u\prod_{h=1}^{n_1}(u+\xi_h)\prod_{h=1}^{n_2}(u^2 + (\xi_{n_1+h} + \bar{\xi}_{n_1+h})u + \xi_{n_1+h}\bar{\xi}_{n_1+h}) \tag{21}$$

Hence, Equation 18 reduces to

$$\ddot{\pi}_r(u) = \frac{|\digamma_r(u)|}{\digamma(u)} = \frac{|\digamma_r(u)|}{u\prod_{h=1}^{n_1}(u+\xi_h)\prod_{h=1}^{n_2}(u^2 + (\xi_{n_1+h} + \bar{\xi}_{n_1+h})u + \xi_{n_1+h}\bar{\xi}_{n_1+h})};\ 1 \leq r \leq 4K-3 \tag{22}$$

The expression in Equation 22 for $\ddot{\pi}_r(u)$ can be represented in partial fraction form as follows:

$$\ddot{\pi}_r(u) = \frac{a_{0,r}}{u} + \sum_{h=1}^{n_1}\frac{a_{h,r}}{(u+\xi_h)} + \sum_{h=1}^{n_2}\frac{b_{h,r}(u) + c_{h,r}}{(u^2 + (\xi_{n_1+h} + \bar{\xi}_{n_1+h})u + \xi_{n_1+h}\bar{\xi}_{n_1+h})}; 1 \leq r \leq 4K-3 \tag{23}$$

The coefficients in the partial fraction representation are computed as follows:

$$a_{0,r} = \frac{|F_r(0)|}{\prod_{h=1}^{n_1}(\xi_h)\prod_{h=1}^{n_2}(\xi_{n_1+h}\bar{\xi}_{n_2+h})} \tag{24}$$

$$a_{h,r} = \frac{|F_r(-\xi_h)|}{(-\xi_h)\prod_{g=1,g\neq h}^{n_1}(\xi_g-\xi_h)\prod_{g=1}^{n_2}({\xi_h}^2+(\xi_{n_1+g}+\bar{\xi}_{n_1+g})(-\xi_h)+\xi_{n_1+g}\bar{\xi}_{n_1+g})}; h = 1, 2, 3, ..., n_1 \tag{25}$$

and

$$\begin{aligned} & b_{h,r}(-\xi_{n_1+h}) + c_{h,r} \\ &= \frac{|F_r(-\xi_{n_1+h})|}{(-\xi_{n_1+h})\prod_{g=1}^{n_1}(\xi_g-\xi_{n_1+h})\prod_{g=1,g\neq h}^{n_2}\left((-\xi_{n_1+h})^2+(\xi_{n_1+g}+\bar{\xi}_{n_1+g})(-\xi_{n_1+h})+\xi_{n_1+h}\bar{\xi}_{n_1+h}\right)}; \\ & h = 1, 2, 3, ..., n_2 \end{aligned} \tag{26}$$

The explicit expression of the transient-state probabilities $\pi_r(t); 1 \leq r \leq 4K-3$ is obtained by taking the inverse Laplace transform of Equation 26. Hence, we have the following expressions for each $\pi_r(t)$:

$$\begin{aligned} & \pi_r(t) \\ &= a_{0,r} + \sum_{h=1}^{n_1} a_{h,r}e^{-\xi_h t} + \sum_{h=1}^{n_2}\left[b_{h,r}e^{-x_h t}\cos y_h t + \frac{c_{h,r}-b_{h,r}x_h}{y_h}e^{-x_h t}\sin y_h t\right]; 1 \leq r \leq 4K-3 \end{aligned} \tag{27}$$

The arbitrary constants $a_{0,r}, a_{h,r}, b_{h,r}$, and $c_{h,r}$ are computed in the above equations (Equations 24-26), and $x_h$ and $y_h$ represent the real and imaginary parts of the respective complex latent root $\xi_{n_1+h}$.

## 5 Performance measures

This research investigates imperfect repair within the framework of the Markovian threshold-based arrival control strategy for MRPs. Our aim is to define reliability and performance indices using transient-state probabilities from the previous section, considering governing parameters. These assessments aim to enhance system reliability and queueing characteristics.

### 5.1 Reliability Measures

In this subsection, we analyze the reliability of the machining system using a threshold-based failed unit arrival control strategy with imperfect repair. This analysis is vital in the broader context of Reliability, Availability, Maintainability, and Safety (RAMS) methodologies, which ensure efficient and reliable system operation. RAMS integrates design elements to meet performance standards, with reliability being a key factor ensuring consistent system operation without failure.

Let $Y$ be the continuous random variable representing the system's time-to-failure. The reliability of a machining system is the probability that it will operate without failure for a specified duration under given conditions. Denoting $P_F(t)$ as the probability of failure at or before time $t$, the reliability of the machining system $R_Y(t)$ is defined as:

$$R_Y(t) = 1 - P_F(t);\ t \geq 0 \tag{28}$$

The Mean time-to-failure $MTTF$ is a crucial reliability metric representing the average operational lifespan of a system before a failure occurs. In our study, we aim to optimize $MTTF$ by recommending strategic preventive, corrective, and predictive maintenance strategies. The $MTTF$ of the machining system is defined as:

$$\begin{aligned} MTTF &= \int_{t=0}^{\infty} R_Y(t)dt = \int_{t=0}^{\infty}(1-P_F(t))dt = \int_{t=0}^{\infty}(1-\pi_{4K-3}(t))dt \\ &= \lim_{u\to 0}\left[\frac{1-a_{0,4K-3}}{u} - \sum_{h=1}^{n_1}\frac{a_{h,4k-3}}{u+\xi_h} - \sum_{h=1}^{n_2}\frac{b_{h,4k-3}u+c_{h,4K-3}}{u^2+(\xi_{n_1+h}+\bar{\xi}_{n_1+h})u+\xi_{n_1+h}\bar{\xi}_{n_1+h}}\right] \\ &= -\sum_{h=1}^{n_1}\frac{a_{h,4K-3}}{\xi_h} - \sum_{h=1}^{n_2}\frac{c_{h,4K-3}}{\xi_{n_1+h}\bar{\xi}_{n_1+h}} \end{aligned} \tag{29}$$

Failure frequency $FF(t)$ quantifies the rate of failures in a system over a specified period.

$$FF(t) = \lambda_{K-1}P_{K-1}(t) \tag{30}$$

### 5.2 Queueing characteristics

Queueing attributes, crucial for refining maintenance strategies and system design, encompass metrics like expected queue length, throughput, available units, waiting time, and delay time, among others.

