# Cubic bipolar fuzzy aggregation operator with priority degree with multi-criteria decision-making

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**Abstract**

Cubic bipolar fuzzy numbers ($CBFN$) are useful for real-world ambiguous data. Prioritised $MCDM$s use priority degrees. Aggregation operators ($AOs$) result from tight priority levels and priority degrees. Thus, "cubic bipolar fuzzy prioritised averaging operator with priority degrees ($CBFPDA$)" and "cubic bipolar fuzzy geometric operator ($CBFPGD$)" are $CBFNs$ prioritised operators. Comparative studies are made. Comparison analysis verifies the proposed method. The comparison study shows the approach works. Comparing the current method to others emphasises its superiority over current operators. Priorities affect object ranking and information fusion. Discussing a 3PRLP optimisation problem's practical implementation is a secondary goal. The recommended 3PRLP reference is evaluated numerically. The best strategy is selected and compared.

**Keywords:** Fuzzy Set, cubic bipolar fuzzy set, aggregation operator, priority degrees, multi-criteria decision making.

## 1 Introduction

Decision making ($DM$) is a vital occurrence in order to choose the best option from the available options. However, due to the inadequate data and inherent human judgments, this process entails ambiguous and hazy information. Classical techniques are unable to determine the best option in the face of ambiguity for these reasons. Zadeh [1, 2] established the notion of fuzzy set ($FS$) to solve such serious challenges, and it has been successfully applied to a wide variety of real-life problems. An intuitionistic fuzzy set gives a membership grade $\mu \in [0,1]$ and a non-membership grade $v \in [0,1]$ to

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each object in the universe [3, 4]. Some extensions of fuzzy sets which are necessary to understand the notion of cubic bipolar fuzzy set are given in Table 1. Many researchers have employed these models successfully in recent decades. All of these models were created in response to the necessity to deal with uncertainty in real-world problems.

Researchers like data aggregation operators. Fuzzy number and interval data improve this model. This model has the most ratings, inaccuracy, and bipolarity. Joy and grief, drug effects and side effects, commodity sweetness and sourness, hopeful and hopeless, etc. can be shown by a $BFS$. They maintain social order. Strategic decisions are subjective, two-sided. Several authors have reported bipolar fuzzy judgements using different methods. Most $MCDM$ problems require quantitative data aggregation. Data aggregation and fusion underpin machine learning, pattern recognition, image processing, and information processing. Information gathered forms an opinion. Crisp integer-based data processing cannot mimic human cognition. These strategies help DMs draw unclear conclusions from incomplete information. DMs need theories to understand ambiguous data values and adapt their DM requirements to the context—pattern recognition or human cognition—to handle real-world ambiguous and fuzzy situations. Riaz and Jamil introduced cubic bipolar fuzzy topology in 2022 [5] and also utilize it in MCDM technique. AOs for IFSs proposed by Xu [6, 7] incorporate averaging and geometric operators. Many experts have made significant contributions to fuzzy set extensions, some important and most relevant are mentioned in the Table 2.

The main contributions of the manuscript are as follows:

- New $AOs$ with priority degree are proposed named as cubic bipolar fuzzy average operator with priority degree ($CBFAPD$) operator and cubic bipolar fuzzy geometric operator with priority degree ($CBFGPD$) operator.

- Certain properties of proposed operators are investigated including, idempotency, boundary, and monotonicity.

- A practical application of $MCDM$ under uncertainty is illustrated using the suggested operators for third party reverse logistic application.

- A numerical example is illustrated to discuss the scientific nature of the proposed MCDM approach to demonstrate its rationality, symmetry, and superiority.

The body of the article is organized as follows:
Section 2 focuses on the fundamentals of $CBFS$, along with their score function, accuracy function, and essential aggregation functions. The article concludes by showcasing some of the developed cubic bipolar fuzzy aggregation operators with priority degree in section 3. We discussed the $MCDM$ strategy as it relates to the selected operators in Section 4. In section 5, we present a case study of

third party reverse logistic providers alongside a numerical illustration. In Section 6, we present the foremost findings of this research.

## 2  Some fundamental notions

In this section, we review some rudiments of cubic bipolar fuzzy fuzzy sets ($CBFSs$) and cubic bipolar fuzzy numbers ($CBFNs$), in addition to the operational laws that govern these concepts, such as inclusion, intersection, union, sum, product, scalar multiplication, and exponents under P(R)-order. We continue our discussion on the concepts of score functions and accuracy functions for the purpose for partial ordering and ranking $CBFNs$.

**Definition 2.1.** [8] Let $V$ be a non-empty set. A $CBFS$ $\mathscr{C}$ in $V$ is defined as follows,

$$\mathscr{C} = \left\{\langle \chi,\, \mathscr{P} = [\mathscr{P}_l,\, \mathscr{P}_u],\, \mathscr{N} = [\mathscr{N}_l,\, \mathscr{N}_u],\, \lambda,\, \mu\rangle|\, \chi \in V\right\}$$

where $[\mathscr{P}_l,\, \mathscr{P}_u] \subseteq [0,\, 1]$ and $[\mathscr{N}_l,\, \mathscr{N}_u] \subseteq [-1,\, 0]$, $\lambda : V \to [0,\, 1]$ and $\mu : V \to [-1,\, 0]$.

**Definition 2.2.** [8] Let $\mathscr{C}_1 = \langle \chi,\, \mathscr{P}_1,\, \mathscr{N}_1,\, \lambda_1,\, \mu_1\rangle$ and $\mathscr{C}_2 = \langle \chi,\, \mathscr{P}_2,\, \mathscr{N}_2,\, \lambda_2,\, \mu_2\rangle$ be two $CBFSs$.Then,

$$
\begin{aligned}
\mathscr{C}_1 \bigoplus_P \mathscr{C}_2 \;=\; & \Big\{\langle \chi,\, [\mathscr{P}_{1l} + \mathscr{P}_{2l} - \mathscr{P}_{1l} * \mathscr{P}_{2l},\, \mathscr{P}_{1u} + \mathscr{P}_{2u} - \mathscr{P}_{1u} * \mathscr{P}_{2u}],\, [-\mathscr{N}_{1l} * \mathscr{N}_{2l},\, -\mathscr{N}_{1u} * \mathscr{N}_{2u}], \\
& \lambda_1 + \lambda_2 - \lambda_1 * \lambda_2,\, -\mu_1 * \mu_2\rangle|\, \chi \in V\Big\}
\end{aligned}
$$

**Definition 2.3.** [8] Let $\mathscr{C}_1 = \langle \chi,\, \mathscr{P}_1,\, \mathscr{N}_1,\, \lambda_1,\, \mu_1\rangle$ and $\mathscr{C}_2 = \langle \chi,\, \mathscr{P}_2,\, \mathscr{N}_2,\, \lambda_2,\, \mu_2\rangle$ be two $CBFSs$.Then,

$$
\begin{aligned}
\mathscr{C}_1 \bigotimes_P \mathscr{C}_2 \;=\; & \Big\{\langle \chi,\, [\mathscr{P}_{1l} * \mathscr{P}_{2l},\, \mathscr{P}_{1u} * \mathscr{P}_{2u}],\, [-(-\mathscr{N}_{1l} - \mathscr{N}_{2l} + \mathscr{N}_{1l} * \mathscr{N}_{2l}), \\
& -(-\mathscr{N}_{1u} - \mathscr{N}_{2u} + \mathscr{N}_{1u} * \mathscr{N}_{2u})],\, \lambda_1 * \lambda_2,\, -(-\mu_1 - \mu_2 - \mu_1 * \mu_2)\rangle|\, \chi \in V\Big\}
\end{aligned}
$$

**Definition 2.4.** [8] Let $\mathscr{C} = \langle \chi,\, \mathscr{P},\, \mathscr{N},\, \lambda,\, \mu\rangle$ be a $CBFS$ and $\alpha > 0$, then $\alpha$-scalar product is expressed as:

$$\mathscr{C}^\alpha = \left\{\langle \chi,\, [(\mathscr{P}_l)^\alpha,\, (\mathscr{P}_u)^\alpha],\, [-\big(1 - (1 - \mathscr{N}_l)^\alpha\big),\, -\big(1 - (1 - \mathscr{N}_u)^\alpha\big)],\, 1 - (1 - \lambda)^\alpha,\, -(-\mu)^\alpha|\, \chi \in V\right\}$$

**Definition 2.5.** [8] Let $\mathscr{C} = \langle \chi,\, \mathscr{P},\, \mathscr{N},\, \lambda,\, \mu\rangle$ be a $CBFS$ and $\alpha > 0$ then $\alpha$-scalar product is defined as:

$$\alpha * \mathscr{C} = \left\{\langle \chi,\, [1 - (1 - \mathscr{P}_l)^\alpha,\, 1 - (1 - \mathscr{P}_u)^\alpha],\, [-(-\mathscr{N}_l)^\alpha,\, -(-\mathscr{N}_u)^\alpha],\, (\lambda)^\alpha,\, -(1 - (1 - \mu)^\alpha)\rangle|\, \chi \in V\right\}$$

**Definition 2.6.** [8] Let $\mathscr{C} = \langle \chi,\, \mathscr{P},\, \mathscr{N},\, \lambda,\, \mu\rangle$ be a $CBFS$ then it's complement is defined as:

$$\mathscr{C}^c = \left\{\langle \chi,\, \mathscr{P}^c,\, \mathscr{N}^c,\, 1 - \lambda,\, 1 - \mu\rangle|\, \chi \in V\right\}$$

## 2.1   Score functions and accuracy functions

Now, we will define score functions and accuracy functions under P(R)-order which will help to order the $CBFNs$. The score functions are often used to rank fuzzy sets in multi-attribute decision making (MADM).

**Definition 2.7.** [8] For a $CBFS$ $\mathscr{C} = \langle \chi, \mathscr{P}, \mathscr{N}, \lambda, \mu \rangle$, the P-order score function for $CBFS$ is defined as:

$$S_P(\mathscr{C}) = \frac{[\mathscr{P}_l + \mathscr{P}_u] + [\mathscr{N}_l + \mathscr{N}_u] - \lambda - \mu}{6}$$

where $S_P(\mathscr{C}_1) \in [-1, 1]$

- If $S_P(\mathscr{C}_1) \le S_P(\mathscr{C}_2)$ then $\mathscr{C}_1 \le \mathscr{C}_2$

- If $S_P(\mathscr{C}_1) = S_P(\mathscr{C}_2)$ then $\mathscr{P}_1 = \mathscr{P}_2; \mathscr{C}_1 = \mathscr{C}_2$

**Definition 2.8.** [8] For a $\mathscr{CBFSC} = \langle \chi, \mathscr{P}, \mathscr{N}, \lambda, \mu \rangle$, the R-order score function for $CBFS$ is defined as:

$$S_R(\mathscr{C}) = \frac{[\mathscr{P}_l + \mathscr{P}_u] + [\mathscr{N}_l + \mathscr{N}_u] + \lambda + \mu}{6}$$

where $S_Q(\mathscr{C}_1) \in [-1, 1]$

- If $S_R(\mathscr{C}_1) \le S_Q(\mathscr{C}_2)$ then $\mathscr{C}_1 \le \mathscr{C}_2$

- If $S_R(\mathscr{C}_1) = S_Q(\mathscr{C}_2)$ then $\mathscr{P}_1 = \mathscr{P}_2; \mathscr{C}_1 = \mathscr{C}_2$

**Definition 2.9.** [8] For a $CBFS$ $\mathscr{C} = \langle \chi, \mathscr{P}, \mathscr{N}, \lambda, \mu \rangle$, the accuracy function for $CBFS$ is defined as:

$$\mathscr{A}(\mathscr{C}) = \frac{[\mathscr{P}_l + \mathscr{P}_u] + [\mathscr{N}_l + \mathscr{N}_u] + \lambda - \mu}{6}$$

where $\mathscr{A}(\mathscr{C}_1) \in [-1, 1]$

- If $\mathscr{A}(\mathscr{C}_1) \le \mathscr{A}(\mathscr{C}_2)$ then $\mathscr{C}_1 \le \mathscr{C}_2$

- If $\mathscr{A}(\mathscr{C}_1) = \mathscr{A}(\mathscr{C}_2)$ then $\mathscr{C}_1 = \mathscr{C}_2$

It's important to remember that $S \in [-1, 1]$. To enable the subsequent research, we design an innovative score function $S(\mathscr{C}) = \frac{3 + \mathscr{P}_l + \mathscr{P}_u + \mathscr{N}_l + \mathscr{N}_u + \lambda + \mu}{6}$. We can see that the score function lies between 0 and 1.

