A Simple Iterative Method for Dynamic Analysis of MDOF Systems with Arbitrary Time-Varying Coefficients Based on Successive Differentiation

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Abstract:
In this article, a novel iterative method is introduced to dynamic analysis of multi-degree-of-freedom systems in which all characteristics of the system could be simultaneously changed with respect to time. The proposed scheme is based on the differentiation of the original equation of motion, on the contrary to the conventional approaches which are usually based on the integration of the motion equation. The numerical investigation of the present method was carried out by analyzing the three systems with one, two and three degrees of freedom. Moreover, the obtained results were comprehensively verified by two distinct approaches: a system with invariant characteristics, and a system with varying mass and stiffness, with closed-form solution. It should be noted that some closed-form solutions of multi-degree-of-freedom (MDOF) time-varying systems were derived and presented for the first time in this article.

Keyword: Free vibration; MDOF systems; Successive differentiation; Time-varying coefficient; Taylor series expansion.

1. Introduction

As is well known, the dynamic behavior of linear and nonlinear structures with time-invariant properties has been comprehensively studied during the past several decades [1–6]. However, in many real phenomena, one or more main characteristics of the structures such as pumps, centrifuges, rockets, turbines, cables and aircrafts are variable. In addition, some new structural vibration control devices such as resettable variable stiffness tuned mass dampers [7], variable stiffness structural joints [8], variable stiffness bracing systems [9] and nonlinear adjustable dynamic vibration absorbers [10] are widely utilized to reduce the structural responses. In general, the governing differential equation of a linear time-varying (LTV) system with N degrees of freedom can be derived by means of spatial discretization of Hamilton's law of variable mass [11] or employing Newton's balance of momentum axiom [12], and is given as:

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$$\boldsymbol{M}(t)\ddot{\boldsymbol{x}}(t)+\boldsymbol{C}(t)\dot{\boldsymbol{x}}(t)+\boldsymbol{K}(t)\boldsymbol{x}(t)=\boldsymbol{F}(t), \tag{1}$$

in which $\boldsymbol{x}(t)$ is the vector of displacement and a dot over a variable represents differentiation with respect to time, therefore $\dot{\boldsymbol{x}}(t)$ is the vector of velocity and $\ddot{\boldsymbol{x}}(t)$ is the acceleration vector. The load vector, $\boldsymbol{F}(t)$, contains external active forces on the original structure, $\boldsymbol{F}_{\text{ext}}(t)$, and additional force caused by the varied mass (expelled or gained). If the relative velocity between the varied mass and the inertial reference frame is represented by $\boldsymbol{u}(t)$ [13], then:

$$\boldsymbol{F}(t)=\boldsymbol{F}_{\text{ext}}(t)+\dot{\boldsymbol{M}}(t)\boldsymbol{u}(t). \tag{2}$$

Furthermore, $\boldsymbol{M}(t)$ is a symmetric and positive definite time-varying (TV) mass matrix, and $\boldsymbol{C}(t)$ and $\boldsymbol{K}(t)$ are symmetric and positive definite or positive semi-definite TV matrices of damping and stiffness, correspondingly. The damping matrix consists of two parts as follows [13]:

$$\boldsymbol{C}(t)=\boldsymbol{C}_{\text{vis}}(t)+\dot{\boldsymbol{M}}(t). \tag{3}$$

Here, $\boldsymbol{C}_{\text{vis}}(t)$ is the viscous damping of the principal system. Considering the free vibration of the system and substituting of Eqs. (2) and (3) into Eq. (1), one obtains:

$$\boldsymbol{M}(t)\ddot{\boldsymbol{x}}(t)+\left(\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\dot{\boldsymbol{x}}(t)-\dot{\boldsymbol{M}}(t)\boldsymbol{u}(t)+\boldsymbol{K}(t)\boldsymbol{x}(t)=\boldsymbol{0}. \tag{4}$$

The initial conditions of the aforementioned systems are known as:

$$\boldsymbol{x}(0)=\begin{Bmatrix} x_1(0)\\ x_2(0)\\ \vdots\\ x_N(0)\end{Bmatrix}=\begin{Bmatrix} x_{1,0}\\ x_{2,0}\\ \vdots\\ x_{N,0}\end{Bmatrix},\quad \dot{\boldsymbol{x}}(0)=\begin{Bmatrix} \dot{x}_1(0)\\ \dot{x}_2(0)\\ \vdots\\ \dot{x}_N(0)\end{Bmatrix}=\begin{Bmatrix} \dot{x}_{1,0}\\ \dot{x}_{2,0}\\ \vdots\\ \dot{x}_{N,0}\end{Bmatrix}. \tag{5}$$

However, many engineering systems have multiple degrees of freedom, some cases such as an open-hearth furnace, the feed equipment of a blast furnace [14] or a pair of spur gears with a low contact ratio [15] can be simplified as a LTV single-degree-of-freedom (SDOF) system. A literature review indicates that in these cases an SDOF model despite its simplicity, can give accurate results [16]. In most of these investigations, analytic solutions for two special cases, namely, $\boldsymbol{u}(t)=\boldsymbol{0}$ and $\boldsymbol{u}(t)=\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t$ were obtained [17]. The first condition is relevant to a case in which the absolute velocity of expelled or gained mass is negligible when compared with the velocity of the original vibrating system. The second condition indicates a case in which the two velocities are equal.

It is worth mentioning, that free oscillation of SDOF systems with periodic time-dependent coefficients has been widely studied in recent decades [18–21]. In general, these equations can be reduced to Mathieu's or Hill's equation [12]. In another research, the free vibration analysis and stability of a linear SDOF oscillator with periodically and stepwise changing TV mass were investigated [22]. Afterward, Núñez and Torres [23] generalized the results of this study to consider forced and nonlinear conditions. They derived the stability criteria by applying adequate change of variables which leads the governing equation to a Newtonian equation. They also used the Floquet theory for linear cases and the Kolmogorov–Arnold–Moser (KAM) method for nonlinear cases. Furthermore, Horssen and Pischanskyy [24] considered both forced and free vibration of a damped linear SDOF oscillator with periodically and stepwise changing TV mass. In this study, the minimal damping rate in which the oscillator always remains stable has been determined. In a more recent study, the exact solution for motion and the criteria of stability of a linear SDOF oscillator with periodic piecewise TV mass for two different cases have been analytically derived by Zukovic and Kovacic [25]. In the first case, mass increases and then decreases linearly in time. In the second case, mass changes trapezoidically.

Despite all these researches, the governing differential equation of many engineering systems has non-periodically TV coefficients [26]. For instance, the feed equipment of a blast furnace can be regarded as an SDOF system with non-periodically TV mass and constant stiffness. A linearly varying length pendulum is another case in which the mass is constant and the stiffness is non-periodically TV. Moreover, deployable/retractable damped cantilever beams due to their application in the robotic field [27], the vibration of strings with TV lengths [28], inherent stability analysis of the systems with variable-stiffness springs [29], the influence of cutting vibration on the machining accuracy of thin-walled parts [30] and time-varying vehicle-bridge interaction systems [31] have received considerable attention in the literature.

The exact analytical solution of Eq. (4) for an SDOF system has been found in some special cases. Konofin [32] investigated a simplified case with a linear stiffness function of time. Prior to this research, Kolenef [33] considered power functions for mass and stiffness, namely, $m(t)=A_1 t^{\mu_1}$ and $k(t)=A_2 t^{\mu_2}$, in which A_i and μ_i are constant. The governing differential equation, in this case, has the following form which does not contain $\dot{m}(t)\dot{x}(t)$:

$$m(t)\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}+k(t)x(t)=0\,. \tag{6}$$

In another study, Li [26] considered an undamped SDOF system with $u(t)=0$ and a general function for mass and stiffness in which mass was an arbitrary continuous TV function, and the stiffness variation was functionally related to the mass variation and vice versa. Hence, the Eq. (4) can be reduced to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]+k(t)x(t)=0\,. \tag{7}$$

The corresponding exact solution of the foregoing equation in six special cases was obtained by using appropriate functional transformation. In this way, Eq. (7) can be transformed into Bessel's equation or other solvable ordinary differential equations with constant coefficients. In another study, Li et al. [12] took into account two different cases for $u(t)$. The former is $u(t)=0$ which results in Eq. (7) and the latter is $u(t)=\mathrm{d}x(t)/\mathrm{d}t$ which gives Eq. (6). Moreover, two different cases for the variation of mass and stiffness were considered as $m(t)=a(1+bt)^c$, $k(t)=\alpha(1+\beta t)^\gamma$ and $m(t)=a\mathrm{e}^{bt}$, $k(t)=\alpha\mathrm{e}^{\beta t}$ which are more appropriate functions in describing certain systems in engineering applications. Later, a more convenient recurrence formula was determined for six important cases of SDOF systems with arbitrary multi-step non-periodic TV mass and functionally related stiffness by Li [14]. In this investigation, the mass and stiffness in each interval can be described by a continuous real function. Next, Li [34] computed the exact solution of six different forced cases of such SDOF systems. Moreover, Li [35] developed the closed-form solution for 10 different cases with continuous TV mass and stiffness. As it is apparent up to now, all aforesaid closed-form solutions were proposed for SDOF systems and there is no exact solution for multi-degree-of-freedom (MDOF) systems in the literature.

