The Tenacity of Generalized Petersen Graphs

A. Khoshnood[1*], D. Moazzami[1*] and A. Ghodousian[1]

[1]Department of Algorithms and Computation, School of Engineering Science, College of Engineering, University of Tehran, 16 Azar, Tehran, 1417614411, Tehran, Iran.

*Corresponding author(s):

Tel: +989132955932, E-mail: asieh.khoshnood@ut.ac.ir;

Tel: +989121395926, E-mail: dmoazzami@ut.ac.ir;

Contributing authors: a.ghodousian@ut.ac.ir;

Abstract. Communication networks can be represented as graphs, where vertices represent network nodes and edges represent connections between them. Various graph theory parameters, such as connectivity, toughness, tenacity, binding number, scattering number, and integrity, were presented to assess the vulnerability of networks. Calculating the values of these vulnerability parameters can be challenging, particularly for certain classes of graphs, such as Generalized Petersen Graphs (GPG), due to their diverse structures. This paper establishes upper and lower bounds for the tenacity of GPG. We demonstrate a lower bound of 1 for the tenacity ($T(GPG(n,k))$), across all values of n and k. Additionally, we explore the tenacity values of GPG and present a general upper bound for the tenacity value in this graph type. By using the relationship between the tenacity parameter and the connectivity $\kappa(G)$ and toughness $t(G)$ parameters, we also update some theorems related to the connectivity and toughness of GPG.

Keywords: Tenacity Parameter, Generalized Peterson graphs, Vulnerability Parameters, Graph Theory, Communication Networks, Algorithm

1. Introduction

For our purposes, a graph G is defined as an ordered pair (V, E), where (V) is a finite set of elements known as vertices, and E is a finite set of elements known as edges. A graph G is considered connected if there exists a path between any two vertices within G. An independent set in G refers to a set of vertices in which no two vertices are adjacent. The vertex independence number of G, denoted as $\alpha(G)$ or simply α, represents the maximum number of vertices in an independent set [1, 2].

The vertex connectivity, denoted as $\kappa(G)$, of a finite, undirected, connected, simple graph G (without loops or multiple edges) is the minimum number of vertices that need to be removed to disconnect the graph or result in the trivial graph K_1.

The tenacity of the graph G, which serves as a measure of its vulnerability [3,4,5], was introduced in prior works [6,7]. Cozzens et al. (1994) calculated the tenacity of the first and second cases of the Harary Graphs but did not provide complete proof for the third case. In another work by Moazzami (2011), a new and comprehensive proof for the third case of the Harary Graphs was presented. Moazzami (1999) compared integrity [8], connectivity [9], binding number [10,11], toughness [12,13,14,15], and tenacity for several classes of graphs, suggesting that tenacity is the most suitable measure for stability or vulnerability as it can differentiate between graphs with varying levels of vulnerability. Subsequent studies on this invariant have been conducted by various researchers [6,7,16-41].

In this paper, we restrict our focus to graphs without loops or multiple edges, using $V(G)$, $E(G)$, and $\omega(G)$ to denote the vertex set, edge set, and the number of components in the graph G, respectively. We specifically consider finite undirected graphs and denote the order of G as $|V(G)|$.

The tenacity of a graph G, denoted $T(G)$, is defined as $T(G) = min\{ \frac{|S| + \tau(G - S)}{\omega(G - S)} \}$, where the minimum is taken over all vertex cutsets S of G. Here, $\tau(G - S)$ represents the number of vertices in the largest component of the graph $G - S$, and $\omega(G - S)$ represents the number of components of $G - S$. A connected graph G is referred to as T-tenacious if $|S| + \tau(G - S) \geq T\omega(G - S)$ holds for any subset S of vertices of G with $\omega(G - S) > 1$. If G is not a complete graph, then there exists the largest T such that G is T-tenacious, and this T is defined as the tenacity of G. Conversely, a complete graph contains no vertex cutsets, and thus, it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is considered a T-set of G if $T(G) = \frac{|S| + \tau(G \quad S)}{\omega(G - S)}$. The Mix-tenacity $T_m(G)$ of a graph G is defined as:

$$T_m(G) = min_{A \subset E(G)} \frac{|A| + m(G - A)}{\omega(G - A)}$$

where $m(G - A)$ denotes the order (the number of vertices) of the largest component of $G - A$, and $\omega(G - A)$ is the number of components of $G - A$.

