Analytical approaches to the (2+1)-dimensional Heisenberg Ferromagnetic Spin Chain equation and their applications for optical devices

Khalid K. Ali[1,*], Abdel-Haleem Abdel-Aty[2,3], M.S.Mehanna[4]

[1] Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt
[2]Department of Physics, College of Sciences, University of Bisha, PO Box 344, Bisha 61922, Saudi Arabia
[3]Physics Department, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
[4]Faculty of Engineering, MTI University, Cairo, Egypt Emails : [1]khalidkaram2012@azhar.edu.eg, [2]amabdelaty@ub.edu.sa, [3]monamehanna@gmail.com
[*]Corresponding author. Te. +201033530861

Abstract

The Heisenberg ferromagnetic spin chain (HFSC) equation has substantial relevance in the fields of optics and electronics, particularly in the advancement of high-density electronic components and faster storage devices. This integrable nonlinear Schrödinger equation characterizes the propagation of nonlinear waves in ferromagnetic spin chain systems. In a recent research paper, two powerful analytical methods, the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method and the extended hyperbolic function method (EHFM), were implemented to solve the (2+1)-dimensional HFSC equation. Most of the results obtained from the study are presented graphically, which can aid in the visualization and interpretation of the gained results. These findings will be useful in the development and optimization of spintronic devices and other electronic components that rely on the behavior of spin systems. The analytical solutions of the HFSC equation yield important insights that enhance our understanding and facilitate the application of magnetism, thermal properties, and topological phenomena across diverse fields such as materials science, condensed matter physics, and quantum technologies. These findings play a crucial role in advancing our knowledge and practical utilization of these phenomena in real-world applications.
Keywords: Heisenberg equation; Solitons; $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$ method; Extended hyperbolic function method.

1 Introduction

In the last decades, the Heisenberg ferromagnetic spin chain equation [1–3] has great applications in soliton theory and condensed matter physics. The study of this non-linear partial differential equation attracts researchers due to its wide applications in different fields, including mathematical physics [4, 5], fluid mechanics [6], the shock wave phenomena in plasma [7], nonlinear electrical transmission lattice occurring in science and engineering [8, 9], optics and quantum information [10, 11], nonlinear optics [12, 13], condensed matter physics [14], geophysical fluid dynamics [15], coastal engineering, wave energy conversion, and fluid dynamics studies [16], plasma physics [17], optical fibers [18], hydrodynamics and solid-state physicss [19], electromagnetic signals [20], chaos theory [21], and others. Many researchers execute different techniques to get the soliton solutions of the non-linear partial differential equations. The present literature concern with the (2+1)-dimensional Heisenberg ferromagnetic spin chain (HFSC) equation

$$iu_t + \alpha_1 u_{xx} + \alpha_2 u_{yy} + \alpha_3 u_{xy} - \alpha_4 \mid u \mid^2 u = 0, \tag{1}$$

$$\alpha_1 = \gamma^4(\Theta + \Theta_2), \alpha_2 = \gamma^4(\Theta_1 + \Theta_2), \alpha_3 = 2\gamma^4\Theta_2, \alpha_3 = 2\gamma^4 B. \tag{2}$$

where:$u = u(x, y, t)$, γ is the grid parameter, Θ and Θ_2 are the coefficients of bilinear exchange interactions in the xy-plane, Θ_2 is the neighboring interaction on the diagonal, B is the uni-axial crystal field anisotropy parameter. The Heisenberg ferromagnetic spin chain equation has important applications in modern magnet theory. It provides a description for the non-linear characteristics of magnets. The field equivalent to the anisotropic and constant external field is formed, when the electromagnetic waves transfer in an isotropic ferromagnetic medium and also dissect the mathematical phenomenon [22, 23]. Many authors gives a traveling wave solutions to the HFSC equation using miscellaneous techniques: Bashar et al. applied the improved F-expansion method and modified simple equation (MSE) [24], the $\exp(\varphi(\xi))$-expansion and the extended tanh-function methods are used by Bashar et al. [25], Osman et al. exerted the new extended FAN sub-equation method [26], Nisar et al. apply the extended (G'/G^2)-expansion method [27], By applying projective Riccati equation and modified F-Expansion methods Aliyu et al. present their solutions [28], Yu-Lan Ma solved by using the bilinear method [29], Bulut et al. used extended sinh-Gordon equation method [30], the complete discrimination system method [31] applied by Han et al., Seadawy et al. applied generalized Riccati mapping and improved auxiliary equation methods [32], Abdul Al Woadud et al. used the modified Kudryashov method [33], Sahooa et al. studied by the modified Khater method [34], Islam et al. applied the unified method [35], Mohammed et al. applied the Jacobi elliptic function method [36], Zahran et al. applied the solitary wave ansatz and the Paul-Painleve approach methods [37], Wang et al. employed the variational method and subequation method [38], Farah M. Al-Askar utilized the $\frac{G'}{G}$ -expansion method and the mapping method [39] and others.

The (2+1)-dimensional Heisenberg ferromagnetic spin chain equation has been applied to a wide variety of problems in optics, including the study of nonlinear optical fibers, the dynamics of liquid crystals, and the propagation of light in photonic crystals. The schedule of the paper is: Section 2 presents the algorithm of the suggested methods [40–46]. In Section 3, we exhibit the application of the methods. Most of the solutions are illustrated by graphs in section 4. The discussion of our results is introduced in Section 5. At the end, Section 6 offers a succinct conclusion.

2 The Strategies of the proposed Techniques

2.1 The $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method [40]

We give the main steps of the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method. Consider the nonlinear partial differential equation

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, ...) = 0, \tag{3}$$

where $u = u(x, y, t)$ represents the complex function to be calculated, while F denotes a polynomial involving both u and its associated partial derivatives.

Step 1: Insert the next wave transformation,

$$u(x, y, t) = e^{i\xi}v(\eta), \;\; \xi = a_1x + b_1y - c_1t, \;\; \eta = a_2x + b_2y - c_2t. \tag{4}$$

where ξ is the envelope phase, a_2 is the wave number in the x-direction, b_2 is the wave numbers in the y-direction, c_2 represents the wave velocity, c_1 denotes the frequency of the pulse and a_1, b_1 are constants. Inserting Equation (4) into Equation (1), then Equation (1) transformed to an ordinary differential equation:

$$H(v', v'', v''', ...) = 0, \tag{5}$$

where H represents a polynomial involving $v(\eta)$ and its corresponding derivatives.

Step 2: Assume that the following represents the solution to Equation (5):

$$v(\eta) = \sum_{i=0}^{N} \mathfrak{m}_i \mathfrak{F}(\eta)^i, \tag{6}$$

where $\mathfrak{F}(\eta) = (\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$, $\mathfrak{G}(\eta)$ fulfill the next ODE,

$$\mathfrak{G}''(\eta) = -\frac{\lambda}{\mathfrak{K}}\mathfrak{G}'(\eta) - \frac{\mu}{\mathfrak{K}^2}\mathfrak{G}(\eta) - \frac{\mu}{\mathfrak{K}^2}\mathfrak{r}, \tag{7}$$

where:$\mathfrak{m}_i$ are unknown constants, $\mathfrak{K}$, μ, λ, and $\mathfrak{r}$ are constants to be determined later. $\mathfrak{F} = \mathfrak{F}(\eta)$ is the solution of the ODE

$$\mathfrak{F}'(\eta) = (\lambda - \mu - 1)\mathfrak{F}(\eta)^2 + \frac{(2\mu - \lambda)}{\mathfrak{K}}\mathfrak{F}(\eta) - \frac{\mu}{\mathfrak{K}^2} \tag{8}$$

Step 3: Finding the positive integer N in Equation (5) requires balancing the highest power nonlinear term with the highest order derivative term.