It should be noticed, however the exact solutions which were found in aforesaid investigations are very useful, but they are limited to only a few classes of systems. In addition, most of previous studies isolated two special cases of free vibration separately, namely, $u(t)=0$ and $u(t)=\mathrm{d}x(t)/\mathrm{d}t$. To address this predicament, Nhleko [17] considered the velocity of the deduced mass proportional to the velocity of the original system, i.e. $u(t)=\kappa\,\mathrm{d}x(t)/\mathrm{d}t$ in which κ is a scalar value. However, a special case with linear TV mass and constant damping and stiffness was considered in this numerical investigation, as follows:

$$m(t)\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+(1-\kappa)\frac{\mathrm{d}m(t)}{\mathrm{d}t}\frac{\mathrm{d}x(t)}{\mathrm{d}t}+c\frac{\mathrm{d}x(t)}{\mathrm{d}t}+kx(t)=0\,. \tag{8}$$

It is worth mentioning, that in the works of Cveticanin [36] and Flores et al. [37] a special case of Eq. (8) with linearly decreasing mass and $u(t)=\mathrm{d}x(t)/\mathrm{d}t$ which means $\kappa=1$ was considered. In the last work, the exact solution was compared to experimental results. Moreover, Hassanpour [38] computed approximate analytical solutions for two other special cases of Eq. (8) with $\kappa=0$, no damping and linearly or exponentially variation of mass which describes the governing equation of load-carrying conveyor belts or resonant biosensors. In another work, three classical perturbation methods have been used in studying the free vibration of LTV systems [39]. The governing equation in this work was taken as $\ddot{u}+\omega^2(t)u=0$.

It is notable that some practical phenomena can be adequately described by differential equations with TV coefficients. Nhleko et al. [40] represented a pseudo-variable mass system for describing the load impulse and the treatment of structure-jumper interaction which is important in some structures such as stadiums. Finally, the responses of the method were verified by experimental data. In another remarkable work, on the basis of simple temporal supply and demand functions, an analogy between volatile stock market dynamics across an economic model and the spring motion in which mass increases linearly in time was established and instructions for better managing oil reserve demand and supply were given by Canessa [41]. In another study, the dynamic behavior of flexible rectangular fluid containers with decreasing liquid was investigated [42]. The virtual mass method (VMM) and finite element method (FEM) were utilized to derive the dynamic equation of the container. It was shown that the decrease of liquid could be treated as additional negative damping and leads to an increase in the frequency of vibration of the system. The vibration control of a flexible beam with a TV mass using piezoelectric actuators has been studied by Ma et al. [43]. In this paper, the linear quadratic Gaussian control strategy was utilized for the suppression of transverse vibration of the system. In a more recent research, the vibration of a rotor with varying mass was described by a two-degree-of-freedom (TDOF) system and the numerical responses were given by a more accurate procedure [44]. In another research, the transverse vibration of moving beams featuring time-varying velocity subjected to a self-excited force moving along with the end, subject to general initial conditions has been studied [45].

As mentioned before, a few classes of systems with TV parameters were analytically studied in the past. To overcome this shortcoming, several numerical methods have been developed over the past decades. The Galerkin method was used to obtain an approximate solution of cantilever beams with TV length by Tabarrok et al. [46]. Sanders [47] studied the vibration of time variable-length strings by means of the finite difference method. In two other investigations, the dynamic behavior of cables with TV length was analyzed by FEM [28, 48]. An efficient transient analysis technique for linear TV structures by employing the Newmark-beta scheme and multi-level sub-structuring method was presented by Zhao and Yu [13]. They verified the proposed method by investigating the free and forced vibration of some structural systems including a steel structure subjected to fire which is a practical example of TV stiffness. In another investigation, Hozhabrossadati and Aftabi Sani [49] found a semi-analytical solution for the governing motion differential equation of structures by applying the differential transform method (DTM). Recently, Chen et al. [50] proposed a new recursive formula of the Wentzel–Kramers–Brillouin (WKB) approximation for linear TV dynamic systems. In this work, the effect of the time step on the calculation accuracy and the effect of the system parameters changing rate were investigated.

In the foregoing section, first, the dynamic equilibrium equation of systems with TV characteristics was reviewed. Then, a brief literature survey was performed and it was shown that the analytical solution for MDOF systems with TV characteristics has not been investigated thoroughly. The remainder of the present article will introduce an iterative method based on Taylor's series and successive differentiation of the governing equation of the system. Subsequently, the effectiveness of the proposed method is verified extensively by analyzing multiple SDOF and MDOF systems and comparing the numerical responses to existing closed-form solutions and additional solutions derived in this study.

2. Methodology

As it is well known, the recursive formulas of many numerical methods, such as the Newmark-beta method and DTM, are based on omitting higher-order derivatives of the Taylor series expansion. In this way, truncation errors may occur when an approximate mathematical procedure is used for solving a specific problem [51]. To overcome this obstacle, in this section, a recursive scheme is presented to allow using higher-order derivatives in the solution series. First, the Taylor series for an infinitely continuously differentiable function of one variable may be written as:

$$\boldsymbol{x}(t)=\boldsymbol{x}(t_0)+(t-t_0)\dot{\boldsymbol{x}}(t_0)+\frac{t-t_0}{2}\ddot{\boldsymbol{x}}(t_0)+\cdots=\sum_{i=0}^{\infty}\frac{(t-t_0)^i}{i!}\frac{\mathrm{d}^i x(t)}{\mathrm{d}t^i}\bigg|_{t=t_0}. \tag{9}$$

In actual applications, the number of terms in the series solution is limited to n. In addition, the point t_0 is usually assumed to be zero. By applying these two conditions to Eq. (9), one can write:

$$\boldsymbol{x}(t)=\sum_{i=0}^{n}\frac{t^i}{i!}\frac{\mathrm{d}^i x(t)}{\mathrm{d}t^i}\bigg|_{t=0}. \tag{10}$$

The first two parameters of the aforementioned relation are initial conditions of the system, and were given in Eq. (5). Recall that the motion equation of the system is as follows:

$$\boldsymbol{M}(t)\ddot{\boldsymbol{x}}(t)+\left(\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\dot{\boldsymbol{x}}(t)-\dot{\boldsymbol{M}}(t)\boldsymbol{u}(t)+\boldsymbol{K}(t)\boldsymbol{x}(t)=\boldsymbol{0}\,. \qquad \text{Rep. (4)}$$

Therefore, the next term, $\ddot{\boldsymbol{x}}(t)$, can be calculated according to Eq. (4) as follows:

$$\ddot{\boldsymbol{x}}(t)=-\boldsymbol{M}(t)^{-1}\left(\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\dot{\boldsymbol{x}}(t)+\boldsymbol{M}(t)^{-1}\dot{\boldsymbol{M}}(t)\boldsymbol{u}(t)-\boldsymbol{M}(t)^{-1}\boldsymbol{K}(t)\boldsymbol{x}(t)\,. \tag{11}$$

Obviously, one can obtain $\ddot{\boldsymbol{x}}(0)$ by setting $t=0$ in the previous equation. Next, differentiating of Eq. (4) with respect to time, t, results in the following relation:

$$\begin{aligned}\boldsymbol{M}(t)\dddot{\boldsymbol{x}}(t)+\left(2\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\ddot{\boldsymbol{x}}(t)+\left(\ddot{\boldsymbol{M}}(t)+\dot{\boldsymbol{C}}_{\text{vis}}(t)+\boldsymbol{K}(t)\right)\dot{\boldsymbol{x}}(t)&\\ -\left(\ddot{\boldsymbol{M}}(t)\boldsymbol{u}(t)+\dot{\boldsymbol{M}}(t)\dot{\boldsymbol{u}}(t)\right)+\dot{\boldsymbol{K}}(t)\boldsymbol{x}(t)=\boldsymbol{0}\,,&\end{aligned} \tag{12}$$

therefor, $\dddot{\boldsymbol{x}}(t)$ can be found as follows:

$$\begin{aligned}\dddot{\boldsymbol{x}}(t)=-\boldsymbol{M}(t)^{-1}&\left(2\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\ddot{\boldsymbol{x}}(t)\\ -\boldsymbol{M}(t)^{-1}&\left(\ddot{\boldsymbol{M}}(t)+\dot{\boldsymbol{C}}_{\text{vis}}(t)+\boldsymbol{K}(t)\right)\dot{\boldsymbol{x}}(t)\\ +\boldsymbol{M}(t)^{-1}&\left(\ddot{\boldsymbol{M}}(t)\boldsymbol{u}(t)+\dot{\boldsymbol{M}}(t)\dot{\boldsymbol{u}}(t)\right)\\ -\boldsymbol{M}(t)^{-1}&\dot{\boldsymbol{K}}(t)\boldsymbol{x}(t)\,.\end{aligned} \tag{13}$$