In [7], Cozzens et al. introduced a parameter called Edge-tenacity, which was later renamed Mix-tenacity by Moazzami in [39]. However, Mix-tenacity is a more appropriate name for this parameter, and it has interesting properties, as seen in the expressions for $T(G)$ and $T_m(G)$.

Since the groundbreaking work of Cozzens, Moazzami, and Stueckle in [6,7], several groups of researchers have studied tenacity and related problems. Piazza et al. used $T_m(G)$ as Edge-tenacity in [37,38], but this parameter is a combination of cutset $A \subset E(G)$ and the number of vertices in the largest component $\tau(G - A)$, which means that the number of edges removed is added to the number of vertices in the remaining graph’s largest component. However, this parameter is not very satisfactory as Edge-tenacity. As a result, Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity $T_e(G)$, in [39]. The Edge-tenacity $T_e(G)$ of a graph G is defined as follows:

$$T_e = min_{A \subset E(G)} \frac{|A| + p(G - A)}{\omega(G - A)}$$

The Edge-tenacity $T_e(G)$ of a graph G is defined as the order (number of edges) of the largest component of $G - A$, denoted by $p(G - A)$, where A is an edge cutset in G, and $\omega(G - A)$ is the number of components of $G - A$. As this new measure of vulnerability considers only edges, it is referred to as Edge-tenacity.

The question of whether it is difficult to recognize T-tenacious graphs have been an open problem for several years. In 1992, Moazzami raised this question in his paper [29]. In 2014, Dadvand et al. showed that recognizing T-tenacious graphs are NP-hard for any fixed positive rational number T [20] .

They proved this by reducing a well-known NP-complete problem called "Independent Set" [1] to the problem of recognizing T-tenacious graphs. The "Independent Set" problem asks whether a given graph has an independent set of a certain size.

Their reduction shows that if there exists an algorithm that can recognize T-tenacious graphs in polynomial time, then there also exists an algorithm that can solve the "Independent Set" problem in polynomial time. Since the "Independent Set" problem is NP-hard, this implies that recognizing T-tenacious graphs is also NP-hard.

Therefore, it is unlikely that there exists a polynomial-time algorithm for recognizing T-tenacious graphs unless $P = NP$. This result highlights the computational complexity of this problem and its potential implications in areas such as network design and optimization.

In [22], Heidari and Moazzami proved that if $NP \neq ZPP$, then for any $\varepsilon > 0$, it is impossible to approximate the tenacity of a graph with n vertices within a factor of $\frac{1}{2}\left(\frac{n-1}{2}\right)^{1-\varepsilon}$ in polynomial time.

2. Preliminaries

In this paper we consider only finite, non-complete, undirected and connected graphs without loops or multiple edges. Consider a graph $G=(V,E)$ with n vertices. For each subset of vertices S of G, $\omega(G-S)$ denotes number of connected components of $G-S$ which is obtained by removing S from G. We define $\tau(G-S)$ to be the number of vertices in the largest component of the graph $G-S$. The tenacity, $T(G)$, of the graph G is defined as

$$T(G)=min\left\{\frac{|A|+\tau(G-A)}{\omega(G-A)}: A\subseteq V, \omega(G-A)\geq 2\right\}.$$

In [7], Cozzens et al. proved several basic results about tenacity.

Proposition 2.1: If G is a spanning subgraph of H, then $T(G)\leq T(H)$.

Proposition 2.2: For any graph G, $T(G)\geq \frac{\kappa(G)+1}{\alpha(G)}$.

Proposition 2.3: If G is not complete, then $T(G)\leq \frac{n-\alpha(G)+1}{\alpha(G)}$, where n is the number of vertices in G.

