Angle and Position Control of a High-Friction Pendulum on Cart – Comparison of SMC and AFSMC Approaches

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Abstract

The study focuses on controlling the angle and position of a high-friction Inverted Pendulum on a moving Cart (IPC) system. An experimental setup with a low-quality, high-friction gearbox is built to make the problem more challenging. The friction force is measured and found to be dependent on the Cart position. A simple position-dependent friction curve is fitted to the experimental measurement and added to the dynamic model of the plant for simulation purposes. Since the IPC dynamic equation is not input-output linearizable, an approximate feedback linearization method is employed, followed by a Sliding Mode Control (SMC) approach. A Direct-Adaptive Fuzzy Sliding Mode Control (AFSMC) approach is then tailored to mimic the feedback linearization part using an adaptive fuzzy engine, reducing the model-based part of the control. The uncertainty bound is estimated online and used in the switching part of the controller to reduce control input chatter. Both SMC and the less model-dependent AFSMC are implemented in simulations and practical implementations. While both methods perform well in the nominal case, the superior performance of AFSMC is revealed when intentionally induced uncertainty and noise are applied to the model and to the Cart position sensor, respectively.

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1. Introduction

The purpose of the present paper is to study and experimentally verify the effectiveness of the SMC and AFSMC approaches in preserving the stability of an IPC with highly nonlinear and unknown friction over the cart movement.

IPC is an Under-actuated Mechanical System (UMS), i.e., a two-degree of freedom system controlled by just one input [**1**]. Furthermore, IPC has a fast and unstable dynamics [**2**] and hence can serve as a benchmark problem for testing various control methods [**3**]. The shortcoming of traditional control methods for control of UMSs was observed about three decades ago [**4**]. IPC is built and studied in various forms, e.g., linear, rotational, single-joint and multi-joint [**5**].

In the case that dynamic equations of the IPC are uncertain, the control problem turns out to be even more challenging [**6**]. Irfan et al. [**7**] proposed a comparative analysis of linear and nonlinear feedback control techniques to obtain better control performance for the inverted pendulum system by considering time, control energy, and tracking error. The implemented control methods are as follows: Linear Quadratic Regulator (LQR), SMC through feedback linearization, Integral Sliding Mode Control (ISMC), and Terminal Sliding Mode Control (TSMC). The designed control laws were examined with various signals, knowing the position of the moving cart and the angle of the inverted pendulum, to determine their regulation performance.

Shokouhi et al. [**8**] applied the SMC approach to deal with high and nonlinear frictions in an inverted pendulum. By applying an additional control force to the moving base of the pendulum, they dealt with the high and non-linear friction between the cart and rail by using an approximate linearization method. In [**9**], Schwab et al. used two different approaches to control the inverted pendulum system without friction on the moving cart: on the one hand, they used SMC to control the pendulum angle and angular velocity, and on the other hand, Predictive Control Model (PCM) was used to control the position of the moving cart. Liu et al. [**10**], Nafa et al. [**11**], Hung et al. [**12**] and Younsi et al. [**13**], designed and implemented a type of AFSMC approach on the inverted pendulum system theoretically. In this research, the existence of friction was ignored and no practical implementation was performed.

A nonlinear friction model was used in [**14**] to include the friction between the moving cart and rail into the system equations. A Linear Quadratic Regulator Controller (LQRC) was proposed and compared with the conventional SMC approach. Experimental verifications, reveal that the suggested SMC approach provides a superior performance.

Based on the control strategy defined in [**15**], an adaptive fuzzy inference system was embedded within the boundary layer to improve the balancing efficiency. In [**16**], Chen et al. considered friction between the moving cart and the rail, as well as friction at the pivot, and applied an AFSMC technique. No experimental verifications are reported, however.

Considering that various approaches have been introduced under the common title of AFSMC in the past, in this paper a special version of direct-AFSMC proposed in [**17**] is employed. Experimental verification and comparison between the SMC and the AFSMC approaches is performed to depict the effectiveness of the AFSMC approach, which is a robust approach with much less reliance on the dynamic model of the system. Main characteristics of the previously introduced approach are summarized in Table 1.

The presence of high and non-linear friction between the cart and rail in real IPC systems poses a challenge. Previous studies employed approaches to find the exact friction model, such as Dahl's model, Bliman-Sorine Model, and LuGre Model [**18**], but these practices are costly, time-consuming, and impose restrictions on simulations and practical implementations. The method proposed in this paper provides a way for researchers to significantly reduce their dependence on friction modelling methods. This approach is novel and highly useful for simulations and practical implementations, as it can inspire researchers in their future investigations.

This paper is organized as follows: In section 1, an introduction and an overview of the control approaches for the inverted pendulum system is presented. In sections 2, dynamic equations with and without the effect of the friction is shortly reviewed and the system model is changed into an approximate feedback linearized form. In section 3, a brief account of SMC is given and in section 4, the propose direct-AFSMC approach is introduced. In section 5, simulation studies for comparing SMC and AFSMC approaches are performed for four cases, namely, Nominal system, Nominal System with Position Sensor Noise, Uncertain System - Friction Added, Uncertain system -Pendulum Length Changed, and the superiority of the proposed AFSMC approach is depicted clearly. Also, at the end of this section, there was a comparative report on uncertainty bound in the AFSMC approach. The specifications of the built experimental setup are discussed in section 7. The performance of the SMC and the AFSMC approaches are compared in section 7 and concluding remarks are given in section 8.

2. Dynamic Modelling

Inverted pendulum systems have been used for more than half a century as a reference system to evaluate control systems due to their nonlinear, unstable, and non-minimum phase characteristics. In Figure 1, the inverted pendulum is connected to the moving cart with a rotating shaft, and the moving cart is connected to a direct current electric servo-motor with a belt. Thus, the IPC has two Degrees of Freedom (DoF), where x is the cart displacement and θ is the pendulum angle position. The purpose of the research is to keep the inverted pendulum near the upright (unstable) equilibrium point by introducing the force $F(t)$ to the moving cart.

Active friction compensation is usually required to improve the dynamics of mechanical systems. Different friction models, such as Dahl's Model, Bliman-Sorine Model, and LuGre Model, are commonly used for friction compensation. However, friction modelling remains a difficult and time-consuming task. Recent efforts have focused on canceling out the friction force effect in mechanical systems through robust online friction compensation procedures. These procedures involve applying a force or torque command equal and opposite to the instantaneous friction force, assuming adequate actuation bandwidth is available. There are two main approaches to active friction compensation: Model-based compensation as considered in [**18**] and Compensation without having an explicit friction force model, as considered in the present study.

A new characterization of the feedback linearized equation by considering total force applied to the moving cart, F_t , is obtained as, by re-defining the total force as

$$F_t = F + \tilde{F}_s + \delta F_s (x_1 + x_2 + sign(x_3)) \tag{1}$$

where F is the control force acting on the moving cart, $\tilde{F}_s$ is the average friction force which can be determined experimentally. δF_s is the bounded uncertain friction force deviated from $\tilde{F}_s$.

Equations of motion of the system by neglecting the friction can be obtained as follows [**8, 19**]:

$$(M+m)\ddot{x} + mL\ddot{\theta}\cos\theta - \sin\theta = F_t \tag{2}$$

$$\cos\theta\ddot{x} + L\ddot{\theta} - g\sin\theta = 0$$

where x is the position of the moving cart relative to the centerline of the rail, θ is the angle of the pendulum with respect to the vertical upright, m is the pendulum mass, M is the moving cart mass, L is the pendulum length and F_t is the total control force applied to the moving cart.

By defining state variables $\boldsymbol{x}(t) = \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, equation (2) is rewritten:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{-mg\cos x_3 \sin x_3 + mLx_4^2 \sin x_3}{M + m(\sin x_3)^2} \\ x_4 \\ \dfrac{(M+m)g\sin x_3 - mLx_4^2\cos x_3 \sin x_3}{\left(M + m(\sin x_3)^2\right)L} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M + m(\sin x_3)^2} \\ 0 \\ \dfrac{-\cos x_3}{\left(M + m(\sin x_3)^2\right)L} \end{bmatrix} \underbrace{\left(F + \tilde{F}_s + \delta F_s\right)}_{F_t} \tag{3}$$

Note in equation (3) that the two-state variables x_1 and x_3 are not directly affected by the total control force, but indirectly through x_2 and x_4. In order to simplify the equations, a virtual control input v is defined in the form of

$$v = \frac{-mg\cos x_3 \sin x_3 + mLx_4^2 \sin x_3 + F_t}{M + m(\sin x_3)^2} \tag{4}$$

Now, equation (3) can be re-written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ \dfrac{g\sin x_3}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\dfrac{\cos x_3}{L} \end{bmatrix} v \tag{5}$$

or in vector form as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(x) + \boldsymbol{g}(x)v \tag{6}$$

2.1. Approximate Input-Output Feedback Linearization

For the SMC approach to be applicable to inverted pendulum, it must be converted to a standard form by the Input-Output Feedback Linearization (IOFL) technique. For this purpose, a suitable diffeomorphism, namely a transformation matrix including the output and its derivatives [**20**] must be obtained as follows:

$$\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x}) = \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} \tag{7}$$