Step 4: Two families of solutions to Equation (8) are presented:

Class 1: When $\Delta = \lambda^2 - 4\mu > 0$,

$$\mathfrak{G} = -\mathfrak{r} + p_1 e^{\frac{1}{2\mathfrak{K}}(-\lambda-\sqrt{\Delta})\eta} + p_2 e^{\frac{1}{2\mathfrak{K}}(-\lambda+\sqrt{\Delta})\eta}, \tag{9}$$

p_1 and p_2 represent arbitrary constants, and they must satisfy the following relation: $\mathfrak{r}^2 + p_1^2 + p_2^2 \neq 0$,then

$$\mathfrak{F}(\eta) = \frac{p_1(\lambda + \sqrt{\Delta}) + p_2(\lambda - \Delta)e^{\frac{\sqrt{\Delta}\eta}{\mathfrak{K}}}}{\mathfrak{K}p_1(\lambda - 2 + \sqrt{\Delta}) + \mathfrak{K}p_2(\lambda - 2 - \Delta)e^{\frac{\sqrt{\Delta}\eta}{\mathfrak{K}}}},$$

$$\mathfrak{F}(\eta) = \frac{[\lambda(p_2 - p_1) - \sqrt{\Delta}(p_2 + p_1)]\sinh(\frac{\sqrt{\Delta}\eta}{2\mathfrak{K}}) + [\lambda(p_2 + p_1) - \sqrt{\Delta}(p_2 - p_1)]\cosh(\frac{\sqrt{\Delta}\eta}{2\mathfrak{K}})}{\mathfrak{K}[(\lambda - 2)(p_2 - p_1) - \sqrt{\Delta}(p_2 + p_1)]\sinh(\frac{\sqrt{\Delta}\eta}{2k}) + \mathfrak{K}[(\lambda - 2)(p_2 + p_1) - \sqrt{\Delta}(p_2 - p_1)]\cosh(\frac{\sqrt{\Delta}\eta}{2\mathfrak{K}})} \tag{10}$$

$$\mathfrak{F}(\eta) = \begin{cases} \frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2k(\lambda-\mu-1)}\tanh(\frac{\sqrt{\Delta}\eta}{2\mathfrak{K}}), & (\lambda - 2)(p_2 - p_1) - \sqrt{\Delta}(p_2 + p_1) = 0, \\ \frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2\mathfrak{K}(\lambda-\mu-1)}\coth(\frac{\sqrt{\Delta}\eta}{2\mathfrak{K}}), & (\lambda - 2)(p_2 + p_1) - \sqrt{\Delta}(p_2 - p_1) = 0. \end{cases} \tag{11}$$

Class 2: When $\Delta = \lambda^2 - 4\mu < 0$,

$$\mathfrak{G} = -\mathfrak{r} + e^{\frac{-\lambda\eta}{2\mathfrak{K}}}(p_1\cos(\frac{\sqrt{-\Delta}\eta}{2k}) + p_2\sin(\frac{\sqrt{-\Delta}\eta}{2\mathfrak{K}})), \tag{12}$$

$$\mathfrak{F}(\eta) = \frac{(\lambda p_1 - \sqrt{-\Delta}p_2)\cos(\frac{\sqrt{-\Delta}\eta}{2\mathfrak{K}}) + (\lambda p_2 + \sqrt{-\Delta}p_1)\sin(\frac{\sqrt{-\Delta}\eta}{2\mathfrak{K}})}{\mathfrak{K}((\lambda - 2)p_1 - \sqrt{-\Delta}p_2)\cos(\frac{\sqrt{-\Delta}\eta}{2k}) + \mathfrak{K}((\lambda - 2)p_2 + \sqrt{-\Delta}p_1)\sin(\frac{\sqrt{-\Delta}\eta}{2\mathfrak{K}})} \tag{13}$$

$$\mathfrak{F}(\eta) = \begin{cases} \frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} + \frac{\sqrt{-\Delta}}{2\mathfrak{K}(\lambda-\mu-1)}\tan(\frac{\sqrt{-\Delta}\eta}{2\mathfrak{K}}), & (\lambda - 2)p_2 + \sqrt{-\Delta}p_1 = 0, \\ \frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} - \frac{\sqrt{-\Delta}}{2\mathfrak{K}(\lambda-\mu-1)}\cot(\frac{\sqrt{-\Delta}\eta}{2\mathfrak{K}}), & (\lambda - 2)p_1 - \sqrt{-\Delta}p_2 = 0. \end{cases} \tag{14}$$

Step 4: After substituting Equation (6) and Equation (8) into Equation (5), the next step involves grouping all coefficients of $\mathfrak{F}(\eta)$ with identical powers and equating them to zero. Solving the resultant system of algebraic equations with the aid of Mathematica program to get the solution of Equation (1).

2.2 The extended hyperbolic function method (EHFM) [41–46]

Herein, we give the essential steps of the extended hyperbolic function method in the next:

Step 1: consider the solution to equation Equation (5) as follows:

$$v(\eta) = \sum_{j=0}^{N} f_j (G(\eta))^j, \tag{15}$$

where $G(\eta)$ satisfies two ODE, the next ODE is the first one:

$$G'(\eta) = G(\eta)\sqrt{\mu G(\eta)^2 + \beta}, \tag{16}$$

where $f_N \neq 0$, $f_j (j = 0, 1, 2, ..., N)$, $\beta, \mu \in R$, are constants to be determined later.

Step 2: From Equation (5), elaborating the homogeneous balance principle as previously discussed to obtain the value of N.

Step 3: substituting Equation (15) and Equation (16) in Equation (5), after aggregating all the coefficients of the $G(\eta)$ with the same power, putting them equal to zero, then solve the gained system of equations using the Mathematica program.

Step 4: The solutions of Equation (17) are eight families given in the next:
Family 1: When $\beta > 0$ and $\mu > 0$,

$$G(\eta) = -\sqrt{\frac{\beta}{\mu}} csch(\sqrt{\beta}(\eta + \eta_0)). \tag{17}$$

Family 2: When $\beta < 0$ and $\mu > 0$,

$$G(\eta) = \sqrt{-\frac{\beta}{\mu}} \sec(\sqrt{-\beta}(\eta + \eta_0)). \tag{18}$$

Family 3: When$\beta > 0$ and $\mu < 0$,

$$G(\eta) = \sqrt{-\frac{\beta}{\mu}} sech(\sqrt{\beta}(\eta + \eta_0)). \tag{19}$$

Family 4: When $\beta < 0$ and $\mu < 0$,

$$G(\eta) = \sqrt{-\frac{\beta}{\mu}} \csc(\sqrt{-\beta}(\eta + \eta_0)). \tag{20}$$

Family 5: When $\beta > 0$ and $\mu = 0$,

$$G(\eta) = \exp(\sqrt{\beta}(\eta + \eta_0)). \tag{21}$$

Family 6: When $\beta < 0$ and $\mu = 0$,

$$G(\eta) = \cos(\sqrt{-\beta}(\eta + \eta_0)) + i\sin(\sqrt{-\beta}(\eta + \eta_0)). \quad (22)$$

Family 7: When $\beta = 0$ and $\mu > 0$,

$$G(\eta) = \pm\frac{1}{(\sqrt{\mu}(\eta + \eta_0))}. \quad (23)$$

Family 8: When $\beta = 0$ and $\mu < 0$,

$$G(\eta) = \pm\frac{i}{(\sqrt{-\mu}(\eta + \eta_0))}. \quad (24)$$

Secondly, the $G(\eta)$ satisfies another ODE, given by:

$$G'(\eta) = \mu G(\eta)^2 + \beta, \quad (25)$$

then, there are six distinct solution families presented for Equation (25):
Family 1: When $\beta\mu > 0$,

$$G(\eta) = sgn(\beta)\sqrt{\frac{\beta}{\mu}}\tan(\sqrt{\beta\mu}(\eta + \eta_0)). \quad (26)$$

Family 2: When $\beta\mu > 0$,

$$G(\eta) = -sgn(\beta)\sqrt{\frac{\beta}{\mu}}\cot(\sqrt{\beta\mu}(\eta + \eta_0)). \quad (27)$$

Family 3: When $\beta\mu < 0$,

$$G(\eta) = sgn(\beta)\sqrt{-\frac{\beta}{\mu}}\tanh(\sqrt{-\beta\mu}(\eta + \eta_0)). \quad (28)$$

Family 4: When $\beta\mu < 0$,

$$G(\eta) = sgn(\beta)\sqrt{-\frac{\beta}{\mu}}\coth(\sqrt{-\beta\mu}(\eta + \eta_0)). \quad (29)$$

Family 5: When $\beta = 0$ and $\mu > 0$,

$$G(\eta) = -\frac{1}{\mu(\eta + \eta_0)}. \quad (30)$$

Family 6: When $\beta < 0$ and $\mu = 0$,

$$G(\eta) = \beta(\eta + \eta_0). \quad (31)$$

Where: sgn is the known sign function.

