On Clustering and Pattern Recognition Techniques Utilizing Bi-parametric Picture Fuzzy (R,S)-Norm Discriminant Information Measure

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Abstract

In the present communication, a very recently proposed bi-parametric (R, S)-norm discriminant measure for picture fuzzy sets has been utilized and different important properties have been discussed. The bi-parametric discriminant measure would give diversification in handling the inexact/incomplete information in terms of obtaining the degree of association and closeness in the data of various applications. The monotonicity of the newly presented discriminant measure in relation to the involved parameters R and S has also been discussed in detail along with its empirical proof. Further, the bi-parametric measure under consideration has been successfully applied in the principle of minimum discriminant information with the help of some illustrative numerical applications in the field of pattern recognition/clustering etc. Additionally, for the validity and efficacy of the presented approach, necessary and detailed comparison studies along with important findings, advantages and limitations have been mentioned.

Keywords: Picture fuzzy information; Decision Science Problems; Machine-Learning; Clustering; Pattern Recognition.

1 Introduction

The involvement of vagueness in decision-making problems is increasing day by day and to handle such complications, the researchers have explored the generic framework of fuzzy sets for making the computation structure more practical in context with real-life problems. Zadeh [1] introduced the concept of fuzzy sets and their entropy measures and then on the basis of cross-entropy measure of probability distributions given by Kullback and Leibler [2], various authors have presented different kinds of divergence/discriminant measures for different extensions of fuzzy sets. Also, various studies handling the uncertainty feature for capturing the inconsistency, impreciseness, and inexactness in a systematic extended fashion by different researchers and notions such as type-2 fuzzy sets [3], rough fuzzy sets [4], neutrosophic sets (NS) [5], intuitionistic fuzzy sets (IFS) [6], Pythagorean fuzzy sets (PyFs) [7], picture fuzzy sets (PFS) [8] and many more have been introduced. Amongst various generalizations of fuzzy sets, the concept of picture fuzzy set's entropy [9] and its properties gained a significant amount of attention and popularity in the research community due to its additional component of uncertainty in a linear way with applications in image processing, machine learning, clustering, electric vehicle charging station, medical diagnosis etc.

It may be noted that various divergence measures [10] are available in the literature that have their own limitations and are not able to address the features encountered by picture fuzzy information measures. However, a lot of studies have been done by various authors in the field of picture fuzzy sets and their applications, but no research has been carried out in the field of bi-parameterizations of the discriminant information measure. For determining the degree of association and proximity in the data of different applications, the bi-parametric discriminant measure would provide suitable and flexible diversification in managing the issues of uncertainty in the picture fuzzy information. The contributions of the present manuscript are listed and enumerated below:

- A very recently proposed (R, S)-norm discriminant measure for the picture fuzzy sets has been utilized.
- Various important properties and monotonicity of the proposed discriminant measure have been studied in detail.
- The implementation of the proposed measure and its properties in the field of machine learning problems, viz. pattern recognition and clustering has been presented with illustrative examples.

The manuscript has been organized as follows. A detailed literature review related to the proposed work has been presented in Section 2. Also, in view of the topics under the proposition, some related fundamental concepts and basic definitions are presented in Section 3. In Section 4, a very recently proposed bi-parametric (R, S)-norm picture fuzzy discriminant measure has been proposed along with some important properties, results, and its validity in accordance with the existing axioms. Further, the monotonic behavior of the proposed measure with respect to the involved parameters has been studied empirically in Section 5. Section 6 comprehensively presents the illustration of the newly provided parametric discriminant measure in machine learning decision-science problems - "pattern recognition and clustering analysis" by solving a numerical example for each. A detailed comparative analysis for the application problems under consideration has been comprehensively outlined in Section 7 Also, the numerical examples show the practical usefulness of the proposed methodology/measure with comparative remarks. Finally, the manuscript is concluded in Section 8.

2 Literature Review

In the field of picture fuzzy sets, various related operations for interval-valued picture fuzzy sets [8] have been provided and applied in the field of decision-making. Wei et al. [11] presented some mathematical models to deal with the problems of MADM in picture fuzzy setup. Further, Wang et al. [12] incorporated the VIKOR method and the mathematical models in an integrated framework so as to obtain the compromise solution in the multi-criteria decision problems. Ashraf et al. [13] (t-norm and t-conorm), Thong and Son [14] (clustering), Jana et al. [15] (Dombi t-norm/t-conorm) studied novel techniques for handling the MADM problem under picture fuzzy information. A novel clustering technique called distributed picture fuzzy clustering technique was provided by Son [16] and in addition to this, picture fuzzy distance measure and analogous hierarchical picture clustering method have also been presented by Son [17] in a generalized way. Further, in order to study the cross-relationship among the criteria and the impact of preferential information, Tian et al. [18] proposed some picture fuzzy weighted operators for solving MCDM problems. Also, some aggregation operators termed Archimedean picture fuzzy linguistic [19] and Einstein weighted/ordered weighted operators [20] have been developed for solving multi-attribute group decision-making problems in picture fuzzy environments. Also, Ejegwa and Zuakwagh [21] used the Fermatean fuzzy composite relations and successfully applied in the problems of pattern recognition. A hybrid technique for assessing analogical skills has also been studied [22]. In addition to this, the trapezoidal fuzzy numbers have also been implemented with the incorporation of some new similarity functions in the decision-making techniques [23], [24].

Ganie et al. [25] presented the new correlation coefficients for picture fuzzy sets and utilized them in some MCDM problems. Singh and Ganie [26] also provided another picture fuzzy correlation coefficient to study the pattern recognition problem and identification of an investment sector based on it. Also, Khan et al.[27] presented some parameterized distance and similarity measures for picture fuzzy sets and applied them to the problem of medical diagnosis. Kadian and Kumar [28] presented a new picture fuzzy divergence measure for the MCDM problem on the basis of Jensen-Tsallis entropy. An innovative form of picture fuzzy distance and similarity measure has been proposed by Ganie et al. [29]. Umar et al. [30] introduced a novel technique of decision-making in machine learning problems with the incorporation of picture fuzzy divergence measures. A novel picture fuzzy entropy has been utilized by Kumar et al. [31] with partial weight information on the basis of the hybrid picture fuzzy methodology. Further, in the picture fuzzy environment aggregation operators based on Schweizer-Sklar norms with unknown weights have been proposed for a new decision-making methodology [32]. Also, for the complex nature of the picture fuzzy sets Hamacher aggregation operators have been implemented for the decision-making techniques [33].