Proposition 2.4: If $m\leq n$ then $T(K_{m,n})=\frac{m+1}{n}$.

Without attempting to obtain the best possible result, we can prove the following relation between $T(G)$ and $t(G)$. This result gives us a number of corollaries.

Theorem 2.1: For any graph G, $T(G)\geq t(G)+\frac{1}{\alpha(G)}$.

Proof: Let $A\subseteq V(G)$ be a t-set and $B\subseteq G$ be a T-set. Then

$$\frac{|B|+\tau(G-B)}{\omega(G-B)}\geq \frac{|B|}{\omega(G-B)}+\frac{1}{\omega(G-B)}\geq \frac{|A|}{\omega(G-A)}+\frac{1}{\alpha(G)}.$$

A graph is an invaluable tool for representing and analyzing relationships, providing numerous applications in problem-solving. Graphs like Tutte, Petersen, and others play a vital role in solving, proving, or disproving specific issues, events, and propositions. The Petersen graph, known for its unique structure with 10 vertices and 15 edges, has been particularly valuable as a counterexample in various open graph problems.

Expanding on this concept, Watkins [42] introduced Generalized Petersen Graphs (GPG), which allows the construction of graphs with 2n vertices, exhibiting a structure akin to Petersen's graph for any natural number n. This development has facilitated the assessment of various graph parameters and vulnerabilities within these graphs, such as Integrity, Toughness, and Tenacity. The primary objective of our study is to explore the distinct states of the GPG and determine their corresponding Tenacity values. Additionally, we aim to analyze the overall state of the graph and establish an upper bound for its Tenacity value. These findings highlight the significance of graph parameters in this field and showcase the potential for further research in this domain.

3. Generalized Petersen Graphs

$GPG(n,k)$ represent graphs consisting of two sets of n vertices denoted as $(u_i, v_i : i = 0,...,n-1)$. These graphs exhibit three sets of edges. The first set $\{(u_i, u_{i+1}) : 0 \le i \le n-1\}$ forms a cycle of n vertices. The second set $\{(u_i, v_i) : 0 \le i \le n-1\}$ connects each of the n vertices u_i to their corresponding vertices v_i. The third set $\{(v_i, v_{i+k}) : 0 \le i \le n-1\}$ establishes connections between the vertices of v_i. The subgraph of $GPG(n,k)$ induced by $\{u_0, u_1, ..., u_{n-1}\}$ is referred to as the outer rim, while the subgraph induced by $\{v_0, v_1, ..., v_{n-1}\}$ is known as the inner rim. An edge of the form (u_i, v_i) is called a spoke. Consequently, the structure of GPG is heavily influenced by the value of k [42].

Figure 1 depicts three examples of $GPG(10,k)$ with different values of $k = 1,2,3$.

4. Vulnerability of the GPG

Extensive research has been conducted on the vulnerability of GPG, focusing on parameters such as Cut Set, Independence Number, and Clique, which are closely associated with their vulnerability. Figure 1 showcases the diverse structures that can be achieved in GPG through the parameter k , posing challenges for analyzing network parameters. To address this challenge, Kilic et al [12]. investigated the Tenacity parameter of a specific GPG, $GPG(n,1)$. Dundar [43] explored the Neighbor-Integrity parameter, while Krnc and Pisanski [44] studied the graph's structure and its connection to Cronoker graphs. The examination of reliability in these graphs was undertaken by Ekinci et al. [45]. Additionally, Ferland [46] evaluated the Toughness parameter in GPG. Several recent studies emphasize the significance of graph parameters and their relationship with the vulnerability of GPG, laying the groundwork for future research in this field [47-51].

Extensive research has been dedicated to exploring the vulnerability of GPG, with a specific focus on parameters such as Cut Set, Independence Number, and Clique, as they are closely associated with graph vulnerability. The parameter k in GPG introduces diverse structures that pose challenges when analyzing network parameters, as demonstrated in Figure 1.