One possible choice for the output signal $y = h(x)$ is proposed [**8**]

$$y = h(x) = x_1 + L\ln\left(\tan x_3 + \sec x_3\right) \tag{8}$$

By differentiating y and by using the Lie derivative $L_{f} h(\boldsymbol{x}) = \frac{\partial h}{\partial x} \boldsymbol{f}(\boldsymbol{x})$, it turns out that

$$y^{(1)} = L_{\boldsymbol{f}}^{1} h(\boldsymbol{x}) + \underbrace{L_{\boldsymbol{g}} L_{\boldsymbol{f}}^{0} h(\boldsymbol{x})}_{0} = x_2 + \frac{L x_4}{\cos x_3} \tag{9}$$

$$y^{(2)} = L_{\boldsymbol{f}}^{2} h(\boldsymbol{x}) + \underbrace{L_{\boldsymbol{g}} L_{\boldsymbol{f}}^{1} h(\boldsymbol{x})}_{0} = \tan x_3 \left(g + \frac{L x_4}{\cos x_3} \right) \tag{10}$$

$$\begin{aligned} y^{(3)} &= L_{\boldsymbol{f}}^{3} h(\boldsymbol{x}) + \underbrace{L_{\boldsymbol{g}} L_{\boldsymbol{f}}^{2} h(\boldsymbol{x})}_{0} \\ &= \left(\frac{2}{(\cos x_3)^3} - \frac{1}{\cos x_3} \right) L x_4^3 + \left(\frac{3g}{(\cos x_3)^2} - 2g \right) x_4 - \underbrace{2 x_4 \tan x_3 v}_{ignore} \end{aligned} \tag{11}$$

$$\begin{aligned} y^{(4)} &= L_{\boldsymbol{f}}^{4} h(\boldsymbol{x}) + L_{g} L_{f}^{3} h(\boldsymbol{x}) v \\ &= f_z(\boldsymbol{x}) + g_z(\boldsymbol{x}) v \end{aligned} \tag{12}$$

wherein,

$$\begin{aligned} f_z(\boldsymbol{x}) &= L x_4^4 \left(\frac{6 \sin x_3}{(\cos x_3)^4} - \frac{\sin x_3}{(\cos x_3)^2} \right) + \frac{6 g \sin x_3}{(\cos x_3)^3} x_4^2 \\ &+ 3 x_4^2 \left(\frac{2 g \sin x_3}{(\cos x_3)^3} - \frac{g \sin x_3}{\cos x_3} \right) + \frac{g \sin x_3}{L} \left(\frac{3g}{(\cos x_3)^2} - 2g \right) \end{aligned} \tag{13}$$

$$g_z(\boldsymbol{x}) = \frac{-6 x_4^2}{(\cos x_3)^2} + 3 x_4^2 - \frac{3g}{L \cos x_3} + \frac{2 g \cos x_3}{L} \tag{14}$$

In equation (11), the term $2x_4 \tan x_3$ becomes very small and may be neglected near the equilibrium point. This condition holds exclusively in the proximity of the equilibrium point. Based on this condition, the reference input must be zero or extremely close to zero. Wherever this approximation is valid, a suitable controller can be designed for v to stabilize the nonlinear system and drive the output y to 0. However, far from the equilibrium point, the relative degree of the system is altered or reduced due to the presence of certain disturbances or uncertainties. This can lead to several consequences in the closed-loop control system with conventional SMC: Loss of uniqueness, Loss of attractivity, Stability issues, and Disturbance rejection issues. This implies that the diffeomorphism obtained with the defined output is a local diffeomorphism. Consequently, such a system is classified as a regulation system rather than a tracking system.

Finally, the input-output linearized form is obtained, i.e.,

$$\dot{\boldsymbol{z}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ y^{(4)} \end{bmatrix} = \begin{bmatrix} L_{\boldsymbol{f}}^1 h(\boldsymbol{x}) \\ L_{\boldsymbol{f}}^2 h(\boldsymbol{x}) \\ L_{\boldsymbol{f}}^3 h(\boldsymbol{x}) \\ f_z(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_z(\boldsymbol{x}) \end{bmatrix} v \tag{15}$$

In vector form, the resulting system is in a suitable form for application of the SMC approach:

$$y^{(4)} = f_z(\boldsymbol{x}) + g_z(\boldsymbol{x}) v \tag{16}$$

Now, from equations (2), (3), and (16) , it can be deduced that

$$F = \left(M + m(\sin x_3)^2\right) v + mg \cos x_3 \sin x_3 - mLx_4^2 \sin x_3 - \tilde{F}_s - \delta F_s \tag{17}$$

Finally, equation (16) turns into the desired form

$$y^{(4)} = f_z(\boldsymbol{x}) + g_z(\boldsymbol{x}) v + \Delta \tag{18}$$

wherein

$$\Delta = g_z(\boldsymbol{x}) v + \frac{\delta F_s}{\left(M + m(\sin x_3)^2\right)} \tag{19}$$

Since $M, m, \sin x_3$, and δF_s are bounded and by equation (14) all components of the function $g_z(\boldsymbol{x})$ are also bounded (except for $x_3 = \pm 90^o$), i.e., there exists an upper bound $\bar{\Delta}$ such that

$$|\Delta| \leq \bar{\Delta} \tag{20}$$

3. SMC Approach

The SMC approach is a well-known technique for control of nonlinear systems under Bounded Unknown External disturbances (BUE-disturbances) and parametric uncertainties [**21**]. Based on the feedback linearized dynamic system (16), the conventional SMC approach is applied to compute the control input v. By considering a regulation problem, the closed-loop error is

$$e = y - y_d \xrightarrow{y_d = 0} e = y \tag{21}$$

and, hence, the sliding function (Sliding variable: $s \to 0$, Sliding surface: $s = 0$) can be proposed as [**8**]

$$s = \left(\frac{d}{dt} + \lambda \right)^3 e = y^{(3)} + 3\lambda y^{(2)} + 3\lambda^2 y^{(1)} + \lambda^{(3)} y \tag{22}$$

where λ is a positive parameter and furthermore

$$\dot{s} = \left(f_z(\boldsymbol{x}) + 3\lambda y^{(3)} + 3\lambda^2 y^{(2)} + \lambda^3 y^{(1)} \right) + g_z(\boldsymbol{x}) v \tag{23}$$

The control input during the reaching phase is denoted by v_{eq}, and during the sliding phase by $v_{sw} = Q.sign(s)$. In the SMC approach, robustness during reaching phase is not guaranteed. In the so-called Constant Rate Reaching Law (CRRL) [**22**], the overall control input is obtained as

$$v = v_{eq} + v_{sw} = \frac{-\left(f_z(\boldsymbol{x}) + 3\lambda y^{(3)} + 3\lambda^2 y^{(2)} + \lambda^3 y^{(1)} \right)}{g_z(\boldsymbol{x})} + Q.sign(s), \quad 0 < Q \tag{24}$$

where $0 < Q$ is a design parameter. It is well-known that by selecting a sufficiently large Q, the closed-loop stability is guaranteed for any given $\bar{\Delta}$ [**8**]. The following approximation is used to reduce excessive control input chattering [**23**]; other researchers have also utilized this approximation [**24, 25**]:

$$sign(s) \cong \frac{s}{\sqrt{s^2+\gamma}}, \quad 0<\gamma \tag{25}$$

The candidate function $V(\boldsymbol{x})$ should be a pseudo-energy and non-incremental function that represents the energy of the dynamic system and it decreases with time. In other words, its derivative must be negative.

$$V(\boldsymbol{x}) = \frac{1}{2}s^2 \tag{26}$$

$$\dot{V}(\boldsymbol{x}) = s\dot{s} = s\left(-Q.sign\left(s\right)\right) \cong -\frac{Qs^2}{\sqrt{s^2+\gamma}} < 0, \quad 0<\gamma \tag{27}$$

It is clear from equation (24) that based on the Lyapunov stability theory, this studied system is stable and the convergence of the sliding mode is guaranteed.

By applying the control strategy (24) to the system (17), the actual control force can be described as

$$F = \left(M + m\left(\sin x_3\right)^2\right)v + mg\cos x_3 \sin x_3 - mLx_4^2 \sin x_3 - \tilde{F}_s \tag{28}$$

4. AFSMC Approach

The indirect-AFSMC approach relies on fuzzy inference system to approximate the uncertain nonlinear system and is based upon these approximations [**26**].

An AFSMC approach is adopted in this paper to address the problem of angular position control and vibration suppression of rotary flexible joint systems. Considering that various methods under the common title of AFSMC approach have been introduced in the literature, in this article, a special version of this approach is used, which requires little information about the dynamic of the system, proposed in [**17**]. As shown in [**6, 27**], an additive fuzzy inference system can uniformly approximate any real continuous function on a compact domain to any degree of accuracy. In the direct-AFSMC approach, instead of identifying the dynamic model of the system, which is a time-consuming process, the control input is estimated directly by a tunable fuzzy inference system.