3 Preliminaries

Here, we present some basic definitions that are in relation to the picture fuzzy set and are available in the literature for ready reference.

Definition 1 ***Intuitionistic Fuzzy Set (IFS)*** *[6]: "An intuitionistic fuzzy set I in X (universe of discourse) is given by $I = \{< x, \rho_I(x), \omega_I(x) > \mid x \in X\}$; where $\rho_I : X \rightarrow [0,1]$ and $\omega_I : X \rightarrow [0,1]$ denote the degree of membership and degree of non-membership respectively and for every $x \in X$ satisfy the condition $0 \leq \rho_I(x) + \omega_I(x) \leq 1$; and the degree of indeterminacy for any IFS I and $x \in X$ is given by $\pi_I(x) = 1 - \rho_I(x) - \omega_I(x)$."*

Definition 2 ***Picture Fuzzy Set(PFS)*** *[8]: "A picture fuzzy set U in X (universe of discourse) is given by*

$$U = \{< x, \rho_U(x), \tau_U(x), \omega_U(x) > \mid x \in X\};$$

where $\rho_U : X \to [0,1]$, $\tau_U : X \to [0,1]$ and $\omega_U : X \to [0,1]$ denote the degree of positive membership, degree of neutral membership and degree of non-membership respectively and for every $x \in X$ satisfy the condition

$$0 \leq \rho_U(x) + \tau_U(x) + \omega_U(x) \leq 1;$$

and the degree of refusal for any picture fuzzy set U and $x \in X$ is given by $\theta_U(x) = 1 - \rho_U(x) - \tau_U(x) - \omega_U(x)$".

Here, it may be noted that $0 \leq \rho_U(x) + \tau_U(x) + \omega_U(x) \leq 1$; where in case of intuitionistic fuzzy set, we have $0 \leq \rho_I(x) + \omega_I(x) \leq 1$; for $\rho_U(x), \tau_U(x), \omega_U(x) \in [0,1]$.

Remark: It may be noted that the concept of neutrality degree can be seen in situations where human opinions involve responses in the form of yes, abstain, no and refusal. The degree of abstain or abstinence means the rejection of yes as well as no.

Definition 3 *[8]: "If $U, V \in PFS(X)$, then the operations can be defined as follows:*

(a) ***Complement:*** $\overline{U} = \{< x, \omega_U(x), \tau_U(x), \rho_U(x) > \mid x \in X\}$;

(b) ***Subsethood:*** $U \subseteq V$ *iff* $\forall x \in X, \rho_U(x) \leq \rho_V(x)$ $\tau_U(x) \geq \tau_V(x)$ *and* $\omega_U(x) \geq \omega_V(x)$;

(c) ***Union:*** $U \cup V = \{< x, \rho_U(x) \vee \rho_V(x), \tau_U(x) \wedge \tau_V(x)$ *and* $\omega_U(x) \wedge \omega_V(x) > \mid x \in X\}$;

(d) ***Intersection:*** $U \cap V = \{< x, \rho_U(x) \wedge \rho_V(x), \tau_U(x) \vee \tau_V(x)$ *and* $\omega_U(x) \vee \omega_V(x) > \mid x \in X\}$.

In this paper, we use $PFS(X)$ to denote the collection of all the PFSs defined on the domain of discourse X."

Definition 4 ***Average Picture Fuzzy Set****[17]: "The average picture fuzzy set of $U_i \in PFS(X), i = 1, 2, ..., n$ is represented by $(U_i)_{av}$ and given by*
$(U_i)_{av} = \{\langle x, \frac{1}{n}\sum_{i=1}^{n} \rho(x), \frac{1}{n}\sum_{i=1}^{n} \tau(x), \frac{1}{n}\sum_{i=1}^{n} \omega(x)\rangle | x \in X\}$."

4 $(R,\, S)$-Norm Picture Fuzzy Discriminant Measure

In the literature of information theory, there are several distance measures, similarity measures, and divergence/dissimilarity measures for different extensions of fuzzy sets, but there are some measures that can be very useful and involve fewer computations at the cost that they do not satisfy the prerequisite standard axioms. In order to overcome such limitations, there is a need to define the notion of discriminant information measures which involves only two axioms. Various researchers have explored the concept of parameterized discriminant information measures and utilized them in different application fields. Some initial related notions [10] are described as follows:

"Let $\triangle_n = \{P = (p_1, p_2, ...p_n),\ p_i \geq 0,\ i = 1, 2, 3, ...n$ *and* $\sum p_i = 1\}$ *be the set of all probability distributions associated with a discrete random variable* X *taking finite values* $x_1, x_{2,}......, x_n$*. For all probability distribution of* $P = (p_1, p_2, \ldots, p_n)$ *and* $Q = (q_1, q_2, \ldots, q_n) \in \triangle_n$*, Joshi and Kumar [10] proposed a divergence measure:*

$$\mathbb{D}_R^S(P, Q) = \frac{R \times S}{S - R}\left[\left(\sum_{i=1}^{n}(p_i^S q_i^{1-S})\right)^{\frac{1}{S}} - \left(\sum_{i=1}^{n}(p_i^R q_i^{1-R}\right)^{\frac{1}{R}}\right]; \tag{4.1}$$

where either $0 < S < 1$ *and* $1 < R < \infty$ *or* $0 < R < 1$ *and* $1 < S < \infty$*."*

It may also be noted that, in an analogous fashion, various parametric discriminant measures for different types of sets such as fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets have been proposed and studied in detail.