To address these challenges, Kilic et al. [12] conducted an in-depth investigation of the Tenacity parameter in a specific type of GPG, $GPG(n,1)$. Dundar [43] explored the Neighbor-Integrity parameter, while Krnc and Pisanski [44] examined the graph's structure and its connection to Cronoker graphs. Ekinci et al. [45] focused on examining reliability within these graphs, and Ferland [46] evaluated the Toughness parameter. These studies underscore the significance of graph parameters and their relationship with the vulnerability of GPG, laying a solid foundation for further research in this field.

4.1. Lower bound

In [46], Ferland presents findings on the toughness of $GPG(n,\mathrm{k})$. The following theorem from [46] establishes a lower bound for the toughness of GPG.

Theorem 4.1.1 ([46]). For $n \geq 5$ and $n \neq 8$, then $t(GPG(n,2)) \geq \frac{5}{4}$.

Also $t(GPG(3,2)) = \frac{3}{2}$, $t(GPG(4,2)) = 1$, and $t(GPG(8,2)) = \frac{5}{4}$.

Corollary 4.1.1 ([28]). For $n \equiv 0$ mod 7, then $t(GPG(n,2)) = \frac{5}{4}$.

By using Theorem 2.1, we have the following results:

Theorem 4.1.2. For $n \geq 5$ and $n \neq 8$, then $T(GPG(n,2)) \geq \frac{5}{4}$.

Also $T(GPG(3,2)) \geq \frac{3}{2}$, $T(GPG(4,2)) \geq 1$, and $T(GPG(8,2)) \geq \frac{5}{4}$.

Corollary 4.1.2. For $n \equiv 0$ mod 7, then $T(GPG(n,2)) \geq \frac{5}{4}$.

For $GPG(n,\mathrm{k})$, Ferland in [46] presents the following theorem:

Theorem 4.1.3 ([28]). For $n \geq 3$ and $1 < n < k$ then $t(GPG(n,k)) \geq 1$.

Moreover, if n is odd and $gcd(n,k) = 1$, then $t(GPG(n,k)) > 1$.

By using Theorem 2.1, we have the following results:

Theorem 4.1.4. For $n \geq 3$ and $1 < n < k$ then $T(GPG(n,k)) \geq 1$.

Moreover, if n is odd and $gcd(n,k) = 1$, then $T(GPG(n,k)) > 1$.

4.2. Upper bound for Tenacity of GPG

GPG, despite having an equal number of vertices, exhibit diverse structures influenced by the parameter k , which poses challenges in calculating various graph and network parameters.

The objective of this paper is to assess the vulnerability of GPG by presenting an upper bound for the Tenacity parameter. To calculate the Tenacity, we require a set of cut vertices that can partition the graph into components with arbitrary numbers of vertices, independent of the value of k . Figure 2 provides an example of such a cut. In the figure, the set S represents the cut vertices, and the graph divides into at least two sets of vertices, denoted as P and Q. Although the vertices in set Q may have multiple connected components, we consider this set as connected to establish the upper bound. Thus, the sets S, P, and Q can be represented as follows:

$$S = \{u_0, v_1, v_2, \ldots, v_{s\ 2}, u_{s\ 1}) \tag{1}$$

$$P = \{u_1, u_2, \ldots, u_{s\ 2}\} \tag{2}$$

$$Q = \{v_0, v_{s\ 1}, v_s, \ldots, v_{n\ 1}, u_s, u_{s+1}, \ldots, u_{n\ 1}\} \tag{3}$$

and $3 \leq s \leq n - 1$.

Based on this division of vertices and considering $s = |S|$, $p = |P|$ and $q = |Q|$ we have:

$$s = p + 2 \tag{4}$$

$$q = 2n - s - p = 2(n - s + 1) \tag{5}$$

Thus, the Tenacity is as follows:

$$T(G) = \min_{S \subset V(G)} \{f(G,S)) \tag{6}$$

$$f(G,S) = \frac{s + \tau(G - S)}{\omega(G - S)} \tag{7}$$

Considering $\omega(G - S) = 2$ and also $\tau(G - S) = max\{p,q\}$ we have:

$$T(G) \le \min_{S \subset V(G)} \left\{ \frac{s + max\{p, q\}}{2} \right\} \tag{8}$$