Similarly, one can obtain $\boldsymbol{x}^{(4)}(t)$, the fourth derivation of $\boldsymbol{x}(t)$ with respect to time, by once again differentiating of Eq. (12), as follows:

$$\begin{aligned}&\boldsymbol{M}(t)\boldsymbol{x}^{(4)}(t)+\left(3\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\dddot{\boldsymbol{x}}(t)+\left(3\ddot{\boldsymbol{M}}(t)+2\dot{\boldsymbol{C}}_{\text{vis}}(t)+\boldsymbol{K}(t)\right)\ddot{\boldsymbol{x}}(t)\\ &+\left(\dddot{\boldsymbol{M}}(t)+\ddot{\boldsymbol{C}}_{\text{vis}}(t)+2\dot{\boldsymbol{K}}(t)\right)\dot{\boldsymbol{x}}(t)-\left(\dddot{\boldsymbol{M}}(t)\boldsymbol{u}(t)+2\ddot{\boldsymbol{M}}(t)\dot{\boldsymbol{u}}(t)+\dot{\boldsymbol{M}}(t)\ddot{\boldsymbol{u}}(t)\right)+\ddot{\boldsymbol{K}}(t)\boldsymbol{x}(t)=\boldsymbol{0}\,.\end{aligned} \tag{14}$$

From this equation, $\boldsymbol{x}^{(4)}(t)$ can be rewritten as:

$$\begin{aligned}\boldsymbol{x}^{(4)}(t)=-\boldsymbol{M}(t)^{-1}&\left(3\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\dddot{\boldsymbol{x}}(t)\\ -\boldsymbol{M}(t)^{-1}&\left(3\ddot{\boldsymbol{M}}(t)+2\dot{\boldsymbol{C}}_{\text{vis}}(t)+\boldsymbol{K}(t)\right)\ddot{\boldsymbol{x}}(t)\\ -\boldsymbol{M}(t)^{-1}&\left(\dddot{\boldsymbol{M}}(t)+\ddot{\boldsymbol{C}}_{\text{vis}}(t)+2\dot{\boldsymbol{K}}(t)\right)\dot{\boldsymbol{x}}(t)\\ +\boldsymbol{M}(t)^{-1}&\left(\dddot{\boldsymbol{M}}(t)\boldsymbol{u}(t)+2\ddot{\boldsymbol{M}}(t)\dot{\boldsymbol{u}}(t)+\dot{\boldsymbol{M}}(t)\ddot{\boldsymbol{u}}(t)\right)\\ -\boldsymbol{M}(t)^{-1}&\ddot{\boldsymbol{K}}(t)\boldsymbol{x}(t)\,.\end{aligned} \tag{15}$$

The procedure can be successively repeated to generate higher-order derivatives of the unknown function which all produce the terms in the solution series. Therefore, the foregoing procedure can be inductively generalized to the ith differentiation of Eq. (4) which may lead to the following expression:

$$\begin{aligned}&\boldsymbol{M}(t)\boldsymbol{x}^{(i+2)}(t)+\left((i+1)\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\boldsymbol{x}^{(i+1)}(t)\\ &+\sum_{j=1}^{i}\left(\binom{i+1}{i-j}\boldsymbol{M}^{(j+1)}(t)+\binom{i}{j}\boldsymbol{C}_{\text{vis}}{}^{(j)}(t)+\binom{i}{j-1}\boldsymbol{K}^{(j-1)}(t)\right)\boldsymbol{x}^{(i-j+1)}(t)\end{aligned} \tag{16}$$

$$-\sum_{j=0}^{i}\binom{i}{j}\boldsymbol{M}^{(i+1-j)}(t)\boldsymbol{u}^{(j)}(t)+\boldsymbol{K}^{(i)}(t)\boldsymbol{x}(t)=\mathbf{0}$$

or

$$\begin{aligned}\boldsymbol{x}^{(i+2)}(t) &= -\boldsymbol{M}(t)^{-1}\left((i+1)\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right)\boldsymbol{x}^{(i+1)}(t)\\ &\quad -\boldsymbol{M}(t)^{-1}\left(\sum_{j=1}^{i}\left(\binom{i+1}{i-j}\boldsymbol{M}^{(j+1)}(t)+\binom{i}{j}\boldsymbol{C}_{\text{vis}}{}^{(j)}(t)+\binom{i}{j-1}\boldsymbol{K}^{(j-1)}(t)\right)\boldsymbol{x}^{(i-j+1)}(t)\right)\\ &\quad +\boldsymbol{M}(t)^{-1}\left(\sum_{j=0}^{i}\binom{i}{j}\boldsymbol{M}^{(i+1-j)}(t)\boldsymbol{u}^{(j)}(t)\right)\\ &\quad -\boldsymbol{M}(t)^{-1}\boldsymbol{K}^{(i)}(t)\boldsymbol{x}(t).\end{aligned} \tag{17}$$

Here, (p) over a variable denotes pth differentiation with respect to time t. This equation can be more simplified into:

$$\boldsymbol{x}^{(i+2)}(t)=\boldsymbol{A}_1(t)\boldsymbol{x}^{(i+1)}(t)+\sum_{j=1}^{i}\boldsymbol{B}_j(t)\boldsymbol{x}^{(i-j+1)}(t)+\sum_{j=0}^{i}\boldsymbol{C}_j(t)\boldsymbol{u}^{(j)}(t)+\boldsymbol{D}_1(t)\boldsymbol{x}(t) \tag{18}$$

in which, $\boldsymbol{A}_1(t)$, $\boldsymbol{B}_j(t)$, $\boldsymbol{C}_j(t)$ and $\boldsymbol{D}_1(t)$ are defined as:

$$\boldsymbol{A}_1(t)=-\boldsymbol{M}(t)^{-1}\left((i+1)\dot{\boldsymbol{M}}(t)+\boldsymbol{C}_{\text{vis}}(t)\right) \tag{19}$$

$$\boldsymbol{B}_j(t)=-\boldsymbol{M}(t)^{-1}\left(\binom{i+1}{i-j}\boldsymbol{M}^{(j+1)}(t)+\binom{i}{j}\boldsymbol{C}_{\text{vis}}{}^{(j)}(t)+\binom{i}{j-1}\boldsymbol{K}^{(j-1)}(t)\right),\quad j=1,2,\ldots,i \tag{20}$$

$$\boldsymbol{C}_j(t)=\boldsymbol{M}(t)^{-1}\left(\binom{i}{j}\boldsymbol{M}^{(i+1-j)}(t)\right),\quad j=0,1,\ldots,i \tag{21}$$

$$\boldsymbol{D}_1(t)=-\boldsymbol{M}(t)^{-1}\boldsymbol{K}^{(i)}(t) \tag{22}$$

This recursive formula gives the value of $\boldsymbol{x}^{(i+2)}(t)$. Finally, by using Taylor's expansion (Eq. (10)), one can obtain a semi-analytical solution of the problem. It is worth noting that here, like some other numerical methods such as DTM, it is implicitly assumed that the matrices $\boldsymbol{M}(t)$, $\boldsymbol{C}(t)$, and $\boldsymbol{K}(t)$ are sufficiently many times continuously differentiable. It should be noted that if one or more elements of the mentioned matrices are piecewise functions, one can divide the total time of the analysis into some shorter subintervals in which all the matrices $\boldsymbol{M}(t)$, $\boldsymbol{C}(t)$, and $\boldsymbol{K}(t)$ are n times continuously differentiable.