**Example 2.10.** Consider two *CBFNs* $\mathscr{C}_1$ and $\mathscr{C}_2$ as:

$$\mathscr{C}_1 = \langle [0.35, \ 0.65], \ [-0.98, \ -0.34], \ 0.40, \ -0.63 \rangle$$

$$\mathscr{C}_2 = \langle [0.25, \ 0.75], \ [-0.92, \ -0.40], \ 0.35, \ -0.77 \rangle$$

and value of scalar is $k = 3$. Calculate union, intersection, ring sum, ring product, scalar power and scalar product under P(R)-order.

1. $\mathscr{C}_1 \cup_P \mathscr{C}_2 = \langle [0.25, \ 0.75], \ [-0.92, \ -0,40], \ 0.40, \ -0.77 \rangle$
2. $\mathscr{C}_1 \cap_P \mathscr{C}_2 = \langle [0.35, \ 0.65], \ [-0.98, \ -0.34], \ 0.35, \ -0.63 \rangle$
3. $\mathscr{C}_1 \oplus_P \mathscr{C}_2 = \langle [0.5125, \ 0.9125], \ [-0.9016, \ -0.1360], \ 0.61, \ -0.4851 \rangle$
4. $\mathscr{C}_1 \otimes_P \mathscr{C}_2 = \langle [0.0875, \ 0.4875], \ [-0.9984, \ -0.6040], \ 0.14, \ -0.9149 \rangle$
5. $\mathscr{C}_1^3 = \langle [0.0429, \ 0.2746], \ [-0.9995, \ -0.7125], \ 0.0640, \ -0.9493 \rangle$ (under P-order)
6. $3 * \mathscr{C}_2 = \langle [0.5781, \ 0,9844], \ [-0.7787, \ -0.0640], \ 0.7254, \ -0.4565 \rangle$ (under P-order)
7. $\mathscr{C}_1^3 = \langle [0.0429, \ 0.2746], \ [-0.9995, \ -0.7125], \ 0.7254, \ -0.4565 \rangle$ (under R-order)
8. $3 * \mathscr{C}_2 = \langle [0.5781, \ 0,9844], \ [-0.7787, \ -0.0640], \ 0.0640, \ -0.9493 \rangle$ (under R-order)

## 2.2 Cubic bipolar fuzzy aggregation operators

In present section, we introduce CBF aggregation operators and CBF weighted aggregation operators.

**Definition 2.11. P-order CBFG operator:** Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], \ [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \ \lambda_k, \ \mu_k \right\rangle$ be collection of CBF elements (CBFEs) then CBFG operator is a mapping $\mathscr{M} : \mathscr{C}^n \to \mathscr{C}$ which we calculate under P-order as follows:

$CBFG_P(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) =$
$$\left\langle \left[ \prod_{k=1}^n (\mathscr{P}_{l_k}), \prod_{k=1}^n (\mathscr{P}_{u_k}) \right], \left[ -(1 - \prod_{k=1}^n (1 - \mathscr{N}_{l_k})), -(1 - \prod_{k=1}^n (1 - \mathscr{N}_{u_k})) \right], \prod_{k=1}^n (\lambda_k), -(1 - \prod_{k=1}^n (1 - \mu_k)) \right\rangle$$

**Definition 2.12. P-order CBFGW operator:** Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \lambda_k, \mu_k \right\rangle$ be collection of CBF elements (CBFEs) and $W = [w_1, w_2, ..., w_n]^T$ be the weight vector, where $\sum_{k=1}^n w_k = 1$ then CBFGWA operator is a mapping $\mathscr{M} : \mathscr{C}^n \to \mathscr{C}$ which we calculate under P-order as follows:

$CBFG_P(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \left\langle \left[ \prod_{k=1}^n (\mathscr{P}_{l_k})^{w_k}, \prod_{k=1}^n (\mathscr{P}_{u_k})^{w_k} \right], \left[ -(1 - \prod_{k=1}^n (1 - (\mathscr{N}_{l_k}))^{w_k}), \right. \right.$

$$\left. \left. -(1 - \prod_{k=1}^n (1 - (\mathscr{N}_{u_k}))^{w_k}) \right], \prod_{k=1}^n (\lambda_k)^{w_k}, -(1 - \prod_{k=1}^n (1 - \mu_k)^{w_k}) \right\rangle$$

**Definition 2.13. P-order CBFA operator:** Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \lambda_k, \mu_k \right\rangle$ be collection of CBF elements (CBFEs) then CBFG operator is a mapping $\mathscr{M} : \mathscr{C}^n \to \mathscr{C}$ which we calculate under P-order as follows:

$$CBFG_P(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \left\langle [1 - \prod_{k=1}^n (1 - \mathscr{P}_{l_k}), 1 - \prod_{k=1}^n (1 - \mathscr{P}_{u_k})], [-\prod_{k=1}^n (\mathscr{N}_{l_k}), -\prod_{k=1}^n (\mathscr{N}_{u_k})], 1 - \prod_{k=1}^n (1 - \lambda_k), -\prod_{k=1}^n (\mu_k) \right\rangle$$

**Definition 2.14. P-order CBFAW operator:** Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \lambda_k, \mu_k \right\rangle$ be collection of CBF elements (CBFEs) and $W = [w_1, w_2, ..., w_n]^T$ be the weight vector, where $\sum_{k=1}^n w_k = 1$ then CBFAW operator is a mapping $\mathscr{M} : \mathscr{C}^n \to \mathscr{C}$ which we calculate under P-order as follows:

$$CBFG_P(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \left\langle [1 - \prod_{k=1}^n (1 - \mathscr{P}_{l_k})^{w_k}, 1 - \prod_{k=1}^n (1 - \mathscr{P}_{u_k})^{w_k}], [-\prod_{k=1}^n (\mathscr{N}_{l_k})^{w_k}, \right.$$
$$\left. -\prod_{k=1}^n (\mathscr{N}_{u_k})^{w_k}], 1 - \prod_{k=1}^n (1 - \lambda_k)^{w_k}, -\prod_{k=1}^n (\mu_k)^{w_k} \right\rangle$$

**Example 2.15.** Consider three *CBFNs*

$$\mathscr{C}_1 = \left\langle [0.25, \ 0.53], [-0.67, \ -0.31], 0.37, \ -0.43 \right\rangle$$

,

$$\mathscr{C}_2 = \left\langle [0.37, \ 0.65], [-0.71, \ -0.39], 0.43, \ -0.65 \right\rangle$$

and

$$\mathscr{C}_3 = \left\langle [0.53, \ 0.87], [-0.83, \ -0.43], 0.65, \ -0.67 \right\rangle.$$

Calculate *CBF* geometric aggregation operators and arithmetics aggregation operators under P(R)-order. Also calculate *CBF* weighted geometric aggregation operators and weighted arithmetics aggregation operators using weights $W = \{0.3, \ 0.3 \ 0.4\}$ under P(R)-order.

**Solution:** By using definitions mentioned above, we have

1. *CBFGA* under P-order: $\left\langle [0.0490, 0.2997], [-0.9837, -0.7601], 0.1034, -0.9342 \right\rangle$

2. *CBFGWA* under P-order: $\left\langle [0.3798, 0.6870], [-0.7565, -0.3840], 0.4849, -0.6043 \right\rangle$

3. *CBFAA* under P-order: $\left\langle [0.7779, 0.9786], [-0.3948, -0.0520], 0.8743, -0.1879 \right\rangle$

4. *CBFAWA* under P-order: $\left\langle [0.4096, 0.7427], [-0.7427, 0.3785], 0.5167, -0.5812 \right\rangle$

# 3 Cubic bipolar fuzzy aggregation operator with priority degree

**Definition 3.1.** Assume that $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j}, \ \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \ \mathscr{N}_{u_j}], \lambda_j, \mu_j \right\rangle$ is the collection of *CBFNs*. A *CBFPDA* operator is defined by the mapping $\Lambda^n \to \Lambda$ is expressed as

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1{}^{d_1}\mathscr{C}_1 \oplus r_2{}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}_n \tag{1}$$

where $r_i{}^{d_i} = \frac{\mathfrak{T}_1{}^{d_i}}{\sum_{k=1}^n \mathfrak{T}_k{}^{d_k}}$ and $\mathfrak{T}_j = \prod_{k=1}^{n-1} (S(\mathscr{C}_k))^{d_k}; \ (j = 1, 2, ..., n)$ and $\mathfrak{T}_1 = 1$.

**Theorem 3.2.** *Assume that* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} , \mathscr{P}_{u_j}] , [\mathscr{N}_{l_j} , \mathscr{N}_{u_j}] , \lambda_j, \mu_j \right\rangle$ *is the collection of CBFNs. A CBFPDA operator is defined by the mapping* $\Lambda^n \to \Lambda$ *is expressed as*

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1{}^{d_1}\mathscr{C}_1 \oplus r_2{}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}_n$$

$$= \left\langle \left[1 - \prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k^d} , 1 - \prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k^d}\right] , \left[ -\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k^d} , -\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{r_k^d}\right] , \right.$$

$$\left. \prod_{k=1}^{n}\lambda_k{}^{r_k^d} , -\left(1 - \prod_{k=1}^{n}(1-\mu_k)^{r_k^d}\right) \right\rangle$$

*Proof.* To prove this theorem, we will use mathematical induction.

$$r_1{}^{d_1}\mathscr{C}_1 = \left\langle \left[1-(1-\mathscr{P}_{l_1})^{r_1^{d_1}} , 1-(1-\mathscr{P}_{u_1})^{r_1^{d_1}}\right] , \left[-(\mathscr{N}_{l_1})^{r_1^{d_1}} , -(\mathscr{N}_{u_1})^{r_1^{d_1}}\right] , \lambda_1{}^{r_1^{d_1}} , -\left(1-(1-\mu_1)^{r_1^{d_1}}\right) \right\rangle$$

$$r_2{}^{d_2}\mathscr{C}_2 = \left\langle \left[1-(1-\mathscr{P}_{l_2})^{r_2^{d_2}} , 1-(1-\mathscr{P}_{u_2})^{r_2^{d_2}}\right] , \left[-(\mathscr{N}_{l_2})^{r_2^{d_2}} , -(\mathscr{N}_{u_2})^{r_2^{d_2}}\right] , \lambda_2{}^{r_2^{d_2}} , -\left(1-(1-\mu_2)^{r_2^{d_2}}\right) \right\rangle$$

$$r_1{}^{d_1}\mathscr{C}_1 \bigoplus r_2{}^{d_2}\mathscr{C}_2 = \left\langle \left[1-(1-\mathscr{P}_{l_1})^{r_1^{d_1}} , 1-(1-\mathscr{P}_{u_1})^{r_1^{d_1}}\right] , \left[-(\mathscr{N}_{l_1})^{r_1^{d_1}} , -(\mathscr{N}_{u_1})^{r_1^{d_1}}\right] , \lambda_1{}^{r_1^{d_1}} , -\left(1-(1-\mu_1)^{r_1^{d_1}}\right) \right\rangle$$

$$\bigoplus \left\langle \left[1-(1-\mathscr{P}_{l_2})^{r_2^{d_2}} , 1-(1-\mathscr{P}_{u_2})^{r_2^{d_2}}\right] , \left[-(\mathscr{N}_{l_2})^{r_2^{d_2}} , -(\mathscr{N}_{u_2})^{r_2^{d_2}}\right] , \lambda_2{}^{r_2^{d_2}} , -\left(1-(1-\mu_2)^{r_2^{d_2}}\right) \right\rangle$$

$$r_1{}^{d_1}\mathscr{C}_1 \bigoplus r_2{}^{d_2}\mathscr{C}_2 = \left\langle \left[1-(1-\mathscr{P}_{l_1})^{r_1^{d_1}} * (1-\mathscr{P}_{l_2})^{r_2^{d_2}} , 1-(1-\mathscr{P}_{u_1})^{r_1^{d_1}} * (1-\mathscr{P}_{u_2})^{r_2^{d_2}}\right] , \left[-(\mathscr{N}_{l_1}^{r_1^{d_1}} * \mathscr{N}_{l_2}^{r_2^{d_2}}) , \right. \right.$$

$$\left. -(\mathscr{N}_{u_1}^{r_1^{d_1}} * \mathscr{N}_{u_2}^{r_2^{d_2}})\right] , \lambda_1{}^{r_1^{d_1}} * \lambda_2{}^{r_2^{d_2}} , -\left(1-(1-\mu_1)^{r_1^{d_1}} * (1-\mu_2)^{r_2^{d_2}}\right) \right\rangle$$

$$= \left\langle \left[1-\prod_{k=1}^{2}(1-\mathscr{P}_{l_k})^{r_k^{d_k}} , 1-\prod_{k=1}^{2}(1-\mathscr{P}_{u_k})^{r_k^{d_k}}\right] , \left[-\prod_{k=1}^{2}(\mathscr{N}_{l_k})^{r_k^{d_k}} , -\prod_{k=1}^{2}(\mathscr{N}_{u_k})^{r_k^{d_k}}\right] , \lambda_k{}^{r_k^{d_k}} , \right.$$

$$\left. 1-\prod_{k=1}^{2}(\mathscr{N}_{l_k})^{r_k^{d_k}}\left(1-\prod_{k=1}^{2}\mu_k\right)^{r_k^{d_k}} \right\rangle$$