As a result, we have:

$$T(G) \le \min_{S \subset V(G)} \left\{ \frac{s + max\{s - 2, 2(n - s + 1)\}}{2} \right\} \tag{9}$$

We consider this problem in two different cases:

First case: $s - 2 \le 2(n - s + 1)$

In this case, we have:

$$s \le 2n - 2s + 4 \Rightarrow s \le \frac{2n + 4}{3} \tag{10}$$

As a result, we will have:

$$\mathrm{T(G)} \le \min_{S \subset V(G)} \left\{ \frac{s + 2(n - s + 1)}{2} \right\} = \min_{S \subset V(G)} \left\{ \frac{2n - s + 2}{2} \right\} = \min_{S \subset V(G)} \left\{ n - \frac{s}{2} + 1 \right\} \tag{11}$$

The value of n is constant in Equation 11. Thus, to minimize Equation 11, it is necessary to maximize the value of s. In Equation 10, the maximum value of s is specified. Therefore, the value of the first upper bound is determined as follows:

$$T(G) \le \min_{S \subset V(G)} \left\{ n - \frac{s}{2} + 1 \right\} = n - \frac{2n + 4}{6} + 1 = \frac{2n + 1}{3} \tag{12}$$

Second case: $s - 2 \ge 2(n - s + 1)$

In this case, we have:

$$s \geq 2n - 2s + 4 \Rightarrow s \geq \frac{2n+4}{3} \tag{13}$$

As a result, we will have:

$$T(G) \leq \min_{S \subset V(G)} \left\{ \frac{s+s-2}{2} \right\} = \min_{S \subset V(G)} \left\{ \frac{2s-2}{2} \right\} = \min_{S \subset V(G)} \{s-1\} \tag{14}$$

To minimize the Relation 14, it is essential to choose the lowest possible value for s. This lower bound for s is determined by the upper bound specified in Relation 13. So we have:

$$T(G) \leq \min_{S \subset V(G)} \left\{ \frac{2n+4}{3} - 1 \right\} = \frac{2n+1}{3} \tag{15}$$

The equality between the values derived from equations Equation 12 and Equation 15 is apparent. Therefore, it is possible to determine the upper bound value for the Tenacity of GPG for all values of k, as indicated by Equation 16.

$$T(G) \leq \left\lceil \frac{2n+1}{3} \right\rceil \tag{16}$$

4.3. Improve the upper bound

To improve the upper bound for the Tenacity of GPG, we consider a scenario where the sets P and Q are fixed, and we introduce c vertices from set P into the cut set S. This process involves removing c vertices from the path in set P, resulting in the creation of $c+1$ components in the graph.

As shown in Equation 2, removing the cut set S divides the graph into two parts: P and Q, where P represents a path. Removing each vertex from this path introduces an additional component to the remaining graph. We exploit this property to improve the upper bound. Initially, we analyze the problem assuming that sets P and Q remain fixed. Subsequently, we strive to achieve the outcome by modifying these sets. With the cut set S held constant, we proceed by adding c vertices from set P to the cut set S:

$$0 \leq c = |C| \leq \left\lfloor \frac{p-1}{2} \right\rfloor \tag{17}$$

Figure 3 is obtained by modifying Figure 2 through the addition of $c = 2$ vertices from set P to set S. As a result, in the optimal scenario, the path P will be divided into approximately $c+1$ components, while, combined with component Q, they will form a total of $c+2$ components. In this case, the tenacity can be expressed using Equation 18.

$$\mathrm{T(G)} \leq \min_{S \subset V(G)} \left\{ \frac{(s+c) + max\left\{q, \frac{p-c}{c}\right)}{c+2} \right\} = \min_{S \subset V(G)} \left\{ \frac{s-2+max\left\{q, \frac{p}{c}-1\right)}{c+2} + 1 \right\} \tag{18}$$