- The expected number of failed units in a system $E_N(t)$ is the average number of units in a failed state at a given time. This statistical measure considers the probability distribution of the failed units.

$$E_N(t) = \sum_{i=0}^{3} \sum_{n=1}^{K-2} nP_{i,n}(t) + \left( \sum_{i=0}^{1} (K-1) P_{i,K-1}(t) \right) + KP_F(t) \tag{31}$$

- System throughput $TP(t)$ is the rate at which failed units are processed, typically measured as the number of failed units processed per unit of time. It is a key indicator of system efficiency and performance.

$$TP(t) = \sum_{n=1}^{K-2} \beta_1 \left[ P_{0,n}(t) + P_{3,n}(t) \right] + \beta_1 P_{0,K-1}(t) \tag{32}$$

- The expected number of standby units in a system $E_S(t)$ is the average count of standby units present over a specific time period. It is a crucial metric for evaluating system readiness and reliability.

$$E_S(t) = \sum_{i=1}^{2} SP_{i,0}(t) + \sum_{i=0}^{3} \sum_{n=1}^{S-1} (S-n) P_{i,n}(t) \tag{33}$$

- The mean number of active units in a system $E_O(t)$ is the average count of operational units present over a specified time period.

$$E_O(t) = M \left[ \sum_{i=1}^{2} P_{i,0}(t) + \sum_{i=0}^{3} \sum_{n=1}^{S} P_{i,n}(t) \right] + \sum_{i=0}^{3} \sum_{n=S+1}^{K-2} (M+S-n) P_{i,n}(t) + \sum_{i=0}^{1} mP_{i,K-1}(t) \tag{34}$$

- The effective failure rate of units $E_F(t)$ is a comprehensive measure that considers various factors like individual component failure rates, redundancy, and repair processes. It provides a holistic view of the system's reliability by accounting for failures and repair effectiveness.

$$E_F(t) = \sum_{i=0}^{3} \sum_{n=1}^{S} (M\lambda + (S-n)\nu) P_{i,n}(t) + \sum_{i=0}^{3} \sum_{n=S+1}^{K-2} (M+S-n) \lambda_d P_{i,n}(t) + \sum_{i=0}^{1} m\lambda_d P_{i,K-1}(t) \tag{35}$$

- The expected waiting time of failed units in a system $E_W(t)$ is the average time a failed unit spends waiting for repair and perfect service. This metric evaluates the efficiency of the repair process.

$$E_W(t) = \frac{E_N(t)}{E_F(t)} \tag{36}$$

- The delay time of a failed unit $E_D(t)$, representing the waiting time before repair, is a critical metric for assessing system performance and reliability. It provides insights into the system's ability to address failures promptly.

$$E_D(t) = \frac{E_N(t)}{\tau(t)} \tag{37}$$

## 6 Cost function

The uninterrupted operation of a machining system is crucial across various sectors like manufacturing, production, and communication. Mathematical modeling and cost analysis are vital for optimal maintenance strategies at minimal costs. The expected total cost function is formulated considering variables like number of failed/standby units, service rates, inspection rates, and setup times. The cost analysis includes holding costs for system maintenance, repair costs for restoring units, inspection costs for quality control, and setup costs for transition expenses. Decision-makers must carefully weigh costs when designing and implementing fault-tolerant systems, balancing fault tolerance with financial implications. Defining unit costs for different system states is crucial for evaluating costs effectively.

$C_H \equiv$ Holding cost per unit time of each failed unit

$C_S \equiv$ Cost per unit time of each standby unit

$C_M \equiv$ Fixed cost per unit time for providing a service with the rate $\mu$

$C_1 \equiv$ Fixed cost per unit time for inspecting an perfect repair with the rate $\beta_1$

$C_2 \equiv$ Fixed cost per unit time for inspecting an imperfect repair with the rate $\beta_2$

$C_3 \equiv$ Fixed cost per unit time for taking a setup time by a system with the rate $\gamma$

Therefore, the expected total cost function for the machining system at time $t$ is given by:

$$E_{TC}(t) = C_H E_N(t) + C_S E_S(t) + C_M \mu + C_1 \beta_1 + C_2 \beta_2 + C_3 \gamma \tag{38}$$

We assume linearity for the unit costs used in the cost function (Equation 38), where they are directly proportional to the governing parameters and derived performance indices.

## 7 Sensitivity analysis

Leveraging differential calculus theory for maxima or minima, we explore the sensitivity of the $R_Y(t)$ and $MTTF$ concerning the system's governing parameters. The studied performance function's variability pattern can be understood by calculating its first derivatives with respect to the decision variable $\Theta$, which represents the system design's governing parameters. Computing the first derivatives of Equation 17, we obtain:

$$\frac{\partial F(u)}{\partial \Theta} \ddot{\Xi}(u) + F(u) \frac{\partial \ddot{\Xi}(u)}{\partial \Theta} = 0 \tag{39}$$

$$\frac{\partial \ddot{\Xi}(u)}{\partial \Theta} = -\left(F(u)\right)^{-1} \frac{\partial F(u)}{\partial \Theta} \ddot{\Xi}(u) \tag{40}$$

Deriving the first derivative of the $R_Y(t)$ from Equation 28, we get:

$$\Phi_\Theta(t) = \frac{\partial R_Y(t)}{\partial \Theta} = 0 - \frac{\partial P_F(t)}{\partial \Theta} = L^{-1}\left(-\frac{\partial \ddot{P}_F(u)}{\partial \Theta}\right) = L^{-1}\left(\frac{\partial \ddot{\pi}_{4K-3}(u)}{\partial \Theta}\right) \tag{41}$$

The chain rule is used to calculate $\frac{\partial \ddot{P}_F(u)}{\partial \Theta}$. This ratio is then used to assess the relative sensitivity analysis of the reliability function.

$$\Omega_\Theta(t) = \frac{\partial R_Y(t)/R_Y(t)}{\partial \Theta/\Theta} = \Phi_\Theta(t). \frac{\Theta}{R_Y(t)} \tag{42}$$

For the sensitivity analysis of $MTTF$, we obtain the first derivative of $MTTF$ with respect to $\Theta$ from Equation 29 as follows:

$$\begin{aligned} \Delta_\Theta &= \frac{\partial\left(MTTF\right)}{\partial \Theta} = \frac{\partial\left(\int_{t=0}^{\infty} R_Y(t)dt\right)}{\partial \Theta} = \lim_{s \to 0}\left[\int_{t=0}^{\infty} \frac{\partial R_Y(t)}{\partial \Theta} e^{-ut} dt\right] = \lim_{u \to 0}\left[-\frac{\partial \ddot{P}_F(u)}{\partial \Theta}\right] \\ &= \lim_{u \to 0}\left[\frac{\partial \ddot{\pi}_{4K-3}(u)}{\partial \Theta}\right] \end{aligned} \tag{43}$$

We calculate the following ratio to determine the relative sensitivity of $MTTF$:

$$\Gamma_\Theta = \frac{\frac{\partial(MTTF)}{MTTF}}{\frac{\partial\Theta}{\Theta}} = \Delta_\Theta \frac{\Theta}{MTTF} \tag{44}$$

The sensitivity analysis of $R_Y(t)$ and $MTTF$ is detailed with numerical illustrations in the following section.

## 8 Special Cases

The model we're studying extends previous research, and these works validate our approach:

Case 1: As $\beta_1$ tends to infinity, our model converges to a Markovian single repairer system with active/standby units and a threshold-based corrective strategy (Jain, Shekhar, an Shukla [37]).

Case 2: When the threshold is $K-1$ with $\beta_1$ approaching infinity, our model behaves like a classical MIP with standby provisioning (*cf.* Wang, Chen, and Yang [38], Gupta [29], Gupta [39]).

Case 3: For $0 < \beta_1 < \infty$, $\beta_2 = 0$, and $\gamma \to \infty$, setting the threshold to $K-1$, $S = 0$, and $\mu = \beta_1$, our model simulates a finite population queueing model with Erlangian service.

Case 4: In Case 3 with $\mu > \beta_1$, the model becomes a finite population single-server queueing model with hyper-exponential service time distribution (Chakravarthy and Agarwal [40]).