Consider the dynamic model of the inverted pendulum with uncertainties due to the model and also the friction force as obtained in equation (18), repeated here for convenience,

$$y^{(4)} = f_z(\boldsymbol{x}) + g_z(\boldsymbol{x})v + \Delta$$

If the model of the system was known completely as in equation (6), then the ideal stabilizing controller v^* could be obtained by feedback linearization method. i.e.,

$$v^* = g_z^{-1}(\boldsymbol{x})\left(-f_z(\boldsymbol{x}) + \sum_{i=1}^{n-1} C_i^{n-1} D^{n-1} \lambda^i e\right) \tag{29}$$

By replacing equation (29) in equation (6), the error dynamics is obtained:

$$\left(D^n + \sum_{i=1}^{n-1} C_i^{n-1} D^{n-1} \lambda^i\right) e = 0 \tag{30}$$

By choosing a Horowitz polynomial with identity coefficients, the error dynamics becomes stable and $\lim_{n \to \infty} e(t) = 0$.

However, since the system is not fully known, the ideal controller v^* be obtained directly. An alternative to the ideal controller (29) can be approximated by a fuzzy inference system by using the universal approximation capability of fuzzy system.

An input-output fuzzy inference system with 4-inputs and 1-output with three fuzzy rules is considered, in which the IF-THEN rules are as follows:

$$Rule\ r: IF\ x_i\ is\ A_1^{\sim r}\ and \ldots and\ x_{n_i}\ is\ A_{n_i}^{\sim r}\ THEN\ y = b^{\sim r} \tag{31}$$

Where $\boldsymbol{x}(t) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ and y, the input and output of the fuzzy inference system, respectively, and $b^{\sim r}$ the single output for the rth base and $A_1^{\sim r} \cdots A_{n_i}^{\sim r}$ is a fuzzy set with Gaussian membership functions.

$$\mu_{A_j^r}\left(x_j\right)=\exp\left(-\left(\frac{x_j-C_j^r}{\sigma_j^r}\right)^2\right) \tag{32}$$

Where C_j^r and σ_j^r are "*center*" and "*width of Gaussian membership function*" respectively [**11, 17**]. Using Singleton fuzzifier, product inference and center average defuzzifier, the output of the fuzzy inference system is obtained as follows:

$$y=\frac{\sum_{r=1}^{n_r} b^{\sim r}\prod_{j=1}^{n_i}\mu_{A_j^r}\left(x_j\right)}{\sum_{r=1}^{n_r}\prod_{i=1}^{n_i}\mu_{A_i^j}\left(x_i\right)} \tag{33}$$

The input value of the firing strength for the *r*th base

$$w^r=\frac{\prod_{j=1}^{n_i}\mu_{A_j^r}\left(x_j\right)}{\sum_{r=1}^{n_r}\prod_{i=1}^{n_i}\mu_{A_i^j}\left(x_i\right)} \tag{34}$$

Therefore, the output of the fuzzy inference system can be rewritten as:

$$y=B^T w \tag{35}$$

where $B=\left[b^{\sim 1} \quad \cdots \quad b^{\sim 3}\right]^T$ and $w=\left[w^{\sim 1} \quad \cdots \quad w^{\sim 3}\right]^T$. Therefore, the ideal controller (29) can be approximated by an ideal fuzzy inference system $v_{fuz}^*\left(s,B^*\right)$ such that

$$v^*=v_{fuz}^*\left(s,B^*\right)+\psi=\hat{B}^{*T}w+\psi \tag{36}$$

where ψ is approximation error or uncertainty which is assumed to be bounded.

$$|\psi|<\Psi \tag{37}$$

Ψ is the approximation of real uncertainty bound and B^* is the optimal parameter vector.

$$B^* \triangleq \arg\min_B\left(|B^T w-v^*|\right) \tag{38}$$

The fuzzy IF-THEN rules of this fuzzy inference system are as follows:

$$Rule\ r: IF\ s\ is\ A^{\sim r}\ THEN\ v_{fuz}^{*} = b^{\sim r} \tag{39}$$

In implementation, the optimal parameter vector B^* and perhaps Ψ are unknown. Let $\hat{v}_{fuz}\left(s,\hat{B}\right)$ be a fuzzy inference system to approximate:

$$\hat{v}_{fuz}\left(s,\hat{B}\right) = \hat{B}^T w \tag{40}$$

where $\hat{B}$ is the estimated value for B^*. The control law for the AFSMC system is considered in the following form:

$$v = \hat{v}_{fuz}\left(s,\hat{B}\right) + v_{sw}\left(s\right) \tag{41}$$

where the fuzzy controller $\hat{v}_{fuz}$ is designed to approximate the ideal controller v^* and $v_{sw}\left(s\right)$ is designed to compensate the difference between the ideal controller and the fuzzy controller. Definition of approximate errors as:

$$\tilde{v}_{fuz} = v^* - \hat{v}_{fuz} \tag{42}$$

and

$$\tilde{B} = B^* - \hat{B} \tag{43}$$

and using equations (36) and (40):

$$\tilde{v}_{fuz} = \tilde{B}^T w + \psi \tag{44}$$

In addition, the estimated uncertainty bound of $\hat{\Psi}$ is as follows:

$$\tilde{\Psi}\left(t\right) = \Psi - \hat{\Psi}\left(t\right) \tag{45}$$

Theorem 1. Consider the system (18) and the controller given by equation (41), where the parameter vector of the fuzzy inference system is adjusted adaptively by

$$\dot{\hat{B}} = -\dot{\tilde{B}} = \alpha_1 s\left(t\right) w \tag{46}$$

and the switching control law is in the form of

$$u_{sw} = \hat{\Psi} sign\left(s(t)\right) sign\left(g_z\left(\boldsymbol{x}\right)\right) \tag{47}$$

with the given rate of change of $\hat{\Psi}$

$$\dot{\hat{\Psi}} = -\dot{\tilde{\Psi}} = \alpha_2 \,|\, s(t) \,|\, sign\left(g_z\left(\boldsymbol{x}\right)\right) \tag{48}$$

Also the positive constant parameters of the learning rates α_1 and α_2 in (46) and (48) are selected to be positive constants, then the closed-loop stability of the system is guaranteed (Proof given in [**28**]).

Figure 2 illustrates the membership functions for the normalized inputs, which are composed of Gaussian functions with variable means and variances. The parameters of these membership functions are chosen so that the sliding variable *s* remains close to zero. The initial output membership functions were arbitrarily chosen as $\hat{B}(0) = \begin{bmatrix}-1 & 0 & 1\end{bmatrix}^T$ and also the initial value is selected as $\Psi(0) = -0.1$. The positive constant parameters of the learning rates were chosen to be $0 < \alpha_1 = 2$ and $0 < \alpha_2 = 4$. It should be mentioned that although the closed-loop performance of the system can be somewhat affected by selected shape of the member function, but the closed-loop stability is guaranteed, as shown in Theorem 1.

Finally, the actual control force applied to the system (28) can be obtained from

$$F = \left(M + m\left(\sin x_3\right)^2\right)v + mg\cos x_3 \sin x_3 - mLx_4^2 \sin x_3 - \tilde{F}_s \tag{49}$$

where $\tilde{F}_s$ is the average friction force which can be determined experimentally.

5. Simulation Studies: SMC vs AFSMC

In equation (11), the term $2x_4 \tan x_3$ could be neglected exclusively in the vicinity of the equilibrium point. Accordingly, the regulation problem is considered in the sequel. For all simulations and experiments, the initial conditions are as follows: $\begin{bmatrix}0.7 & 0 & 8\pi/180 & 0\end{bmatrix}^T$ and in equation (27) the parameter γ is set to 20. Further specifications are provided in Table 2.