3 Applications

Substituting by the wave transformation Equation (4) in Equation (1), we acquire the real part equation:

$$c_1 v(\eta) - v(\eta)(a_1^2\alpha_1 + b_1^2\alpha_2 + a_1b_1\alpha_3 + \alpha_4 v(\eta)^2) + (a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)v''(\eta) = 0. \tag{32}$$

Balancing v'' with v^3 in Equation (32), we get $3N = N + 2$, then $N = 1$,
and the imaginary part equation is given by:

$$(-c_2 + 2a_1a_2\alpha_1 + 2b_1b_2\alpha_2 + a_2b_1\alpha_3 + a_1b_2\alpha_3)v'(\eta) = 0. \tag{33}$$

We get:

$$c_2 = 2a_1a_2\alpha_1 + 2b_1b_2\alpha_2 + a_2b_1\alpha_3 + a_1b_2\alpha_3. \tag{34}$$

3.1 The $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method

Utilizing Equation (6), we express the solution to Equation (32) as:

$$v(\eta) = \mathfrak{m}_0 + \mathfrak{m}_1\mathfrak{F}(\eta). \tag{35}$$

Substituting Equation (35) in Equation (32), then equating to zero all the coefficients with the same powers of $\mathfrak{F}(\eta)$, we acquire the following system of equations:

$$a_2\alpha_3b_2\lambda\mu\mathfrak{m}_1 - 2a_2\alpha_3b_2\mu^2\mathfrak{m}_1 - a_1\alpha_3b_1\mathfrak{m}_0\mathfrak{K}^3 + a_2^2\alpha_1\lambda\mu\mathfrak{m}_1 - 2a_2^2\alpha_1\mu^2\mathfrak{m}_1 - a_1^2\alpha_1\mathfrak{m}_0\mathfrak{K}^3$$
$$+\alpha_2b_2^2\lambda\mu\mathfrak{m}_1 - 2\alpha_2b_2^2\mu^2\mathfrak{m}_1 - \mathfrak{m}_0\mathfrak{K}^3\left(\alpha_2b_1^2 + \alpha_4\mathfrak{m}_0^2\right) + c_1\mathfrak{m}_0\mathfrak{K}^3 = 0,$$
$$-a_1\alpha_3b_1\mathfrak{m}_1\mathfrak{K}^3 + a_2\alpha_3b_2\lambda^2\mathfrak{m}_1\mathfrak{K} - 4a_2\alpha_3b_2\lambda\mu\mathfrak{m}_1\mathfrak{K} - 2a_2\alpha_3b_2\mu\mathfrak{m}_1\mathfrak{K}(\lambda-\mu-1)$$
$$+4a_2\alpha_3b_2\mu^2\mathfrak{m}_1\mathfrak{K} - a_1^2\alpha_1\mathfrak{m}_1\mathfrak{K}^3 + a_2^2\alpha_1\lambda^2\mathfrak{m}_1\mathfrak{K} - 4a_2^2\alpha_1\lambda\mu\mathfrak{m}_1\mathfrak{K} - 2a_2^2\alpha_1\mu\mathfrak{m}_1\mathfrak{K}(\lambda-\mu-1)$$
$$+4a_2^2\alpha_1\mu^2\mathfrak{m}_1\mathfrak{K} - \alpha_2b_1^2\mathfrak{m}_1\mathfrak{K}^3 + \alpha_2b_2^2\lambda^2\mathfrak{m}_1\mathfrak{K} - 4\alpha_2b_2^2\lambda\mu\mathfrak{m}_1\mathfrak{K}$$
$$-2\alpha_2b_2^2\mu\mathfrak{m}_1\mathfrak{K}(\lambda-\mu-1) + 4\alpha_2b_2^2\mu^2\mathfrak{m}_1\mathfrak{K} + c_1\mathfrak{m}_1\mathfrak{K}^3 - 3\alpha_4\mathfrak{m}_0^2\mathfrak{m}_1\mathfrak{K}^3 = 0,$$
$$-3a_2\alpha_3b_2\lambda\mathfrak{m}_1\mathfrak{K}^2(\lambda-\mu-1) + 6a_2\alpha_3b_2\mu\mathfrak{m}_1\mathfrak{K}^2(\lambda-\mu-1) - 3a_2^2\alpha_1\lambda\mathfrak{m}_1\mathfrak{K}^2(\lambda-\mu-1) + 6a_2^2\alpha_1$$
$$\mu\mathfrak{m}_1\mathfrak{K}^2(\lambda-\mu-1) - 3\alpha_2b_2^2\lambda\mathfrak{m}_1\mathfrak{K}^2(\lambda-\mu-1) + 6\alpha_2b_2^2\mu\mathfrak{m}_1\mathfrak{K}^2(\lambda-\mu-1) - 3\alpha_4\mathfrak{m}_0\mathfrak{m}_1^2\mathfrak{K}^3 = 0,$$
$$2a_2\alpha_3b_2\mathfrak{m}_1\mathfrak{K}^3(\lambda-\mu-1)^2 + 2a_2^2\alpha_1\mathfrak{m}_1\mathfrak{K}^3(\lambda-\mu-1)^2 + 2\alpha_2b_2^2\mathfrak{m}_1\mathfrak{K}^3(\lambda-\mu-1)^2 - \alpha_4\mathfrak{m}_1^3\mathfrak{K}^3 = 0.$$

We acheive two sets of solutions:
Set 1:

$$\mathfrak{m}_0 = -\frac{(\lambda-2\mu)\sqrt{\alpha_1a_2^2 + \alpha_3a_2b_2 + \alpha_2b_2^2}}{\sqrt{2}\sqrt{\alpha_4}\mathfrak{K}}, \quad \mathfrak{m}_1 = \frac{\sqrt{2}(\lambda-\mu-1)\sqrt{\alpha_1a_2^2 + \alpha_3a_2b_2 + \alpha_2b_2^2}}{\sqrt{\alpha_4}},$$
$$c_1 = \alpha_1a_1^2 + \alpha_3a_1b_1 + \frac{\left(\lambda^2-4\mu\right)\left(\alpha_1a_2^2 + \alpha_3a_2b_2 + \alpha_2b_2^2\right)}{2\mathfrak{K}^2} + \alpha_2b_1^2 \tag{36}$$

Set 2:

$$\mathfrak{m}_0 = \frac{(\lambda-2\mu)\sqrt{\alpha_1a_2^2 + \alpha_3a_2b_2 + \alpha_2b_2^2}}{\sqrt{2}\sqrt{\alpha_4}\mathfrak{K}}, \quad \mathfrak{m}_1 = \frac{\sqrt{2}(-\lambda+\mu+1)\sqrt{\alpha_1a_2^2 + \alpha_3a_2b_2 + \alpha_2b_2^2}}{\sqrt{\alpha_4}},$$
$$c_1 = \alpha_1a_1^2 + \alpha_3a_1b_1 + \frac{\left(\lambda^2-4\mu\right)\left(\alpha_1a_2^2 + \alpha_3a_2b_2 + \alpha_2b_2^2\right)}{2\mathfrak{K}^2} + \alpha_2b_1^2 \tag{37}$$

We obtain the subsequent sets of solution families:
Family 1: When $\Delta = \lambda^2 - 4\mu > 0$,

$$u(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\times \\ \left(\mathfrak{m}_0 + \mathfrak{m}_1\left(\frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2\mathfrak{K}(\lambda-\mu-1)}\tanh\left(\frac{\sqrt{\Delta}(a_2x+b_2y-c_2t)}{2\mathfrak{K}}\right)\right)\right), \tag{38}$$

Subject to the constraint: $(\lambda-2)(p_2-p_1) - \sqrt{\Delta}(p_2+p_1) = 0$,

$$u(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\times \\ \left(\mathfrak{m}_0 + \mathfrak{m}_1\left(\frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2\mathfrak{K}(\lambda-\mu-1)}\coth\left(\frac{\sqrt{\Delta}(a_2x+b_2y-c_2t)}{2\mathfrak{K}}\right)\right)\right), \tag{39}$$

Subject to the constraint: $(\lambda-2)(p_2+p_1) - \sqrt{\Delta}(p_2-p_1) = 0$.
Family 2: When $\Delta = \lambda^2 - 4\mu < 0$,

$$u(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\times \\ \left(\mathfrak{m}_0 + \mathfrak{m}_1\left(\frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} + \frac{\sqrt{-\Delta}}{2\mathfrak{K}(\lambda-\mu-1)}\tan\left(\frac{\sqrt{-\Delta}(a_2x+b_2y-c_2t)}{2\mathfrak{K}}\right)\right)\right), \tag{40}$$

Subject to the constraint: $(\lambda-2)p_2 + \sqrt{-\Delta}p_1 = 0$,

$$u(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\times \\ \left(\mathfrak{m}_0 + \mathfrak{m}_1\left(\frac{\lambda-2\mu}{2\mathfrak{K}(\lambda-\mu-1)} + \frac{\sqrt{-\Delta}}{2\mathfrak{K}(\lambda-\mu-1)}\cot\left(\frac{\sqrt{-\Delta}(a_2x+b_2y-c_2t)}{2\mathfrak{K}}\right)\right)\right), \tag{41}$$

Subject to the constraint: $(\lambda-2)p_1 - \sqrt{-\Delta}p_2 = 0$.