Which shows that Equation (1) is true for $n=2$, now let (1) holds for $n=k$, i.e.,
$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_k) =$

$$\left\langle \left[1-\prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k^{d_k}} , 1-\prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k^{d_k}}\right] , \left[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k^{d_k}} , -\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{r_k^{d_k}}\right] , \prod_{k=1}^{n}\lambda_k{}^{r_k^{d_k}} , -\left(1-\prod_{k=1}^{n}(1-\mu_k)^{r_k^{d_k}}\right) \right\rangle$$

(2)

Now, we will show the Equation (1) holds for $n=k+1$, by using the *CBFS* operational laws
$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_{k+1}) = CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_k) \bigoplus \mathscr{C}_{k+1}$

$$= \left\langle \left[1-\prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k^{d_k}} , 1-\prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k^{d_k}}\right] , \left[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k^{d_k}} , -\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{r_k^{d_k}}\right] , \prod_{k=1}^{n}\lambda_k{}^{r_k^{d_k}} , \right.$$

$$\left. -\left(1-\prod_{k=1}^{n}(1-\mu_k)^{r_k^{d_k}}\right) \right\rangle \bigoplus \left\langle \left[1-(1-\mathscr{P}_{l_{k+1}})^{r_{k+1}^{d_{k+1}}} , 1-(1-\mathscr{P}_{u_{k+1}})^{r_{k+1}^{d_{k+1}}}\right] , \left[-(\mathscr{N}_{l_{k+1}})^{r_{k+1}^{d_{k+1}}} , -(\mathscr{N}_{u_{k+1}})^{r_{k+1}^{d_{k+1}}}\right] , \right.$$

$$\left. \lambda_{k+1}{}^{r_{k+1}^{d_{k+1}}} , -\left(1-(1-\mu_{k+1})^{r_{k+1}^{d_{k+1}}}\right) \right\rangle$$

$$= \left\langle \left[1 - \prod_{k=1}^{n+1}(1-\mathscr{P}_{l_k})^{r_k{}^{d_k}} \ , \ 1 - \prod_{k=1}^{n+1}(1-\mathscr{P}_{u_k})^{r_k{}^{d_k}}\right] \ , \ \left[-\prod_{k=1}^{n+1}(\mathscr{N}_{l_k})^{r_k{}^{d_k}} \ , \ -\prod_{k=1}^{n+1}(\mathscr{N}_{u_k})^{r_k{}^{d_k}}\right] \ , \ \prod_{k=1}^{n+1}\lambda_k{}^{r_k{}^{d_k}} \ , \right.$$

$$\left. -\left(1 - \prod_{k=1}^{n+1}(1-\mu_k)^{r_k{}^{d_k}}\right) \right\rangle$$

This proves that $n = k + 1$, Equation (1) holds, then

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = \left\langle \left[1 - \prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k{}^{d_k}} \ , \ 1 - \prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k{}^{d_k}}\right] \ , \ \left[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k{}^{d_k}} \ , \ -\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{r_k{}^{d_k}}\right] \ , \right.$$

$$\left. \prod_{k=1}^{n}\lambda_k{}^{r_k{}^{d_k}} \ , \ -\left(1 - \prod_{k=1}^{n}(1-\mu_k)^{r_k{}^{d_k}}\right) \right\rangle \tag{3}$$

$\square$

**Example 3.3.** Consider four *CBFNs* $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, and $\mathscr{C}_4$ as:

$$\mathscr{C}_1 = \left\langle [0.7391 \ , \ 0.8756] \ , \ [-0.7659 \ , \ -0.4631] \ , \ 0.7929 \ , \ -0.5745 \right\rangle$$

$$\mathscr{C}_2 = \left\langle [0.9431 \ , \ 0.9996] \ , \ [-0.3743 \ , \ -0.1329] \ , \ 0.9567 \ , \ -0.2729 \right\rangle$$

$$\mathscr{C}_3 = \left\langle [0.1457 \ , \ 0.9192] \ , \ [-0.7954 \ , \ -0.2343] \ , \ 0.7351 \ , \ -0.5827 \right\rangle$$

$$\mathscr{C}_4 = \left\langle [0.5299 \ , \ 0,8153] \ , \ [-0.8137 \ , \ -0.7143] \ , \ 0.6979 \ , \ -0.7799 \right\rangle$$

Calculate cubic bipolar fuzzy aggregation operator with priority degree $d = (4, 1, 1)$.

**Solution:** Firstly we will calculate score values and of each *CBFN*

$S(\mathscr{C}_1) = 0.6007$ ; $S(\mathscr{C}_2) = 0.8532$; $S(\mathscr{C}_3) = 0.5312$; $S(\mathscr{C}_4) = 0.4557$

$\mathfrak{T}_1 = 1.0000$; $\quad \mathfrak{T}_2 = 0.6007$; $\quad \mathfrak{T}_3 = 0.5125$; $\quad \mathfrak{T}_4 = 0.2722$

$r_1^{d_1} = 0.4192$; $\quad r_2^{d_2} = 0.2561$; $\quad r_3^{d_3} = 0.2185$; $\quad r_4^{d_4} = 0.1160$

By using formula (1), we have

$$CBDAPD(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \left\langle [0.7581 \ , \ 0.9733] \ , \ [-0.6458 \ , \ -0.3025] \ , \ 0.8045 \ , \ -1.0000 \right\rangle$$

Here, we have some essential elements amongst *CBFPDA's* operator.

**Theorem 3.4.** *(Idempotency) Assume that* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} \ , \ \mathscr{P}_{u_j}] \ , \ [\mathscr{N}_{l_j} \ , \ \mathscr{N}_{u_j}] \ , \ \lambda_j, \ \mu_j \right\rangle$ *is the collection of CBFNs. A CBFPDA operator is defined by the mapping* $\Lambda^n \rightarrow \Lambda$ *is expressed as*

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1^{d_1}\mathscr{C}_1 \oplus r_2^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n^{d_n}\mathscr{C}_n \tag{4}$$

*where* $r_i^{d_i} = \frac{\mathfrak{T}_1^{d_i}}{\sum_{k=1}^{n}\mathfrak{T}_k^{d_k}}$ *and* $\mathfrak{T}_j = \prod_{k=1}^{n-1}\left(S(\mathscr{C}_k)\right)^{d_k}$; *(j = 1, 2, ..., n) and* $\mathfrak{T}_1 = 1$ *and* $S(\mathscr{C}_k)$ *is the score function of* $k^{th}$ *CBFN. If* $\mathscr{C}_j = \mathscr{C}$ $\forall j$ *then* $CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = \mathscr{C}$

*Proof.* Consider the Equation (1)

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1{}^{d_1}\mathscr{C}_1 \oplus r_2{}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}_n$$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1{}^{d_1}\mathscr{C} \oplus r_2{}^{d_2}\mathscr{C} \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}$$

$$= \frac{\mathfrak{T}_1{}^{d_1}}{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}\mathscr{C} \oplus \frac{\mathfrak{T}_2{}^{d_2}}{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}\mathscr{C} \oplus \cdots \oplus \frac{\mathfrak{T}_n{}^{d_n}}{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}\mathscr{C}$$

$$= \left(\frac{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}\mathscr{C}\right)$$

$$= \mathbb{1}\mathscr{C}$$

$$= \mathscr{C}$$

$\square$

**Theorem 3.5.** *(Monotonicity) Consider that*

$$\mathscr{C}_j = \left\langle \left[\mathscr{P}_{l_j}, \mathscr{P}_{u_j}\right], \left[\mathscr{N}_{l_j}, \mathscr{N}_{u_j}\right], \lambda_j, \mu_j \right\rangle$$

*and*

$$\mathscr{C}_j^* = \left\langle \left[\mathscr{P}_{l_j}^*, \mathscr{P}_{u_j}^*\right], \left[\mathscr{N}_{l_j}^*, \mathscr{N}_{u_j}^*\right], \lambda_j^*, \mu_j^* \right\rangle$$

*are the families of CBFNs, where* $\mathfrak{T}_j = \prod_{k=1}^{j-1}\left(S(\mathscr{C}_k)\right)^{d_k}$ *and* $\mathfrak{T}_j^* = \prod_{k=1}^{j-1}\left(S(\mathscr{C}_k^*)\right)^{d_k}$ ; $(j = 2, 3, ..., n)$ , $\mathfrak{T}_1 = 1 = \mathfrak{T}_1^*$.

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) \leq CBFPDA(\mathscr{C}_1^*, \mathscr{C}_2^*, ..., \mathscr{C}_n^*)$$

*Proof.* Consider the elements of *CBFNs* and develop relation between them:

If $\mathscr{P}_{l_j} \leq \mathscr{P}_{l_j}^*$;
$\Rightarrow 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j})^{r_j{}^{d_j}} \leq 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j}^*)^{r_j{}^{d_j}}$ ;

If $\mathscr{N}_{l_j} \geq \mathscr{N}_{l_j}^*$

If $\Rightarrow \mathscr{P}_{u_j} \leq \mathscr{P}_{u_j}^*$
$\Rightarrow 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{u_j})^{r_j{}^{d_j}} \leq 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{u_j}^*)^{r_j{}^{d_j}}$

If $\mathscr{N}_{u_j} \geq \mathscr{N}_{u_j}^*$

If $\lambda_j \leq \lambda_j^* \Rightarrow \prod_{j=1}^{k}\left(\lambda_j\right)^{r_j{}^{d_j}} \leq \prod_{j=1}^{k}\left(\lambda_j^*\right)^{r_j{}^{d_j}}$
If $\mu_j \geq \mu_j^*$
$\Rightarrow (1 - \mu_j)^{r_j{}^{d_j}} \geq (1 - \mu_j^*)^{r_j{}^{d_j}}$
$\Rightarrow 1 - \prod_{j=1}^{k}(1 - \mu_j)^{r_j{}^{d_j}} \geq 1 - \prod_{j=1}^{k}(1 - \mu_j^*)^{r_j{}^{d_j}}$
By combining all above generated inequalities, we have

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) \leq CBFPDA(\mathscr{C}_1^*, \mathscr{C}_2^*, ..., \mathscr{C}_n^*)$$

$\square$

**Theorem 3.6.** *(Boundedness) Consider that* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} , \mathscr{P}_{u_j}] , [\mathscr{N}_{l_j} , \mathscr{N}_{u_j}] , \lambda_j , \mu_j \right\rangle$ *is the family of CBFNs and* $\mathscr{C}^- = \min_j(\mathscr{C}_j)$ *and* $\mathscr{C}^+ = \max_j(\mathscr{C}_j)$ *then* $\mathscr{C}^- \leq \mathscr{C}_j \leq \mathscr{C}^+$

*Proof.*

$$\min_j(\mathscr{P}_{l_j}) \leq \mathscr{P}_{l_j} \leq \max_j(\mathscr{P}_{l_j})$$

$$\Rightarrow \min_j(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}} \geq (1-\mathscr{P}_{l_j})^{r_j{}^{d_j}} \geq \max_j(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}}$$

$$\Rightarrow \prod_{j=1}^{k} \min_j(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}} \geq \prod_{j=1}^{k}(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}} \geq \prod_{j=1}^{k} \max_j(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}}$$

$$\Rightarrow \min_j\left(1-\prod_{j=1}^{k}(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}}\right) \leq 1-\prod_{j=1}^{k}(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}} \leq \max_j\left(1-\prod_{j=1}^{k}(1-\mathscr{P}_{l_j})^{r_j{}^{d_j}}\right) \tag{5}$$