## 9 Numerical Results

In this section, we conduct numerical experiments to explore reliability, queueing, and parameter sensitivity, aiming to provide deeper insights complementing the theoretical analysis. Due to the complexity of continuous-time Markov chains in machining systems with real-time constraints, our investigation relies heavily on numerical methods. Our research analyzes the impact of various parameters on $R_Y(t)$ and $MTTF$. We model a scenario similar to a data center's server infrastructure, where active units are operational servers and warm standby units are available for immediate use in case of failure. Failures and repairs follow exponential distributions with constant rates. Using MATLAB R2020b, we conduct numerical experiments with default parameters: $M = 12$, $S = 5$, $m = 2$, $\lambda = 0.4$, $\nu = 0.35$, $\lambda_d = 0.8$, $\mu = 16$, $\beta_1 = 8$, $\beta_2 = 1$, and $\gamma = 4$. Our model aims to enhance server infrastructure reliability by strategically provisioning standbys and controlling failed unit arrivals, minimizing downtime and ensuring uninterrupted service.

Figure 2 depicts $R_Y(t)$ (Equation 28) over varying $t$ and system parameters. Across all subgraphs (i)-(x), a decreasing trend in $R_Y(t)$ is observed over time. The results indicate that $R_Y(t)$ decreases with increasing failure rates of active/standby units and the inspection rate of imperfect repair. Conversely, $R_Y(t)$ improves with higher repair rates for failed units and the inspection rate of perfect repair, as shown in Figure 2 (vii)-(viii). Moreover, $R_Y(t)$ is enhanced with more standby units and reduced setup time. These findings support a preventive maintenance policy, emphasizing the importance of standby provisioning and controlled failed unit arrivals. Figure 3 presents the $R_Y(t)$ sensitivities, comparing the impacts of different parameters. The sensitivities of $\lambda_d$, $\beta_1$, and $\mu$ are notably higher than those of $\lambda$, $\nu$, $\beta_2$, and $\gamma$, making them crucial for enhancing $R_Y(t)$ while managing costs. The sensitivity order at any time $t$ is $\lambda_d > \lambda > \beta_1 > \beta_2 > \nu > \mu > \gamma$, highlighting the importance of preventive and corrective measures to improve system reliability and performance.

Reliability is crucial for the effectiveness of machining systems, with $MTTF$ being a key metric (Equation 29). Figure 4 illustrates the variability of $MTTF$ through a bar plot. Smaller values of $M$ lead to a more significant decline in $MTTF$, indicating higher sensitivity. Conversely, larger $M$ values show reduced sensitivity to changes in failure rates. Higher inspection and repair rates correlate with increased $MTTF$, especially in systems with fewer active units. Balancing the number of active and standby units is essential for designing a reliable yet cost-effective system. Table 1 summarizes the sensitivity and relative sensitivity of $MTTF$ (Equations 43 and 44) to different parameters. The analysis reveals that $\lambda_d$ has the highest sensitivity, followed by $\lambda$, $\nu$, $\beta_1$, $\beta_2$, $\mu$, and $\gamma$. This ranking suggests that prioritizing preventive measures over corrective actions is crucial for improving system performance and mitigating degradation.

Figure 5 shows the temporal evolution of the $E_N(t)$ (Equation 31). Each sub-graph (i)-(x) represents different system parameters. Initially, $E_N(t)$ rises over time in each sub-graph but stabilizes and eventually maintains a constant level after a specific time $t$. The influence of the threshold-based failed unit arrival controlled policy is evident. $E_N(t)$ correlates positively with an increased count of active/standby units and their failure rates, while it decreases with higher rates of repair and inspection for perfect repair. Figure 6 displays the system's throughput (Equation 32), as it varies over time with different system parameters. The throughput increases with more active units, higher failure rates, and increased inspection rates for imperfect repair, as seen in Figure 6. These factors contribute to a higher number of failed units being repaired, thus increasing the throughput. Parameters leading to higher throughput are crucial for system enhancement. Conversely, throughput decreases with higher service rates, increased standby units, and longer setup times. A notable trend in Figure 6(viii) shows that a significantly large inspection rate for perfect repair, $\beta_1$, leads to decreased throughput. Conversely, a lower inspection rate initially yields lower throughput but becomes more prominent over time.

Figure 7 depicts a surface plot showing the relationship between the expected total cost (Equation 33), time, and system parameters like $\mu$, $\beta_1$, $\lambda$, and $\lambda_d$ for aforementioned default system parameters and following unit costs $C_H = 90, C_S = 60, C_M = 10, C_1 = 13, C_2 = 2, C_3 = 2$. The plot indicates that increasing the service or inspection rate for perfect repairs reduces the overall cost by repairing more failed units. Conversely, higher failure rates of active or deteriorating units lead to increased costs (Figure 7(iii)-7(iv)). To minimize costs, proactive preventive measures should be taken to prevent deterioration and reduce delays in MRPs.

Tables 2-6 present a comprehensive overview of system characteristics with varying parameters. In our experiments, default values are set to the previously mentioned values for parameters and costs. Results show that $R_Y(t)$ decreases while $E_N(t)$ increases over time, regardless of parameter variations. $MTTF$ decreases monotonically with all parameters except $\mu$. Our conclusions from the numerical illustrations are as follows:

- Preventive measures should delay both active and standby unit failures, with instantaneous and perfect switching to maintain system function.
- Repairs should emphasize high inspection rates for optimal system thresholds and capacities, minimizing lost failed units.
- Optimization of standby units within cost constraints is crucial to prevent degradation during short-mode operation.

## 10 Conclusion

This study presents a Markovian model of a FTS, integrating real-time paradigms for practical applicability. It is one of the initial attempts to quantitatively assess machining system reliability, considering controlled failed unit arrival policies and imperfect repair. The study utilizes efficient numerical computation techniques such as Laplace transform, eigenvalue, and linear algebra to calculate transient-state probabilities, reliability measures, and queueing characteristics. Sensitivity analysis identifies critical parameters. However, the model could be enhanced by integrating differentiated working vacations, working breakdowns, common-cause failures, switching failures, and switching delays. Further research on system analysis, design, and optimization to derive optimal decision parameters would be valuable.