It should be pointed out that to rely on the accuracy of the results in the numerical examples, first, an appropriate number of terms in the solution polynomial, n, was supposed. Then, the response was compared with the results of the procedure when the number of terms in the solution series was taken $n+1$. Afterward, if the difference between two sets of responses does not meet the convergence criteria, the number of terms in the solution series should be added one more time. This process must be continued until accurate results are achieved. It is noteworthy that a realistic criterion to evaluate the convergence of the results is utilizing the root mean square error ($RMSE$) which can be derived from the following relation:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}\left(x_i^{n+1}(t_{end}) - x_i^{n}(t_{end})\right)^2}{N}} = \frac{\left\|\boldsymbol{x}^{n+1}(t_{end}) - \boldsymbol{x}^{n}(t_{end})\right\|_2}{\sqrt{N}} \leq \varepsilon \tag{23}$$

here, $x_i^{n+1}(t_{end})$ and $x_i^{n}(t_{end})$ correspond to the displacement of the i th degree of freedom at the final time of analysis in which the number of terms in the solution polynomial in Eq. (10) is taken $n+1$ and n, respectively. Moreover, ε is the convergence tolerance and should be a sufficiently small value. Recall, that N is the number of degrees of freedom of the system and the operator $\| \, . \, \|_2$ stands for Euclidean norm. If the foregoing equation is satisfied, then we get accurate results. The mentioned computation procedure is summarized in a flowchart shown in Fig. 14.

3. Numerical investigations

To illustrate the robustness of the developed method, the free vibration of four SDOF and MDOF systems with non-periodically time-variable parameters was investigated. In each numerical example, after giving all parameters, the results are graphically presented in the form of time history and hysteresis plots.

3.1. An undamped and invariant SDOF mass-spring system

The closed-form solution for the response of the well-known equation of motion of an undamped and invariant SDOF mass-spring system which is shown in Fig. 15, may be found as [52]:

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \,, \tag{24}$$

here, x_0 and $\dot{x}_0$ are the initial displacement and velocity, correspondingly; and ω_n is equal to $\sqrt{k/m}$ and represents the natural circular frequency of the system.

To clarify the proposed procedure, a few derivatives of the unknown function in the solution series are obtained in detail. $x(0)$ and $\dot{x}(0)$ are the initial conditions and are equal to x_0 and $\dot{x}_0$, respectively. By setting $i=0$ and $t=0$ in Eq. (17) one can obtain:

$$\ddot{x}(0) = -\frac{1}{m_0}(kx_0) = -\omega_n^2 x_0 \,. \tag{25}$$

In the same way, setting $i=1$ and $t=0$ in the mentioned equation leads to:

$$\dddot{x}(0) = -\frac{1}{m_0}\left(\binom{1}{0} k\dot{x}_0\right) = -\omega_n^2 \dot{x}_0 \,. \tag{26}$$

Likewise, one can obtain the next derivative by setting $i=2$ and $t=0$ in the recursive equation, as below:

$$x^{(4)}(0) = -\frac{1}{m_0}\left(\binom{2}{0}k\ddot{x}_0\right) = \omega_n^4 x_0\ . \tag{27}$$

This procedure can be continued until sufficient derivatives of the unknown function are obtained. Finally, the substitution of the Eqs. (25) through (27) and other obtained values in Eq. (10) lead to the response series as:

$$\bar{x}(t) = x_0(1-\frac{1}{2}\omega_n^2 t^2 + \frac{1}{24}\omega_n^4 t^4 - \cdots) + \frac{\dot{x}_0}{\omega_n}(\omega_n t - \frac{1}{6}\omega_n^3 t^3 + \frac{1}{120}\omega_n^5 t^5 - \cdots)\ . \tag{28}$$

It is obvious that the two parentheses on the right side of the foregoing relation are the Taylor series expansion of $\cos\omega_n t$ and $\sin\omega_n t$, respectively. $x(t)$ and $\bar{x}(t)$ are depicted in Fig. 16 for $x_0 = 0.1\,\mathrm{m}$, $\dot{x}_0 = 0.1\ \mathrm{m/s}$, $m = 1000\ \mathrm{kg}$, $k = 40000\ \mathrm{N/m}$ and $n = 80$. Since both curves have a good agreement, the robustness of the proposed method is clearly indicated.

3.2. An undamped SDOF mass-spring system with variable parameters

The second example is an undamped SDOF mass-spring system with time-variable mass and constant stiffness which is shown in Fig. 17. The governing differential equation of the system can be obtained by setting $m(t) = m_0 \mathrm{e}^{-\beta t}$ and $k(t) = constant$ in Eq. (7). Exact solution of this equation could be easily found using the classical methods of solving differential equations [53] as:

$$x(t) = \mathrm{e}^{\frac{\beta t}{2}}\left[C_1 J_1(\frac{2\varpi}{\beta}\mathrm{e}^{\frac{\beta t}{2}}) + C_2 Y_1(\frac{2\varpi}{\beta}\mathrm{e}^{\frac{\beta t}{2}})\right], \tag{29}$$

where J_α and Y_α represent the first kind and the second kind of Bessel functions of order α, correspondingly. Besides, ϖ is equal to $\sqrt{k/m_0}$. Moreover, C_1 and C_2 stand for constants of integration. By applying the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ to determine the integration constants, one can obtain:

$$x(t) = \frac{\pi}{\beta}\mathrm{e}^{\frac{\beta t}{2}}\left[J_1(\frac{2\varpi}{\beta}\mathrm{e}^{\frac{\beta t}{2}})\left[x_0\varpi Y_0(\frac{2\varpi}{\beta}) - \dot{x}_0 Y_1(\frac{2\varpi}{\beta})\right] - Y_1(\frac{2\varpi}{\beta}\mathrm{e}^{\frac{\beta t}{2}})\left[x_0\varpi J_0(\frac{2\varpi}{\beta}) - \dot{x}_0 J_1(\frac{2\varpi}{\beta})\right]\right]. \tag{30}$$

For a better understanding of the proposed arithmetic operations, a few derivatives of the unknown function in the solution series were obtained in detail. $x(0)$ and $\dot{x}(0)$ are the initial conditions and are equal to x_0 and $\dot{x}_0$, respectively. In the next step, by setting $i = 0$ and $t = 0$ in Eq. (17) one can write:

$$\ddot{x}(0) = -\frac{1}{m_0}(-m_0\beta\dot{x}_0) - \frac{1}{m_0}(kx_0) = \beta\dot{x}_0 - \varpi^2 x_0\ . \tag{31}$$

Likewise, by setting $i = 1$ and $t = 0$ in the mentioned equation, one obtains:

$$\dddot{x}(0) = -\frac{1}{m_0}(-2m_0\beta\ddot{x}_0) - \frac{1}{m_0}(m_0\beta^2 + k)\dot{x}_0 = (\beta^2 - \varpi^2)\dot{x}_0 - 2\varpi^2\beta x_0\ . \tag{32}$$

The proposed procedure can be continued until sufficient derivatives of the unknown function are determined. Finally, the substitution of Eq. (31) and Eq. (32) and other derived values in Eq. (10) leads to the following series:

$$\bar{x}(t) = x_0 + \dot{x}_0 t + \frac{1}{2}[\beta \dot{x}_0 - \varpi^2 x_0]t^2 + \frac{1}{6}[(\beta^2 - \varpi^2)\dot{x}_0 - 2\varpi^2 \beta x_0]t^3 + \cdots \tag{33}$$

The displacement variation with respect to time is depicted in Fig. 18 for $x_0 = 0.1\,\mathrm{m}$, $\dot{x}_0 = 0.1\ \mathrm{m/s}$, $m_0 = 1000\ \mathrm{kg}$, $k = 40000\ \mathrm{N/m}$, $\beta = \pm 0.25$ and $\beta = \pm 0.5$ and $n = 140$. It can be seen that both numerical and exact solutions almost coincide in all four cases. In addition, all solutions remain pseudo-periodic. It is notable, that the pseudo-period decreases and the amplitude of the response increases as time elapses when β is positive and vice versa. As it is concluded in the previous research [26], the parameter $\dot{m}\dot{x}$ in the governing equation can be considered as viscous damping. Therefore, it can be assumed as negative damping when β is positive and vice versa. Furthermore, the phase plots for the mass-spring systems are also presented in Fig. 19 by utilizing both numerical and exact responses. To clear phase plots more, we also depict the phase plots in $(t, x(t), \dot{x}(t))$ space in Fig. 20 for $\beta = 0.25$ and $\beta = 0.5$.

3.3. An undamped two-DOF system with variable mass and stiffness

The third example is an undamped two-degree-of-freedom system with TV mass and stiffness which is presented in Fig. 21.