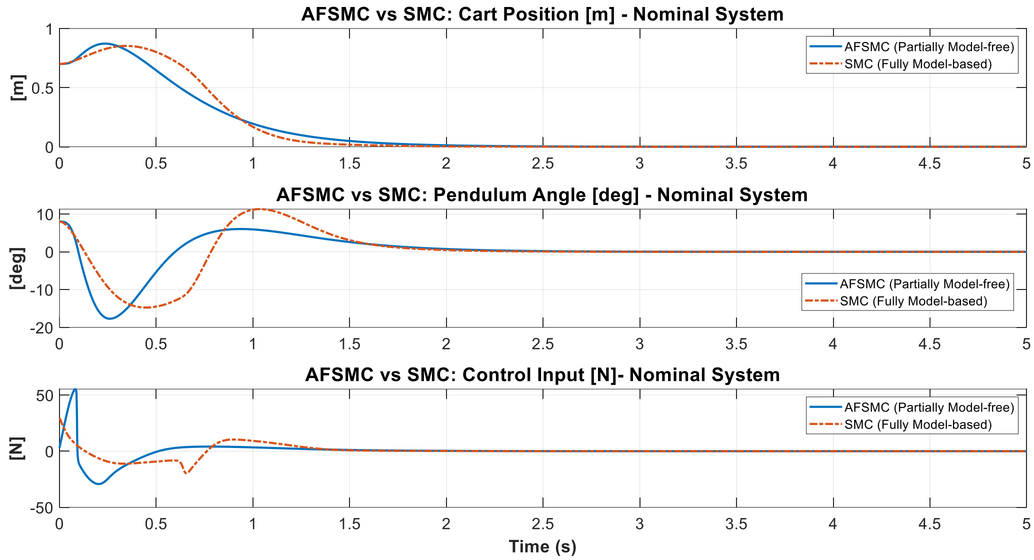

Figure 3 to

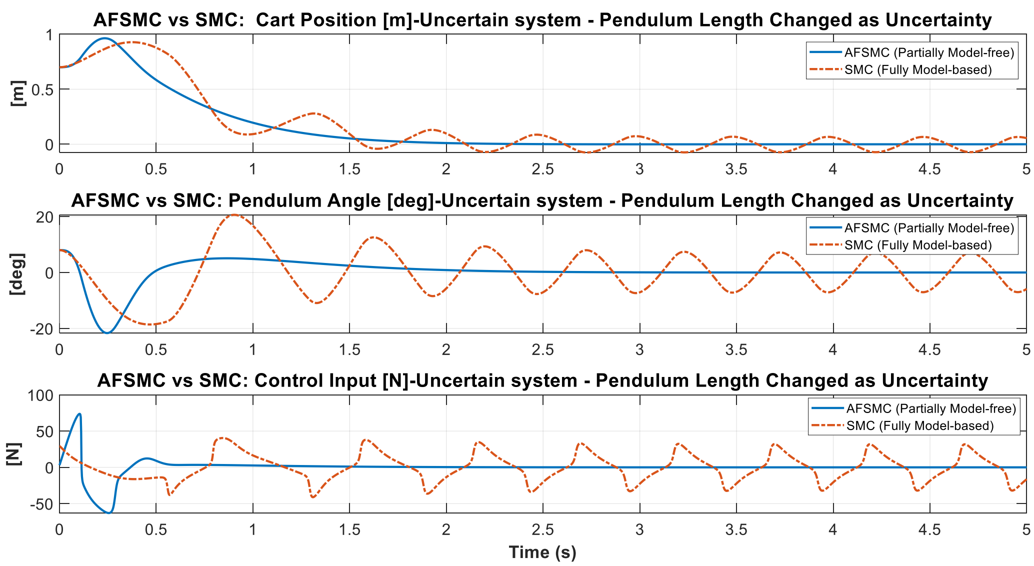

Figure 6 present the simulation results for the cases shown in Table 2. Control of Nominal System

Evaluating control designs on the nominal model allows for understanding the basic behavior and performance of SMC and AFSMC without initially being encumbered by complexities such as uncertainties, disturbances, or unmodeled dynamics

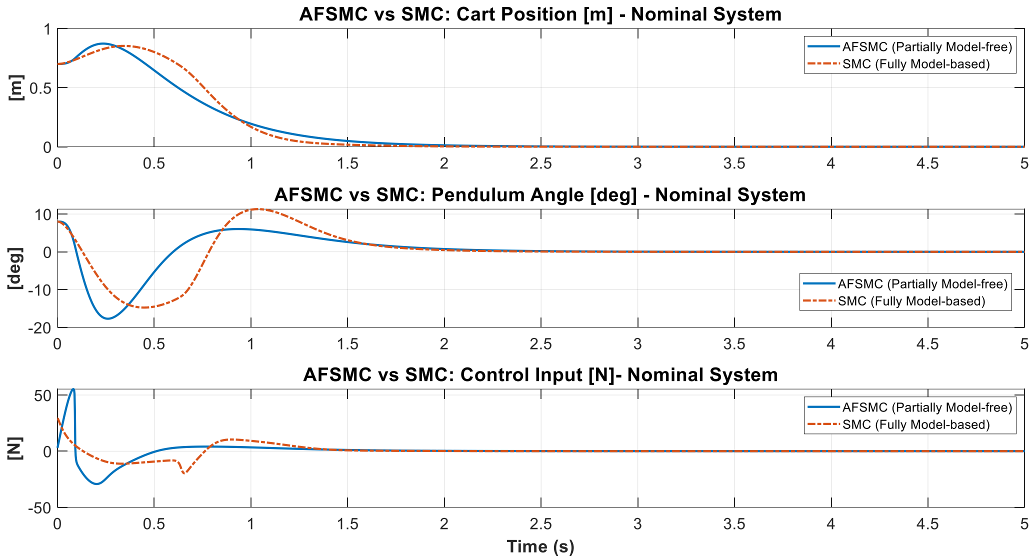

Figure 3 depicts the comparable response of SMC and AFSMC approaches. It is important to note that SMC approach is strongly model-based for the feedback linearized part of the control input. While, while for the AFSMC approach, the equilibrium control part is estimated adaptively. One of the reasons for the close operation of SMC and AFSMC is the lack of friction, noise, and parametric uncertainty changes in the nominal system.

5.1. Nominal System with Position Sensor Noise

In this section, by applying a sinusoidal type noise

$$n(t) = 0.003 \sum_{n=30,50,70} \sin(t)$$

to the cart position sensor, the simulation results for SMC and AFSMC approaches are compared in

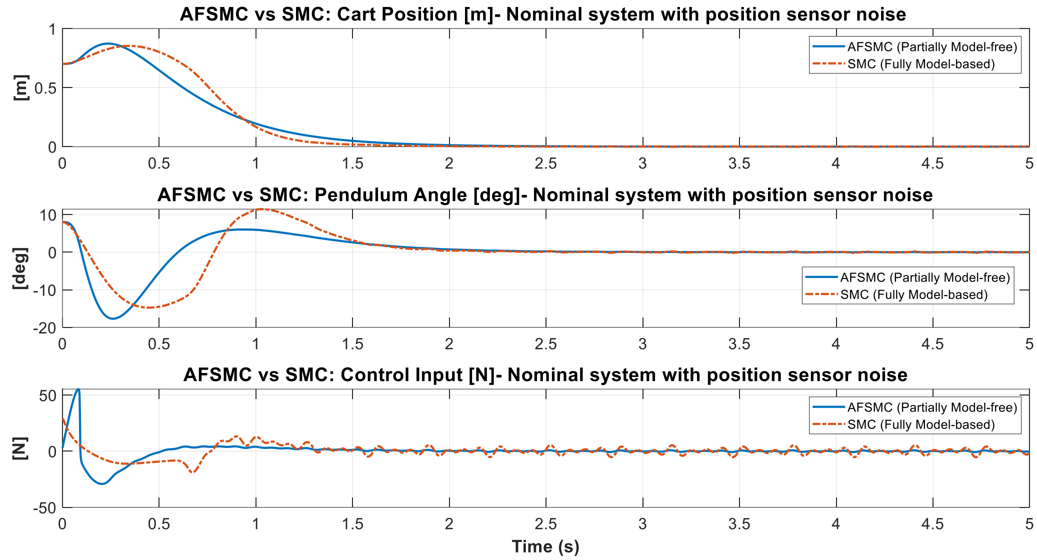

Figure 4, depicting the superior performance of the AFSCM.

For the SMC, the values $Q = 8$ and $\lambda = 6$ in equations (23), (24), and (25) were obtained by trials and errors to just retain the closed-loop stability while for the AFSMC case, the values of $\lambda = 4$, $\alpha_1 = 2$, and $\alpha_2 = 4$ were determined more easily, considering the guaranteed closed-loop stability property devised in Theorem 1.

Uncertain System 1: Nominal System + Friction added

According to the experimental implementation reported in [**8**], the approximate friction value $11.1216 - 2.8256\,cos(32.34x_1) - 2.57\,sin(32.34x_1)$ was considered for the simulation purposes and added to the equations of plant.

As shown in

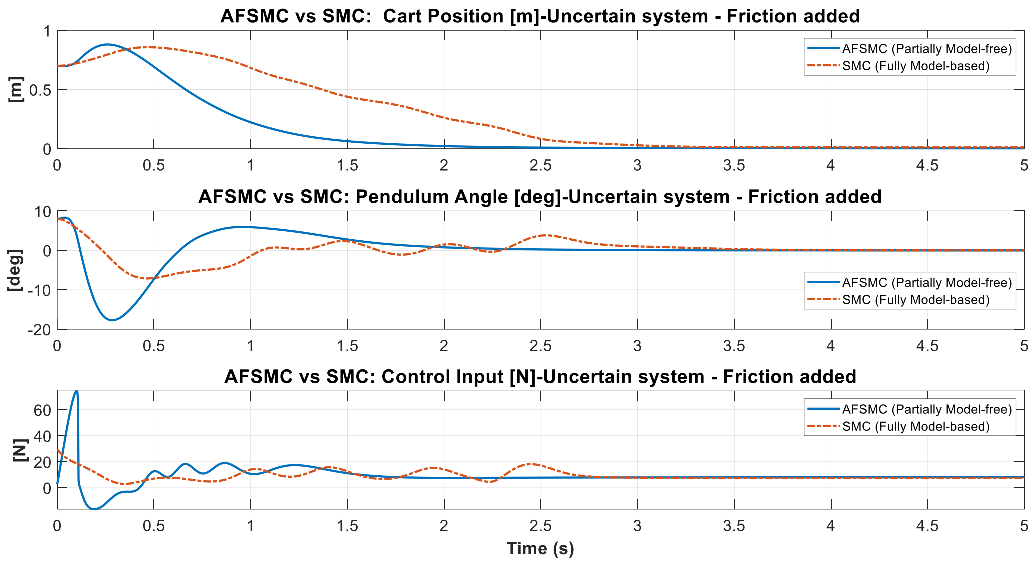

Figure 5, AFSMC provides a relatively better performance in controlling the system and overcoming the friction uncertainty, although, despite the AFSMC, the SMC is utilizing the full information of the dynamical model to generate the control input.