### Author's Declaration

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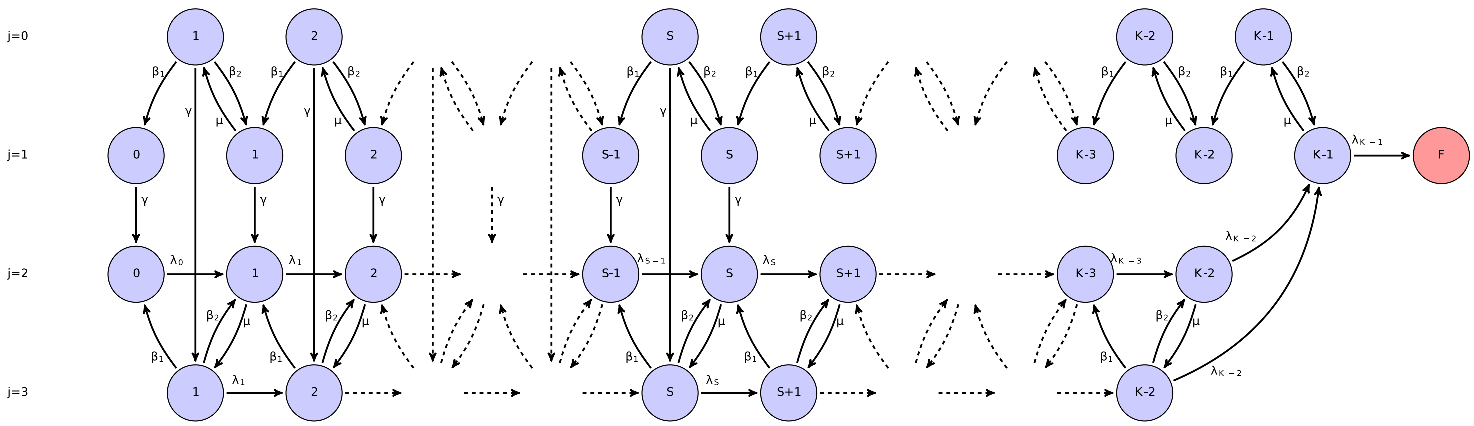


Figure 1: Transition diagram

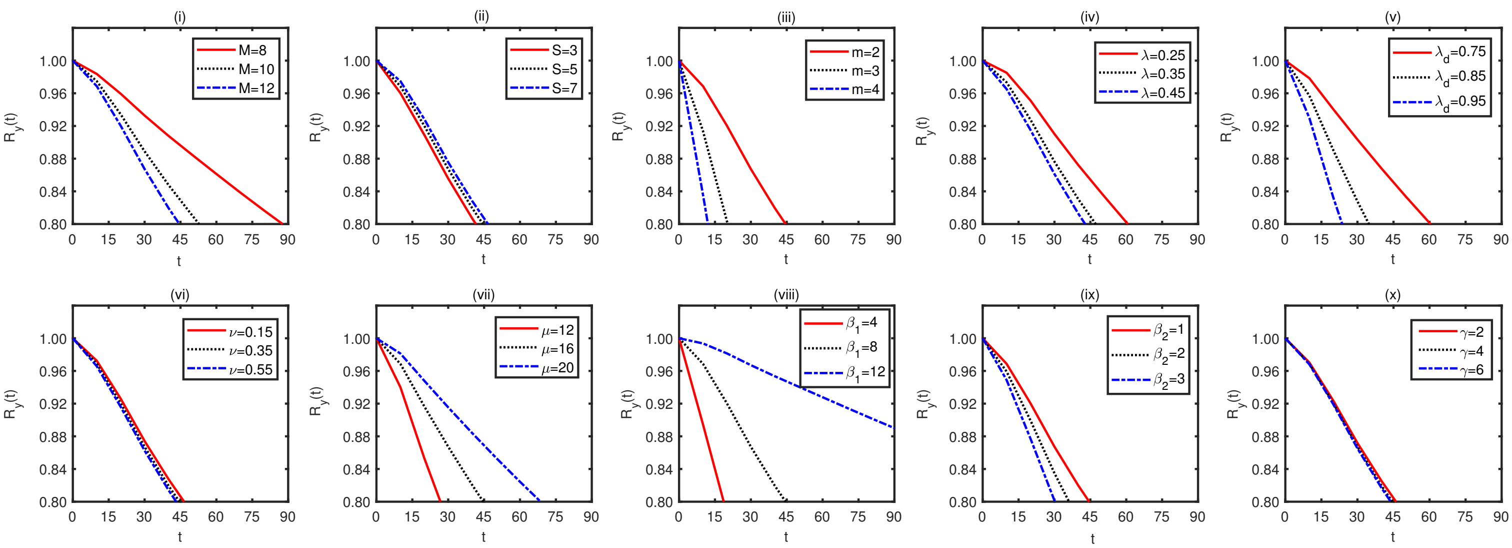


Figure 2: $R_Y(t)$ wrt $t$ for different parameters

- Table 3: Performance evaluations for different parameters $S$, $\nu$ and $t$
- Table 4: Performance evaluations for different parameters $m$, $\lambda_d$ and $t$
- Table 5: Performance evaluations for different parameters $\mu$, $\beta_1$ and $t$
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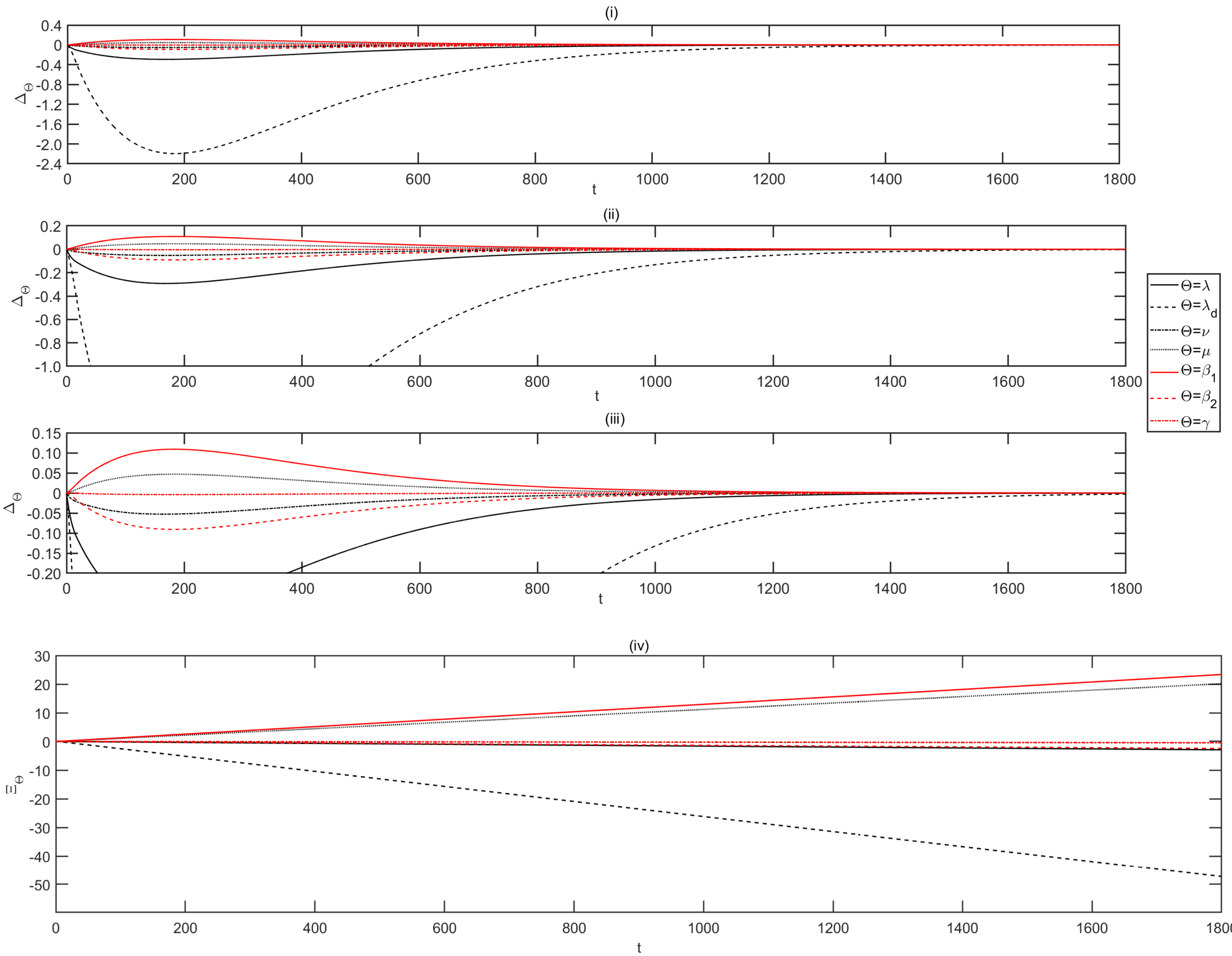