Similarly,

$$\min_j\left(1-\prod_{j=1}^{k}(1-\mathscr{P}_{u_j})^{r_j{}^{d_j}}\right) \leq 1-\prod_{j=1}^{k}(1-\mathscr{P}_{u_j})^{r_j{}^{d_j}} \leq \max_j\left(1-\prod_{j=1}^{k}(1-\mathscr{P}_{u_j})^{r_j{}^{d_j}}\right) \tag{6}$$

$$\min_j \mathscr{N}_{l_j} \geq \mathscr{N}_{l_j} \geq \max_j \mathscr{N}_{l_j}$$

$$-\min_j\left(\prod_{j=1}^{k}(\mathscr{N}_{l_j})^{r_j{}^{d_j}}\right) \geq -\prod_{j=1}^{k}(\mathscr{N}_{l_j})^{r_j{}^{d_j}} \geq -\max_j\left(\prod_{j=1}^{k}(\mathscr{N}_{l_j})^{r_j{}^{d_j}}\right) \tag{7}$$

Similarly,

$$-\min_j\left(\prod_{j=1}^{k}(\mathscr{N}_{u_j})^{r_j{}^{d_j}}\right) \geq -\prod_{j=1}^{k}(\mathscr{N}_{u_j})^{r_j{}^{d_j}} \geq -\max_j\left(\prod_{j=1}^{k}(\mathscr{N}_{u_j})^{r_j{}^{d_j}}\right) \tag{8}$$

$$\min_j\left(\lambda_j\right) \leq \lambda_j \leq \max_j\left(\lambda_j\right)$$

$$\min_j\left(\lambda_j^{r_j{}^{d_j}}\right) \leq \lambda_j^{r_j{}^{d_j}} \leq \max_j\left(\lambda_j^{r_j{}^{d_j}}\right)$$

$$\min_j\left(\prod_{j=1}^{k}\lambda_j^{r_j{}^{d_j}}\right) \leq \prod_{j=1}^{k}\lambda_j^{r_j{}^{d_j}} \leq \max_j\left(\prod_{j=1}^{k}\lambda_j^{r_j{}^{d_j}}\right) \tag{9}$$

by combining Equations (5)-(9), we have

$$\min_j \mathscr{C}_j \leq \mathscr{C}_j \leq \max_j \mathscr{C}_j$$

$\square$

**Corollary 3.7.** *Consider* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} , \mathscr{P}_{u_j}] , [\mathscr{N}_{l_j} , \mathscr{N}_{u_j}] , \lambda_j , \mu_j \right\rangle$ *is the assemblage of largest CBFNs i.e.* $\mathscr{C}_j = \langle [1, 1], [-1, -1], 1, -1 \rangle$ *for all j, then*

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \langle [1, 1], [-1, -1], 1, -1 \rangle$$

*Proof.* The proof od Corollary similar to the Theorem 3.4. □

**Corollary 3.8.** *Consider* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} , \mathscr{P}_{u_j}] , [\mathscr{N}_{l_j} , \mathscr{N}_{u_j}] , \lambda_j , \mu_j \right\rangle$ *is the assemblage of smallest CBFNs i.e.* $\mathscr{C}_j = \langle [0, 0], [0, 0], 0, 0 \rangle$ *for all j, then*

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \langle [0, 0], [0, 0], 0, 0 \rangle$$

*Proof.* Here, $\mathscr{C}_j = \langle [0, 0], [0, 0], 0, 0 \rangle$ then by the definition of the score function, we have $S(\mathscr{C}_j) = 0$. Since, $r_i{}^{d_i} = \frac{\mathfrak{T}_1{}^{d_i}}{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}$ and $\mathfrak{T}_j = \prod_{k=1}^{n-1} (S(\mathscr{C}_k))^{d_k}$; $(j = 1, 2, ..., n)$ and $\mathfrak{T}_1 = 1$ and $\mathfrak{T}_j = 0$ for $j = 2, 3, ..., n$.

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1{}^{d_1}\mathscr{C}_1 \oplus r_2{}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}_n$$
$$= 1 \cdot \mathscr{C}_1 \oplus 0 \cdot \mathscr{C}_2 \oplus \cdots \oplus 0 \cdot \mathscr{C}_n$$
$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \mathscr{C}_1$$

□

**Theorem 3.9.** *Consider* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} , \mathscr{P}_{u_j}] , [\mathscr{N}_{l_j} , \mathscr{N}_{u_j}] , \lambda_j , \mu_j \right\rangle$ *and* $\beta_j = \left\langle [\mathscr{A}_{l_j} , \mathscr{A}_{u_j}] , [\mathscr{B}_{l_j} , \mathscr{B}_{u_j}] , \omega_j , \eta_j \right\rangle$ *are two collection of CBFNs, if r > 0 and* $\beta = \left\langle [\mathscr{A}_l , \mathscr{A}_u] , [\mathscr{B}_l , \mathscr{B}_u] , \omega , \eta \right\rangle$ *is a CBFN, then*

1. $CBFPDA(\mathscr{C}_1 \oplus \beta, \mathscr{C}_2 \oplus \beta, \cdots, \mathscr{C}_n \oplus \beta) = CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) \oplus \beta$

2. $CBFPDA(r\mathscr{C}_1, r\mathscr{C}_2, \cdots, r\mathscr{C}_n) = rCBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n)$

3. $CBFPDA(\mathscr{C}_1 \oplus \beta_1, \mathscr{C}_2 \oplus \beta_2, \cdots, \mathscr{C}_n \oplus \beta_n) = CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) \oplus CBFPDA(\beta_1, \beta_2, \cdots, \beta_n)$

4. $CBFPDA(r\mathscr{C}_1 \oplus \beta, r\mathscr{C}_2 \oplus \beta, \cdots, r\mathscr{C}_n \oplus \beta) = rCBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) \oplus \beta$

*Proof.* This is trivial by Definition. □

**Property 1:** Assume that $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} , \mathscr{P}_{u_j}] , [\mathscr{N}_{l_j} , \mathscr{N}_{u_j}] , \lambda_j , \mu_j \right\rangle$ is the collection of *CBFNs*, then we have $\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (1, 1, \cdots, 1)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = CBFW(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n)$.

*Proof.* Given that $(d_1, d_2, \cdots, d_{n-1}) \to (1, 1, \cdots, 1)$ from this we have $r_j{}^{d_j} = \prod_{k=1}^{j-1} \left(S(\mathscr{C}_k)\right)^{d_k} \longrightarrow \prod_{k=1}^{j-1} \left(S(\mathscr{C}_k)\right)$
$r_j{}^{d_j} = r_j$

By this, we obtained

$$r_j{}^{d_j} \to r_j$$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1{}^{d_1}\mathscr{C}_1 \oplus r_2{}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}_n$$

$$\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (1,1,\cdots,1)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \lim_{(d_1, d_2, \cdots, d_{n-1}) \to (1,1,\cdots,1)} \left(r_1{}^{d_1}\mathscr{C}_1 \oplus r_2{}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}_n\right)$$

$$\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (1,1,\cdots,1)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1\mathscr{C}_1 \oplus r_2\mathscr{C}_2 \oplus \cdots \oplus r_n\mathscr{C}_n$$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = CBFW(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n)$$

$\square$

**Property 2:** Assume that $\mathscr{C}_j = \left\langle \left[\mathscr{P}_{l_j}, \mathscr{P}_{u_j}\right], \left[\mathscr{N}_{l_j}, \mathscr{N}_{u_j}\right], \lambda_j, \mu_j \right\rangle$ is the collection of *CBFNs* and $S(\mathscr{C}_j) \neq 0 \forall j$, then we have $\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (0,0,\cdots,0)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = \frac{1}{k}(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n)$.

*Proof.* Given that $(d_1, d_2, \cdots, d_{n-1}) \to (0, 0, \cdots, 0)$ by applying the limit we have, $\left(S(C_j)\right)^{d_j} = 1 \ \forall j$ and $r_j{}^{d_j} = \frac{1}{k}$.

$$\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (0,0,\cdots,0)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \lim_{(d_1, d_2, \cdots, d_{n-1}) \to (0,0,\cdots,0)} \left(r_1{}^{d_1}\mathscr{C}_1 \oplus r_2{}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus r_n{}^{d_n}\mathscr{C}_n\right)$$

$$= \frac{1}{k}\mathscr{C}_1 \oplus \frac{1}{k}\mathscr{C}_2 \oplus \cdots \oplus \frac{1}{k}\mathscr{C}_n$$

$$= \frac{1}{k}\left(\mathscr{C}_1 \oplus \mathscr{C}_2 \oplus \cdots \oplus \mathscr{C}_k\right)$$

Hence proved. $\square$

**Property 3:** Assume that $\mathscr{C}_j = \left\langle \left[\mathscr{P}_{l_j}, \mathscr{P}_{u_j}\right], \left[\mathscr{N}_{l_j}, \mathscr{N}_{u_j}\right], \lambda_j, \mu_j \right\rangle$ is the collection of *CBFNs* and $S(\mathscr{C}_j) \neq 0$ or $S(\mathscr{C}_j) \neq 1 \forall j$, then we have $\lim_{d_1 \to +\infty} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = \mathscr{C}_1$.

*Proof.* By applying the limit $d_1 \to +\infty$ for each $g = 2, 3, \cdots k$, we have

$$\mathfrak{T}_j = \lim_{d_1 \to +\infty} \prod_{k=1}^{j-1} \left(S(\mathscr{C}_k)\right)^{d_k} \quad = \left(S(\mathscr{C}_1)\right)^{\infty} \cdot \left(S(\mathscr{C}_2)\right)^{d_2} \cdot \left(S(\mathscr{C}_3)\right)^{d_3} \cdots \left(S(\mathscr{C}_{k-1})\right)^{d_{k-1}} = 0; \quad \text{as} 0 < S(\mathscr{C}_{k-1}) < 1$$

$\sum \mathfrak{T}_j^{(d)} = \mathfrak{T}_1, \; r_1{}^{d_1} = \frac{\mathfrak{T}_1^{(d)}}{\sum_j \mathfrak{T}_j^{(d)}} = 1$

$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \mathscr{C}_1$  □

**Example 3.10.** Consider four *CBFNs* $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ and $\mathscr{C}_4$ listed below:

$$\mathscr{C}_1 = \langle [0.35, \; 0.65], \; [-0.75, \; -0.25], \; 0.45, \; -0.65 \rangle$$

$$\mathscr{C}_2 = \langle [0.15, \; 0.55], \; [-0.71, \; -0.37], \; 0.40, \; -0.60 \rangle$$

$$\mathscr{C}_3 = \langle [0.25, \; 0.75], \; [-0.65, \; -0.35], \; 0.50, \; -0.55 \rangle$$

$$\mathscr{C}_4 = \langle [0.50, \; 0.90], \; [-0.55, \; -0.25], \; 0.75, \; -0.40 \rangle.$$

Now we will calculate the score functions $S(\mathscr{C}_1) = 0.4667, \; S(\mathscr{C}_2) = 0.4033, \; S(\mathscr{C}_3) = 0.4917, \; S(\mathscr{C}_4) = 0.6583$ and $T_1 = 1$

1. for $(d_1, d_2, d_3) = (1, 1, 1) \; T_2 = 0.4667; \; T_3 = 0.1882, \; T_4 = 0.0925$

$\sum_j T_j = 1.7474$

$\mathfrak{T}_1 = 0.5723, \; \mathfrak{T}_2 = 0.2671, \; \mathfrak{T}_3 = 0.1077, \; \mathfrak{T}_4 = 0.0529$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3006, 0.6622], \; [-0.7160, \; -0.2878], \; 0.4531, \; -0.6166 \rangle$$

2. for $(d_1, d_2, d_3) = (6, 1, 1) \; T_2 = 0.0103; \; T_3 = 0.0042, \; T_4 = 0.0020$

$\sum_j T_j = 1.0165$

$\mathfrak{T}_1 = 0.9838, \; \mathfrak{T}_2 = 0.0101, \; \mathfrak{T}_3 = 0.0041, \; \mathfrak{T}_4 = 0.0020$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3482, 0.6505], \; [-0.74687, \; -0.2513], \; 0.4501, \; -0.6488 \rangle$$

3. for $(d_1, d_2, d_3) = (8, 1, 1) \; T_2 = 0.0023; \; T_3 = 0.0009, \; T_4 = 0.0004$

$\sum_j T_j = 1.0036$

$\mathfrak{T}_1 = 0.9964, \; \mathfrak{T}_2 = 0.0023, \; \mathfrak{T}_3 = 0.0009, \; \mathfrak{T}_4 = 0.0004$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3496, 0.6501], \; [-0.74697, \; -0.2503], \; 0.4500, \; -0.6498 \rangle$$

4. for $(d_1, d_2, d_3) = (10, 1, 1) \; T_2 = 0.0005; \; T_3 = 0.0002, \; T_4 = 0.0001$

$\sum_j T_j = 1.0008$

$\mathfrak{T}_1 = 0.9992, \; \mathfrak{T}_2 = 0.0005, \; \mathfrak{T}_3 = 0.0002, \; \mathfrak{T}_4 = 0.0001$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3500, 0.6500], \; [-0.7500, \; -0.2500], \; 0.4500, \; -0.6500 \rangle$$

Hence proved as $d_1 \to \infty$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \mathscr{C}_1.$$

# 4  Cubic bipolar fuzzy geometric operator with priority degree

**Definition 4.1.** Assume that $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} \, , \, \mathscr{P}_{u_j}] \, , \, [\mathscr{N}_{l_j} \, , \, \mathscr{N}_{u_j}] \, , \, \lambda_j \, , \, \mu_j \right\rangle$ is the collection of $CBFNs$ and $CBFPDG : M^n \to M$ be a mapping defined as

$$CBFPDG(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = \mathscr{C}_1{}^{r_1{}^{d_1}} \bigotimes \mathscr{C}_2{}^{r_2{}^{d_2}} \bigotimes \cdots \mathscr{C}_n{}^{r_n{}^{d_n}}$$

where $\mathfrak{T}_j = \prod_{k=1}^{j-1} \left( S(\mathscr{C}_k) \right)^{d_k}$ and $\mathfrak{T}_1 = 1$ such that $S(\mathscr{C}_k)$ is the score function of $k^{th}$ $CBFN$.