Similarly, on the basis of the above proposed theoretic probabilistic information measure (4.1), Dhumras and Bajaj [34] presented a new bi-parametric picture fuzzy discriminant measure for two PFSs $U, V \in PFS(X)$ as follows:

$$\mathbb{I}_R^S(U, V) = \frac{R \times S}{n(S - R)}\sum_{i=1}^{n}\left[\begin{array}{c} \left(\rho_U(x_i)^S \rho_V(x_i)^{(1-S)} + \tau_U(x_i)^S \tau_V(x_i)^{(1-S)} + \omega_U(x_i)^S \omega_V(x_i)^{(1-S)} + \theta_U(x_i)^S \theta_V(x_i)^{(1-S)}\right)^{\frac{1}{S}} \\ -\left(\rho_U(x_i)^R \rho_V(x_i)^{(1-R)} + \tau_U(x_i)^R \tau_V(x_i)^{(1-R)} + \omega_U(x_i)^R \omega_V(x_i)^{(1-R)} + \theta_U(x_i)^R \theta_V(x_i)^{(1-R)}\right)^{\frac{1}{R}} \end{array}\right]; \tag{4.2}$$

where either $0 < S < 1$ & $1 < R < \infty$ or $0 < R < 1$ & $1 < S < \infty$. $\mathbb{I}_R^S(U, V)$ is not showing the symmetric nature with respect to the argument sets. In accordance, it can be structured as follows:

$$\mathbb{J}_R^S(U, V) = \mathbb{I}_R^S(U, V) + \mathbb{I}_R^S(V, U). \tag{4.3}$$

Guiwu Wei [35] presented the concept of discriminant information measure for PFSs and defined "*picture fuzzy cross entropy*" as $\mathbb{I}_{PFS}(U, V)$ which satisfies two axioms -

$\mathbb{I}_{PFS}(U,V) \geq 0$ and $\mathbb{I}_{PFS}(U,V) = 0$ if and only if $U = V$. For fuzzy sets $\mathbb{I}_{FS}(U,V) \neq \mathbb{I}_{FS}(\overline{U},\overline{V})$. However, for PFSs, $\mathbb{I}_{PFS}(U,V) = \mathbb{I}_{PFS}(\overline{U},\overline{V})$ holds.

5 Monotone Behavior of Proposed Discriminant Measure

The monotonic behavior of the presented bi-parametric picture fuzzy (R, S)-norm discriminant measure has been studied in detail for proper understanding and clarification. For the sake of this, the following 4 pairs of picture fuzzy sets

$$A = (U_1, U_2),\ B = (U_3, U_4),\ C = (U_5, U_6) \text{ and } D = (U_7, U_8)$$

over the universe of discourse $X = \{x_1, x_2, x_3\}$:

$$U_1 = \{(x_1, 0.1, 0.3, 0.5), (x_2, 0.2, 0.4, 0.2), (x_3, 0.1, 0.2, 0.4)\};$$
$$U_2 = \{(x_1, 0.2, 0.3, 0.2), (x_2, 0.2, 0.1, 0.2), (x_3, 0.1, 0.3, 0.4)\}.$$

$$U_3 = \{(x_1, 0.1, 0.1, 0.5), (x_2, 0.2, 0.2, 0.3), (x_3, 0.1, 0.4, 0.4)\};$$
$$U_4 = \{(x_1, 0.1, 0.3, 0.2), (x_2, 0.2, 0.4, 0.3), (x_3, 0.1, 0.2, 0.5)\}.$$

$$U_5 = \{(x_1, 0.1, 0.2, 0.5), (x_2, 0.2, 0.2, 0.2), (x_3, 0.2, 0.4, 0.4)\};$$
$$U_6 = \{(x_1, 0.2, 0.3, 0.1), (x_2, 0.2, 0.1, 0.1), (x_3, 0.1, 0.3, 0.5)\}.$$

$$U_7 = \{(x_1, 0.1, 0.1, 0.7), (x_2, 0.2, 0.3, 0.2), (x_3, 0.1, 0.4, 0.2)\};$$
$$U_8 = \{(x_1, 0.5, 0.2, 0.1), (x_2, 0.2, 0.4, 0.2), (x_3, 0.2, 0.4, 0.1)\}.$$

Without loss of generality, we can first fix up the values of S as $S = 0.15, 0.25, 1, 4, 15, 60, 150$, then the value of R is translated in a range, say, $(0.001 \leq R \leq 295)$ and then the values of the proposed measure are being tabulated for these pairs of picture fuzzy sets in consideration. Based on the Table 1, we plot the charts that clearly reflect the monotone behavior of the information measure in connection with the parameters as shown in Figure 1.

6 Applications of Proposed Discriminant Measure in Pattern Recognition Problems

We utilize the newly proposed bi-parametric (R, S)-Norm discriminant information measure for PFSs in the different application areas of decision-making machine learning problems.

6.1 Pattern Recognition

In the problems of pattern recognition, a pattern that is completely unknown is grouped into the already designed patterns by making use of various information measures like discriminant/divergence measure, similarity measure, and distance measure. The methodology to solve a general problem of pattern recognition, we present the procedural steps with the help of Figure 2.

Now we consider a numerical illustration of pattern recognition which is already present in the literature. Also, the results are contrasted with those of the existing measures to test the predominance of the proposed discriminant measure.
Remark: With reference to the dataset applicable in the proposed methodology, it is being observed that the overall time complexity of the proposed technique is of linear order, i.e., $O(n)$; where n is the number of the point in the dataset.

6.1.1 Numerical Illustration [36]

In order to have a classification in the unknown pattern V with the prior known patterns $U_j (j = 1, 2, ..., n)$ by making use of different fuzzy measures. Suppose we consider 3 patterns which are already known be U_1, U_2 and U_3. In the picture fuzzy representation of these patterns in $X = \{x_1, x_2, x_3\}$ given as:

$$U_1 = \{(x_1, 0.4, 0.3, 0.1), (x_2, 0.5, 0.3, 0.2), (x_3, 0.4, 0.3, 0), (x_4, 0.7, 0, 0.2), (x_5, 0.6, 0.1, 0.1)\};$$
$$U_2 = \{(x_1, 0.7, 0.1, 0.1), (x_2, 0.2, 0.3, 0.4), (x_3, 0.2, 0.1, 0.5), (x_4, 0.1, 0.5, 0.2), (x_5, 0.3, 0.3, 0.3)\};$$
$$U_3 = \{(x_1, 0.1, 0.3, 0.4), (x_2, 0.4, 0.3, 0.1), (x_3, 0.3, 0.4, 0.2), (x_4, 0.2, 0.5, 0.3), (x_5, 0.5, 0.3, 0.1)\}.$$

Suppose there is an unclassified sample pattern V which is provided in the picture fuzzy representation as:

$$V = \{(x_1, 0.6, 0.2, 0.1), (x_2, 0.3, 0.4, 0.2), (x_3, 0.4, 0.3, 0.2), (x_4, 0.7, 0.1, 0.0), (x_5, 0.4, 0.2, 0.2)\}.$$

Now, in order to find out the pattern to which the unknown pattern V belongs, we shall make use of the principle of minimum discriminant information as given by [37], the procedure for allotment of V to U_{j^*} is determined by

$$j^* = arg\min_j(\mathbb{I}_R^S(U_j, V)). \tag{6.1}$$

Now, using proposed discriminant measure (4.2), we get the values of $arg\min_j(\mathbb{I}_R^S(U_j, V))$ tabulated in Table 2:

Hence, from Table 2, it can be seen that the completely unknown pattern V is clubbed to pattern U_1.