Figure 3: Sensitivity and relative sensitivity of $R_Y(t)$

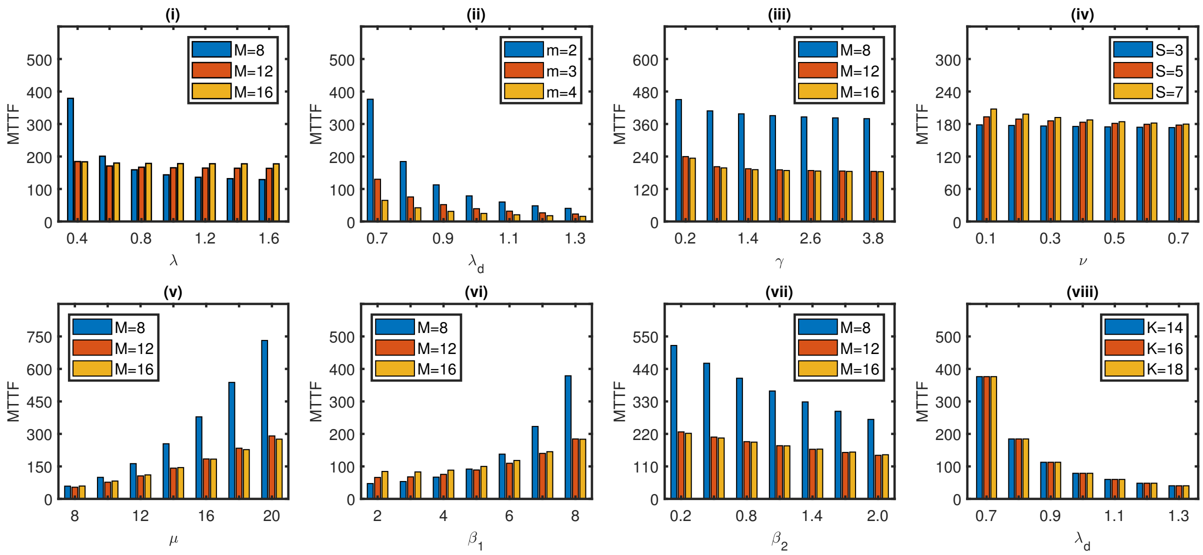

Figure 4: $MTTF$ wrt $t$ for different parameters

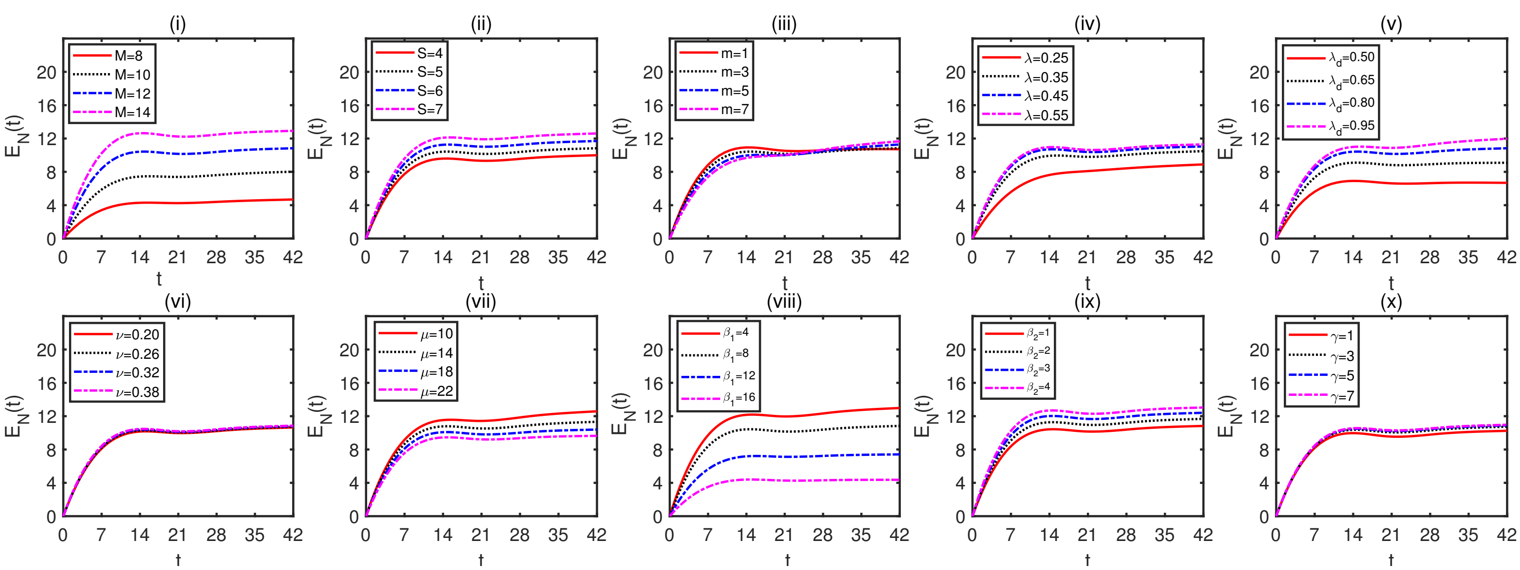

Figure 5: $E_N(t)$ wrt $t$ for different parameters

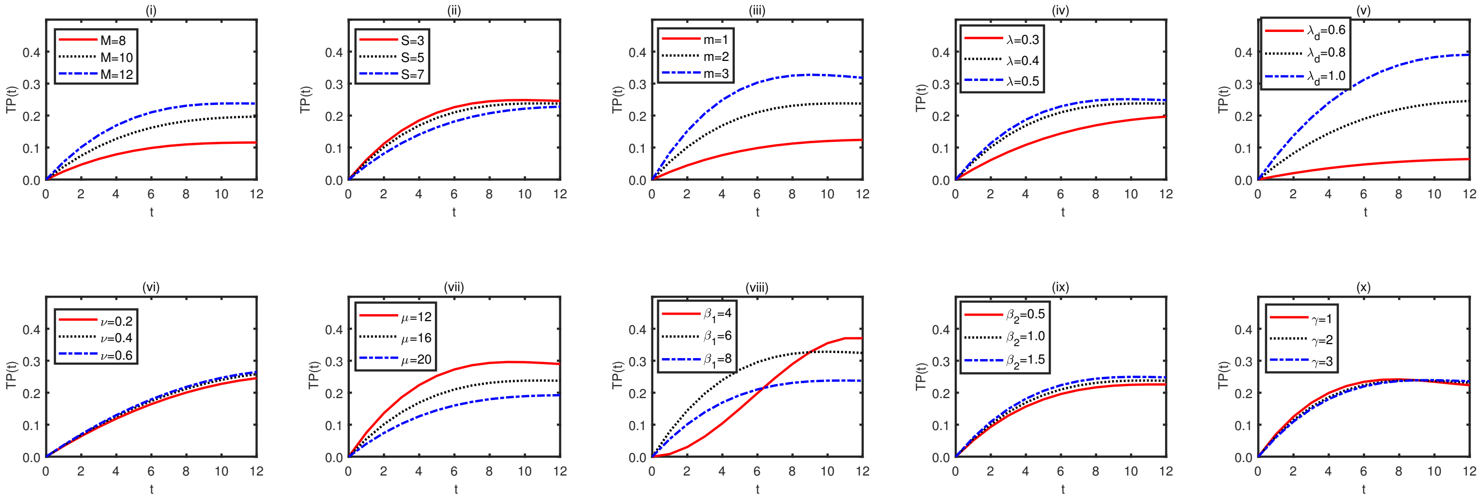


Figure 6: $TP(t)$ wrt $t$ for different parameters

Table 1: Sensitivity and relative sensitivity of $MTTF$

| | $M, S, m$ | $\boldsymbol{\Theta}$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $\lambda$ | $\lambda_d$ | $\nu$ | $\mu$ | $\beta_1$ | $\beta_2$ | $\gamma$ |
| $\Delta_\Theta$ | 10, 5, 2 | -501.53683 | -1412.45511 | -109.38124 | 33.27704 | 86.35596 | -63.59000 | -2.59526 |
| | 12, 5, 2 | -142.75361 | -1077.72249 | -25.66233 | 22.98489 | 53.53459 | -44.55887 | -1.92651 |
| | 14, 5, 2 | -61.45667 | -1019.16402 | -9.50638 | 20.99768 | 46.71424 | -41.16135 | -1.55964 |
| | | | | | | | | |
| | 12, 3, 2 | -64.87038 | -1036.59368 | -8.63333 | 21.25930 | 48.58546 | -41.55342 | -1.23844 |
| | 12, 5, 2 | -142.75361 | -1077.72249 | -25.66233 | 22.98489 | 53.53459 | -44.55887 | -1.92651 |
| | 12, 7, 2 | -203.79150 | -1098.62437 | -44.13549 | 24.13662 | 57.02288 | -46.53853 | -2.38493 |