The mass and stiffness variations are expressed in the following equations.

$$\boldsymbol{M}(t) = \begin{bmatrix} 2m_0 \mathrm{e}^{-\beta t} & 0 \\ 0 & m_0 \mathrm{e}^{-\beta t} \end{bmatrix}, \tag{34}$$

$$\boldsymbol{K}(t) = \begin{bmatrix} 3k_0 \mathrm{e}^{-\gamma t} & -k_0 \mathrm{e}^{-\gamma t} \\ -k_0 \mathrm{e}^{-\gamma t} & k_0 \mathrm{e}^{-\gamma t} \end{bmatrix}. \tag{35}$$

The governing differential equation of the system can be obtained by setting the following mass and stiffness matrices into Eq. (4) and considering $\boldsymbol{u}(t) = \boldsymbol{0}$, which means the absolute velocity of the separated mass is negligible. Before utilizing the proposed method, the exact solution of the system depicted in Fig. 21 could be obtained by the pseudo-modal analysis described in the following. The governing differential equation of such a system with initial displacement $\{x_{1,0} \quad x_{2,0}\}^{\mathrm{T}}$ and zero initial velocity can be turned into two decoupled ordinary differential equations by the pseudo-modal analysis. Therefore, the system response can be evaluated through pseudo-modal expansion as:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{y}(t), \tag{36}$$

in which $\boldsymbol{\Phi}(t)$ and $\boldsymbol{y}(t)$ are the normal modal matrix and modal displacements, correspondingly. In this case, the modal matrix can be found as:

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \tag{37}$$

hence, decoupled mass, damping and stiffness matrices can be calculated as:

$$\boldsymbol{M}^*(t)=\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\Phi}=\begin{bmatrix}6m_0\mathrm{e}^{-\beta t} & 0\\ 0 & 3m_0\mathrm{e}^{-\beta t}\end{bmatrix}, \tag{38}$$

$$\boldsymbol{C}^*(t)=\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{C}_{\mathrm{vis}}+\dot{\boldsymbol{M}})\boldsymbol{\Phi}=\begin{bmatrix}-6m_0\beta\mathrm{e}^{-\beta t} & 0\\ 0 & -3m_0\beta\mathrm{e}^{-\beta t}\end{bmatrix}, \tag{39}$$

$$\boldsymbol{K}^*(t)=\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\Phi}=\begin{bmatrix}3k_0\mathrm{e}^{-\gamma t} & 0\\ 0 & 6k_0\mathrm{e}^{-\gamma t}\end{bmatrix}. \tag{40}$$

Also, the initial modal displacement and velocity are:

$$\boldsymbol{y}(0)=\boldsymbol{\Phi}^{-1}\boldsymbol{x}(0)=\begin{Bmatrix}(x_{1,0}+x_{2,0})/3\\ (2x_{1,0}-x_{2,0})/3\end{Bmatrix},\quad \dot{\boldsymbol{y}}(0)=\boldsymbol{\Phi}^{-1}\dot{\boldsymbol{x}}(0)=\begin{Bmatrix}0\\0\end{Bmatrix}. \tag{41}$$

Consequently, two decomposed differential equations could be written in the following forms:

$$\begin{aligned}&6m_0\mathrm{e}^{-\beta t}\ddot{y}_1(t)-6m_0\beta\mathrm{e}^{-\beta t}\dot{y}_1(t)+3k_0\mathrm{e}^{-\gamma t}y_1(t)=0,\\ &3m_0\mathrm{e}^{-\beta t}\ddot{y}_2(t)-3m_0\beta\mathrm{e}^{-\beta t}\dot{y}_2(t)+6k_0\mathrm{e}^{-\gamma t}y_2(t)=0.\end{aligned} \tag{42}$$

Similar to the previous example, the exact solution of these two equations can be calculated as:

$$y_1(t)=\mathrm{e}^{\frac{\beta t}{2}}\,\bar{\omega}_1^{\frac{\beta}{\beta-\gamma}}(\beta-\gamma)^{-\frac{\beta}{\beta-\gamma}}\begin{bmatrix}C_1\,\Gamma(-\dfrac{\gamma}{\beta-\gamma})\,J_{-\frac{\beta}{\beta-\gamma}}\left(\dfrac{2\bar{\omega}_1(2\beta+\gamma)}{(\beta-\gamma)\times|2\beta+\gamma|}\mathrm{e}^{\frac{\beta-\gamma}{2}t}\right)\\ +C_2\,\Gamma(\dfrac{2\beta-\gamma}{\beta-\gamma})\,J_{\frac{\beta}{\beta-\gamma}}\left(\dfrac{2\bar{\omega}_1(2\beta+\gamma)}{(\beta-\gamma)\times|2\beta+\gamma|}\mathrm{e}^{\frac{\beta-\gamma}{2}t}\right)\end{bmatrix}, \tag{43}$$

and

$$y_2(t)=\mathrm{e}^{\frac{\beta t}{2}}\,\bar{\omega}_2^{\frac{\beta}{\beta-\gamma}}(\beta-\gamma)^{-\frac{\beta}{\beta-\gamma}}\begin{bmatrix}C_3\,\Gamma(-\dfrac{\gamma}{\beta-\gamma})\,J_{-\frac{\beta}{\beta-\gamma}}\left(\dfrac{2\bar{\omega}_2(2\beta+\gamma)}{(\beta-\gamma)\times|2\beta+\gamma|}\mathrm{e}^{\frac{\beta-\gamma}{2}t}\right)\\ +C_4\,\Gamma(\dfrac{2\beta-\gamma}{\beta-\gamma})\,J_{\frac{\beta}{\beta-\gamma}}\left(\dfrac{2\bar{\omega}_2(2\beta+\gamma)}{(\beta-\gamma)\times|2\beta+\gamma|}\mathrm{e}^{\frac{\beta-\gamma}{2}t}\right)\end{bmatrix}, \tag{44}$$

in which C_1 through C_4 are constants of integration and can be found by applying the initial conditions to the problem. In addition, $\bar{\omega}_1=\sqrt{k_0/(2m_0)}$, $\bar{\omega}_2=\sqrt{2k_0/m_0}$, and Γ represent the gamma function. The free vibration of the system for $\boldsymbol{x}_0=\{-0.5\quad 2\}^{\mathrm{T}}$ m, $\dot{\boldsymbol{x}}_0=\boldsymbol{0}$ m/s, $m_0=1000$ kg, $k_0=40000$ N/m, $\gamma=-0.1$ and $\beta=-0.25$ is presented in Fig. 22. The numerical solutions were obtained by using the first 100 terms of the series. Once again, numerical and exact solutions coincide very well, for both time history and hysteresis plots. In addition, both numerical and exact phase plots of the first and second degrees of freedom of the mass-spring system are depicted in Fig. 10.

As mentioned before, the parameter $\dot{m}\dot{x}$ in the motion equation can be regarded as viscous damping. Here, this hypothesis was examined again. To do this, numerical phase plots of the first and second degrees of freedom for two values of $\beta = -0.25$ and $\beta = -0.15$ are plotted in Fig. 24. All other parameters in both plots are similar to the previously given values. It should be noted that as a result of damping, amplitudes of the displacement and velocity decrease as time elapses while the system loses energy. Additionally, this phenomenon results in the phase plots spiraling inward [54].

3.4. A three-DOF system with time-variable mass and stiffness

Finally, an undamped three-DOF system with time-variable mass and stiffness which is shown in Fig. 25 is taken into account.

The time-varied mass and stiffness matrices of the system are as follows:

$$\boldsymbol{M}(t) = \begin{bmatrix} m_0 \mathrm{e}^{-\beta_1 t} & 0 & 0 \\ 0 & m_0 \mathrm{e}^{-\beta_2 t} & 0 \\ 0 & 0 & m_0 \mathrm{e}^{-\beta_3 t} \end{bmatrix}, \tag{45}$$

$$\boldsymbol{K}(t) = \begin{bmatrix} 3k_0 \mathrm{e}^{-\gamma_1 t} + 2k_0 \mathrm{e}^{-\gamma_2 t} & -2k_0 \mathrm{e}^{-\gamma_2 t} & 0 \\ -2k_0 \mathrm{e}^{-\gamma_2 t} & 2k_0 \mathrm{e}^{-\gamma_2 t} + k_0 \mathrm{e}^{-\gamma_3 t} & -k_0 \mathrm{e}^{-\gamma_3 t} \\ 0 & -k_0 \mathrm{e}^{-\gamma_3 t} & k_0 \mathrm{e}^{-\gamma_3 t} \end{bmatrix}. \tag{46}$$

Initial conditions of the system were considered as $\boldsymbol{x}(0) = \{1 \quad -0.5 \quad 2\}^{\mathrm{T}}$ m and $\dot{\boldsymbol{x}}_0 = \boldsymbol{0}$ m/s. Also, $\boldsymbol{u}(t) = \boldsymbol{0}$ was taken into account. Firstly, to verify the numerical solution, the displacement curves are presented for $m_0 = 1000$ kg, $k_0 = 40000$ N/m, $\gamma_1 = \gamma_2 = \gamma_3 = -0.1$ and $\beta_1 = \beta_2 = \beta_3 = -0.25$ in Table 2. The exact solution is obtained in the same way as the previous example. Once again, both numerical and exact plots have excellent agreement. The diagrams were obtained by taking the first 120 terms of the series. Afterward, the effect of parameters β and γ on the structure was investigated and the numerical response for $\beta_1 = -0.25$, $\beta_2 = -0.15$, $\beta_3 = -0.05$, $\gamma_1 = -0.3$, $\gamma_2 = -0.15$ and $\gamma_3 = -0.1$ were obtained.