6.1.2 Numerical Illustration [38]

Let us consider another numerical illustration regarding pattern recognition for more clarity on the proposed notion. Consider three patterns that are already known U_1, U_2 and U_3. In the picture fuzzy representation of these patterns in $X = \{x_1, x_2, x_3\}$ given as:

$$U_1 = \{(x_1, 0.4, 0.4, 0.1), (x_2, 0.7, 0.15, 0.1), (x_3, 0.3, 0.3, 0.2)\};$$
$$U_2 = \{(x_1, 0.5, 0.3, 0.1), (x_2, 0.7, 0.2, 0.05), (x_3, 0.5, 0.3, 0.1)\};$$
$$U_3 = \{(x_1, 0.4, 0.5, 0.1), (x_2, 0.7, 0.1, 0.1), (x_3, 0.4, 0.3, 0.2)\}.$$

Suppose there is an unclassified sample pattern V which is provided in the picture fuzzy representation as:

$$V = \{(x_1, 0.1, 0.1, 0.4), (x_2, 0.8, 0.05, 0.05), (x_3, 0.05, 0.8, 0.05)\}.$$

Now, in order to find out the pattern to which the unknown pattern V belongs, we shall make use of the principle of minimum discriminant information as given by [37], the procedure for allotment of V to U_{j^*} is determined by

$$j^* = arg\min_j(\mathbb{I}_R^S(U_j, V)). \tag{6.2}$$

Now, using the proposed discriminant measure (4.2), we get the values of $arg\min_j(\mathbb{I}_R^S(U_j, V))$ tabulated in Table 3:

Hence, from Table 3, it may be noted that the completely unclassified pattern V is clubbed to pattern U_1.

6.2 Clustering

In this section, the proposed discriminant measure is applied for clustering on various PFSs. The method of clustering for PFSs has been introduced by Singh[39] on similar lines as introduced by Xu et al.[40] for IFSs.
In order to examine the applicability of the proposed discriminant measure, its practicality is given by the help of a numerical illustration.
Note: The various steps of the methodology involved in clustering has been shown in Figure 3.

6.2.1 Numerical Illustration [41]

A collection of data for building materials is provided in Table 4 which is composed of 5 main materials: "sealant, floor varnish, wall paint, carpet, and chloride flooring" which are to be evaluated against eight criteria. The main aim behind this collection of data is to verify the method of clustering on a collection of data having a large number of objects.

The methodology of clustering involves the following steps as:
Step 1: In the first step, all of the PFSs $C_j (j = 1, 2, 3, 4, 5)$ is counted as single cluster

$$\{C_1\}, \{C_2\}, \{C_3\}, \{C_4\}, \{C_5\}.$$

Step 2: In the next step, draw a comparison between each of the PFSs C_j by making use of the proposed discriminant measure (4.2) (R=S=1) as:

$$\begin{aligned}
\mathbb{I}(C_1, C_2) = \mathbb{I}(C_2, C_1) = 0.1471\\
\mathbb{I}(C_1, C_3) = \mathbb{I}(C_3, C_1) = 0.1835\\
\mathbb{I}(C_1, C_4) = \mathbb{I}(C_4, C_1) = 0.1358\\
\mathbb{I}(C_1, C_5) = \mathbb{I}(C_5, C_1) = 0.1234\\
\mathbb{I}(C_2, C_3) = \mathbb{I}(C_3, C_2) = 0.1005\\
\mathbb{I}(C_2, C_4) = \mathbb{I}(C_4, C_2) = 0.1693\\
\mathbb{I}(C_2, C_5) = \mathbb{I}(C_5, C_2) = 0.4454\\
\mathbb{I}(C_3, C_4) = \mathbb{I}(C_4, C_3) = 0.1307\\
\mathbb{I}(C_3, C_5) = \mathbb{I}(C_5, C_3) = 0.1603\\
\mathbb{I}(C_4, C_5) = \mathbb{I}(C_5, C_4) = 0.1202
\end{aligned}$$

since,

$$\mathbb{I}(C_1, C_5) = \min\{\mathbb{I}(C_1, C_2), \mathbb{I}(C_1, C_3), \mathbb{I}(C_1, C_4), \mathbb{I}(C_1, C_5)\} = 0.1234;$$

$$\mathbb{I}(C_2, C_3) = \min\{\mathbb{I}(C_2, C_1), \mathbb{I}(C_2, C_3), \mathbb{I}(C_2, C_4), \mathbb{I}(C_2, C_5)\} = 0.1005;$$

$$\mathbb{I}(C_4, C_5) = \min\{\mathbb{I}(C_4, C_1), \mathbb{I}(C_4, C_2), \mathbb{I}(C_4, C_3), \mathbb{I}(C_4, C_5)\} = 0.1202.$$

Now, in every step only two clusters can be grouped, the PFSs $C_j (j = 1, 2, 3, 4, 5)$ can be grouped into the following three clusters as

$$\{C_1\}, \{C_2, C_3\}, \{C_4, C_5\}.$$

Step 3: Here, the centers of each grouped cluster are computed with the definition 4 and shown in Table 5.

Now, again compare each of the grouped clusters with their centers by making use of the proposed discriminant measure given by equation (4.2) and we get

$$\mathbb{I}(\mathbb{C}(C_1), \mathbb{C}(C_2, C_3)) = \mathbb{I}(\mathbb{C}(C_2, C_3), \mathbb{C}(C_1)) = 0.0305$$

$$\mathbb{I}(\mathbb{C}(C_1), \mathbb{C}(C_4, C_5)) = \mathbb{I}(\mathbb{C}(C_4, C_5), \mathbb{C}(C_1)) = 0.0094$$

$$\mathbb{I}(\mathbb{C}(C_2, C_3), \mathbb{C}(C_4, C_5)) = \mathbb{I}(\mathbb{C}(C_4, C_5), \mathbb{C}(C_2, C_3)) = 0.0061$$

$$\mathbb{I}(\mathbb{C}(C_2, C_3), \mathbb{C}(C_4, C_5)) = \min\{\mathbb{I}(\mathbb{C}(C_1), \mathbb{C}(C_2, C_3)), \mathbb{I}(\mathbb{C}(C_1), \mathbb{C}(C_4, C_5)), \mathbb{I}(\mathbb{C}(C_2, C_3), \mathbb{C}(C_4, C_5))\}$$

Hence, on similar lines, the PFSs $C_j (j = 1, 2, 3, 4, 5)$ can be grouped into the following clusters as:

$$\{C_1\}, \{C_2, C_3, C_4, C_5\}$$

Step 4: In the final step, the above two clusters can be further grouped into a single cluster as:

$$\{C_1, C_2, C_3, C_4, C_5\}$$

The overall process of the above clustering is shown in the following Table 6:

Remarks: It appears to be prominent that the values of the discriminant information measure computed above are somewhat promising and more reliable which establishes and affirms the effectiveness of the proposed technique.