| | | | | | | | | |
| | 12, 5, 1 | -385.97641 | -7082.77422 | -67.59066 | 139.34465 | 377.53003 | -287.92860 | -3.85303 |
| | 12, 5, 2 | -142.75361 | -1077.72249 | -25.66233 | 22.98489 | 53.53459 | -44.55887 | -1.92651 |
| | 12, 5, 3 | -90.11140 | -335.08105 | -16.46149 | 7.89276 | 15.98151 | -14.65419 | -1.28434 |
| $\Gamma_\Theta$ | 10, 5, 2 | -0.91374 | -5.14666 | -0.17437 | 2.42508 | 3.14661 | -0.28963 | -0.04728 |
| | 12, 5, 2 | -0.30951 | -4.67327 | -0.04868 | 1.99336 | 2.32139 | -0.24152 | -0.04177 |
| | 14, 5, 2 | -0.13584 | -4.50548 | -0.01839 | 1.85652 | 2.06513 | -0.22746 | -0.03447 |
| | | | | | | | | |
| | 12, 3, 2 | -0.14750 | -4.71394 | -0.01718 | 1.93355 | 2.20944 | -0.23621 | -0.02816 |
| | 12, 5, 2 | -0.30951 | -4.67327 | -0.04868 | 1.99336 | 2.32139 | -0.24152 | -0.04177 |
| | 12, 7, 2 | -0.43000 | -4.63621 | -0.08149 | 2.03714 | 2.40637 | -0.24549 | -0.05032 |
| | | | | | | | | |
| | 12, 5, 1 | -0.17196 | -6.31086 | -0.02635 | 2.48316 | 3.36385 | -0.32069 | -0.01717 |
| | 12, 5, 2 | -0.30951 | -4.67327 | -0.04868 | 1.99336 | 2.32139 | -0.24152 | -0.04177 |
| | 12, 5, 3 | -0.47725 | -3.54930 | -0.07629 | 1.67206 | 1.69282 | -0.19403 | -0.06802 |