It is noteworthy to mention that, like other numerical methods, this scheme requires appropriate convergence criteria. The number of terms in the solution series, i.e. n, is the key parameter that affects the convergence. Selecting a small value for n may affect the convergence, but in some problems with large t, the results may never converge. This phenomenon arises from the essence of the numerical methods, especially the schemes that are based on Taylor series expansion which could be converged around the base expansion point, i.e. t_0. To remedy this, the total time of the analysis should be divided into shorter subintervals and the solution series should be written at the beginning of each subinterval [55]. Fig. 26 displays the convergence of the response by increasing the value of n in the last numerical example.

4. Conclusion

This article proposed an iterative numerical method, based on successive differentiation, for dynamic analysis of time-varying multi-degree-of-freedom systems. The solution process was thoroughly described and the required equations were derived step by step. After that, the proposed method was first validated by the results of two single-degree-of-freedom systems with invariant and variant coefficients. Then, the verification process was continued by the help of two multi-degree-of-freedom systems with time-varying mass and stiffness. It is worth mentioning that the closed-form solution for an undamped two DOF system with variable mass and stiffness was also extracted for the first time. Based on the numerical investigations performed in this study, the following concluding remarks could be offered:

- The proposed method could be systematically applied to any equation of motion for time-varying discrete dynamical systems. Also, the required formulations could be generalized in a simple and straight process.
- It was demonstrated that time-varying mass could be considered as viscous damping. It means when the rate of variation of mass is positive, pseudo-period increases and the amplitude of the response decreases as time elapses and vice versa.
- Similar to the other numerical methods, the convergence test should be carried out, and the sensitive parameters should be detected. For the proposed method, the final results are strongly dependent on the number of differentiating of the original equation. It is noticeable that the number of differentiations is approximately equal to the number of terms in the solution series. It is important to note that to have converged results for the problems with large time, the total time of the analysis should be divided into smaller time steps.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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List of notations

$\boldsymbol{A}_1(t)$	Defined by Eq. (19)	$\boldsymbol{K}(t)$	Stiffness matrix	$\dot{\boldsymbol{x}}(t)$	Velocity vector
$\boldsymbol{B}_j(t)$	Defined by Eq. (20)	$\boldsymbol{K}^*(t)$	Decoupled stiffness matrix	$\dot{\boldsymbol{x}}(0)$	Initial velocity vector
$\boldsymbol{C}(t)$	Damping matrix	$\boldsymbol{M}(t)$	Mass matrix	$\ddot{\boldsymbol{x}}(t)$	Acceleration vector
C_i	Integration constant	$\boldsymbol{M}^*(t)$	Decoupled mass matrix	$\boldsymbol{y}(t)$	Modal displacement vector
$\boldsymbol{C}_j(t)$	Defined by Eq. (21)	N	Number of degrees of freedom	Y_α	Second kind of Bessel functions of order α
$\boldsymbol{C}_{\text{vis}}(t)$	Viscous damping matrix	n	Number of terms in the series solution	ε	Convergence tolerance
$\boldsymbol{C}^*(t)$	Decoupled damping matrix	$RMSE$	Root mean square error	κ	Scalar value
$\boldsymbol{D}_1(t)$	Defined by Eq. (22)	t	Time	$\boldsymbol{\Phi}(t)$	Normal modal matrix
$\boldsymbol{F}(t)$	Load vector	$\boldsymbol{u}(t)$	Relative velocity of varied mass vector	ω	Circular frequency
$\boldsymbol{F}_{\text{ext}}(t)$	External force vector	$\boldsymbol{x}(t)$	Displacement vector	ω_n	Natural circular frequency
J_α	First kind of Bessel functions of order α	$\boldsymbol{x}(0)$	Initial displacement vector		

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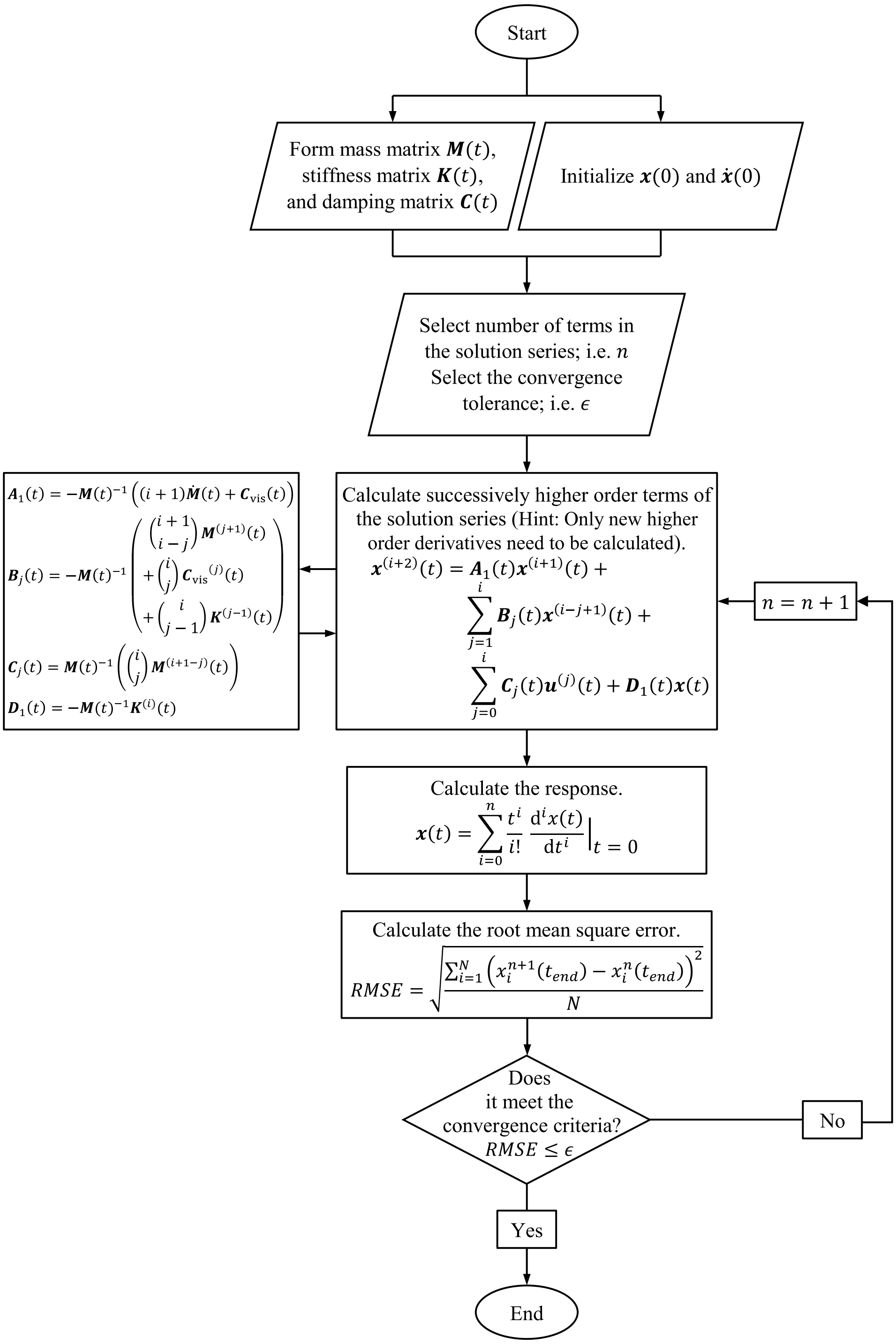


Fig. 14 Flowchart of the proposed method

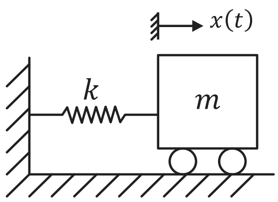


Fig. 15 Undamped and invariant mass-spring system

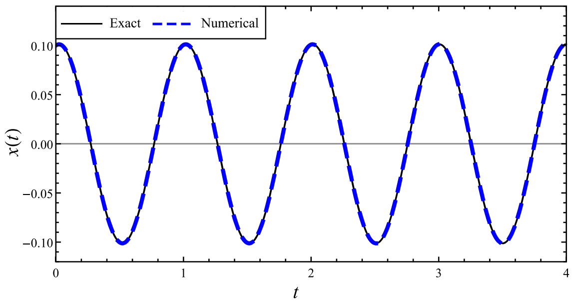


Fig. 16. The response of an undamped and invariant mass-spring system with $n=80$