7 Comparative Analysis & Discussion

In this section, we have presented a detailed comparative analysis based on the numerical examples solved in the previous section. In this process, the necessary respective comparative remarks have been listed along with advantages and limitations.

Comparative Remarks: (Referring to the numerical illustration presented in subsection 6.1.1)

In order to check the predominance of the proposed discriminant measure over the existing measures a tabular comparison has been carried out in Table 7.

Now, from Table 7, it can be clearly seen that the proposed discriminant measure (4.2) and the other existing measures give the same results and the completely unknown pattern V is grouped to U_1 with the help of these measures. This shows the consistency of the result with more prominent classifier score values.

Comparative Remarks: (Referring to the numerical illustration presented in subsection 6.1.2)

In order to check the predominance of the proposed discriminant measure over the existing measures a tabular comparison has been carried out in Table 8:

Now, from Table 8, it can be clearly seen that the newly presented discriminant measure (4.2) and the other existing measures yield similar results and the completely unclassified pattern V is grouped to U_1 with the help of these measures.

Comparative Remarks: (Referring to the numerical illustration presented in subsection 6.2.1)

In order for the validation of the proposed discriminant measure, a comparison is drawn with the two methods, i.e., hierarchical picture fuzzy clustering method **(HPC)** given by Son [17] and intuitionistic hierarchical clustering method **(IHC)** given by Xu [42] in the Table 9 and Table 10. From the comparative analysis, it can be clearly seen that the technique of clustering by making use of the proposed discriminant measure is good and is of great effectiveness. There is no restriction on the number of attributes(which can be increased) in the collection of datasets.

Advantages and Limitations: On comparing our proposed technique with the existing ones, we find that our method is equally consistent with the importance given to the decision-makers by assigning their weights. In addition, the suggested approach concurrently recommends the exact input and strategy, which is not the case with other methods. Additionally, compared to other ways that involve a complex process to reach a conclusion, the suggested method is easier to apply and involves fewer computations. The limitations of the presented methodology lie in the fact that it won't be able to cover the "sub-parametrization features" inherited in information related to some decision-making problem. As a future work, it may be suggested that the notion of picture fuzzy hypersoft sets may accordingly be implemented.

As a result, the performance of the suggested method is really good. A characteristic comparison table explaining the advantages and features of the proposed measures and techniques is given in Table 11 for understanding the motivation and the necessity of the proposed techniques:

8 Conclusions and Scope for Future Work

The proposed bi-parametric measure for PFSs has been feasibly utilized and the monotonicity of the discriminant measure in R and S has been successfully established. The discriminant measure's efficiency has been better than the existing measures due to the parameters and uncertainty components (degree of abstain and refusal) which are very necessary for the practical problems. Further, the discriminant measure has been implemented in the machine learning decision science problems which are related to the field of "pattern recognition and clustering". The methodology for each application has been outlined separately and illustrated with the help of a numerical example for each. The outcomes in each of the applications considered in the manuscript are in line with the existing practices in use but with less computational effort. The suggested methodology assesses each choice in relation to each criterion separately by assigning considerable importance/weightage while deciding the preference in case of a decision-making dilemma. When compared to various existing strategies, the proposition advises the particular input & straightforward procedure simultaneously.

In the future, the notion of useful bi-parametric discriminant measures for the more generalized T-spherical fuzzy sets on the basis of the utility distribution information measures discussed by Hooda et al. [43] can be proposed. This proposition will be supported by the concept of integrated ambiguity and information improvement measures. Additionally, the constrained optimization of these information measures may be discussed in detail with the possible scope of applications in various other types of decision-making problems. Besides the above-stated possibilities, these deliberations may be analogously devised by taking soft sets, bipolar sets [44], complex fuzzy sets, bipolar soft sets [45] and spherical and T-spherical fuzzy sets [46] [47] into account as per the necessity raised due to uncertainty of the information.

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Appendices

Figure Captions

Table Captions

- Table 5: Centers of grouped clusters
- Table 6: Clustering results of proposed measure
- Table 7: Comparative Analysis with Some Existing Measures
- Table 8: Comparative Analysis with Some Existing Measures
- Table 9: Clustering results of existing measures
- Table 10: Clustering results of existing measures
- Table 11: Characteristic Comparison with the Existing Techniques