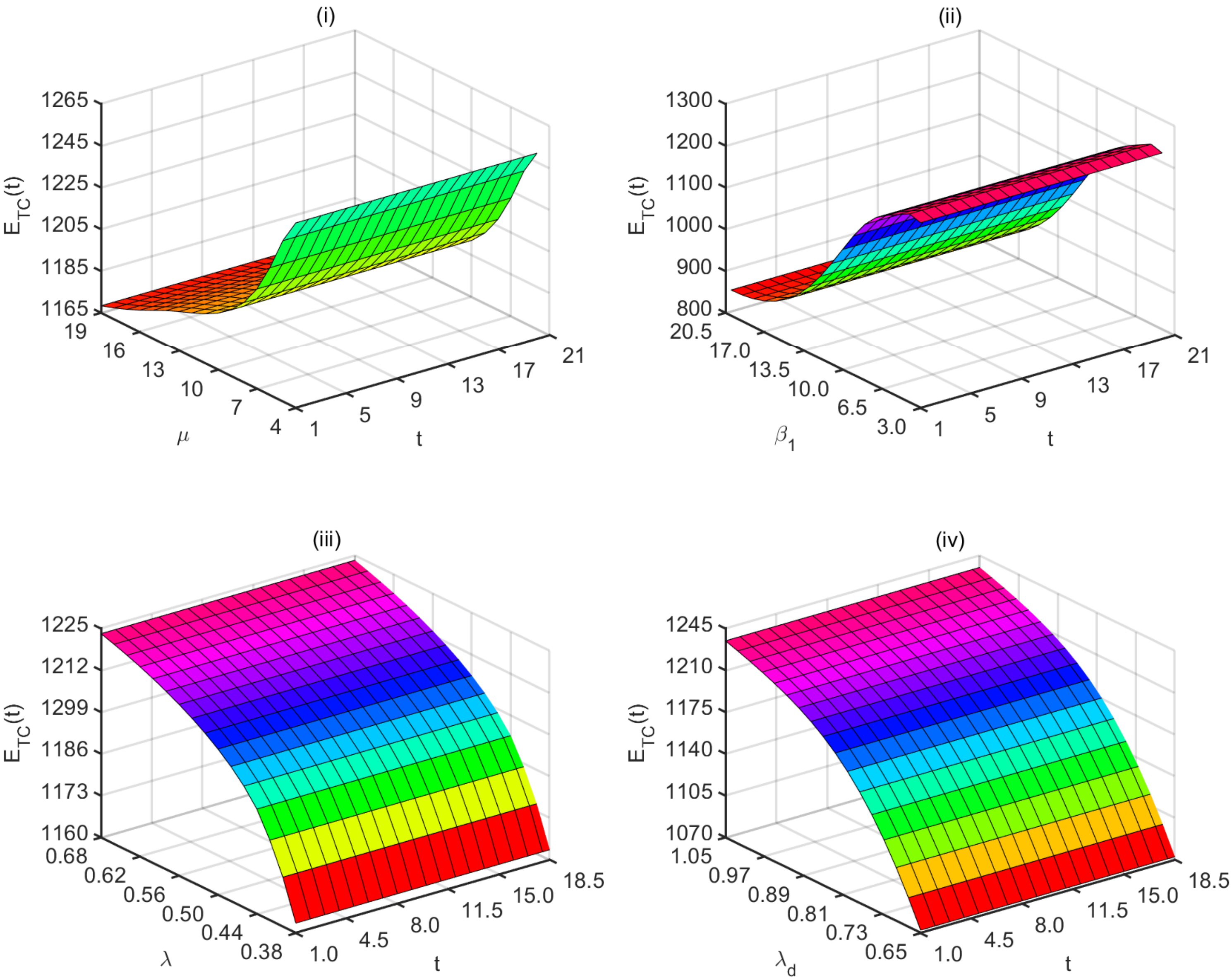


Figure 7: $E_{TC}(t)$ wrt $t$ for different parameters

Table 2: Performance evaluations for different parameters $M$, $\lambda$ and $t$

| $M$ | $\lambda$ | $t$ | $R_Y(t)$ | $MTTF$ | $E_N(t)$ | $T_P(t)$ | $E_S(t)$ | $E_O(t)$ | $E_F(t)$ | $E_W(t)$ | $FF(t)$ | $E_D(t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 0.4 | 10 | 0.98846 | 541.82368 | 3.18858 | 0.09225 | 2.44401 | 7.40995 | 4.27817 | 1.34172 | 0.01847 | 0.02893 |
| | | 25 | 0.96140 | 541.82368 | 3.45954 | 0.09210 | 2.35823 | 7.19634 | 4.15894 | 1.20217 | 0.06175 | 0.02662 |
| | | 40 | 0.93499 | 541.82368 | 3.69422 | 0.08958 | 2.29342 | 6.99862 | 4.04468 | 1.09487 | 0.10401 | 0.02425 |
| | 0.7 | 10 | 0.96772 | 225.37488 | 5.65095 | 0.22361 | 0.83744 | 6.54157 | 5.14780 | 0.91096 | 0.05165 | 0.03957 |
| | | 25 | 0.90478 | 225.37488 | 6.06542 | 0.20926 | 0.78214 | 6.11541 | 4.81234 | 0.79341 | 0.15235 | 0.03450 |
| | | 40 | 0.84594 | 225.37488 | 6.45138 | 0.19565 | 0.73128 | 5.71769 | 4.49937 | 0.69743 | 0.24649 | 0.03033 |
| 16 | 0.4 | 10 | 0.97205 | 227.77787 | 13.08031 | 0.22078 | 0.09419 | 7.86172 | 5.99480 | 0.45831 | 0.04472 | 0.01688 |
| | | 25 | 0.90932 | 227.77787 | 13.59297 | 0.20899 | 0.08115 | 7.29524 | 5.58012 | 0.41051 | 0.14508 | 0.01537 |
| | | 40 | 0.85054 | 227.77787 | 14.00721 | 0.19548 | 0.07590 | 6.82359 | 5.21938 | 0.37262 | 0.23913 | 0.01396 |
| | 0.7 | 10 | 0.96758 | 222.37667 | 13.31460 | 0.22225 | 0.03697 | 7.68001 | 6.10746 | 0.45870 | 0.05188 | 0.01669 |
| | | 25 | 0.90389 | 222.37667 | 13.82931 | 0.21205 | 0.03130 | 7.10308 | 5.65094 | 0.40862 | 0.15378 | 0.01533 |
| | | 40 | 0.84429 | 222.37667 | 14.23615 | 0.19807 | 0.02924 | 6.63476 | 5.27836 | 0.37077 | 0.24914 | 0.01391 |

Table 3: Performance evaluations for different parameters $S$, $\nu$ and $t$

| $S$ | $\nu$ | $t$ | $R_Y(t)$ | $MTTF$ | $E_N(t)$ | $T_P(t)$ | $E_S(t)$ | $E_O(t)$ | $E_F(t)$ | $E_W(t)$ | $FF(t)$ | $E_D(t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.1 | 10 | 0.97044 | 224.63368 | 7.41111 | 0.22645 | 0.19273 | 7.42878 | 5.36248 | 0.72357 | 0.04730 | 0.03055 |
| | | 25 | 0.90693 | 224.63368 | 7.83340 | 0.21104 | 0.18191 | 6.94973 | 5.01274 | 0.63992 | 0.14891 | 0.02694 |
| | | 40 | 0.84760 | 224.63368 | 8.23681 | 0.19723 | 0.17001 | 6.49508 | 4.68481 | 0.56876 | 0.24384 | 0.02395 |
| | 0.3 | 10 | 0.96926 | 221.81462 | 7.49025 | 0.22850 | 0.16751 | 7.37400 | 5.39580 | 0.72038 | 0.04918 | 0.03051 |
| | | 25 | 0.90511 | 221.81462 | 7.91811 | 0.21317 | 0.15700 | 6.88837 | 5.03921 | 0.63642 | 0.15183 | 0.02692 |
| | | 40 | 0.84520 | 221.81462 | 8.32064 | 0.19906 | 0.14661 | 6.43245 | 4.70568 | 0.56554 | 0.24768 | 0.02392 |
| 7 | 0.1 | 10 | 0.98773 | 267.92154 | 8.43309 | 0.14452 | 1.61272 | 9.00390 | 5.43958 | 0.64503 | 0.01963 | 0.01714 |
| | | 25 | 0.93697 | 267.92154 | 10.25991 | 0.18304 | 0.91557 | 7.82173 | 5.15178 | 0.50213 | 0.10086 | 0.01784 |
| | | 40 | 0.88473 | 267.92154 | 10.76116 | 0.17524 | 0.83246 | 7.34807 | 4.86412 | 0.45201 | 0.18444 | 0.01628 |
| | 0.3 | 10 | 0.98259 | 244.32256 | 10.04759 | 0.18664 | 0.80206 | 8.19701 | 5.59375 | 0.55673 | 0.02785 | 0.01858 |
| | | 25 | 0.92421 | 244.32256 | 11.10095 | 0.19965 | 0.51674 | 7.36713 | 5.19938 | 0.46837 | 0.12126 | 0.01798 |
| | | 40 | 0.86789 | 244.32256 | 11.52706 | 0.18772 | 0.48292 | 6.91483 | 4.88193 | 0.42352 | 0.21137 | 0.01629 |