Given that s is constant, the values of q and p also remain constant. As a result, Equation 18 achieves its minimum when c attains its highest value. The maximum value for c can be determined using Equation 17. By leveraging equations Equation 17 and Equation 18, the minimum value can be expressed as demonstrated in Equation 19.

$$T(G) \leq \min_{S\subset V(G)} \left\{ \frac{s - 2 + max\left\{ q, \frac{p}{\frac{p-1}{2}} - 1 \right\}}{\frac{p-1}{2} + 2} + 1 \right\} = \min_{S\subset V(G)} \left\{ \frac{2s - 4 + 2max\left\{ q, \frac{2}{p-1} + 1 \right\}}{p+3} + 1 \right\} \tag{19}$$

Let's delve into a detailed analysis of Equation 19. Figures 2 and 3 illustrate that removing two vertices from a set Q is equivalent to adding one vertex to a set P and one vertex to set S. Equation 19 implies that reducing the number of vertices in Q leads to a smaller value for the equation. Consequently, we expect that the minimum value of Equation 19 will be achieved when Q has its minimum size. Moreover, maximizing the value of c ideally results in the largest component in the set P containing either one or two vertices, depending on whether P has an odd or even number of vertices. Therefore, we need to consider two scenarios: one for odd n and another for even n.

Let's start with the case where n is an odd number. In this scenario, we can observe the following:

- The cut set consists of

$$S \cup C = \left\{ \{v_{2k}\}, \{u_t\} : k = 0, \ldots, \frac{n-1}{2}, t = 1, \ldots, n-2 \right\}$$

- The number of vertices in the cut set is equal to

$$s + c = |S \cup C| = \frac{n+1}{2} + n - 2 = \frac{3n-3}{2}$$

- The largest component may have 2 vertices (indicating that Q is connected), and the smallest components are

$$\Omega = \left\{ \{u_0, u_{n-1}\}, \{v_{2k+1}\} : k = 0, \ldots, \frac{n-3}{2} \right\}$$

- The minimum number of components is equal to

$$\omega(G - S - C) = |\,\Omega\,| = \frac{n+1}{2}$$

Hence, the new upper bound for the tenacity relation will be expressed in the form of relation Equation 20:

$$T_{\text{Discrete}}(G) = T(G) \leq \frac{\frac{3n-3}{2} + 2}{\frac{n+1}{2}} = \frac{3n+1}{n+1} = 3 - \frac{2}{n+1} \tag{20}$$

Now let's consider the second case where n is even. In this scenario, we have:

- The cut set consists of

$$S \cup C = \left\{ \{v_{2k}\}, \{u_t\} : k = 0 \ldots \frac{n-2}{2}, t = 1 \ldots n-1 \right\}$$

- The number of vertices in the cut set is equal to

$$s + c = |\,S \cup C\,| = \frac{n}{2} + n - 1 = \frac{3n-2}{2}$$

- The largest component has 1 vertex, and the components are

$$\Omega = \left\{ \{u_0\}, \{v_{2k+1}\} : k = 0 \ldots \frac{n-2}{2} \right\}$$

- The number of components is equal to

$$\omega(G - S - C) = |\,\Omega\,| = \frac{n+2}{2}$$

Therefore, in this case, the new upper bound for the tenacity relation will be expressed in the form of relation Equation 21:

$$T(G) \le \frac{\frac{3n-2}{2}+1}{\frac{n+2}{2}} = \frac{3n}{n+2} = 3 - \frac{6}{n+2} \tag{21}$$

It is evident that as the value of n increases, the values obtained in equations Equation 20 and Equation 21 are significantly smaller compared to the value calculated in Equation 16. Hence, equations Equation 20 and Equation 21 provide a tighter upper bound for the Tenacity value in GPG.