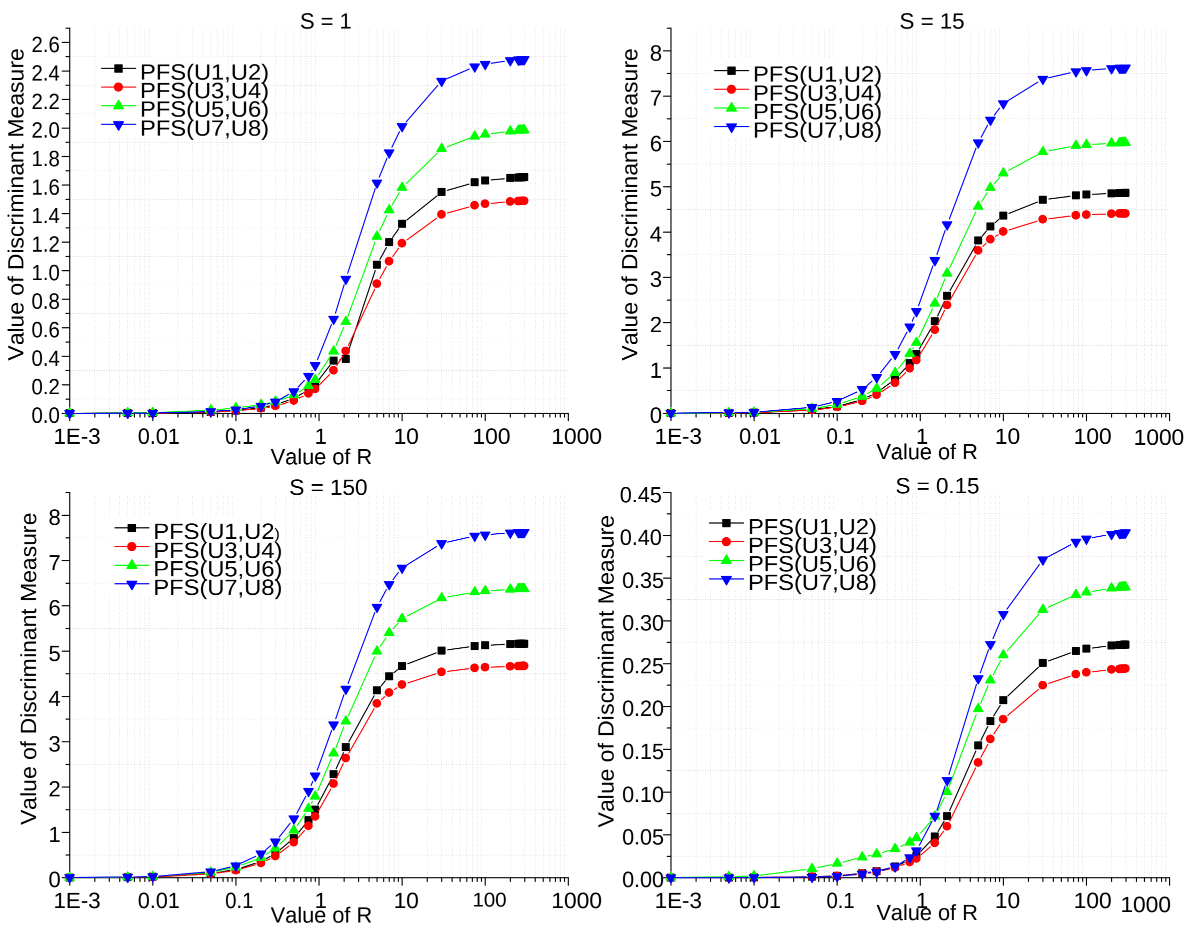

Figure 1: Monotonicity Property of Bi-parametric Discriminant Measure

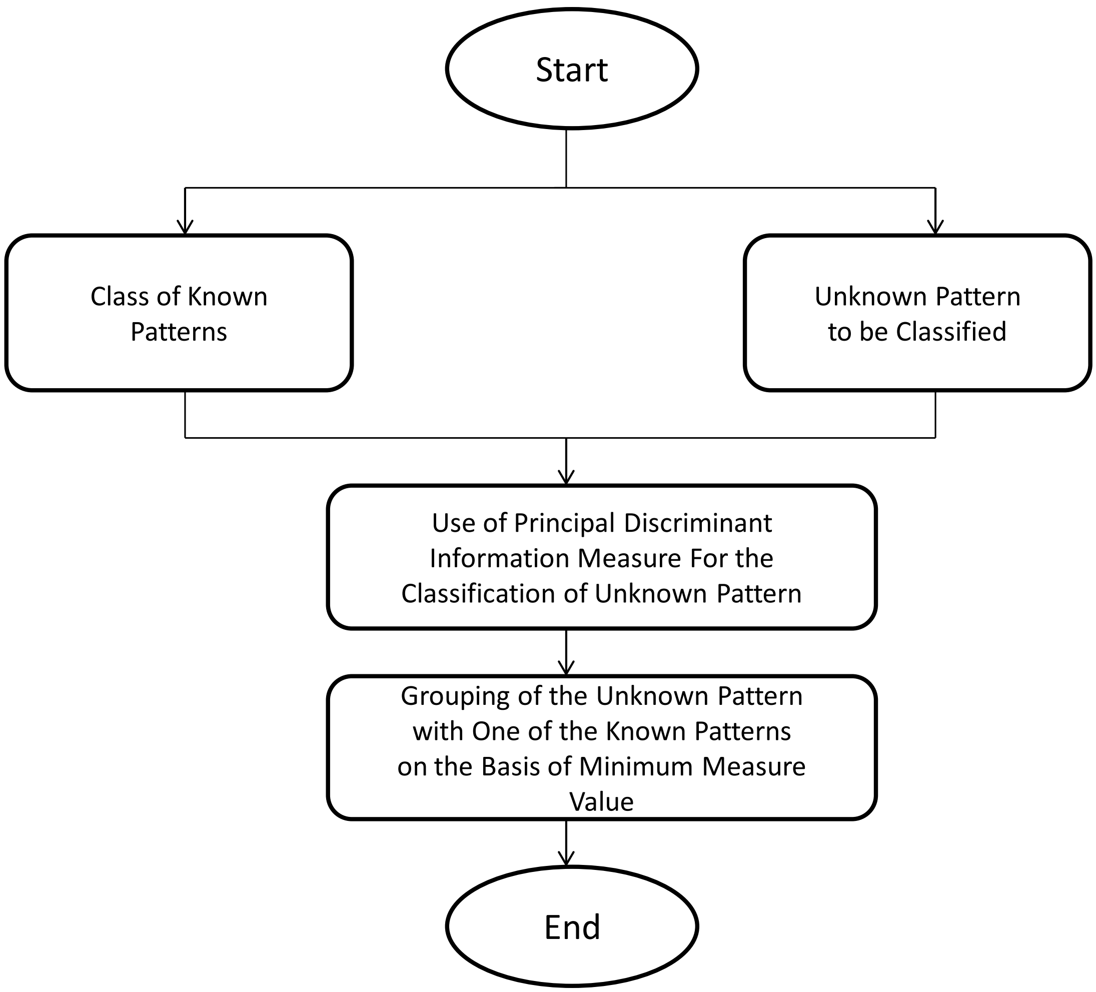

Figure 2: Procedural Steps for Solving a Pattern Recognition Problem

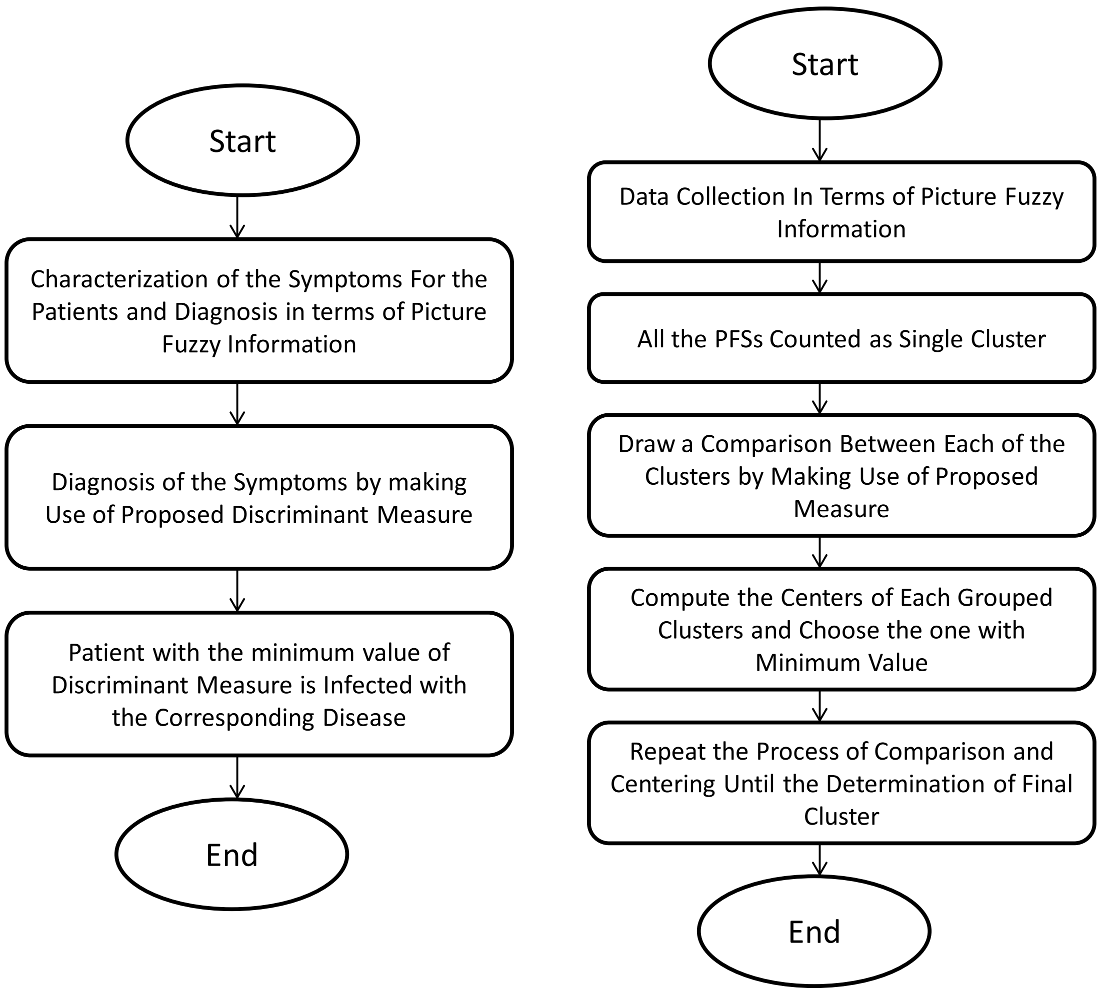

Figure 3: Procedural Steps of Solving the Clustering Problem

Table 1: Values of R, S Norm Discriminant Measure

No	R	$S = 0.15$				$S = 1$				$S = 15$				$S = 150$			
		A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D
1	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.003
2	0.005	0.000	0.000	0.001	0.000	0.001	0.001	0.002	0.001	0.008	0.007	0.01	0.011	0.009	0.008	0.012	0.013
3	0.01	0.000	0.000	0.002	0.000	0.002	0.002	0.005	0.002	0.015	0.014	0.021	0.022	0.018	0.016	0.024	0.027
4	0.05	0.001	0.001	0.01	0.001	0.009	0.008	0.022	0.012	0.076	0.068	0.102	0.113	0.091	0.081	0.119	0.134
5	0.1	0.002	0.002	0.017	0.002	0.019	0.017	0.038	0.024	0.152	0.136	0.195	0.225	0.181	0.162	0.229	0.267
6	0.2	0.005	0.005	0.024	0.004	0.039	0.034	0.061	0.051	0.304	0.272	0.374	0.452	0.358	0.321	0.441	0.532
7	0.3	0.007	0.007	0.028	0.007	0.06	0.051	0.082	0.081	0.454	0.406	0.549	0.677	0.533	0.478	0.646	0.793
8	0.5	0.013	0.012	0.034	0.013	0.104	0.089	0.127	0.152	0.748	0.669	0.894	1.125	0.871	0.781	1.046	1.305
9	0.75	0.021	0.018	0.041	0.023	0.165	0.138	0.191	0.261	1.103	0.988	1.314	1.665	1.272	1.142	1.523	1.911
10	0.9	0.026	0.022	0.046	0.032	0.204	0.169	0.234	0.336	1.306	1.172	1.557	1.972	1.498	1.347	1.796	2.251