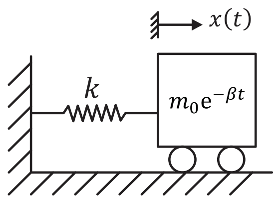
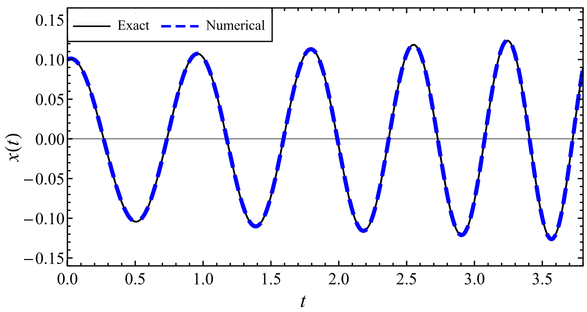
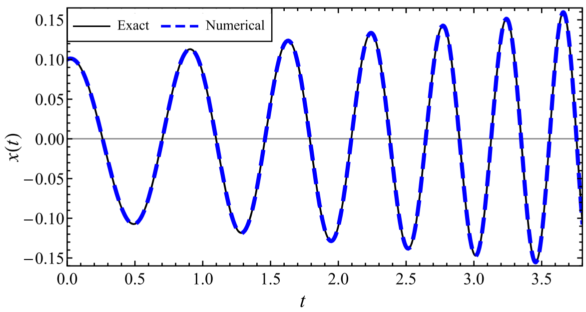


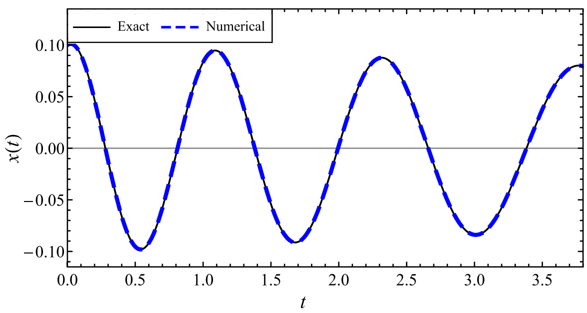
Fig. 17 Undamped time-variable SDOF mass-spring system



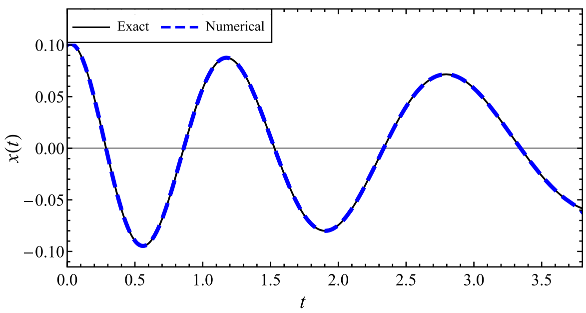
(a) $\beta=0.25$



(b) $\beta=0.5$

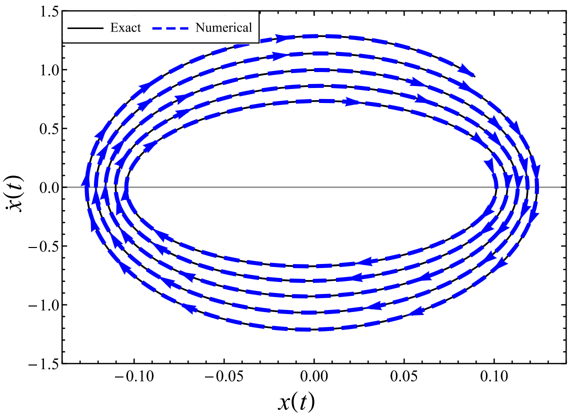


(c) $\beta=-0.25$

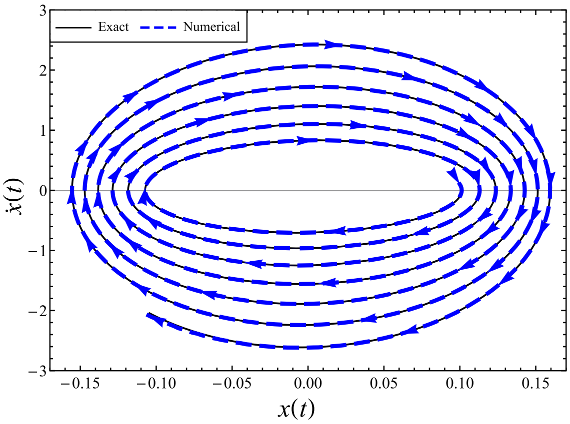


(d) $\beta=-0.5$

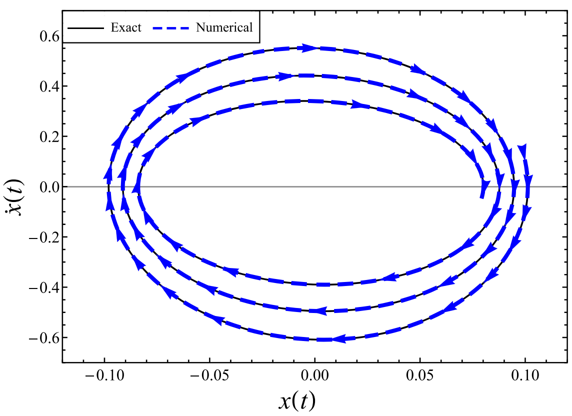
Fig. 18 The response of an undamped SDOF mass-spring for various values of β and $n=140$



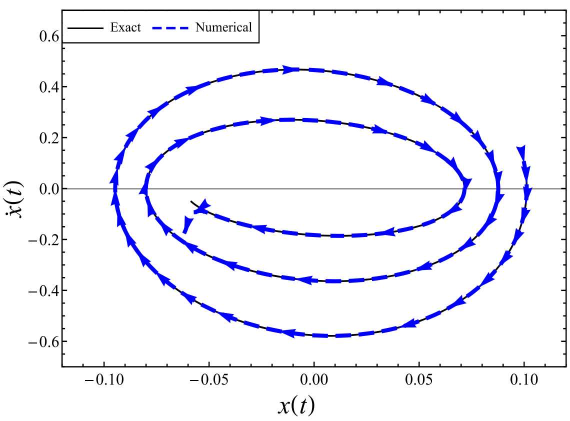
(a) $\beta = 0.25$



(b) $\beta = 0.5$

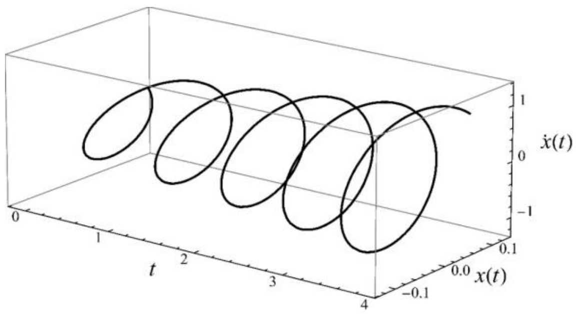


(c) $\beta = -0.25$

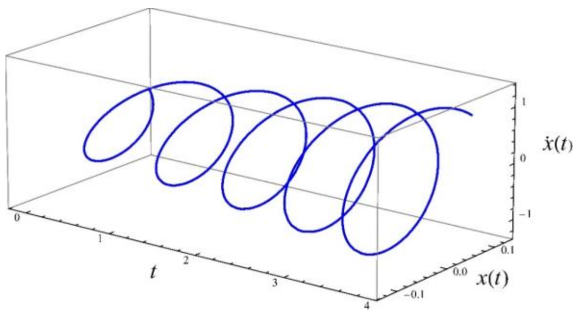


(d) $\beta = -0.5$

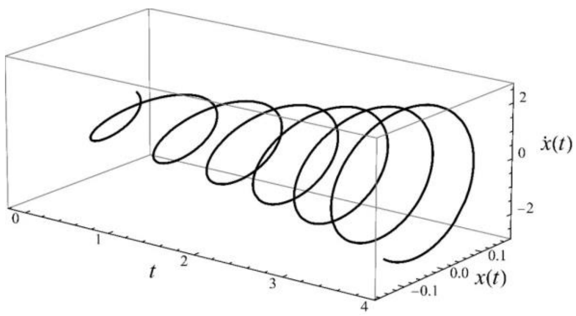
Fig. 19 Phase plots of an undamped SDOF mass-spring for various values of β and $n = 140$