Figures 4 and 5 show the upper bounds ($T(GPG(7, 4)) \le 2.75$ and $T(GPG(10, 3)) \le 2.5$) and optimal cuts ($T(GPG(7, 4)) = 2$ and $T(GPG(10, 3)) = 1.1$) for graphs GPG (7, 4) and GPG (10, 3), respectively. As can be seen, it is difficult to distinguish the cut set to calculate the Tenacity values in these two graphs, but in the case of the upper band, the cut set follows a rule that is not dependent on the values of k for GPG (n, k) graphs. Now, by using the upper bands and Proposition 2.2 and Theorem 2.1, With the premise that for all $n \ge 1$, $3 - \frac{6}{n+2} < 3 - \frac{2}{n+2}$, this upper bound can be used as corollaries 4.3.1 and 4.3.2.

Corollary 4.3.1: $3 - \frac{2}{n+1} \ge T(GPG(n,k)) \ge \frac{\kappa(GPG(n,k))+1}{\alpha(GPG(n,k))}$

Corollary 4.3.2: $3 - \frac{2}{n+1} \ge T(GPG(n,k)) \ge t(GPG(n,k)) + \frac{1}{\alpha(GPG(n,k))}$

In addition, the bands obtained in theorems 4.1.2 and 4.1.4 will also be updated in the form of corollaries 4.3.3 and 4.3.4.

Corollary 4.3.3: For $n \equiv 0 \mod 7$, then $3 - \frac{6}{n+2} \geq T(GPG(n,2)) \geq \frac{5}{4}$.

Corollary 4.3.4: For $n \geq 3$ and $1 < n < k$, then $3 - \frac{2}{n+1} \geq T(GPG(n,k)) \geq 1$.

5. Conclusion

Determining the values of key graph parameters can be a formidable challenge, particularly when dealing with specialized or complex network structures. In this work, we have demonstrated an effective approach to overcome these difficulties by leveraging upper bounds and approximations as a foundation, and then systematically refining and improving these initial estimates.

Much like the process of developing and enhancing approximate or randomized algorithms, our methodology has yielded promising results in the context of calculating the Tenacity of GPG. By establishing an upper bound for the Tenacity of these graphs and drawing upon previous theorems and propositions relating Tenacity to other important parameters such as toughness and connectivity, we were able to derive upper bounds for these related measures as well.

The versatility of this approach holds significant promise, as it can be readily extended to the calculation of diverse graph parameters across a wide range of graph classes. Through iterative refinement, whereby we incrementally increase lower bounds and decrease upper bounds, we can steadily converge on more accurate and precise values for the complex structural characteristics of specialized networks like the GPG.

Moving forward, this robust framework opens up avenues for further exploration and optimization. By continually enhancing our understanding of the relationships between various graph-theoretic measures, we can unlock new insights and accelerate the analysis of intricate network topologies. Ultimately, this work contributes a powerful tool for the graph theory community, empowering researchers and practitioners to tackle the challenging task of parameter calculation with greater efficiency and precision.

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Biographies

Asieh Khoshnood was born in Iran in 1985. She is a PhD Candidate at the Department of Algorithms and Computation, School of Engineering Science, University of Tehran. He holds an MSc degree in Computer Engineering from the Islamic Azad University and also a BSc degree in the same field from the Shahid Bahonar University of Kerman. His research interest is Network Vulnerability and Graph Theory.

Dara Moazzami He is a theoretical computer scientist and professor of Mathematics at the University of Tehran. He received a PhD in Applied Mathematics from the University of Northeastern- Boston MA. U.S.A. in 1992, under the supervision of Professor Margaret Cozzens, and a B.Sc in Pure Mathematics in 1976, and M.Sc in Applied Mathematics in1978, from University of Quebec – Montreal Canada. His research area is in algorithms and complexity theory, specifically Vulnerability in Networks and tenacity parameter. The concept of graph tenacity was introduced by him as a measure of network vulnerability and reliability. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed.