3.2 The extended hyperbolic function method

Equation (15) introduce the solution as follows:

$$v(\eta) = f_0 + f_1G(\eta). \tag{42}$$

The first form
Substituting Equation (42) and Equation (16) in Equation (32), then grouping terms of like powers and setting all coefficients to zero leads to the following set of equations:

$$\begin{aligned} &c_1f_0 - a_1^2f_0\alpha_1 - b_1^2f_0\alpha_2 - a_1b_1f_0\alpha_3 - f_0^3\alpha_4 = 0, \\ &c_1f_1 - a_1^2f_1\alpha_1 + \beta a_2^2f_1\alpha_1 - b_1^2f_1\alpha_2 + \beta b_2^2f_1\alpha_2 - a_1b_1f_1\alpha_3 + \beta a_2b_2f_1\alpha_3 \\ &-3f_0^2f_1\alpha_4 = 0, \quad -3f_0f_1^2\alpha_4 = 0, \\ &2\mu a_2^2f_1\alpha_1 + 2\mu b_2^2f_1\alpha_2 + 2\mu a_2b_2f_1\alpha_3 - f_1^3\alpha_4 = 0. \end{aligned} \tag{43}$$

Solving this system yields two sets of solutions:
Set 1:

$$\begin{aligned} &f_0 = 0, \quad f_1 = -\frac{\sqrt{2\mu(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha_4}}, \\ &c_1 = a_1^2\alpha_1 - \beta a_2^2\alpha_1 + b_1^2\alpha_2 - \beta b_2^2\alpha_2 + a_1b_1\alpha_3 - \beta a_2b_2\alpha_3. \end{aligned} \tag{44}$$

Set 2:

$$f_0 = 0, \quad f_1 = \frac{\sqrt{2\mu(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_{_4}},$$
$$c_1 = a_1^2\alpha_1 - \beta a_2^2\alpha_1 + b_1^2\alpha_2 - \beta b_2^2\alpha_2 + a_1b_1\alpha_3 - \beta a_2b_2\alpha_3. \tag{45}$$

Therefore, we derive the subsequent sets of solution families as follows:

Family 1: When $\beta > 0$ and $\mu > 0$,

$$u_1(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\pm\Bigg(\frac{\sqrt{2\mu(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_{_4}}\Bigg)$$
$$\Bigg(\sqrt{\frac{\beta}{\mu}}csch(\sqrt{\beta}((a_2x + b_2y - c_2t) + \eta_0))\Bigg)\Bigg). \tag{46}$$

Family 2: When $\beta < 0$ and $\mu > 0$,

$$u_2(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\Bigg(\frac{\sqrt{2\mu(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_{_4}}\Bigg)$$
$$\Bigg(\sqrt{-\frac{\beta}{\mu}}\sec(\sqrt{-\beta}((a_2x + b_2y - c_2t) + \eta_0))\Bigg)\Bigg). \tag{47}$$

Family 3: When $\beta > 0$ and $\mu < 0$,

$$u_3(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\pm\Bigg(\frac{\sqrt{2\mu(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_{_4}}\Bigg)$$
$$\Bigg(\sqrt{-\frac{\beta}{\mu}}sech(\sqrt{\beta}((a_2x + b_2y - c_2t) + \eta_0))\Bigg)\Bigg). \tag{48}$$

Family 4: When $\beta < 0$ and $\mu < 0$,

$$u_4(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\pm\Bigg(\frac{\sqrt{2\mu(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_{_4}}\Bigg)\Bigg(\sqrt{-\frac{\beta}{\mu}}\csc(\sqrt{-\beta}(\eta + \eta_0))\Bigg)\Bigg). \tag{49}$$

Family 7: When $\beta = 0$ and $\mu > 0$,

$$u_7(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\pm\Bigg(\frac{\sqrt{2\mu(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_{_4}}\Bigg)\Bigg(\frac{1}{(\sqrt{\mu}((a_2x + b_2y - c_2t) + \eta_0))}\Bigg)\Bigg). \tag{50}$$

The Second form

Substituting Equation (42) and Equation (25) in Equation (32), then collecting terms of like powers and setting all the coefficients equal to zero, we acquire the following set of equations:

$$c_1f_0 - a_1^2f_0\alpha_1 - b_1^2f_0\alpha_2 - a_1b_1f_0\alpha_3 - f_0^3\alpha_4 = 0,$$
$$c_1f_1 - a_1^2f_1\alpha_1 + 2\beta\mu a_2^2f_1\alpha_1 - b_1^2f_1\alpha_2 + 2\beta\mu b_2^2f_1\alpha_2 - a_1b_1f_1\alpha_3 + 2\beta\mu a_2b_2f_1\alpha_3$$
$$- 3f_0^2f_1\alpha_4 = 0, \quad 3f_0f_1^2\alpha_4 = 0, \tag{51}$$
$$2\mu^2a_2^2f_1\alpha_1 + 2\mu^2b_2^2f_1\alpha_2 + 2\mu^2a_2b_2f_1\alpha_3 - f_1^3\alpha_4 = 0.$$

Solving this system yields two sets of solutions:
Set 1:

$$\begin{aligned} &f_0 = 0,\ \ f_1 = -\frac{\mu\sqrt{2(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_4}, \\ &c_1 = a_1^2\alpha_1 - 2\beta\mu a_2^2\alpha_1 + b_1^2\alpha_2 - 2\beta\mu b_2^2\alpha_2 + a_1b_1\alpha_3 - 2\beta\mu a_2b_2\alpha_3. \end{aligned} \tag{52}$$

Set 2:

$$\begin{aligned} &f_0 = 0,\ \ f_1 = \frac{\mu\sqrt{2(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_4}, \\ &c_1 = a_1^2\alpha_1 - 2\beta\mu a_2^2\alpha_1 + b_1^2\alpha_2 - 2\beta\mu b_2^2\alpha_2 + a_1b_1\alpha_3 - 2\beta\mu a_2b_2\alpha_3. \end{aligned} \tag{53}$$

Therefore, the solitary wave solutions of Equation (1) are given by the next formulas:

Family 1: When $\beta\mu > 0$,

$$\begin{aligned} u_1(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\left(\frac{\mu\sqrt{2(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_4}\right) \\ \left(sgn(\beta)\sqrt{\frac{\beta}{\mu}}\tan(\sqrt{\beta\mu}((a_2x + b_2y - c_2t) + \eta_0))\right)\Bigg). \end{aligned} \tag{54}$$

Family 2: When $\beta\mu > 0$,

$$\begin{aligned} u_2(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\left(\frac{\mu\sqrt{2(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_4}\right) \\ \left(sgn(\beta)\sqrt{\frac{\beta}{\mu}}\cot(\sqrt{\beta\mu}((a_2x + b_2y - c_2t) + \eta_0))\right)\Bigg). \end{aligned} \tag{55}$$

Family 3:When $\beta\mu < 0$,

$$\begin{aligned} u_3(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\left(\frac{\mu\sqrt{2(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_4}\right) \\ \left(sgn(\beta)\sqrt{-\frac{\beta}{\mu}}\tanh(\sqrt{-\beta\mu}((a_2x + b_2y - c_2t) + \eta_0))\right)\Bigg). \end{aligned} \tag{56}$$

Family 4: When $\beta\mu < 0$,

$$\begin{aligned} u_4(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\Bigg(\mp\left(\frac{\mu\sqrt{2(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_4}\right) \\ \left(sgn(\beta)\sqrt{-\frac{\beta}{\mu}}\coth(\sqrt{-\beta\mu}((a_2x + b_2y - c_2t) + \eta_0))\right)\Bigg). \end{aligned} \tag{57}$$

Family 5: When$\beta = 0$ and $\mu > 0$,

$$u_5(x,y,t) = e^{i(a_1x+b_1y-c_1t)}\left(\pm\left(\frac{\mu\sqrt{2(a_2^2\alpha_1 + b_2^2\alpha_2 + a_2b_2\alpha_3)}}{\sqrt{\alpha}_4}\right)\left(\frac{1}{\mu(\eta+\eta_0)}\right)\right). \tag{58}$$