11	1.5	0.048	0.040	0.071	0.072	0.369	0.301	0.434	0.661	2.031	1.841	2.426	3.015	2.287	2.073	2.746	3.377
12	2.1	0.072	0.059	0.1	0.114	0.37	0.435	0.642	0.941	2.592	2.386	3.087	3.752	2.88	2.637	3.452	4.165
13	5	0.154	0.134	0.2	0.233	0.1.041	0.907	1.239	1.617	3.814	3.593	4.564	5.741	4.134	3.842	4.994	5.977
14	7	0.183	0.162	0.231	0.273	1.199	1.065	1.425	1.828	4.122	3.839	4.974	6.25	4.44	4.084	5.404	6.475
15	10	0.207	0.185	0.26	0.308	1.329	1.19	1.582	2.012	4.361	4.011	5.299	7.038	4.672	4.257	5.717	6.84
16	30	0.251	0.225	0.313	0.372	1.55	1.394	1.855	2.329	4.712	4.28	5.772	7.038	5.01	4.537	6.171	7.377
17	75	0.265	0.238	0.331	0.392	1.619	1.457	1.941	2.431	4.811	4.366	5.904	7.039	5.112	4.626	6.306	7.542
18	100	0.267	0.239	0.334	0.396	1.631	1.468	1.956	2.448	4.828	4.38	5.927	7.041	5.129	4.641	6.329	7.57
19	200	0.271	0.243	0.338	0.401	1.649	1.484	1.978	2.474	4.853	4.402	5.96	7.059	5.155	4.663	6.362	7.614
20	250	0.272	0.244	0.339	0.402	1.652	1.487	1.982	2.479	4.858	4.406	5.967	7.119	5.16	4.669	6.371	7.621
21	260	0.272	0.244	0.339	0.403	1.653	1.488	1.983	2.48	4.859	4.407	5.968	7.133	5.161	4.669	6.372	7.622
22	270	0.272	0.244	0.339	0.403	1.654	1.488	1.984	2.481	4.859	4.408	5.969	7.134	5.162	4.770	6.373	7.623
23	280	0.272	0.244	0.339	0.403	1.654	1.489	1.984	2.481	4.860	4.408	5.969	7.135	5.162	4.770	6.374	7.624
24	290	0.272	0.244	0.339	0.403	1.654	1.489	1.985	2.482	4.861	4.409	5.971	7.136	5.163	4.771	6.375	7.626
25	295	0.272	0.244	0.34	0.403	1.655	1.489	1.985	2.482	4.861	4.409	5.971	7.137	5.163	4.771	6.375	7.627