5.2. Uncertain System 2: Nominal System with Pendulum Length Changed

One of the commercial applications of the IPC, is for design and production of the so called two-wheel Segway [**29**]. In order to study the robustness against the height of the human driver, the effect of the length of the pendulum on the performance can considered as a parametric uncertainty. With such motivation, in our study, the length of the inverted pendulum, L, is increased from 0.3729 to 0.5 meter (about 34% change). The behaviors of SMC and AFSMC are then examined

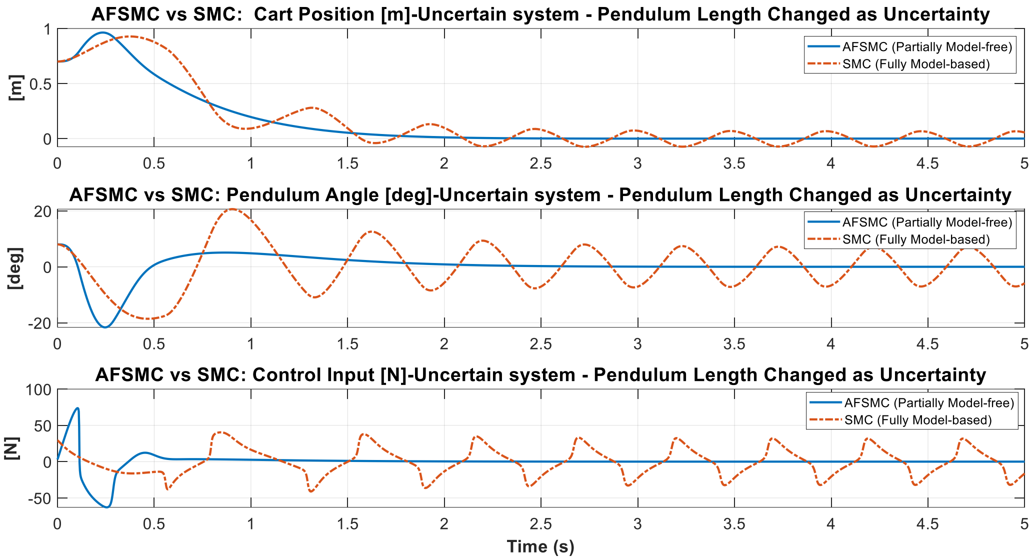

Figure 6.

The superior performance of AFSMC over SMC is evident in the situations where the parameters of the system model change during the control process.

5.3. Uncertainty Bound Estimation: AFSMC Approach

In the conventional SMC, information about the upper bound of uncertainties is a crucial requirement for designing a robust controller. While it is commonly assumed that this information is available, obtaining it in practical scenarios might be challenging. The adaptive approach devised in the AFSMC approach, provides a solution for an indirect estimation of such an upper bound, leading to improved control performance, robustness, and reduced chattering effects. Based on (45) and (48) the time history of the estimated

uncertainty bound, $\hat{\Psi}$, defined as the difference between the equilibrium control generated by the AFSMC and the one generated based on the fully model-based feedback linearization in the SMC approach, is shown in

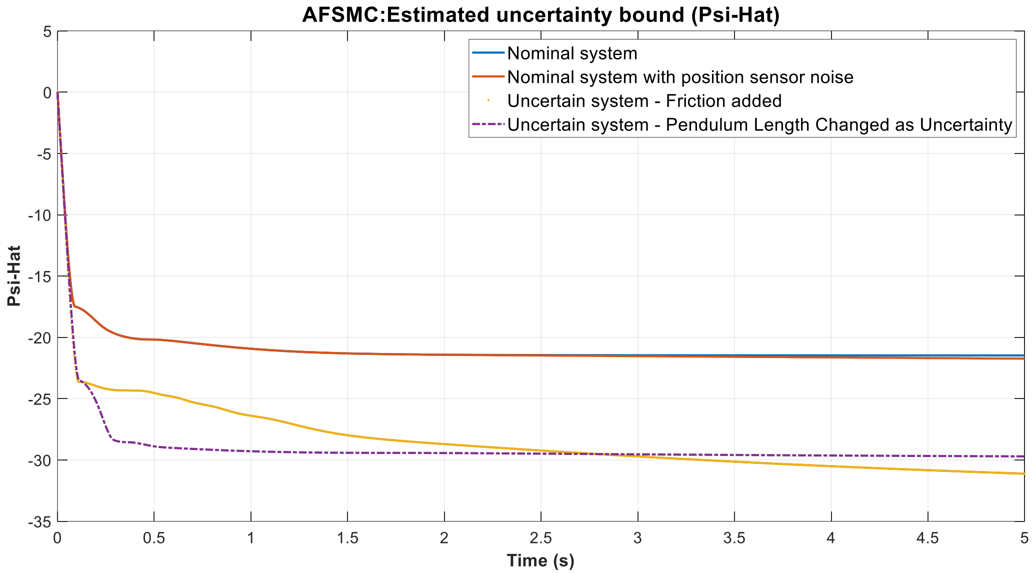

Figure 7.

The key idea is to utilize an adaptive fuzzy system to estimate the bound of uncertainties affecting the system. This estimated bound is then incorporated into the AFSMC switching gain computation to ensure the reachability condition is satisfied despite uncertainties. The adaptive fuzzy approach provides an effective way to learn these bounds online without requiring explicit uncertainty models.

6. Experimental Setup

To validate the proposed approach, an experimental setup consisting of an inverted pendulum on a moving cart is designed and built. A belt drives the moving cart over a 2-meter guide rail. The belt is by a DC servo motor and a planetary gearbox. The rail on which the moving cart moves is deliberately designed to bring in a high and non-linear friction between the cart and rail. Figure 8 depicts the general structure of the setup. Further specifications are shown in Table 3.

7. Experimental Validation

In the experimental implementation, the design parameters for the SMC are set as $\lambda = 6$ and $Q = 8$. Similarly, for the AFSMC, the parameter values are chosen as: $\lambda = 4$, $\alpha_1 = 2$, and $\alpha_2 = 4$ The initial conditions are set equal to $\begin{bmatrix} 0 & 0 & 15\pi/180 & 0 \end{bmatrix}^T$.

The control behavior of the closed-loop system is investigated with two control approaches, SMC and AFSMC. It is shown that in the absence of an accurate mathematical model of the friction, both the SMC and AFSMC approaches can maintain the position of the moving cart and the inverted pendulum to an acceptable level near the equilibrium point.

Figure 9 shows the comparison between the position of the moving cart in two control approaches. The dotted line diagram is for the SMC approaches and the solid-line diagram is for the AFSMC approaches. In this figure, it can be seen that both controllers are able to keep the moving cart near the equilibrium point after 3 seconds. Although the AFSMC approaches is less dependent on the model, it provides a relatively better performance.

Figure 10 shows the comparison between the inverted pendulum angle in two control approaches, SMC and AFSMC. The initial angle was about 15 degrees, and the moving cart was able to keep the inverted pendulum almost in the upright direction after several turns of going back and forth near the equilibrium point. It can be seen that the fluctuations with the SMC is considerably more than the AFSMC.

Figure 11 compares the time history of control inputs for the SMC and AFSMC cases. It can be seen that the presence of nonlinear friction between the moving cart and the rail prevents the control force to converge to zero value.

8. Discussion and Conclusion

In this study, an attempt was made to study the effectiveness of the SMC approach as well as the recently developed direct-AFSMC approach in controlling an inverted pendulum on a moving cart in the presence of highly nonlinear and uncertain friction force. The problem with the dynamic of an inverted pendulum is that it cannot be completely feedback linearized and hence transformed into the regular form so that the SMC-like approach becomes applicable. Instead, the system was approximately linearized and the ignored dynamic was considered as uncertainty which can be more or less handled by the control input.

The comparison of the performance results of these two controllers showed that, although the existence of nonlinear unknown variable friction factor has negative effects on the performance of the closed-loop system, in general, both control approaches were able to control the inverted pendulum and the position of the moving cart to an acceptable extent. It was experimentally revealed that unlike the SMC approach, whose control logic is mainly dependent on the dynamic model, the AFSMC approach used in this paper is considerably less independent on the pendulum dynamic model, and even in the presence of high-friction and large parametric uncertainty, provided a better performance.

Nomenclature

Abbreviation	Term
(AFSMC)	Adaptive-Fuzzy Sliding Mode Control
(CRR)	Constant Rate Reaching
(Direct-AFSMC)	Direct-Adaptive Fuzzy Sliding Mode Control
(DoF)	Degrees of Freedom
(IOFL)	Input-Output Feedback Linearization
(IPC)	Inverted Pendulum on a Moving Cart
(ISMC)	Integral Sliding Mode Control
(LQR)	Linear Quadratic Regulator
(LQRC)	Linear Quadratic Regulator Controller
(PCM)	Predictive Control Model
(SMC)	Sliding Mode Control
(TSMC)	Terminal Sliding Mode Control
(UMS)	Under-actuated Mechanical System

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Biography

Farbood Shokouhi is a Ph.D. candidate in Mechanical and Control Engineering at Iran University of Science and Technology (IUST) in Iran. His research focuses on motion and control, with a particular emphasis on robotics, especially parallel robots, and robust control in nonlinear systems. Shokouhi has published several articles in these areas.