The $CBFPDG$ operator is explained in the theorem mentioned below whose prove follows the $CBFN's$ operational laws.

**Theorem 4.2.** *Let* $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k} \, , \, \mathscr{P}_{u_k}] \, , \, [\mathscr{N}_{l_k} \, , \, \mathscr{N}_{u_k}] \, , \, \lambda_k \, , \, \mu_k \right\rangle$ *be the collection of CBFNs, we can find* *CBFPDG by the mapping:*
*CBFPDG*$(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) =$

$$\left\langle \left[ \prod_{j=1}^{n} (\mathscr{P}_{l_j})^{r_j{}^{d_j}} \, , \, \prod_{j=1}^{n} (\mathscr{P}_{u_j})^{r_j{}^{d_j}} \right] , \left[ -\left(1 - \prod_{j=1}^{n}(1 - \mathscr{N}_{l_j})^{r_j{}^{d_j}} \, , \, -\left(1 - \prod_{j=1}^{n}(1 - \mathscr{N}_{u_j})^{r_j{}^{d_j}}\right) \right] , 1 - \prod_{j=1}^{n}(1 - \lambda_j)^{r_j{}^{d_j}} \, , \, -\prod_{j=1}^{n}(\mu_j)^{r_j{}^{d_j}} \right\rangle$$

*Proof.* Proof is similar to Theorem (3.2)  □

**Theorem 4.3.** *(Idempotency) Assume that* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} \, , \, \mathscr{P}_{u_j}] \, , \, [\mathscr{N}_{l_j} \, , \, \mathscr{N}_{u_j}] \, , \, \lambda_j, \mu_j \right\rangle$ *is the collection* *of CBFNs. A CBFPDA operator is defined by the mapping* $\Lambda^n \to \Lambda$ *is expressed as*

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1{}^{d_1}\mathscr{C}_1 \bigotimes r_2{}^{d_2}\mathscr{C}_2 \bigotimes \cdots \bigotimes r_n{}^{d_n}\mathscr{C}_n \tag{10}$$

*where* $r_i{}^{d_i} = \frac{\mathfrak{T}_1{}^{d_i}}{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}$ *and* $\mathfrak{T}_j = \prod_{k=1}^{n-1} \left( S(\mathscr{C}_k) \right)^{d_k}$; $(j = 1, 2, ..., n)$ *and* $\mathfrak{T}_1 = 1$ *and* $S(\mathscr{C}_k)$ *is the score* *function of* $k^{th}$ *CBFN. If* $\mathscr{C}_j - \mathscr{C} \quad \forall j$ *then CBFPDA*$(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = \mathscr{C}$

*Proof.* Proof is similar to Theorem (3.4)  □

**Theorem 4.4.** *(Monotonicity) Consider that*

$$\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j} \, , \, \mathscr{P}_{u_j}] \, , \, [\mathscr{N}_{l_j} \, , \, \mathscr{N}_{u_j}] \, , \, \lambda_j \, , \, \mu_j \right\rangle$$

*and*

$$\mathscr{C}_j^* = \left\langle [\mathscr{P}_{l_j}^* \, , \, \mathscr{P}_{u_j}^*] \, , \, [\mathscr{N}_{l_j}^* \, , \, \mathscr{N}_{u_j}^*] \, , \, \lambda_j^* \, , \, \mu_j^* \right\rangle$$

*are the families of CBFNs, where* $\mathfrak{T}_j = \prod_{k=1}^{j-1} \left( S(\mathscr{C}_k) \right)^{d_k}$ *and* $\mathfrak{T}_j^* = \prod_{k=1}^{j-1} \left( S(\mathscr{C}_k^*) \right)^{d_k}$; $(j = 2, 3, ..., n)$, $\mathfrak{T}_1 = 1 = \mathfrak{T}_1^*$.

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) \leq CBFPDA(\mathscr{C}_1^*, \mathscr{C}_2^*, ..., \mathscr{C}_n^*)$$

*Proof.* Proof is similar to Theorem (3.5) □

**Theorem 4.5.** *(Boundedness) Consider that* $\mathscr{C}_j = \left\langle \left[ \mathscr{P}_{l_j} , \mathscr{P}_{u_j} \right] , \left[ \mathscr{N}_{l_j} , \mathscr{N}_{u_j} \right] , \lambda_j , \mu_j \right\rangle$ *is the family of CBFNs and* $\mathscr{C}^- = \min_j(\mathscr{C}_j)$ *and* $\mathscr{C}^+ = \max_j(\mathscr{C}_j)$ *then* $\mathscr{C}^- \leq \mathscr{C}_j \leq \mathscr{C}^+$

*Proof.* Proof is similar to Theorem (3.6) □

# 5 Methodology for $MCDM$ using profounded $AOs$

Let $\mathbb{M} = \{M_1, M_2, ..., M_n\}$ be the collection of alternatives and $\mathbb{C} = \{C_1, C_2, ..., C_m\}$ is the assemblage of criterions, priorities are assigned between the criterions. $C_i \geq_{d_k} C_j$ indicates criteria $C_i$ is superior than criteria $C_j$ with degree $d_k$. Consider $\mathbb{D} = \{D_1, D_2, ..., D_l\}$ is set of decision makers $(DM)$. Priorities are assigned between the DMs provided by strict priority orientation, $D_1 >_{d_1} D_2 >_{d_2} > \cdots >_{d_{l-1}} D_l$. *DMs* give a matrix according to their own opinions and viewpoints $\mathbb{D}^l = \left(\mathscr{C}_{ij}^l\right)_{n \times m}$ for the alternative $M_i$ and criteria $C_j$ by the $D_l$ decision maker.

The suggested operators will be implemented to the MCDM, which will require the preceding steps.

**Algorithm**

**Step 1:**

Obtain the decision matrix $\mathbb{D}^l = (\mathscr{P}_{ij})_{m \times n}$, where all entries of matrix are $CBFNs$ assigned by the standpoints of the decision makers.

Indicators of $(\tau_c)$ cost and $(\tau_b)$ benefit are the two types of criterion described in the decision matrix. If all indicators are of the same type, then normalisation is not necessary; however, in MCGDM, there may be two distinct criteria types. As a result of applying the normalisation formula presented in Equation 11, the matrix was modified to become the transforming response matrix, with the notation $\mathbb{D}^l = (\mathscr{Q}_{ij})_{m \times n}$.

$$(\mathscr{Q}_{ij})_{m \times n} = \begin{cases} (\mathscr{P}_{ij})^c, & j \in \tau_c \\ (\mathscr{P}_{ij}), & j \in \tau_b \end{cases} \tag{11}$$

**Step 2:**

Using Equations, combine all of the independent $CBF$ decision matrices into one combined evaluation matrix of the alternatives using one of the provided $AOs$ (1,2).

**Step 3:**

Aggregate the $CBFNs$ for each alternatives by using $CBFAPD$ (or $CBFGPD$) operator.

**Step 4:**

Calculate the score values of all accumulative $CBFNs$ alternatives assessments.

**Step 5:**

Rank the all score values of alternatives and choose the highest one as best alternative.

Pictorial structure of the algorithm is viewed in Figure 1

# 6 Case Study

In this section, an algorithm for solving the $MCDM$ problem in a cubic bipolar environment is proposed.

Reverse logistics (RLs) recycles or reuses goods. Supply chains supply consumers. Supply chain experts measure efficiency with on-time delivery (OTD). Supply chain metrics include order-to-delivery time. Service delivery completes the supply chain. Receiving the wrong item, a damaged item, a product that doesn't match the company's logo, or no longer needing the item are all valid reasons for a refund or exchange. The product must be returned, disassembled, inspected, recycled, and repaired. They require frequent supply chain reversals. RLs benefit consumers and industry. Reusing, recycling, and repairing are RLs. Manufacturers can reuse assets. Recycling companies would benefit from RLs. RLs only save materials. E-commerce boosts RLs.

Online retailers expect 414 million in 2018 after replacing shopping carts. Online returns exceed 30%, compared to 8.89±1% in brick-and-mortar stores. Supply chains struggled with logistics costs as product returns increased. Thus, reverse logistics system setup requires care. Material reversal metrics. Returns, product types, dollars, and lost profits are included. Return risk metrics can identify issues and grow the business. Reverse logistics pays off. Supply chain turnaround? RLs must be "forward" logistics-efficient (customer support, storage, system integration, etc). Supply chain returns optimisation boosts output, customer satisfaction, and savings.

1. RLs reduces shipping costs and resells goods that would have been thrown out if returned. Profit margins will boost if recycled and resold materials generate revenue and the system works well.

2. Your company's return policy can affect customer perceptions. It's possible that the advertised product's defective part caused the bad result. Fixing mistakes is as important as closing deals. Resolve product issues with customers. Customer loyalty can be increased by giving customers multiple return options. You may be able to return an item to a physical shop without the original receipt or packaging and receive a full refund regardless of the reason.

3. Customer satisfaction would increase if you had a well-organized return and replacement system. It speeds up repairing, refurbishing, and reusing products to avoid buying new ones.

4. RLs can help you recycle, resell, or reuse products that would otherwise go to landfills. This raises the brand's social and environmental responsibilities and profits. Remanufacturing or refurbishing extends product life.

Growth adds customers, sites, and manufacturing processes. Some companies lack capital and overhead. Businesses should enhance functionality outside their systems. 3PRLP aid. An outside agency provides 3PRLPs for cost savings, productivity, and capability development. 3PRLP services are intermittent or permanent. Business needs 3PRLP. Companies outgrow storage. 3PRLP warehouse management aids storage. Infrastructure, vehicle, and shipping costs may hurt other businesses. 3PRLP's large fleets of specialised trucks and facilities are cheaper. Strong 3PRLPs help US firms enter Canada. 3PRLPs help companies with customer support, delivery times, refunds, order tracking, technical services, stock management, and more. Businesses may increase 3PRLP associate value. Supply chain specialists aim to increase productivity, speed processes, and lower logistics costs, including transportation. Figure 2 shows logistical cost breakdown.

This shortlisting technique is an MCDM assignment for the 3PRLP. Studies show that 3PRLP selection is of scholarly and commercial interest. MCDMs have proliferated in recent years. Models were developed for 3PRLP evaluation. Realistic RLs outsourced assessments are often ambiguous and imprecise due to partial ignorance, imprecise assessment, and partial or unavailable decision-making for further facts. "Outsourcing" first appeared in the American Glossary in 1981 as "outside resourcing." Outsourcing logistics is a major company achievement. A logistics contract provider outsources many companies at once, creating economic balance and lowering costs. Yang et al. noted that cost reduction is very seldom the main goal of MCDM outsourcing [9, 10, 11, 12].

Numerous academics have described a number of 3PRLP outsourcing modules, including:
(1) the advantages and disadvantages of working with a third-party logistics provider; and (2) selecting 3PRLPs for a long-term collaboration.
The second module selects 3PRLPs for decision-making based on attainability. 3PRLP reduces environmental risks, resource issues, and product life to maximise profits. Sustainable development principles are encouraged and even required in supply chain management 3PRLP configurations in developing nations. Choosing a 3PRLP is well-studied. Researchers have debated the most important 3PRLP selection criteria for 20 years. Researchers surveyed these crucial factors. The best 3PRLP is chosen based on the six criteria listed in Table 3.