Amin Ghodousia He is Professor of Mathematics at University of Tehran. He received a PhD in Optimization and network theory from the Amirkabir University of Technolog in 2008, and a M.Sc in Applied Mathematics from the Amirkabir University of Technolog in 2003, and B.Sc in Applied Mathematics from The University of Guilan in 2001. His research area is in Optimization Algorithms and Graph Theory

Figure 1. Three different *GPG* with 20 vertices and different *k*

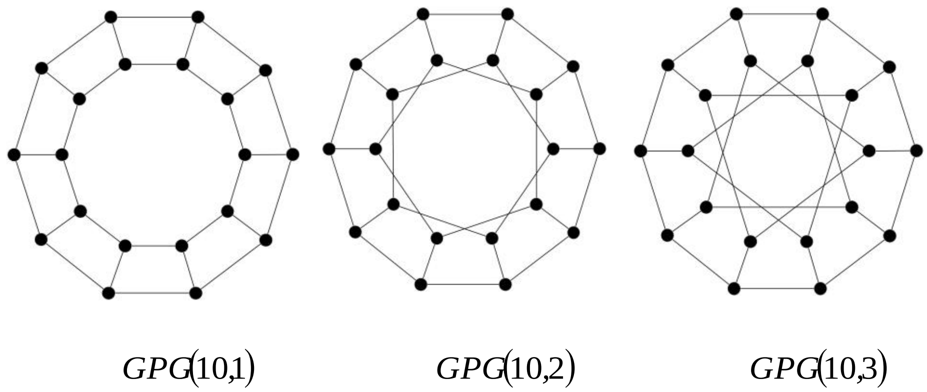

Figure 2. $GPG(10,1)$ with its cut set S (Dashed area)

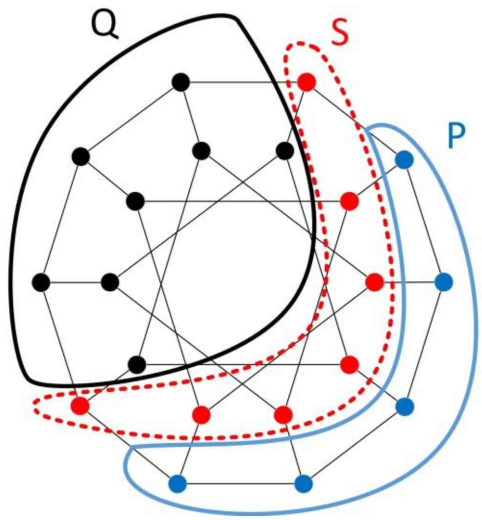

Fig. 3: $GPG(10,3)$ with its cut set S plus some vertices in P

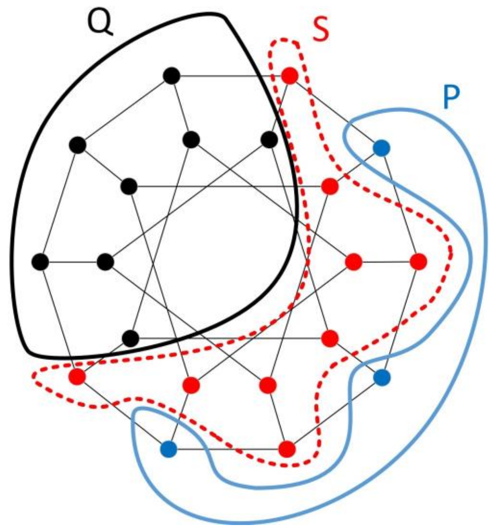

Figure 4. Upper bound and optimal cut for GPG (7, 4)

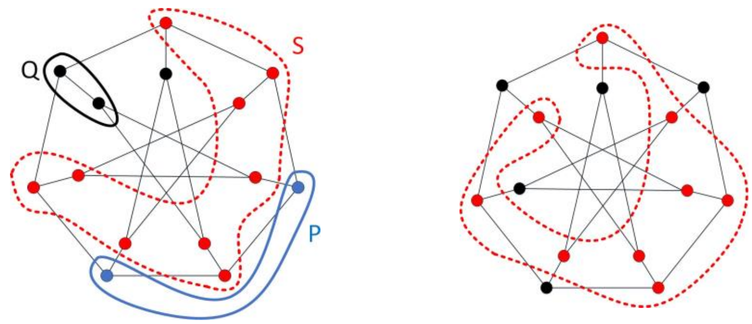

Figure 5. Upper bound and optimal cut for GPG (10, 3)