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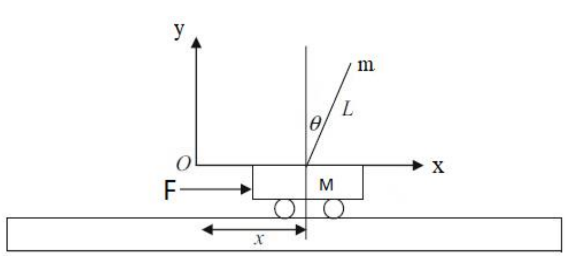


Figure 1. The structure of the inverted pendulum system and moving cart

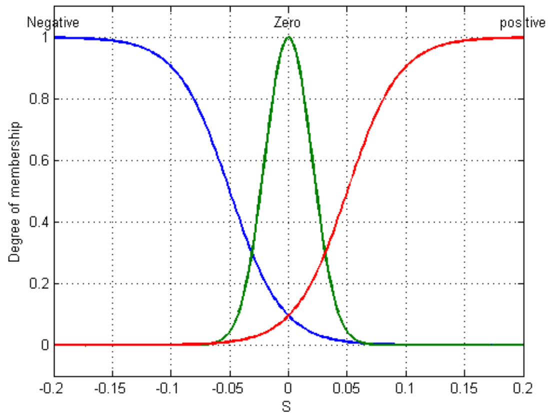


Figure 2 . Input fuzzy membership functions

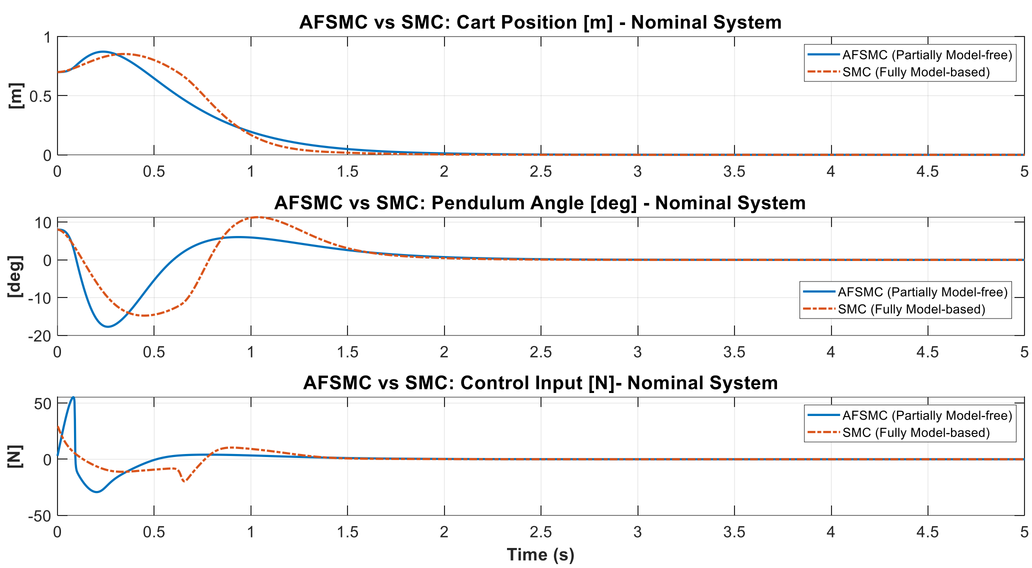


Figure 3 . Comparative Response for Nominal System

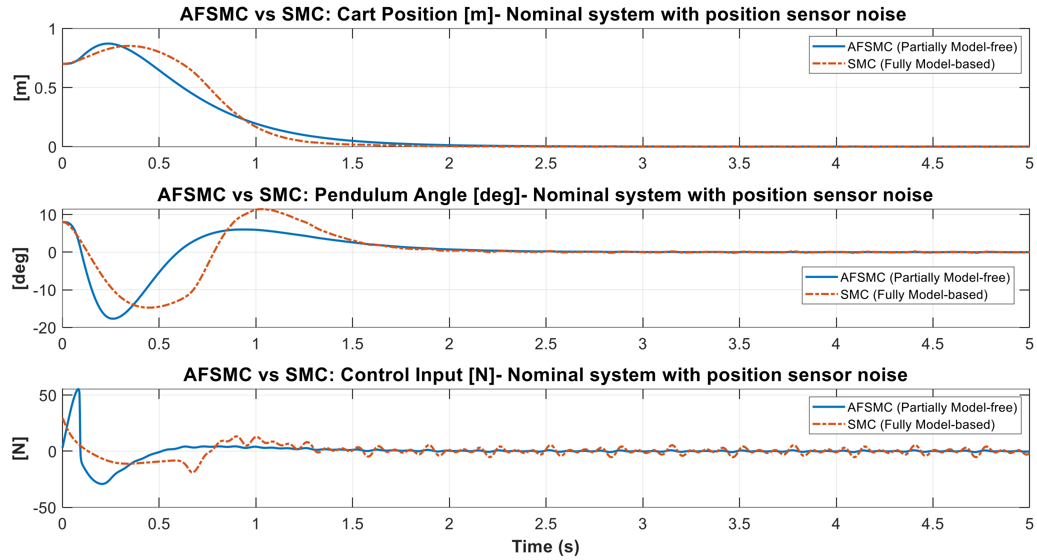


Figure 4 . Comparative Response for Nominal System with Position Sensor Noise

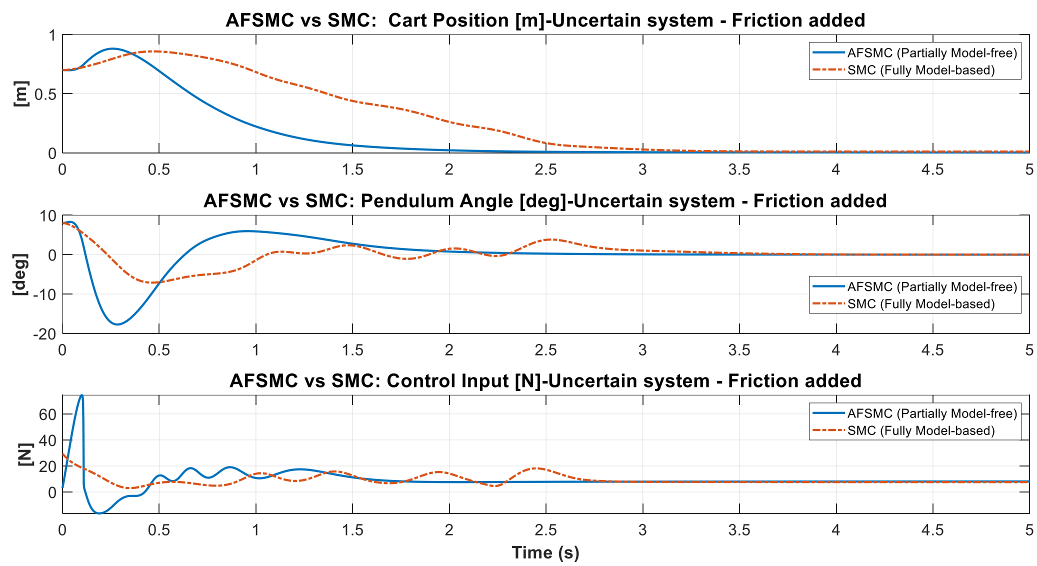


Figure 5 . Comparative Response for System with Friction Force

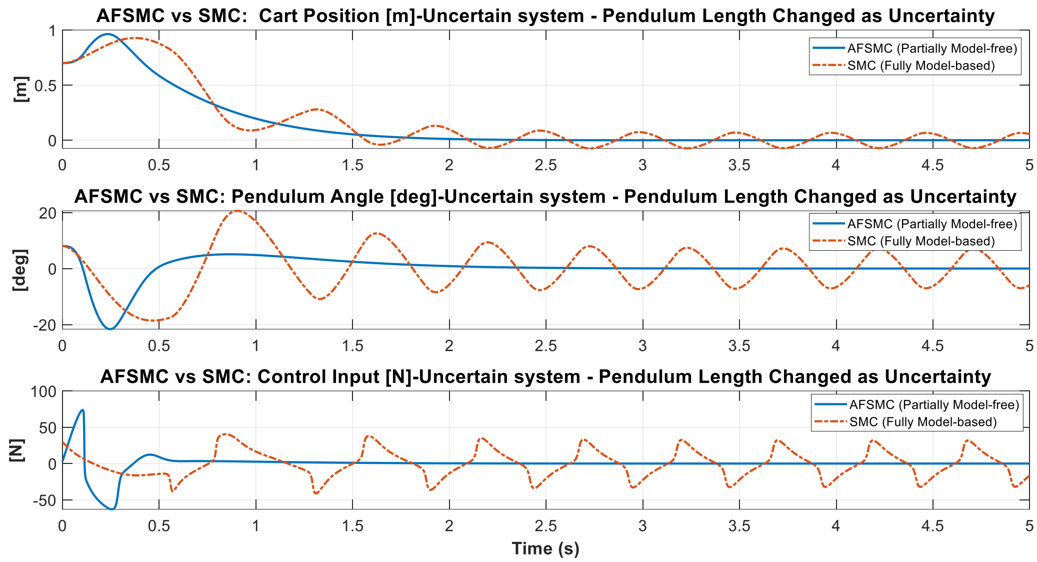


Figure 6 . Comparative Response with Pendulum Length Changed

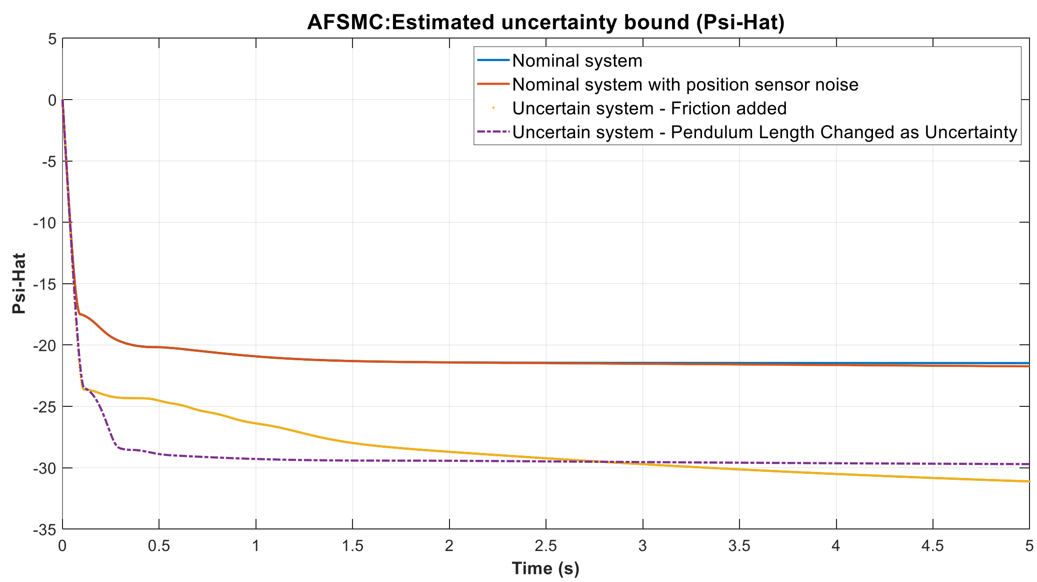


Figure 7 . Estimated Uncertainty Bound in AFSMC

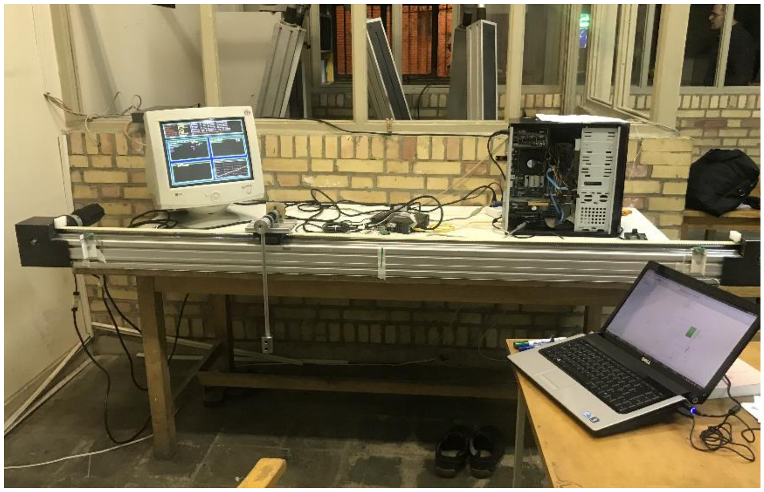


Figure 8 . Inverted pendulum made with high-friction between the moving cart and the rail