4 Graphical illustrations

In the present section, we exhibit our solutions graphically. The graphs are presented as follows: The graph of Equation (38) for $u(x,y,t)$ Equation (38) with set 1 (36) applying the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method is shown in Figure 1 at $\mathfrak{K}=2, \lambda=\sqrt{6}, \mu=1.3, a_1=1, a_2=1, b_1=1, b_2=0.1, \alpha_1=0.1, \alpha_2=0.005, \alpha_3=0.005, \alpha_4=0.004, p_2=-3.02046p_1$. Equation (38) for $u(x,y,t)$ with set 2 Equation (37) are plotted in Figure 2 using the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method at $\mathfrak{K}=2, \lambda=\sqrt{6}, \mu=1.3, a_1=1, a_2=1, b_1=1, b_2=1, \alpha_1=0.1, \alpha_2=0.1, \alpha_3=0.005, \alpha_4=0.004, p_2=-3.02046p_1$. Figure 3 presents the graph of Equation (40) for $u(x,y,t)$ with set 1 Equation (36) employing the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method at $\mathfrak{K}=2, \lambda=0.01, \mu=0.3, a_1=0.1, a_2=0.1, b_1=0.1, b_2=0.01, \alpha_1=0.3, \alpha_2=0.02, \alpha_3=0.4, \alpha_4=0.5, p_2=0.550452p_1$. Figure 4 shows the graph of Equation (40) for $u(x,y,t)$ with set 2 Equation (37) using the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method at $\mathfrak{K}=2, \lambda=0.01, \mu=0.3, a_1=0.1, a_2=0.1, b_1=0.1, b_2=0.01, \alpha_1=0.3, \alpha_2=0.02, \alpha_3=0.4, \alpha_4=0.1, p_2=0.550452p_1$. Figure 5 shows the graph of (48) for $u_3(x,y,t)$ with set 1 Equation (44) applying the extended hyperbolic function method at $\alpha_2=0.3, \beta=0.1, \alpha_3=0.005, \mu=-0.4, \alpha_1=0.005, a_1=1, b_1=1, \alpha_4=0.05, b_2=1, \eta_0=0.3, a_2=1$. Figure 6 presents the graph of Equation (48) for $u_3(x,y,t)$ with set 2 Equation (45) using the extended hyperbolic function method at $\beta=0.1, \alpha_3=0.005, \mu=-0.6, \alpha_1=0.005, a_1=1, \alpha_2=0.3, a_2=1, b_1=1, b_2=1, \eta_0=0.3, \alpha_4=0.05$. Figure 7 shows the graph of Equation (56) for $u_3(x,y,t)$ with set 1 Equation (52) applying the extended hyperbolic function method at $\beta=-0.1, \alpha_2=0.5, \mu=0.4, \alpha_1=0.1, a_2=1, \alpha_3=0.3, b_1=1, \alpha_4=0.6, b_2=1, \eta_0=0.8, a_1=1$. Finally, Figure 8 presents the graph of (56) for $u_3(x,y,t)$ with set 2 Equation (53) using the extended hyperbolic function method at $\alpha_2=0.5, \beta=-0.1, \mu=0.4, \alpha_1=0.1, a_2=1, \alpha_3=0.3, b_1=1, \alpha_4=0.6, b_2=1, \eta_0=0.8, a_1=1$.

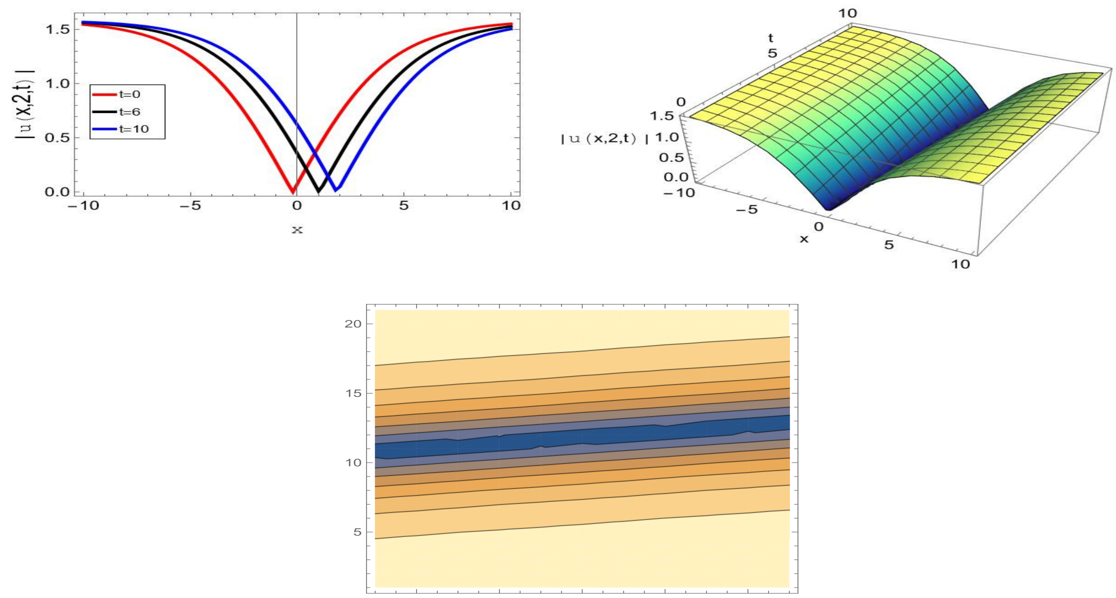

Figure 1: Solution Equation (38) for $u(x,y,t)$(Set 1, Equation (36)) obtained using the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method with parameters: $\mathfrak{K}=2, \lambda=\sqrt{6}, \mu=1.3, a_1=1, a_2=1, b_1=1, b_2=0.1, \alpha_1=0.1, \alpha_2=0.005, \alpha_3=0.005, \alpha_4=0.004, p_2=-3.02046p_1$.

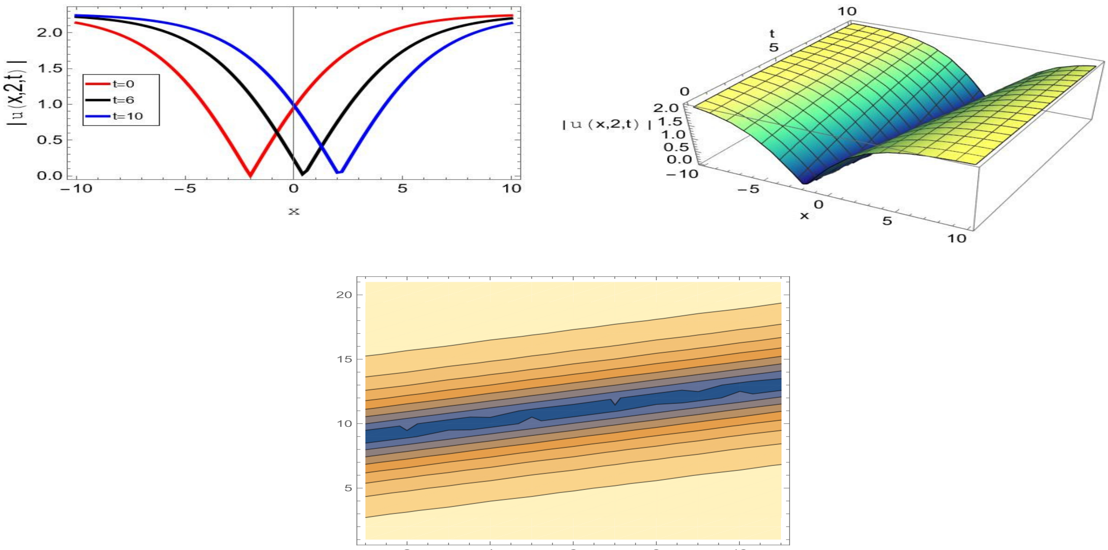

Figure 2: Solution Equation (38) for $u(x, y, t)$(Set 2, Equation (37)) obtained using the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method with parameters: $\mathfrak{K} = 2, \lambda = \sqrt{6}, \mu = 1.3, a_1 = 1, a_2 = 1, b_1 = 1, b_2 = 1, \alpha_1 = 0.1, \alpha_2 = 0.1, \alpha_3 = 0.005, \alpha_4 = 0.004, p_2 = -3.02046p_1$.