Table 4: Performance evaluations for different parameters $m$, $\lambda_d$ and $t$

| $m$ | $\lambda_d$ | $t$ | $R_Y(t)$ | $MTTF$ | $E_N(t)$ | $T_P(t)$ | $E_S(t)$ | $E_O(t)$ | $E_F(t)$ | $E_W(t)$ | $FF(t)$ | $E_D(t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.80 | 10 | 0.99615 | 1225.10702 | 9.41232 | 0.10018 | 0.28664 | 7.36624 | 5.39825 | 0.57353 | 0.00308 | 0.01064 |
| | | 25 | 0.98413 | 1225.10702 | 9.78649 | 0.10662 | 0.20365 | 7.07454 | 5.27795 | 0.53931 | 0.01270 | 0.01089 |
| | | 40 | 0.97210 | 1225.10702 | 9.87540 | 0.10534 | 0.20094 | 6.98755 | 5.21330 | 0.52791 | 0.02232 | 0.01067 |
| | 1.10 | 10 | 0.97333 | 229.55458 | 10.58896 | 0.33263 | 0.27209 | 6.20249 | 5.96139 | 0.56298 | 0.02934 | 0.03141 |
| | | 25 | 0.91086 | 229.55458 | 11.02444 | 0.31270 | 0.24685 | 5.78819 | 5.57756 | 0.50593 | 0.09805 | 0.02836 |
| | | 40 | 0.85238 | 229.55458 | 11.40811 | 0.29262 | 0.23100 | 5.41655 | 5.21944 | 0.45752 | 0.16238 | 0.02565 |
| 3 | 0.80 | 10 | 0.93055 | 91.18102 | 8.43583 | 0.31082 | 0.51005 | 7.97519 | 5.48570 | 0.65029 | 0.16668 | 0.03685 |
| | | 25 | 0.78465 | 91.18102 | 9.49022 | 0.26414 | 0.42043 | 6.70925 | 4.62400 | 0.48724 | 0.51685 | 0.02783 |
| | | 40 | 0.66153 | 91.18102 | 10.35473 | 0.22269 | 0.35446 | 5.65653 | 3.89847 | 0.37649 | 0.81233 | 0.02151 |
| | 1.10 | 10 | 0.80274 | 36.17703 | 9.36952 | 0.43603 | 0.53177 | 6.75540 | 5.75924 | 0.61468 | 0.65095 | 0.04654 |
| | | 25 | 0.51670 | 36.17703 | 11.37737 | 0.28075 | 0.34154 | 4.34746 | 3.70749 | 0.32587 | 1.59488 | 0.02468 |
| | | 40 | 0.33258 | 36.17703 | 12.66829 | 0.18071 | 0.21983 | 2.79825 | 2.38634 | 0.18837 | 2.20250 | 0.01426 |

Table 5: Performance evaluations for different parameters $\mu$, $\beta_1$ and $t$

| $\mu$ | $\beta_1$ | $t$ | $R_Y(t)$ | $MTTF$ | $E_N(t)$ | $T_P(t)$ | $E_S(t)$ | $E_O(t)$ | $E_F(t)$ | $E_W(t)$ | $FF(t)$ | $E_D(t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 4 | 10 | 0.86035 | 57.65581 | 11.06680 | 0.34604 | 0.07488 | 5.77903 | 4.30223 | 0.38875 | 0.22343 | 0.03127 |
| | | 25 | 0.65964 | 57.65581 | 12.20387 | 0.26341 | 0.05796 | 4.44409 | 3.30621 | 0.27091 | 0.54457 | 0.02158 |
| | | 40 | 0.50580 | 57.65581 | 13.09365 | 0.20259 | 0.04426 | 3.40337 | 2.53267 | 0.19343 | 0.79073 | 0.01547 |
| | 12 | 10 | 0.96939 | 172.67989 | 8.87112 | 0.19255 | 0.40356 | 7.73761 | 5.50311 | 0.62034 | 0.04898 | 0.02171 |
| | | 25 | 0.88720 | 172.67989 | 9.63133 | 0.18414 | 0.31893 | 6.97634 | 5.01571 | 0.52077 | 0.18047 | 0.01912 |
| | | 40 | 0.81152 | 172.67989 | 10.17482 | 0.16844 | 0.29166 | 6.38108 | 4.58781 | 0.45090 | 0.30157 | 0.01656 |
| 22 | 4 | 10 | 0.93925 | 129.25142 | 10.38952 | 0.36735 | 0.13399 | 6.49430 | 4.77457 | 0.45956 | 0.09720 | 0.03536 |
| | | 25 | 0.83509 | 129.25142 | 11.02459 | 0.32820 | 0.11831 | 5.76204 | 4.23881 | 0.38449 | 0.26386 | 0.02977 |
| | | 40 | 0.74245 | 129.25142 | 11.57724 | 0.29188 | 0.10514 | 5.12217 | 3.76825 | 0.32549 | 0.41207 | 0.02521 |
| | 12 | 10 | 0.99892 | 4664.74338 | 3.80928 | 0.01801 | 2.28720 | 10.95531 | 6.21547 | 1.63167 | 0.00172 | 0.00473 |
| | | 25 | 0.99577 | 4664.74338 | 3.99093 | 0.01967 | 2.20845 | 10.84950 | 6.20045 | 1.55363 | 0.00676 | 0.00493 |
| | | 40 | 0.99257 | 4664.74338 | 4.03091 | 0.01963 | 2.20066 | 10.81394 | 6.18057 | 1.53329 | 0.01188 | 0.00487 |

Table 6: Performance evaluations for different parameters $\beta_1$, $\beta_2$ and $t$

| $\beta_1$ | $\beta_2$ | $t$ | $R_Y(t)$ | $MTTF$ | $E_N(t)$ | $T_P(t)$ | $E_S(t)$ | $E_O(t)$ | $E_F(t)$ | $E_W(t)$ | $FF(t)$ | $E_D(t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 0.1 | 10 | 0.94987 | 134.15670 | 9.98867 | 0.32047 | 0.20131 | 6.82790 | 4.99127 | 0.49969 | 0.08020 | 0.03208 |
| | | 25 | 0.84738 | 134.15670 | 10.64981 | 0.28674 | 0.17980 | 6.07843 | 4.44717 | 0.41758 | 0.24419 | 0.02692 |
| | | 40 | 0.75587 | 134.15670 | 11.22758 | 0.25577 | 0.16039 | 5.42201 | 3.96692 | 0.35332 | 0.39061 | 0.02278 |
| | 1.1 | 10 | 0.93566 | 107.43943 | 10.78592 | 0.32889 | 0.18643 | 6.66884 | 4.87617 | 0.45209 | 0.10295 | 0.03049 |
| | | 25 | 0.81096 | 107.43943 | 11.50211 | 0.28682 | 0.16069 | 5.75961 | 4.21727 | 0.36665 | 0.30247 | 0.02494 |
| | | 40 | 0.70276 | 107.43943 | 12.10223 | 0.24855 | 0.13925 | 4.99115 | 3.65458 | 0.30198 | 0.47559 | 0.02054 |
| 14 | 0.1 | 10 | 0.99806 | 2448.48951 | 4.33914 | 0.02482 | 2.01960 | 10.68440 | 6.20550 | 1.43012 | 0.00310 | 0.00572 |
| | | 25 | 0.99211 | 2448.48951 | 4.63162 | 0.02770 | 1.90059 | 10.50518 | 6.17371 | 1.33295 | 0.01262 | 0.00598 |
| | | 40 | 0.98604 | 2448.48951 | 4.70436 | 0.02758 | 1.88744 | 10.43926 | 6.13601 | 1.30433 | 0.02234 | 0.00586 |
| | 1.1 | 10 | 0.99696 | 1541.70141 | 5.30705 | 0.03508 | 1.78134 | 10.40038 | 6.18059 | 1.16460 | 0.00486 | 0.00661 |
| | | 25 | 0.98753 | 1541.70141 | 5.68336 | 0.03919 | 1.64005 | 10.15689 | 6.12462 | 1.07764 | 0.01994 | 0.00690 |
| | | 40 | 0.97794 | 1541.70141 | 5.78764 | 0.03887 | 1.62228 | 10.05609 | 6.06517 | 1.04795 | 0.03529 | 0.00672 |