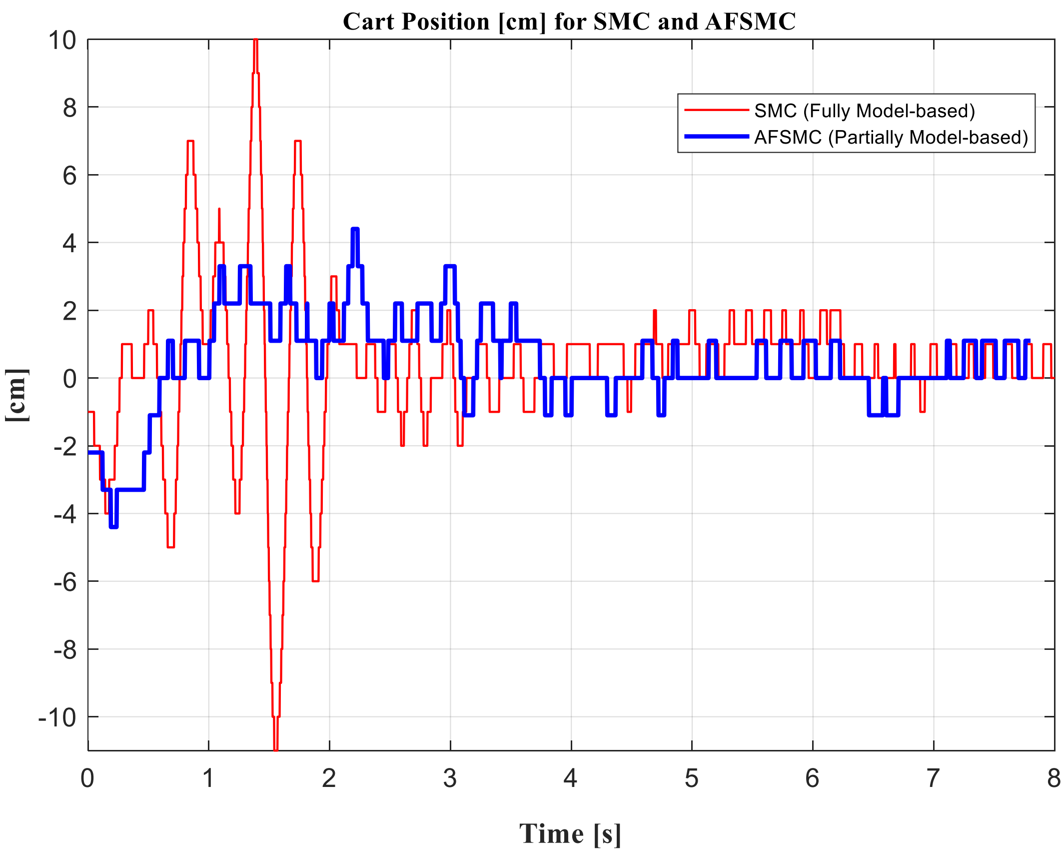


Figure 9. Moving cart position

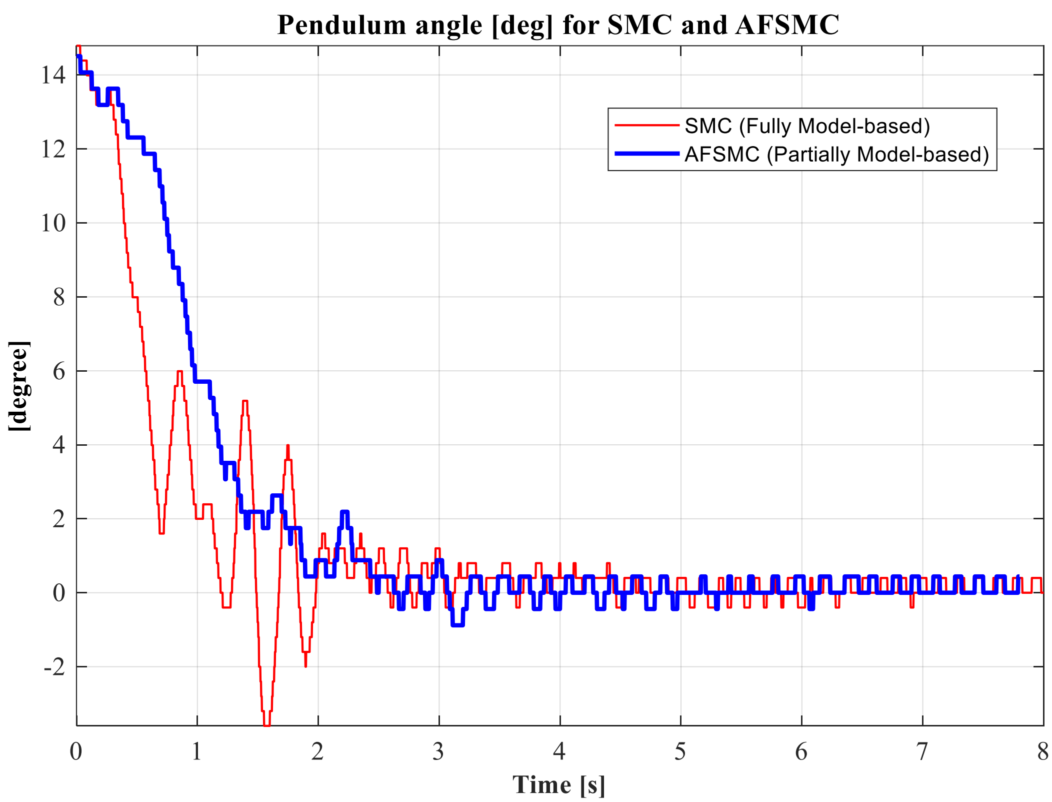


Figure 10. Pendulum angle

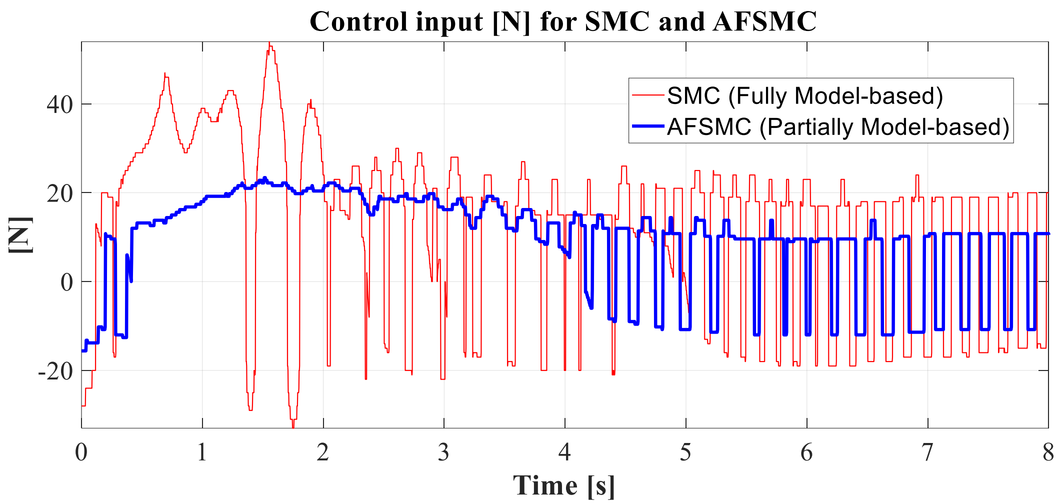


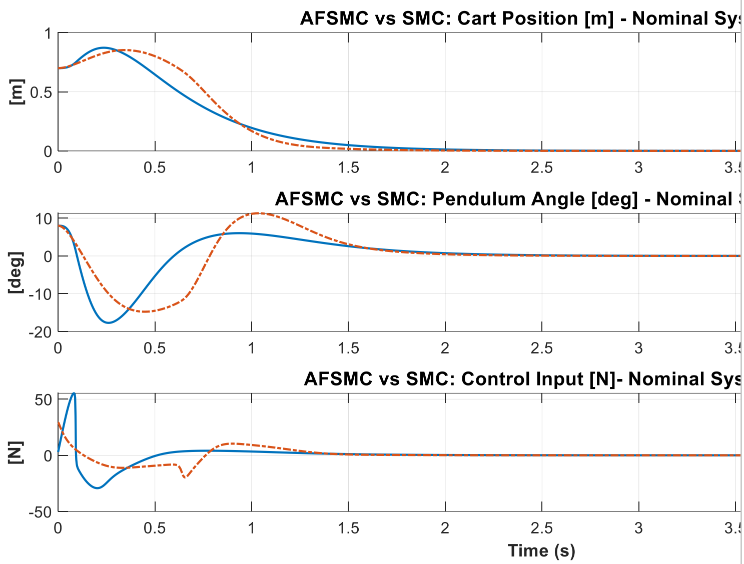
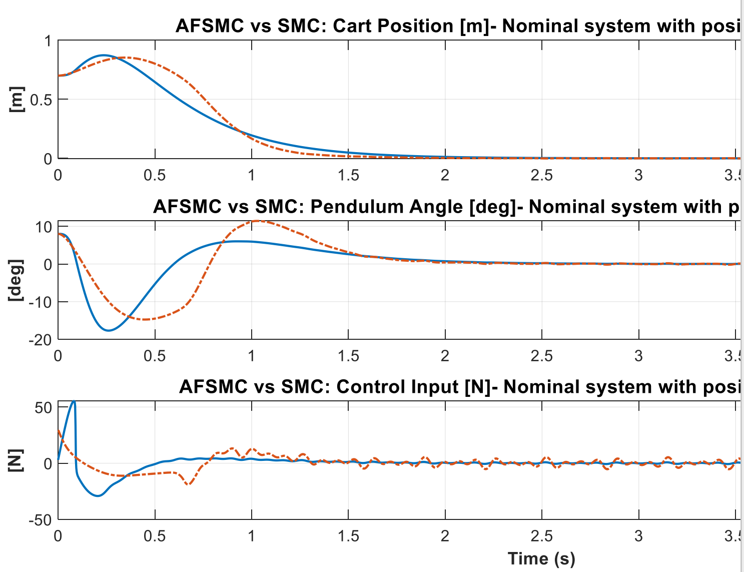
Figure 11. Control Input

Table 1. Summary of selected literature for control of an IPC.

Reference No.	Experimental Verification	Control Approach	Friction force Considered	Angle control only (θ)	Position and angle control (x, θ)
[7]	-	LQR, SMC, ISMC, TSMC	✓	-	✓
[8]	✓	SMC	✓	✓	✓
[9]	-	SMC + MPC	-	-	✓
[14]	-	FSMC + LQRC	✓	✓	✓
[10, 11, 12, 13]	-	AFSMC	-	✓	✓
[6, 15, 16]	-	AFSMC	✓	✓	✓
Proposed Approach	✓	AFSMC	✓	✓	✓

Table 2. A Guide to Simulation System Categories