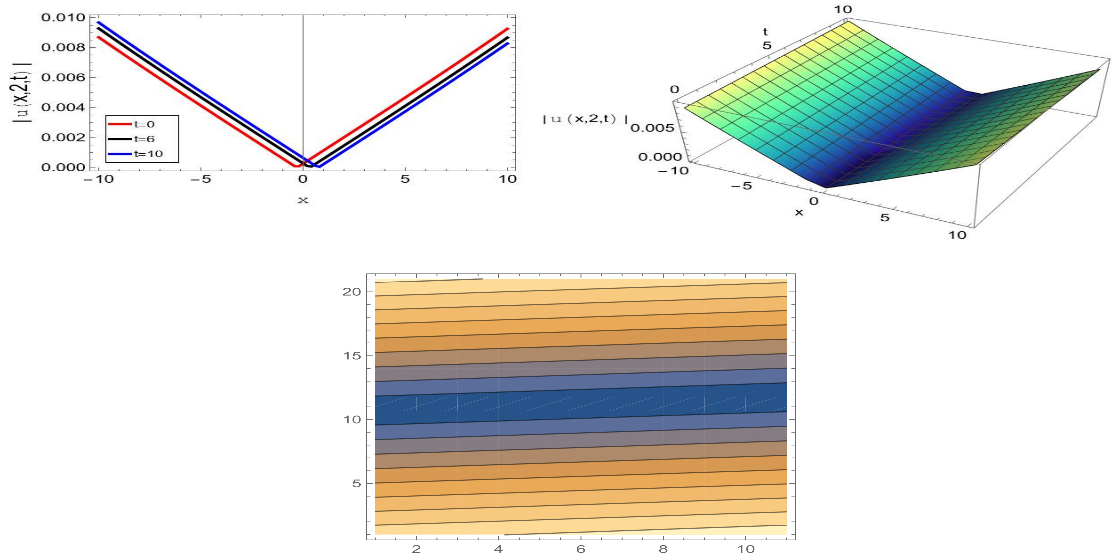

Figure 3: Solution Equation (40) for $u(x, y, t)$(Set 1, Equation (36)) obtained using the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method with parameters: $\mathfrak{K} = 2, \lambda = 0.01, \mu = 0.3, a_1 = 0.1, a_2 = 0.1, b_1 = 0.1, b_2 = 0.01, \alpha_1 = 0.3, \alpha_2 = 0.02, \alpha_3 = 0.4, \alpha_4 = 0.5, p_2 = 0.550452p_1$.

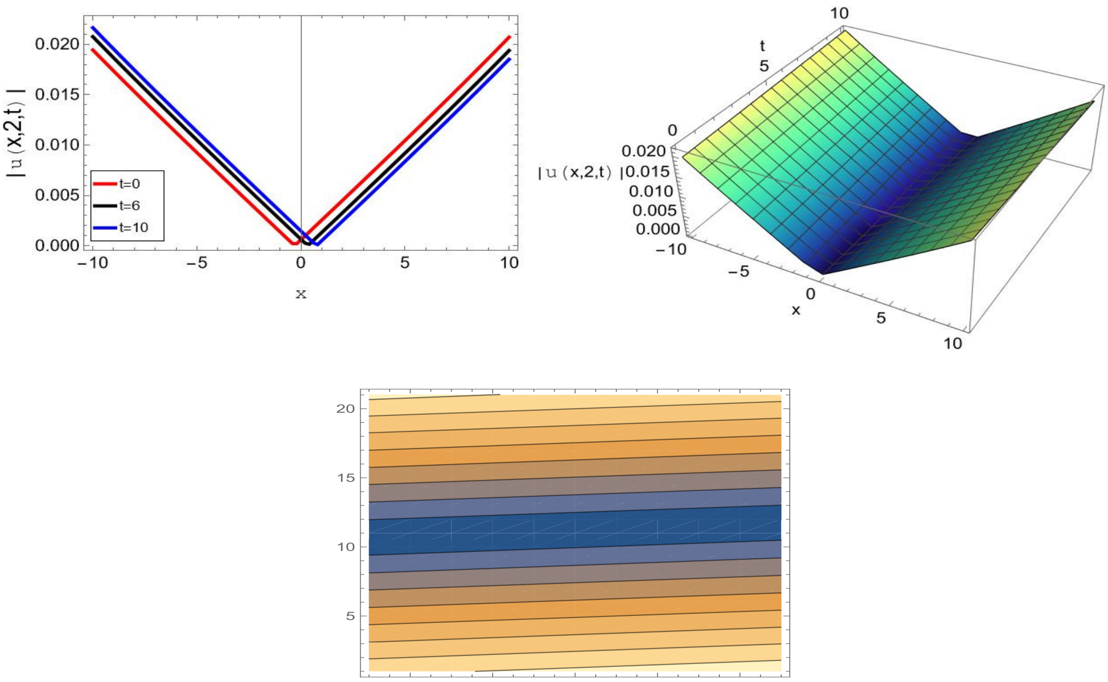

Figure 4: Solution Equation (39) for $u(x, y, t)$(Set 2, Equation (37)) obtained using the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method with parameters: $\mathfrak{K} = 2, \lambda = 0.01, \mu = 0.3, a_1 = 0.1, a_2 = 0.1, b_1 = 0.1, b_2 = 0.01, \alpha_1 = 0.3, \alpha_2 = 0.02, \alpha_3 = 0.4, \alpha_4 = 0.1, p_2 = 0.550452p_1$.

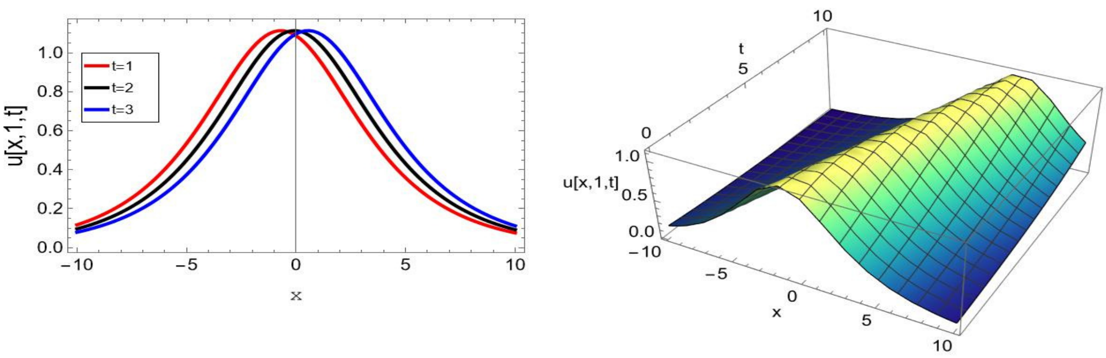

Figure 5: Solution Equation (48) for $u_3(x, y, t)$ (Set 1, Equation (44)) obtained using the Extended Hyperbolic Function method with parameters: $\alpha_2 = 0.3$, $\beta = 0.1$, $\alpha_3 = 0.005$, $\mu = -0.4$, $\alpha_1 = 0.005$, $a_1 = 1$, $b_1 = 1$, $\alpha_4 = 0.05$, $b_2 = 1$, and $\eta_0 = 0.3$, $a_2 = 1$.

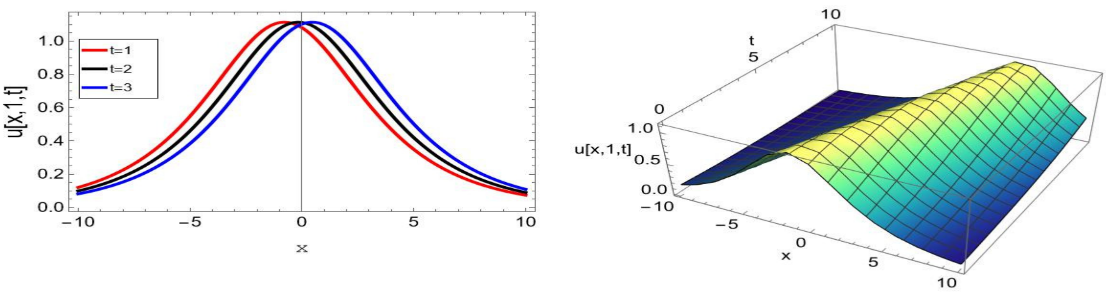

Figure 6: Solution Equation (48) for $u_3(x,y,t)$ (Set 2, Equation (45)) obtained using the extended hyperbolic function method with parameters: $\beta = 0.1, \alpha_3 = 0.005, \mu = -0.6, \alpha_1 = 0.005, a_1 = 1, \alpha_2 = 0.3, a_2 = 1, b_1 = 1, b_2 = 1, \eta_0 = 0.3, \alpha_4 = 0.05$.

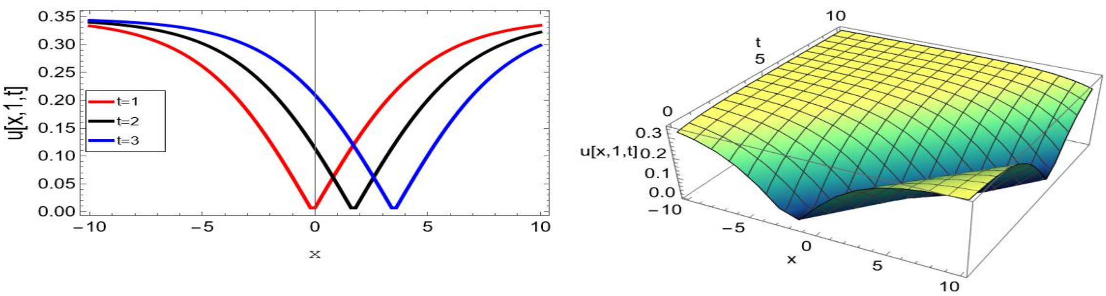

Figure 7: Solution Equation (56) for $u_3(x,y,t)$(Set 1, Equation (52)) obtained using the extended hyperbolic function method with parameters: $\beta = -0.1, \alpha_2 = 0.5, \mu = 0.4, \alpha_1 = 0.1, a_2 = 1, \alpha_3 = 0.3, b_1 = 1, \alpha_4 = 0.6, b_2 = 1, \eta_0 = 0.8, a_1 = 1$.