## 6.1 Problem formulation

Company preference determines the 3PRLP selection criterion. The criterion selected diverse sources. Several researchers have spent the last two decades identifying 3PRLP analysis and selection criteria. Researchers identified key factors by surveying. In this article, we use the six criterion for selecting best 3PRLP given in Table 3 and Table 4.

### 6.1.1 Parameters

Selection is a difficult problem to solve, criteria and alternatives play a vital role in resolving it. This problem formulation considers the following criteria and alternatives.

### 6.1.2 Assumption

We have four decision makers ($DMs$) {$D_1$ , $D_2$ , $D_3$ , $D_4$} will assign linguistic values from the Table 5 according to their own interest, experience and knowledge to the above mentioned criteria's and alternatives in the Table 6.

### 6.1.3 Calculations

**Step 1:** The decision matrices are obtained by the decision makers represented in the Tables (7,8,9,10). The normalized decision matrices are obtained by the decision makers in which each entry represent the viewpoint of decision makers toward the criterions and alternatives shown in the Tables (11,12,13,14).
**Step 2:** To aggregate the decision matrix we will follow these steps:

1. Calculate the score functions for all the decision matrices and shown in Table 15.

2. Calculate the $r_i{}^{d_i} = \frac{\mathfrak{T}_1{}^{d_i}}{\sum_{k=1}^{n} \mathfrak{T}_k{}^{d_k}}$ and $\mathfrak{T}_j = \prod_{k=1}^{n-1} \big(S(\mathscr{C}_k)\big)^{d_k}$.

3. Accumulate the decision matrices by using Equation 1.

$$
\mathfrak{T}_{ij}^{(1)} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

$$
\mathfrak{T}_{ij}^{(2)} = \begin{pmatrix}
0.15835 & 0.15835 & 0.37835 & 0.60925 & 0.87840 & 0.02668 \\
0.02668 & 0.37835 & 0.60925 & 0.87840 & 0.02668 & 0.02668 \\
0.37835 & 0.60925 & 0.87840 & 0.37835 & 0.87840 & 0.87840 \\
0.02668 & 0.15835 & 0.37835 & 0.60925 & 0.87840 & 0.02668 \\
0.02668 & 0.37835 & 0.60925 & 0.87840 & 0.37835 & 0.87840 \\
0.15835 & 0.37835 & 0.60925 & 0.87840 & 0.37835 & 0.87840
\end{pmatrix}
$$

$$
\mathfrak{T}_{ij}^{(3)} = \begin{pmatrix}
0.05939 & 0.09647 & 0.32943 & 0.09647 & 0.02344 & 0.00422 \\
0.01001 & 0.22849 & 0.53517 & 0.32943 & 0.02344 & 0.02344 \\
0.01001 & 0.01625 & 0.32943 & 0.22849 & 0.77159 & 0.32943 \\
0.00163 & 0.13909 & 0.22849 & 0.53517 & 0.02344 & 0.01001 \\
0.00163 & 0.32943 & 0.22849 & 0.53517 & 0.32943 & 0.02344 \\
0.09647 & 0.32943 & 0.22849 & 0.53517 & 0.32943 & 0.13909
\end{pmatrix}
$$

$$
\mathfrak{T}_{ij}^{(4)} = \begin{pmatrix}
0.03618 & 0.08474 & 0.00879 & 0.00257 & 0.00371 & 0.00158 \\
0.00610 & 0.20069 & 0.01428 & 0.05217 & 0.00879 & 0.01428 \\
0.00879 & 0.00609 & 0.20071 & 0.20071 & 0.12218 & 0.20071 \\
0.00143 & 0.00371 & 0.03618 & 0.01428 & 0.00371 & 0.00159 \\
0.00099 & 0.05217 & 0.20071 & 0.08474 & 0.20071 & 0.00371 \\
0.08474 & 0.12355 & 0.20071 & 0.20113 & 0.20071 & 0.02202
\end{pmatrix}
$$

Accumulative matrix is given in Equation (12)

$$
\mathfrak{T}_{ij} = \begin{pmatrix}
1 & 0.36820 & 0.10160 & 0.02290 \\
1 & 0.32380 & 0.19080 & 0.04940 \\
1 & 0.66580 & 0.28090 & 0.12320 \\
1 & 0.34570 & 0.15630 & 0.01010 \\
1 & 0.52380 & 0.24130 & 0.09050 \\
1 & 0.54570 & 0.27630 & 0.13880
\end{pmatrix}
\tag{12}
$$

$$r_i^{(d)} = \begin{pmatrix} 0.6699 & 0.2467 & 0.0681 & 0.0153 \\ 0.6394 & 0.2070 & 0.1220 & 0.0316 \\ 0.4831 & 0.3217 & 0,1357 & 0.0595 \\ 0.6613 & 0,2286 & 0.1034 & 0.0067 \\ 0.5389 & 0.2823 & 0.1300 & 0.0488 \\ 0.5100 & 0.2783 & 0.1409 & 0.0708 \end{pmatrix} \tag{13}$$

**Step 3:** Perform row wise accumulation to combine the values of criterions for each alternatives shown in Table 16 and Figure 3.

**Step 4:** Calculate the score function for each values of Table 17 and list the values in Table 18 where the priority degree is ordered as

$$(d_1, d_2, d_3) = (4, 1, 1)$$

. **Step 5:** Rank the alternative in ascending order and select the best alternative as optimal solution.

$$M_6 \geq M_3 \geq M_5 \geq M_2 \geq M_4 \geq M_1$$

# 7 Pros and Cons of CBFPDA

Every MCDM technique have some advantages and disadvantages similarly our proposed method has both strengths and weaknesses. Few important of them are listed below:

- The main advantage of proposed method calculate the weights of criteria automatically. Which is more efficient than the weights given by decision makers.

- Easy to adopt and compute.

- The weight vectors should be non-negative.

- Usually there is interaction between the membership and non-membership grades in the aggregation operators but in the proposed method the membership and non-membership are independent.

# 8 Comparison analysis

We obtained $M_6 \geq M_3 \geq M_5 \geq M_2 \geq M_4 \geq M_1$ rating by our proposed method, to validate our optimal alternative, we run the same problem by the existing operators. The validity of our suggested aggregation operators is demonstrated by the fact that we obtain the same optimal decision shown in Table 19 and illustrate in Figure 4.

# 9 Conclusion

This study addresses data ambiguity using positive and negative membership grades and interval values with $CBFNs$. $CBF$ combines $CS$ and $BFS$ models. We defined the cubic bipolar fuzzy prioritised averaging (geometric) operator with priority degrees using strict priority orders. Priority degree theories will help merge massive $CBF$ data. Priority degree hypotheses have been extensively researched and will help integrate multiple $CBF$ data sets. A $CBF$ group MCDM method was developed based on the prioritised AOs. An example illustrates the suggested approach, and the results are compared to other AOs. We also analyse how priorities affect results. Priority levels affect results, making the idea appealing. DM's freedom to choose the priority degree vector makes this method more resilient and difficult. The $CBF$ framework has a group $MCDM$ strategy based on the prioritised $AOs$. An analogy illustrates the proposed method, which is compared to many contemporary $AOs$. Priority degrees affect aggregated results. The $DM$ can choose the priority degree vector based on priorities and problem complexity, strengthening the suggested solution. We applied $MCDM$ to demonstrate the proposed method.

Future work may use fuzzy judgements to implement the suggested work in practise. AOs and MCDM would improve decision-making, medical diagnosis, pattern recognition, computational intelligence, and artificial intelligence. We'll also work on objective priority degree methods soon.

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Table 1: Some extensions of fuzzy sets

| Fuzzy models | Researchers | Constraints |
|---|---|---|
| Fuzzy set ($FS$) | Zadeh [1] | Membership values |
| Interval-valued fuzzy set ($IVFS$) | Zadeh [2] | Interval grading |
| Intuitionistic fuzzy set ($IFS$) | Atanassov [3, 4] | $\mu + v \leq 1$ |
| Pythagorean fuzzy set ($PFS$) | Yager [13, 14] | $\mu^2 + v^2 \leq 1$ |
| Fermatean fuzzy set ($FFS$) | Senapati and Yager [15, 16] | $\mu^3 + v^3 \leq 1$ |
| q-Rung orthopair fuzzy set ($q$-$ROF$) | Yager [17] | $\mu^q + v^q \leq 1, q \geq 1$ |
| Bipolar fuzzy set ($BFS$) | Zhang [18, 19] | Positive grading $\mu^+ \in [0,1]$ and negative grading $\mu^- \in [-1,0]$ |
| Cubic set ($CS$) | Jun, Kim and Yung [20] | Interval and fuzzy grading |
| Cubic bipolar fuzzy set ($CBFS$) | Riaz and Tehrim [21] | Hybrid model of $CS$ and $BFS$ |

Table 2: Some basic aggregation operators (AOs)

| Aggregation operators | Fuzzy models | Researchers | References |
|---|---|---|---|
| Ordered weighted averaging AO | Crisp | Yager (1988) | [22] |
| Geometric AO | IFS | Xu and Yager (2006) | [23] |
| AO | IFS | Xu (2007) | [24] |
| Geometric Einstein AO | IFS | Wang and Liu (2011) | [25] |
| Hamacher AO | BFS | Wei G., Alsaadi F. E., Hayat T. et al. (2017) | [26] |
| Bonferroni mean AO | Cubic IFS | Kaur and Garg (2018) | [27] |
| Dombi AO | neutrosophic cubic sets | Shi and Ye (2018) | [28] |
| Dombi AO | Pythagorean | Akram, Dudek and Dar (2019) | [29] |
| Cubic fuzzy AO | Pythagorean | Khan, Khan, Shahzad et al. (2019) | [30] |
| Priority degree AO | q-Rung orthopair FS | Riaz, Fareed, Shakeel et al. (2021) | [31] |
| Prioritized AO | q-Rung orthopair FS | Riaz, Pamucar, Farid et al. (2020) | [32] |
| Prioritized weighted AO | complex spherical | Akram, Khan, Alcantud et al. (2021) | [33] |
| Prioritized AOs with priority degree | complex intuitionistic | Garg and Rani (2021) | [34] |
| Prioritized AO with priority degree | q-Rung orthopair | Riaz, Farid, Shakeel et al. (2021) | [35] |
| Einstein prioritized AO | linear Diophantine | Farid, Riaz, Khan et al. (2022) | [36] |
| Einstein prioritized AO | single-valued neutrosophic | Farid, Garg, Riaz et al. (2022) | [37] |
| Prioritized interactive AO | q-rung orthopair | Farid and Riaz (2022) | [38] |
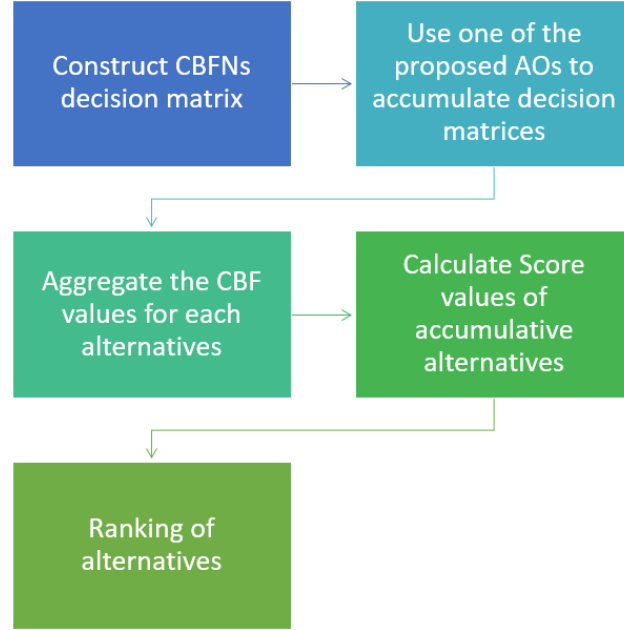
Figure 1: Pictorial structure of the algorithm