	Name Syste m	Posit ion Sens or	Frict ion Adde d	Pendu lum Lengt h

		Noise Added		Changed
 Figure 3	Nominal System	-	-	-
 Figure 4	Nominal System with Position Sensor Noise	✓	-	-

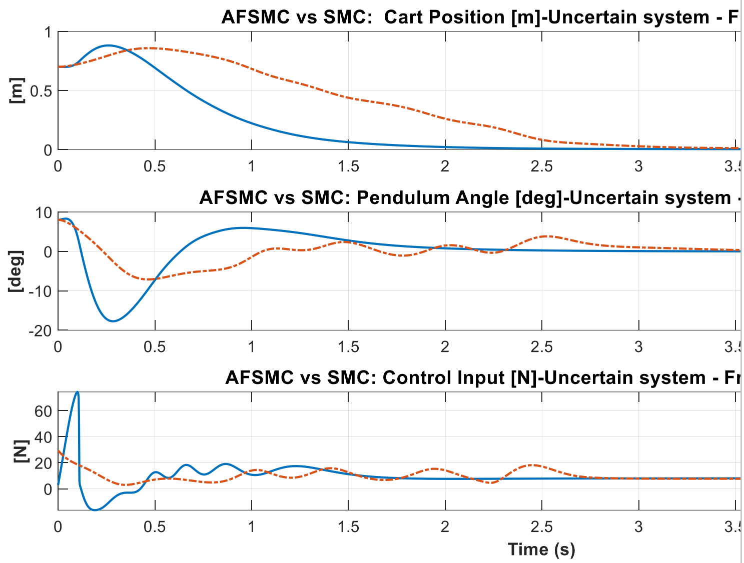
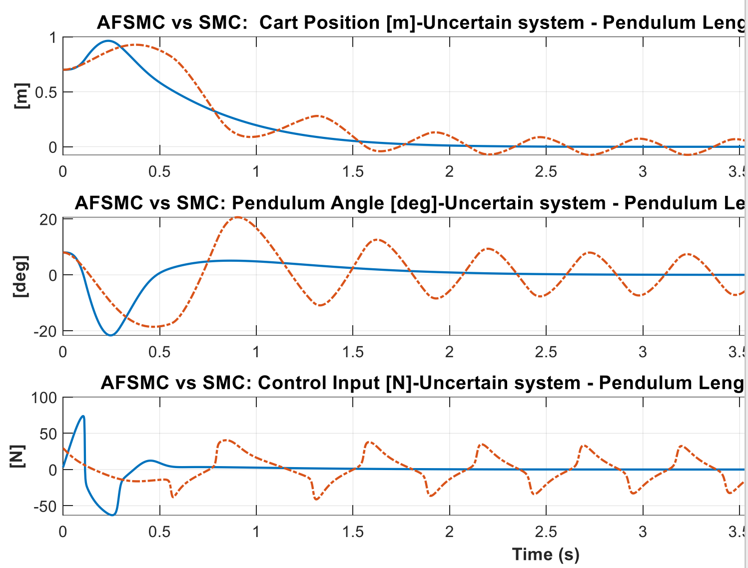
 **Figure 5**	Uncertain System 1: Nominal System + Friction Added	-	✓	-
 **Figure 6**	Uncertain system 2: Nominal System + Pendulum Length Changed	-	-	✓

Table 3. Laboratory system physical characteristics		
Name	**Symbol**	**Quantity and unit of measurement**
Pendulum mass	m	0.650 [Kg]

Moving cart mass	M	2.1 [Kg]
Pendulum length	L	0.3729 [m]
Monetary radius of pendulum base drive belt	R_p	0.03184 [m]
Constant current electric motor torque constant	K_t	0.49 [N.m/A]
Planetary gearbox conversion ratio	Kg	5