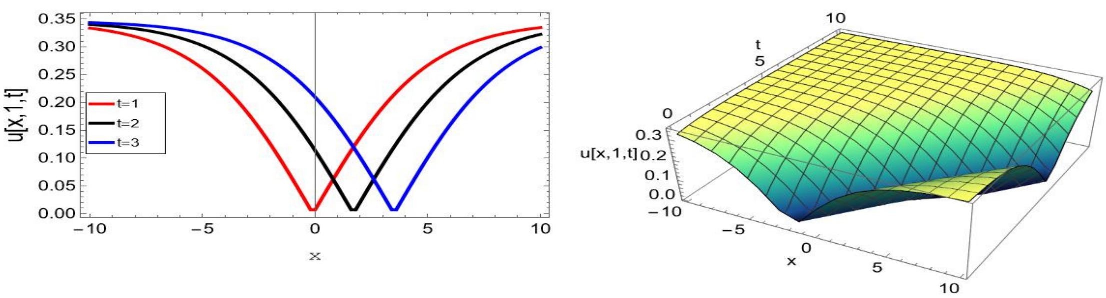

Figure 8: Solution Equation (56) for $u_3(x,y,t)$(Set 2, Equation (53)) obtained using the extended hyperbolic function method with parameters: $\alpha_2 = 0.5, \beta = -0.1, \mu = 0.4, \alpha_1 = 0.1, a_2 = 1, \alpha_3 = 0.3, b_1 = 1, \alpha_4 = 0.6, b_2 = 1, \eta_0 = 0.8, a_1 = 1$.

5 Discussion

The (2+1)-dimensional Heisenberg ferromagnetic spin chain (HFSC) equation is an important nonlinear Schrödinger equation that can be used to model the dynamics of spin systems in magnetic materials. Analytical solutions of this equation can provide insights into the fundamental physics of spin systems and their interactions with light, which can be useful in the design and optimization of various devices in optics and related fields.
The $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method is a versatile and systematic approach for solving a wide range of nonlinear partial differential equations. This method is applicable to diverse problems, with the flexibility to determine key parameters, providing insights into the system's behavior. It categorizes solutions into distinct classes based on the discriminant, making it a valuable tool for understanding complex physical and mathematical phenomena. The extended hyperbolic function method excels in solving nonlinear partial differential equations, its capability to handle diverse nonlinearities makes it a valuable tool for obtaining precise mathematical solutions. These solutions can be used to design and optimize various devices in optics and related fields.
Graphs are formidable tools for illustrating and clarifying solution representations. With the specified parameter values indicated alongside each graph, the profiles of the two-dimensional graphs exhibit a characteristic bell-shaped form and the wave moves to the right as time increases in Figures 5 and 6. In Figures 1,2,3,4, 7 and 8 the reverse wave moves to the right as time progresses.

6 Conclusion

The Heisenberg ferromagnetic spin chain equation has important applications in optics and related fields, including spintronics, quantum dots, and quantum optics. The model can aid in the design and optimization of various devices and can provide insights into the fundamental physics of spin systems and their interactions with light. In this study, We have scrutinized the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation by using two strong analytical methods, the first one is the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method and the second one is the extended hyperbolic function method (EHFM). Our methods carry new soliton solutions with various types as: bright soliton solutions, dark soliton solutions, periodic solutions, Singular solutions, and others. Choosing appropriate values for the parameters we present most of our solutions graphically to show the power of the proposed methods.
The obtained solutions in this research have many forms: hyperbolic, trigonometric, power, exponential, and rational functions, and the solutions presented analytically in other researches have also many forms, and therefore it is difficult to make a comparison in tables. However, a comparison with recent references [30, 32] was made during the solution through graphics, which proved the compatibility of our results with others.
In future research we can applying the $(\frac{\mathfrak{G}'}{\mathfrak{K}\mathfrak{G}'+\mathfrak{G}+\mathfrak{r}})$-expansion method and the extended hyperbolic function technique (EHFM) to:
1-Extending the equation to higher dimensions, including (3+1) or beyond. This expansion would significantly enhance our understanding of the equation's behavior in more complex systems, shedding light on emergent phenomena and collective interactions within the spin chain. 2-Solve the nonlinear modified Gardner (mG) equation, which holds particular importance in comprehending the intricate dynamics of quantum electron-positron ion magneto plasmas [47].

References

[1] Mikeska, H.J., Steiner, M., H.J.”Solitary excitations in one-dimensional magnets”, Advances in Physics, **40(3)**, pp.343, (1991).

[2] Zakharov,V.E., Takhtadzhyan, L.A. ”Equivalence of the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet”, Theoretical and Mathematical Physics, **38**, pp. 17-23, (1979).

[3] Nakamura, K., Sasada,T., ”Gauge equivalence between one-dimensional Heisenberg ferromagnets with single-site anisotropy and nonlinear Schrödinger equations”, Journal of Physics C: Solid State Physics, **15(26)**, pp.915, (1982).

[4] Riaz, M.B, Jhangeer, A., Atangana, A., et al., ”Supernonlinear wave, associated analytical solitons, and sensitivity analysis in a two-component Maxwellian plasma”, Journal of King Saud University-Science, **34(5)**, pp.102108, (2022).

[5] Islam, S.M.R., Akbulut, A., Arafat, S.M Y., ”Exact solutions of the different dimensional CBS equations in mathematical physics”, Partial Differential Equations in Applied Mathematics, **5**, pp.100320, (2022).

[6] Samina, S., Jhangeer, A., Chen, Z., A study of phase portraits, ”multistability and velocity profile of magneto-hydrodynamic Jeffery?Hamel flow nanofluid”, Chinese Journal of Physics, **80**, pp.397-413, (2022).

[7] Almusawa, H., Jhangeer, A., ”Nonlinear self-adjointness, conserved quantities and Lie symmetry of dust size distribution on a shock wave in quantum dusty plasma”, Communications in Nonlinear Science and Numerical Simulation, **114**, pp.106660, (2022).

[8] Almusawa, H., Jhangeer, A., ”A Study of the Soliton Solutions with an Intrinsic Fractional Discrete Nonlinear Electrical Transmission Line, Fractal and Fractional”, **6(6)**, pp.334, (2022).

[9] Bashar, Md.H., Inc, M., Islam, S.M.R., et al., ”Soliton solutions and fractional effects to the time-fractional modified equal width equation”, Alexandria Engineering Journal, **61(12)**, pp.12539-12547, (2022).

[10] Rtaz, M.B., Awrejcewicz, J., Jhangeer, A., et al., ”Appraisal Of Analytical Solutions For(2+1)-Dimensional Nonlinear Chiral Schrödinger Equation”, Fractals, **30(05)**, pp.2240159, (2022).

[11] Islam, S.M.R, Bashar, Md.H., Arafat, S.M Y., et al., ”Effect of the free parameters on the Biswas-Arshed model with a unified technique”, Chinese Journal of Physics, **77**, pp.2501-2519, (2022).

[12] Jhangeer, A., Almusawa, H., Ur Rahman, R., ”Fractional derivative-based performance analysis to Caudrey-Dodd-Gibbon-Sawada-Kotera equation”, Results in Physics, **36**, pp.105356, (2022).

[13] Khan, K., Mudaliar, R.K., Islam, S.M.R.,”Traveling Waves in Two Distinct Equations: The (1+1)-Dimensional cKdV-mKdV Equation and The sinh-Gordon Equation”, International Journal of Applied and Computational Mathematics, **9**, pp.21, (2023).

[14] Ren, B., He, Z.W. , Sun, Y.Li , et al., ”Dynamics of Mixed Lump-Soliton Solutions According to a (2+1)-Dimensional Coupled Nonlinear Partial Differential Like Equation”, Journal of the Korean Physical Society, **74(8)**, pp.744-750, (2019).

[15] Ren, B., "Interaction solutions for mKP equation with nonlocal symmetry reductions and CTE method", Physica Scripta, **90(6)**, pp. 065206. (2015).