Table 3: Set of criterions [39]

| $\mathscr{C}_i$ | Criterion | Explanation |
|---|---|---|
| $\mathscr{C}_1$ | Time Deliver | client-required products or services on time. On-time delivery is the percentage of work completed within the customer's requested or company-committed timeframe. Long delivery times don't help times or by declining difficult business |
| $\mathscr{C}_2$ | Experience | The factory's past service or product achievements will be examined. |
| $\mathscr{C}_3$ | Reliability | This ensures that products and services are reliable and improve customer satisfaction. |
| $\mathscr{C}_4$ | Knowledge sharing | Traditional information sharing involves sender-receiver data exchanges. These exchanges use hundreds of open and proprietary protocols, message formats, and file types. Information sharing is a platform that regulates data and information exchange between clients and providers to ensure privacy, security, and data quality. |
| $\mathscr{C}_5$ | Reputation | Dependent on how others define one's identity. Decentralized, unplanned social control maintains social order through reputation. |
| $\mathscr{C}_6$ | Flexibility | System marketability. "Response capability" is a system's ability to quickly and cost-effectively respond to internal or external changes that affect value delivery. Thus, flexibility is how easily a system adapts to uncertainty to maintain or increase value. |
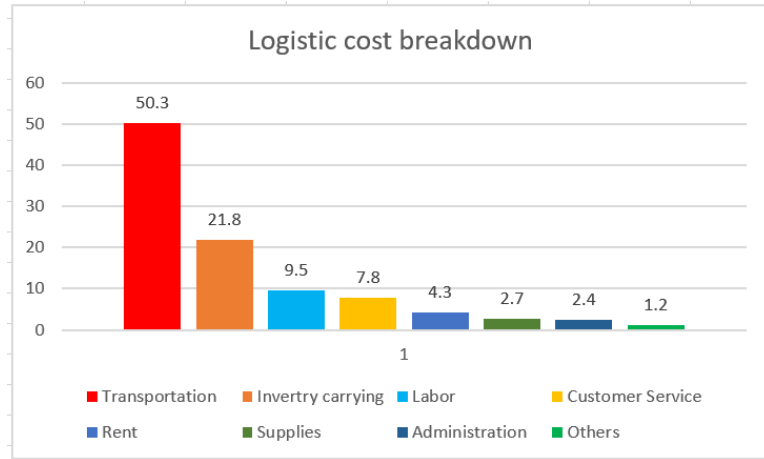
Figure 2: Logistic cost breakdown

Table 4: Set of alternatives

| $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|---|---|---|---|---|---|
| Company 1 | Company 2 | Company 3 | Company 4 | Company 5 | Company 6 |

Table 5: Linguistic variable and their associated fuzzy values

| Sr. No. | Linguistic variable | Signs and code | $CBF$ values |
|---|---|---|---|
| 1 | Very low | ★ | $[0.0000 , 0.1000] , [-1.0000 , -0.9900] , 0.0500, -0.9999$ |
| 2 | Low | ★★ | $[0.1000 , 0.3000] , [-0.9900 , -0.7900] , 0.1500, -0.8199$ |
| 3 | Satisfactory | ★★★ | $[0.3001 , 0.5000] , [-0.7900 , -0.5900] , 0.4500, -0.6199$ |
| 4 | High | ★★★★ | $[0.5001 , 0.8000] , [-0.5901 , -0.3900] , 0.7500, -0.4145$ |
| 5 | Very high | ★★★★★ | $[0.8001 , 1.0000] , [-0.3901 , -0.0001] , 0.9755, -0.1150$ |

Table 6: Linguistic terms for alternatives w.r.t criterions

| | Strategies | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|---|
| $M_1$ | $\mathscr{C}_1$ | ★★ | ★★★ | ★★★★ | ★★★ |
| | $\mathscr{C}_2$ | ★★ | ★★★★ | ★★★★★ | ★★★★ |
| | $\mathscr{C}_3$ | ★★★ | ★★★★★ | ★ | ★★★★★ |
| | $\mathscr{C}_4$ | ★★★★ | ★★ | ★ | ★★ |
| | $\mathscr{C}_5$ | ★★★★★ | ★ | ★★ | ★★ |
| | $\mathscr{C}_6$ | ★ | ★★ | ★★★ | ★★★ |
| $M_2$ | $\mathscr{C}_1$ | ★ | ★★★ | ★★★★ | ★★★★ |
| | $\mathscr{C}_2$ | ★★★ | ★★★★ | ★★★★★ | ★★★★★ |
| | $\mathscr{C}_3$ | ★★★★ | ★★★★★ | ★ | ★★★ |
| | $\mathscr{C}_4$ | ★★★★★ | ★★★ | ★★ | ★★★★ |
| | $\mathscr{C}_5$ | ★ | ★★★★★ | ★★★ | ★★★★★ |
| | $\mathscr{C}_6$ | ★ | ★★★★★ | ★★★★ | ★★★ |
| $M_3$ | $\mathscr{C}_1$ | ★★★ | ★ | ★★★★★ | ★★★★ |
| | $\mathscr{C}_2$ | ★★★★ | ★ | ★★★ | ★★★★★ |
| | $\mathscr{C}_3$ | ★★★★★ | ★★★ | ★★★★ | ★★ |
| | $\mathscr{C}_4$ | ★★★ | ★★★★ | ★★★★★ | ★★ |
| | $\mathscr{C}_5$ | ★★★★★ | ★★★★★ | ★★ | ★★★ |
| | $\mathscr{C}_6$ | ★★★★★ | ★★★ | ★★★★ | ★ |
| $M_4$ | $\mathscr{C}_1$ | ★ | ★★★★ | ★★★★★ | ★★ |
| | $\mathscr{C}_2$ | ★★ | ★★★★★ | ★ | ★★★★ |
| | $\mathscr{C}_3$ | ★★★ | ★★★★ | ★★ | ★★★★★ |
| | $\mathscr{C}_4$ | ★★★★ | ★★★★★ | ★ | ★★ |
| | $\mathscr{C}_5$ | ★★★★★ | ★ | ★★ | ★★★★ |
| | $\mathscr{C}_6$ | ★ | ★★★ | ★★ | ★★★ |
| $M_5$ | $\mathscr{C}_1$ | ★ | ★★★★ | ★★★★ | ★★★★ |
| | $\mathscr{C}_2$ | ★★★ | ★★★★★ | ★★ | ★★★★★ |
| | $\mathscr{C}_3$ | ★★★★ | ★★★ | ★★★★★ | ★ |
| | $\mathscr{C}_4$ | ★★★★★ | ★★★★ | ★★ | ★★★★ |
| | $\mathscr{C}_5$ | ★★★ | ★★★★★ | ★★★★ | ★★ |
| | $\mathscr{C}_6$ | ★★★★★ | ★ | ★★ | ★★★★ |
| $M_6$ | $\mathscr{C}_1$ | ★★ | ★★★★ | ★★★★★ | ★★★★ |
| | $\mathscr{C}_2$ | ★★★ | ★★★★★ | ★★★ | ★★★★★ |
| | $\mathscr{C}_3$ | ★★★★ | ★★★ | ★★★★★ | ★ |
| | $\mathscr{C}_4$ | ★★★★★ | ★★★★ | ★★★ | ★★★★ |
| | $\mathscr{C}_5$ | ★★★ | ★★★★★ | ★★★★ | ★★★ |
| | $\mathscr{C}_6$ | ★★★★★ | ★★ | ★★ | ★★★★ |

Note that: The star sign indicated the linguistic terms as declared in Table 5.

Table 7: Decision matrix by $D_1$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.7000 , 0.9000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_2$ | [0.0000 , 1.0000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_3$ | [0.5000 , 0.6999], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_4$ | [0.9000 , 1.0000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_5$ | [0.9000 , 1.0000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_6$ | [0.7000 , 0.9000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_2$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_3$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_4$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_5$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_2$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_3$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_4$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_5$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |

Table 8: Decision matrix by $D_2$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.5000 , 0.6999], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_2$ | [0.5000 , 0.6999], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_3$ | [0.9000 , 1.0000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_4$ | [0.2000 , 0.6999], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_5$ | [0.2000 , 0.6999], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.2000 , 0.6999], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_2$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_3$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_4$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_5$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_2$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_3$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_4$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_5$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_6$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |

Table 9: Decision matrix by $D_3$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.2000 , 0.6999], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_2$ | [0.2000 , 0.6999], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_3$ | [0.0000 , 0.1999], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_4$ | [0.0000 , 0.1999], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_5$ | [0.2000 , 0.6999], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_6$ | [0.0000 , 0.1999], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_2$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_3$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_4$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_5$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_6$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_2$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_3$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_4$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_5$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_6$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |

Table 10: Decision matrix by $D_4$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.5000 , 0.69999] , [−0.7900 , −0.5900] , 0.4500, −0.6199 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |
| $M_2$ | [0.2000 , 0.4999] , [−0.5901 , −0.3900] , 0.7500, −0.4145 | [0.8001 , 1.0000] , [−0.3901 , −0.0001] , 0.9755, −0.1150 |
| $M_3$ | [0.2000 , 0.4999] , [−0.5901 , −0.3900] , 0.7500, −0.4145 | [0.8001 , 1.0000] , [−0.3901 , −0.0001] , 0.9755, −0.1150 |
| $M_4$ | [0.4000 , 0.9000] , [−0.9900 , −0.7900] , 0.1500, −0.8199 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |
| $M_5$ | [0.2000 , 0.6999] , [−0.5901 , −0.3900] , 0.7500, −0.4145 | [0.8001 , 1.0000] , [−0.3901 , −0.0001] , 0.9755, −0.1150 |
| $M_6$ | [0.5000 , 0.6999] , [−0.7900 , −0.5900] , 0.4500, −0.6199 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.8001 , 1.0000] , [−0.3901 , −0.0001] , 0.9755, −0.1150 | [0.1000 , 0.3000] , [−0.9900 , −0.7900] , 0.1500, −0.8199 |
| $M_2$ | [0.3001 , 0.5000] , [−0.7900 , −0.5900] , 0.4500, −0.6199 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |
| $M_3$ | [0.1000 , 0.3000] , [−0.9900 , −0.7900] , 0.1500, −0.8199 | [0.1000 , 0.3000] , [−0.9900 , −0.7900] , 0.1500, −0.8199 |
| $M_4$ | [0.8001 , 1.0000] , [−0.3901 , −0.0001] , 0.9755, −0.1150 | [0.1000 , 0.3000] , [−0.9900 , −0.7900] , 0.1500, −0.8199 |
| $M_5$ | [0.0000 , 0.1000] , [−1.0000 , −0.9900] , 0.0500, −0.9999 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |
| $M_6$ | [0.0000 , 0.1000] , [−1.0000 , −0.9900] , 0.0500, −0.9999 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.1000 , 0.3000] , [−0.9900 , −0.7900] , 0.1500, −0.8199 | [0.3001 , 0.5000] , [−0.7900 , −0.5900] , 0.4500, −0.6199 |
| $M_2$ | [0.8001 , 1.0000] , [−0.3901 , −0.0001] , 0.9755, −0.1150 | [0.3001 , 0.5000] , [−0.7900 , −0.5900] , 0.4500, −0.6199 |
| $M_3$ | [0.3001 , 0.5000] , [−0.7900 , −0.5900] , 0.4500, −0.6199 | [0.0000 , 0.1000] , [−1.0000 , −0.9900] , 0.0500, −0.9999 |
| $M_4$ | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 | [0.3001 , 0.5000] , [−0.7900 , −0.5900] , 0.4500, −0.6199 |
| $M_5$ | [0.1000 , 0.3000] , [−0.9900 , −0.7900] , 0.1500, −0.8199 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |
| $M_6$ | [0.3001 , 0.5000] , [−0.7900 , −0.5900] , 0.4500, −0.6199 | [0.5001 , 0.8000] , [−0.5901 , −0.3900] , 0.7500, −0.4145 |

Table 11: Normalized decision matrix by $D_1$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_2$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_3$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_4$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_5$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_6$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_2$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_3$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_4$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_5$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_2$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_3$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_4$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_5$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |

Table 12: Normalized decision matrix by $D_2$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_2$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_3$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_4$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_5$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_2$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_3$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_4$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_5$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_2$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_3$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_4$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_5$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_6$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |

Table 13: Normalized decision matrix by $D_3$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_2$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_3$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_4$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_5$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_6$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_2$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_3$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_4$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_5$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_6$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_2$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_3$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_4$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_5$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_6$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |

Table 14: Normalized decision matrix by $D_4$

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_2$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_3$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_4$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_5$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 |
| $M_6$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_2$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_3$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_4$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 |
| $M_5$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_6$ | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_2$ | [0.8001 , 1.0000], [−0.3901 , −0.0001], 0.9755, −0.1150 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_3$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.0000 , 0.1000], [−1.0000 , −0.9900], 0.0500, −0.9999 |
| $M_4$ | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 |
| $M_5$ | [0.1000 , 0.3000], [−0.9900 , −0.7900], 0.1500, −0.8199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |
| $M_6$ | [0.3001 , 0.5000], [−0.7900 , −0.5900], 0.4500, −0.6199 | [0.5001 , 0.8000], [−0.5901 , −0.3900], 0.7500, −0.4145 |