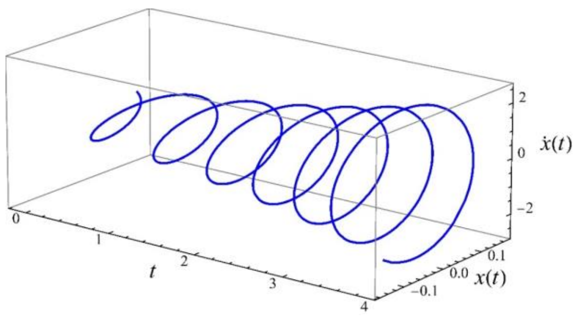
(a) $\beta = 0.25$ - Exact response



(b) $\beta = 0.25$ - Numerical response



(c) $\beta = 0.5$ - Exact response



(d) $\beta = 0.5$ - Numerical response

Fig. 20 Phase plots in 3D space of an undamped SDOF mass-spring with respect to $\beta = 0.25$ and $\beta = 0.5$, and $n = 140$

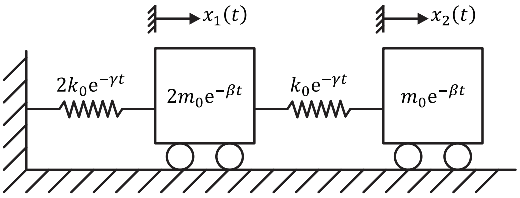


Fig. 21 Two-degree-of-freedom mass-spring system

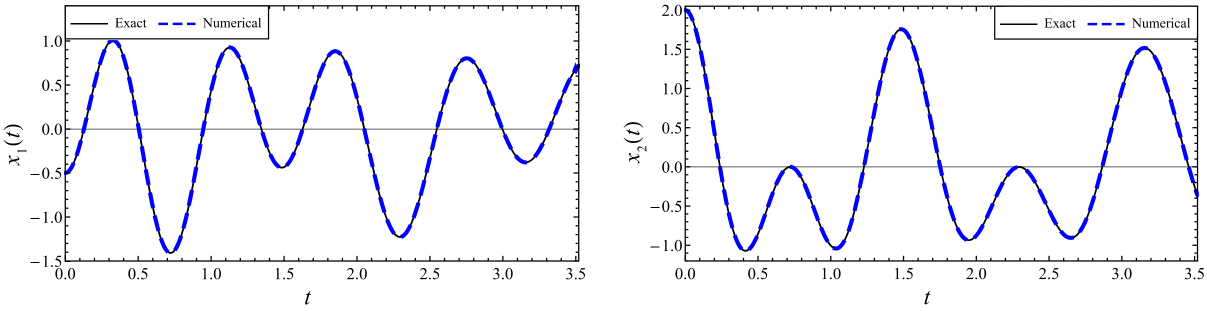


Fig. 22 Response of two DOF system with $n = 100$

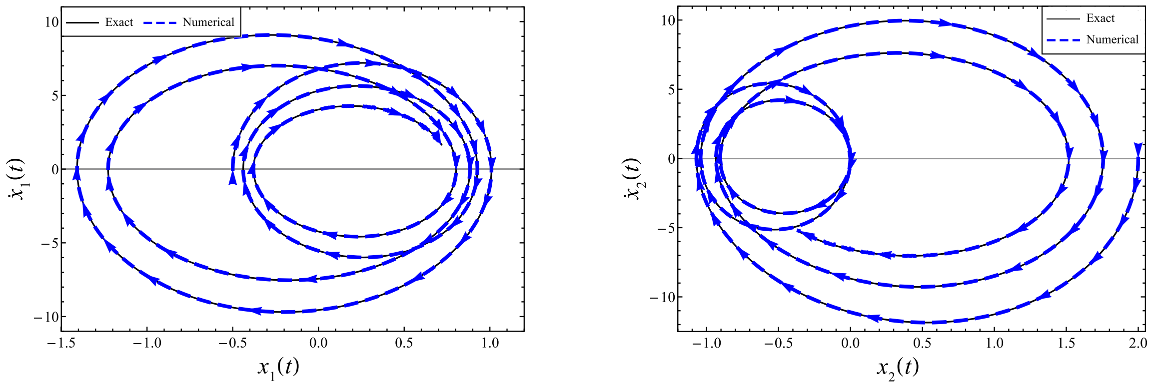


Fig. 23 Phase plots of two DOF system with $n = 100$

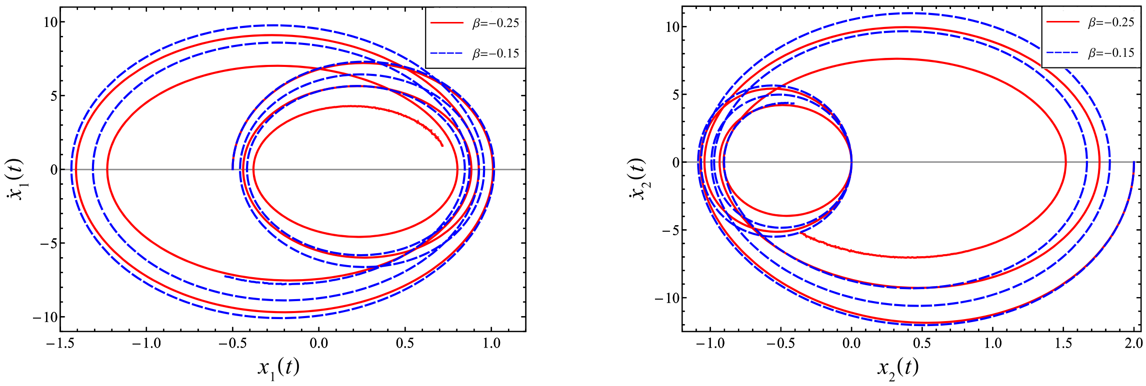


Fig. 24 Phase plots of two DOF system for various values of β and $n = 100$

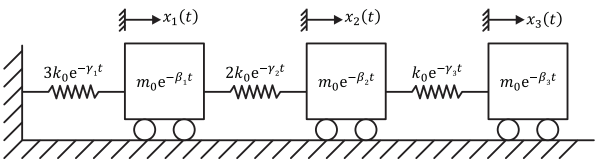


Fig. 25 Three DOF mass-spring system

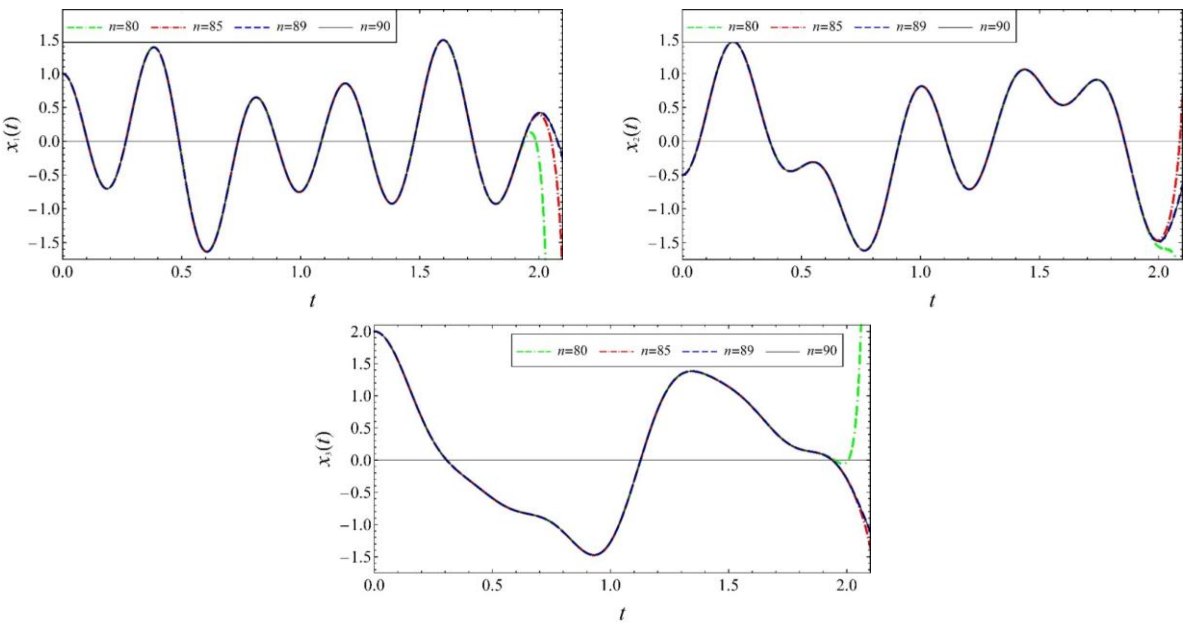


Fig. 26 Convergence test for three DOF system

Tables

Table 2 Response of three DOF system with different parameters and $n = 120$

$\beta_1 = \beta_2 = \beta_3 = -0.25$ $\gamma_1 = \gamma_2 = \gamma_3 = -0.1$	$\beta_1 = -0.25$, $\beta_2 = -0.15$, $\beta_3 = -0.05$ $\gamma_1 = -0.3$, $\gamma_2 = -0.15$, $\gamma_3 = -0.1$