[16] Ren, B., Cheng, X.P., "CTE Solvability, Nonlocal Symmetry and Explicit Solutions of Modified Boussinesq System", Communications in Theoretical Physics, **66(1)**, pp.84-92, (2016).

[17] Ahmad, J., Akram, S., Noor, K., et al., "Soliton solutions of fractional extended nonlinear Schrödinger equation arising in plasma physics and nonlinear optical fiber", Scientific Reports, **13(1)**, pp.10877, (2023).

[18] Ahmad, J., Rani, S., Bin Turki, N., et al., "Novel resonant multi-soliton solutions of time fractional coupled nonlinear Schrödinger equation in optical fiber via an analytical method", Results in Physics, **52(4)**, pp.106761, (2023).

[19] Akram, S., Ahmad, J., Bin Turki, N., et al., "On the exploration of soliton solutions of the nonlinear Manakov system and its sensitivity analysis", Results in Physics, **52(6)**, pp.106772, (2023).

[20] Ahmad, J., Mustafa, Z., Ur-Rehman, S., et al., "Solitary wave structures for the stochastic Nizhnik-Novikov-Veselov system via modified generalized rational exponential function method", Results in Physics, **52(5)**, pp.106776, (2023).

[21] Islam, S.M.R., Khan, K., Akbar, M.A., "Optical soliton solutions, bifurcation, and stability analysis of the Chen-Lee-Liu model", Results in Physics, **51**, pp.106620, (2023).

[22] Anitha, T., Latha, M.M., Vasanthi, C.C., "Dromions in (2+1)-dimensional ferromagnetic spin chain with bilinear and biquadratic interactions", Physica A: Statistical Mechanics and its Applications, **415**, pp.105-115, (2014).

[23] Tang, G., Wang, S., Wang, G., "Solitons and complexitons solutions of an integrable model of (2+1)-dimensional Heisenberg ferromagnetic spin chain, Nonlinear Dynamics", **88**, pp.2319-2327, (2017).

[24] Bashar, Md.H., Islam, S.M.R., "Exact solutions to the (2+1)-Dimensional Heisenberg ferromagnetic spin chain equation by using modified simple equation and improve F-expansion methods", Physics Open, **5**, pp.100027, (2020).

[25] Bashar, Md.H., Islam, S.M.R., Kumar, D., "Construction of traveling wave solutions of the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation", Partial Differential Equations in Applied Mathematics, **4**, pp.100040, (2021).

[26] Osman, M.S., Tariq, K.U., Bekir, A., et al., "Investigation of soliton solutions with different wave structures to the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation", Communications in Theoretical Physics, **72(3)**, pp.035002, (2020).

[27] Nisar, K.S., Inc, M., Jhangeer, A., et al., "New soliton solutions of Heisenberg ferromagnetic spin chain model", Pramana-J.Physics, **96(28)**, (2022).

[28] Aliyu, A.I., Li, Y., Inc, M., et al., "Solitons and complexitons to the(2+1)-dimensional Heisenberg ferromagnetic spin chain model", International Journal of Modern Physics B, **33(30)**, pp. 1950368, (2019).

[29] Ma, Yu.L., Lump wave phase transition for the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation, Optik, **231**, pp.166505, (2021).

[30] Bulut, H., Sulaiman, T.A., Baskonus, H.M., "Dark, bright and other soliton solutions to the Heisenberg ferromagnetic spin chain equation", Superlattices and Microstructures, **123**, pp.12-19, (2018).

[31] Han, T., Wen, J., Li, Z., et al., "New Traveling Wave Solutions for the (2+1)-Dimensional Heisenberg Ferromagnetic Spin Chain Equation", Hindawi, Mathematical Problems in Engineering, **2022**, pp.1-9, (2022).

[32] Seadawy, A.R., Nasreen, N., Lu, D., et al., "Arising Wave propagation in nonlinear media for the(2+1)-dimensional Heisenberg Ferromagnetic Spin Chain dynamical Model", Physica A: Statistical Mechanics and its Applications, **538**, pp.122846, (2020).

[33] Abdul Al Woadud, K.M., Kumar, D., Islam, Md.J., et al., "Extraction of Solitary Wave Features to the Heisenberg Ferromagnetic Spin Chain and the Complex Klein-Gordon Equations", International Journal of Applied and Computational Mathematics, **5**, pp.1-11, (2019).

[34] Sahooa, S., Tripathy, A., "New exact solitary solutions of the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation", The European Physical Journal Plus, **137(3)**, (2022).

[35] Islam, S., Halder, B., Ali, A.R., "Optical and rogue type soliton solutions of the (2+1) dimensional nonlinear Heisenberg ferromagnetic spin chains equation", Scientific Reports, **13**, pp.1-10, (2023).

[36] Mohammed, W.W., Al-Askar, F.M., Cesarano, C., et al., "Wiener Process Effects on the Solutions of the Fractional (2 + 1)-Dimensional Heisenberg Ferromagnetic Spin Chain Equation", Mathematics, **10(12)**, pp.2043, (2022).

[37] Zahran, E.H.M., Bekir, A., "Enormous soliton solutions to a (2+1)-dimensional Heisenberg ferromagnetic spin chain equation", Chinese Journal of Physics, **77**, pp.1236-1252, (2022).

[38] Wang, K.J., Shi, F., Wang, G.D., "Abundant Soliton Structures to the (2+1)-Dimensional Heisenberg Ferromagnetic Spin Chain Dynamical Model", Advances in Mathematical Physics, **2023**, pp.1-9, (2023).

[39] Al-Askar, F.M., "The solitary wave solutions of the stochastic Heisenberg ferromagnetic spin chain equation using two different analytical methods", Frontiers in Physics, **2023**, pp.1-9, (2023).

[40] Hong, B., "Assorted exact explicit solutions for the generalized Atangana's fractional BBM-Burgers equation with the dissipative term", Frontiers in Physics, **10**, pp.1-10, (2022).

[41] Shang, Y., Huang, Y., Yuan, W., "The extended hyperbolic functions method and new exact solutions to the Zakharov equations", Applied Mathematics and Computation, **200**, pp.110-122, (2008).

[42] Ur Rehman, H., Imran, M.A., Ullah, N., et al., "Exact solutions of (2+1)-dimensional Schrödinger's hyperbolic equation using different techniques", Numerical Methods for Partial Differential Equations, **2020**, pp.1-20, (2020).

[43] Nestor, S.S., Houwe, A., Betchewe, G., et al., "A series of abundant new optical solitons to the conformable space-time fractional perturbed nonlinear Schrödinger equation", Physica Scripta, **95**, pp.1-9, (2020).

[44] Ur Rehman, H., Iqbal, I., Hashemi, M.S., et al., "Analysis of cubic-quartic-nonlinear Schrödinger's equation with cubic-quintic-septic-nonic form of self-phase modulation through different techniques", Optik, **287**, pp.171028, (2023).

[45] Shi, Da , Ur Rehman, H., Iqbal, I., et al., "Analytical study of the dynamics in the double-chain model of DNA", Results in Physics, **52**, pp.106787, (2023).

[46] Ur Rehman, H., Awan, A.U., Tag-ElDin, E.M., et al., Haider, R., "Extended hyperbolic function method for the (2+1)-dimensional nonlinear soliton equation", Results in Physics, **40**, pp.105802, (2022).

[47] Jhangeer, A., Hussain, A., Junaid-U-Rehman, M., et al., Quasi-periodic, chaotic and travelling wave structures of modified Gardner equation, Chaos, Solitons and Fractals, **143**, pp.110578, (2021).

Biographies

Khalid K. Ali
I have authored/co-authored over 160 scientific papers in top-ranked International Journals and Conference Proceedings. My research interests include Numerical Analysis, Finite Difference Methods, Finite Element Methods, Computational Mathematics, Partial differential equations, Integral equations, Integro-differential equation, and q-calculus.

Abdel-Haleem Abdel-Aty
Professor (Assistant) at University of Bisha Saudi Arabia. I have authored/co-authored over 256 scientific papers in top-ranked International Journals and Conference Proceedings. My research interests include theories of quantum measurement, nanomechanical modeling, highly non-classical light, practical information security, and optical implementations of quantum information tasks. His current research interests include quantum resources, optical and atomic implementations of quantum information tasks and protocols, mathematical modeling, differential equations and applications in science and engineering and classifications algorithms.

M.S.Mehanna
Professor (Assistant) at Modern Technology and Information University. I have authored/co-authored over 27 scientific papers in top-ranked International Journals and Conference Proceedings. My research interests include Computational Mathematics, Partial differential equations, Integral equations, Integro-differential equation and applications in science, engineering and classifications algorithms.