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Table 15: Score functions for $D_i$

$D_1$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $M_1$ | 0.15835 | 0.15835 | 0.37503 | 0.60925 | 0.87840 | 0.02668 |
| $M_2$ | 0.02668 | 0.37503 | 0.60925 | 0.87840 | 0.02668 | 0.02668 |
| $M_3$ | 0.37503 | 0.60925 | 0.87840 | 0.37503 | 0.87840 | 0.87840 |
| $M_4$ | 0.02668 | 0.15835 | 0.37503 | 0.60925 | 0.87840 | 0.02668 |
| $M_5$ | 0.02668 | 0.37503 | 0.60935 | 0.87840 | 0.37503 | 0.87840 |
| $M_6$ | 0.15835 | 0.37503 | 0.60925 | 0.87840 | 0.37503 | 0.87840 |

$D_2$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $M_1$ | 0.37503 | 0.60925 | 0.87840 | 0.15835 | 0.02668 | 0.15835 |
| $M_2$ | 0.37503 | 0.60925 | 0.87840 | 0.37503 | 0.87840 | 0.87840 |
| $M_3$ | 0.02668 | 0.02668 | 0.37503 | 0.60925 | 0.87840 | 0.37503 |
| $M_4$ | 0.60925 | 0.87840 | 0.60925 | 0.87840 | 0.02668 | 0.37503 |
| $M_5$ | 0.60925 | 0.87840 | 0.37503 | 0.60925 | 0.87840 | 0.02668 |
| $M_6$ | 0.60925 | 0.87840 | 0.37503 | 0.60925 | 0.87840 | 0.15835 |

$D_3$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $M_1$ | 0.60925 | 0.87840 | 0.02668 | 0.02668 | 0.15835 | 0.37503 |
| $M_2$ | 0.60925 | 0.87840 | 0.02668 | 0.15835 | 0.37503 | 0.60925 |
| $M_3$ | 0.87840 | 0.37503 | 0.60925 | 0.87840 | 0.15835 | 0.60925 |
| $M_4$ | 0.87840 | 0.02668 | 0.15835 | 0.02668 | 0.15835 | 0.15835 |
| $M_5$ | 0.60925 | 0.15835 | 0.87840 | 0.15835 | 0.60925 | 0.15835 |
| $M_6$ | 0.87840 | 0.37503 | 0.87840 | 0.37503 | 0.60925 | 0.15835 |

$D_4$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $M_1$ | 0.37503 | 0.60925 | 0.87840 | 0.15835 | 0.15835 | 0.37503 |
| $M_2$ | 0.60925 | 0.87840 | 0.37503 | 0.60925 | 0.87840 | 0.37503 |
| $M_3$ | 0.60925 | 0.87840 | 0.15835 | 0.15835 | 0.37503 | 0.02668 |
| $M_4$ | 0.15835 | 0.60925 | 0.87840 | 0.15835 | 0.60925 | 0.37503 |
| $M_5$ | 0.60925 | 0.87840 | 0.02668 | 0.60925 | 0.15835 | 0.60925 |
| $M_6$ | 0.60925 | 0.87840 | 0.02668 | 0.60925 | 0.37503 | 0.60925 |

Table 16: Aggregated decision matrix

| | $\mathscr{C}_1$ | $\mathscr{C}_2$ |
|---|---|---|
| $M_1$ | [0.1905 , 0.4115], [−0.9009 , −0.6975], 0.2232 , −0.7627 | [0.3036 , 1.0000], [−0.8114 , −0.3564], 0.2598 , −0.7266 |
| $M_2$ | [0.1650 , 0.6335], [−0.8783 , −0.7716], 0.1194 , −0.9999 | [0.4615 , 1.0000], [−0.6673 , −0.1427], 0.5633 , −0.5267 |
| $M_3$ | [0.6491 , 1.0000], [−0.7611 , −0.2093], 0.2541 , −0.9692 | [0.2714 , 1.0000], [−0.5990 , −0.3404], 0.2979 , −0.9653 |
| $M_4$ | [0.7220 , 1.0000], [−0.8041 , −0.3086], 0.1272 , −0.9980 | [0.3575 , 1.0000], [−0.7982 , −0.1034], 0.2076 , −0.8792 |
| $M_5$ | [0.2736 , 0.5502], [−0.7841 , −0.6443], 0.1734 , −0.9881 | [0.5224 , 1.0000], [−0.6440 , −0.0346], 0.5040 , −0.5437 |
| $M_6$ | [0.3807 , 1.0000], [−0.7496 , −0.1987], 0.3180 , −0.1795 | [0.5166 , 1.0000], [−0.6382 , −0.0498], 0.5740 , −0.5072 |

| | $\mathscr{C}_3$ | $\mathscr{C}_4$ |
|---|---|---|
| $M_1$ | [0.4838 , 1.0000], [−0.6673 , −0.0628], 0.4745 , −0.7294 | [0.3886 , 0.6923], [−0.7005 , −0.5000], 0.4091 , −0.7619 |
| $M_2$ | [0.5452 , 1.0000], [−0.5830 , −0.0799], 0.5614 , −0.7817 | [0.6795 , 1.0000], [−0.5247 , −0.0023], 0.6559 , −0.3901 |
| $M_3$ | [0.3705 , 1.0000], [−0.5473 , −0.4647], 0.6565 , −0.4201 | [0.4621 , 1.0000], [−0.6624 , −0.1618], 0.5518 , −0.4686 |
| $M_4$ | [0.3404 , 1.0000], [−0.7529 , −0.5219], 0.4538 , −0.6094 | [0.5627 , 1.0000], [−0.5689 , −0.0652], 0.5955 , −0.7397 |
| $M_5$ | [0.4952 , 1.0000], [−0.6230 , −0.1566], 0.5887 , −0.5001 | [0.7458 , 1.0000], [−0.4493 , −0.0085], 0.7550 , −0.2948 |
| $M_6$ | [0.4952 , 1.0000], [−0.6230 , −0.1566], 0.5887 , −0.6419 | [0.6813 , 1.0000], [−0.4904 , −0.0048], 0.8086 , −0.3085 |

| | $\mathscr{C}_5$ | $\mathscr{C}_6$ |
|---|---|---|
| $M_1$ | [0.6629 , 1.0000], [−0.5318 , −0.0020], 0.4010 , −0.9177 | [0.0542 , 0.1946], [−0.9781 , −0.8968], 0.0788 , −0.9987 |
| $M_2$ | [0.3480 , 1.0000], [−0.7762 , −0.1035], 0.1328 , −0.9976 | [0.3489 , 1.0000], [−0.7659 , −0.1294], 0.1379 , −0.9975 |
| $M_3$ | [0.7358 , 1.0000], [−0.4616 , −0.0006], 0.7226 , −0.6781 | [0.6272 , 1.0000], [−0.5476 , −0.0087], 0.6150 , −0.6287 |
| $M_4$ | [0.6605 , 1.0000], [−0.5341 , −0.0022], 0.4068 , −0.9062 | [0.0905 , 0.2364], [0.9451 , −0.8563], 0.0939 , −0.9985 |
| $M_5$ | [0.3218 , 0.5488], [−0.7690 , −0.5671], 0.4558 , −0.6123 | [0.5995 , 1.0000], [−0.5861 , −0.0065], 0.1329 , −0.9458 |
| $M_6$ | [0.5297 , 1.0000], [−0.6232 , −0.0482], 0.5983 , −0.4896 | [0.6113 , 1.0000], [−0.5844 , −0.0061], 0.4450 , −0.5501 |

Table 17: Accumulative decision matrix

| Alternatives | Fuzzy values |
|---|---|
| $M_1$ | [0.9725 , 1] , [-0.1777 , 0] , 0.0004 , -1 |
| $M_2$ | [0.9722 , 1] , [-0.1066 , 0] , 0.0005 , -1 |
| $M_3$ | [0.9915 , 1] , [-0.0418 , 0] , 0.0122 , -1 |
| $M_4$ | [0.9841 , 1] , [-0.1388 , 0] , 0.0003 , -1 |
| $M_5$ | [0.9879 , 1] , [-0.0637 , 0] , 0.0024 , -1 |
| $M_6$ | [0.9912 , 1] , [-0.0532 , 0] , 0.0223 , -0.9910 |

Table 18: Score function of alternatives

| Alternative | Score value | Ranking |
|---|---|---|
| $M_1$ | 0.6325 | 6th |
| $M_2$ | 0.6444 | 4th |
| $M_3$ | 0.6603 | 2nd |
| $M_4$ | 0.6409 | 5th |
| $M_5$ | 0.6544 | 3rd |
| $M_6$ | 0.6616 | 1st |

Table 19: Comparison between proposed methods and existing techniques

| Existing techniques | Ranking | Optimal result |
|---|---|---|
| CBF ordered weighted geometric AO [40] | $M_6 \geq M_5 \geq M_3 \geq M_2 \geq M_4 \geq M_1$ | $M_6$ |
| CBF averaging AO [21] | $M_6 \geq M_5 \geq M_3 \geq M_2 \geq M_4 \geq M_1$ | $M_6$ |
| CBF Dombi averaging AO [41] | $M_6 \geq M_3 \geq M_4 \geq M_5 \geq M_2 \geq M_1$ | $M_6$ |
| CBF geometric AO [8] | $M_6 \geq M_3 \geq M_2 \geq M_4 \geq M_1 \geq M_5$ | $M_6$ |
| CBF TOPSIS [42] | $M_6 \geq M_3 \geq M_5 \geq M_2 \geq M_4 \geq M_1$ | $M_6$ |
| CBF ELECTRE-I [42] | $M_6$ | $M_6$ |

**Biographies**

**Nimra Jamil** received her BSc and MSc degrees in Mathematics from the University of the Punjab, Lahore in 2011 and 2013, respectively. She received M.Phil degree in Mathematics from the COM-SATS University Islamabad, Lahore Campus, in 2016. She is currently PhD scholar at Department of Mathematics, University of the Punjab, Lahore, Pakistan. She is the author of 04 SCI research papers and her research interests include bipolar fuzzy sets, cubic sets, cubic bipolar fuzzy sets, cubic m-polar fuzzy sets, multi-criteria decision-making problems, aggregation operators, information measures, information fusion, machine learning, artificial intelligence, and topological data analy-
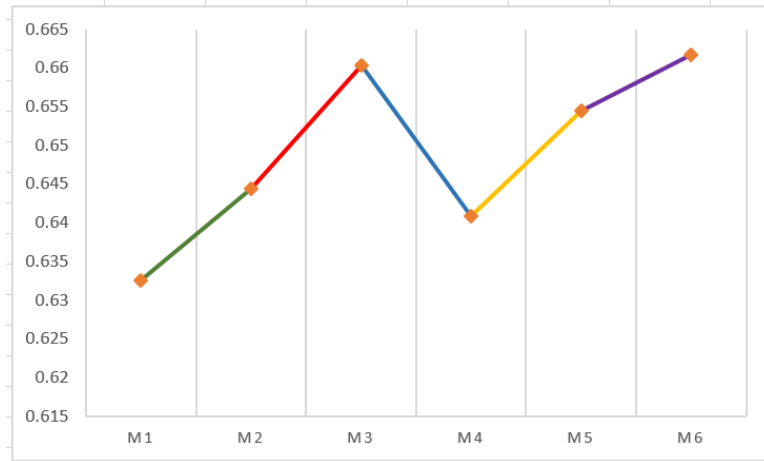
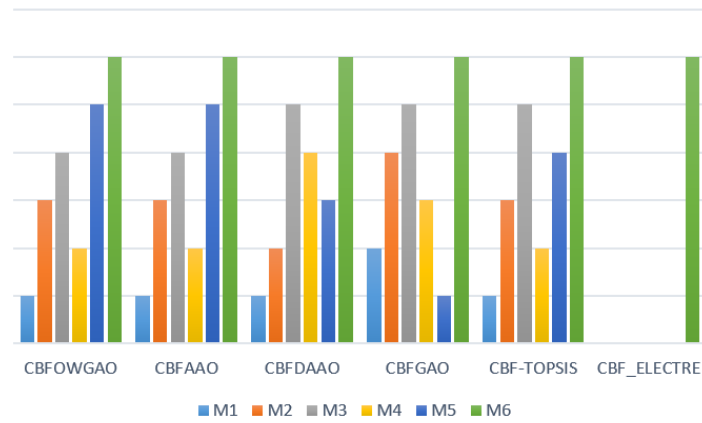Figure 3: Score functions of alternatives



Figure 4: Comparison analysis

sis.

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