Table 2: Values of $\mathbb{I}_R^S(U_j, V)$, with $j \in \{1, 2, 3\}$

	R	S	U_1	U_2	U_3
V	0.1	10	**0.1159**	0.1292	0.1569
V	0.9	10	**0.6952**	1.0771	1.2706

Table 3: Values of $\mathbb{I}_R^S(U_j, V)$, with $j \in \{1, 2, 3\}$

	R	S	U_1	U_2	U_3
V	0.1	10	**0.1637**	0.2019	0.2512
V	0.9	10	**1.4326**	1.7279	2.2817

Table 4: Collection of data for building materials

	"Sealant"(C_1)	**"Floor varnish"**(C_2)	**"Wall paint"**(C_3)	**"Carpet"**(C_4)	**"Chloride flooring"**(C_5)
A_1	$(0.9, 0.0, 0.0)$	$(0.5, 0.2, 0.2)$	$(0.45, 0.1, 0.25)$	$(1.0, 0.0, 0.0)$	$(0.9, 0.0, 0.0)$
A_2	$(0.1, 0.3, 0.5)$	$(0.6, 0.1, 0.05)$	$(0.6, 0.1, 0.2)$	$(1.0, 0.0, 0.0)$	$(0.9, 0.1, 0.0)$
A_3	$(0.5, 0.1, 0.2)$	$(1.0, 0.0, 0.0)$	$(0.9, 0.0, 0.0)$	$(0.85, 0.05, 0.05)$	$(0.8, 0.0, 0.1)$
A_4	$(0.2, 0.0, 0.0)$	$(0.15, 0.3, 0.35)$	$(0.1, 0.5, 0.3)$	$(0.75, 0.15, 0.0)$	$(0.7, 0.1, 0.1)$
A_5	$(0.4, 0.15, 0.2)$	$(0.0, 0.3, 0.5)$	$(0.2, 0.3, 0.4)$	$(0.2, 0.2, 0.6)$	$(0.5, 0.05, 0.1)$
A_6	$(0.1, 0.4, 0.5)$	$(0.7, 0.05, 0.1)$	$(0.6, 0.1, 0.1)$	$(0.15, 0.25, 0.6)$	$(0.3, 0.35, 0.3)$
A_7	$(0.3, 0.3, 0.2)$	$(0.5, 0.1, 0.2)$	$(0.15, 0.4, 0.4)$	$(0.1, 0.3, 0.4)$	$(0.15, 0.25, 0.5)$
A_8	$(0.5, 0.1, 0.0)$	$(0.65, 0.1, 0.1)$	$(0.2, 0.05, 0.1)$	$(0.3, 0.3, 0.4)$	$(0.4, 0.2, 0.1)$

Table 5: Centers of grouped clusters

	$\mathbb{C}(C_1)$	$\mathbb{C}(C_2, C_3)$	$\mathbb{C}(C_4, C_5)$
A_1	$(0.475, 0.15, 0.225)$	$(0.95, 0.0, 0.0)$	$(0.9, 0.0, 0.0)$
A_2	$(0.6, 0.1, 0.125)$	$(0.95, 0.05, 0.0)$	$(0.1, 0.3, 0.5)$
A_3	$(0.95, 0.0, 0.0)$	$(0.825, 0.025, 0.075)$	$(0.5, 0.1, 0.2)$
A_4	$(0.125, 0.4, 0.325)$	$(0.725, 0.125, 0.05)$	$(0.2, 0.0, 0.0)$
A_5	$(0.1, 0.3, 0.45)$	$(0.35, 0.125, 0.35)$	$(0.4, 0.15, 0.2)$
A_6	$(0.65, 0.075, 0.1)$	$(0.225, 0.3, 0.45)$	$(0.1, 0.4, 0.5)$
A_7	$(0.325, 0.25, 0.3)$	$(0.125, 0.275, 0.45)$	$(0.3, 0.3, 0.2)$
A_8	$(0.425, 0.075, 0.35)$	$(0.35, 0.25, 0.25)$	$(0.5, 0.1, 0.0)$

Table 6: Clustering results of proposed measure

	Clusters
Stage 1	{“Sealant”}, {“Floor varnish”}, {“Wall paint”}, {“Carpet”}, {“Chloride flooring”}
Stage 2	{“Sealant”}, {“Floor varnish”,“Wall paint”}, {“Carpet”,“Chloride flooring”}
Stage 3	{“Sealant”}, {“Floor varnish”,“Wall paint”,“Carpet”,“Chloride flooring”}
Stage 4	{“Sealant”,“Floor varnish”,“Wall paint”,“Carpet”,“Chloride flooring”}

Table 7: Comparative Analysis with Some Existing Measures

Method	U_1	U_2	U_3	Results
Ganie et al. [25]	0.4929	0.1621	-0.7676	U_1
Ganie et al. [25]	0.2270	0.1608	-0.6793	U_1
Thao [38]	0.5586	-0.0480	-0.7284	U_1
Dutta [48]	0.2000	0.3000	0.3400	U_1
Dutta [48]	0.1789	0.2933	0.3162	U_1
Singh [39]	0.9168	0.7625	0.7138	U_1
Singh [39]	0.8838	0.7500	0.6739	U_1
Umar et al. [30]	0.0901	0.1600	0.1785	U_1
Proposed (R=0.1, S=10)	0.1159	0.1292	0.1569	U_1
Proposed (R=0.9, S=10)	0.6952	1.0771	1.2706	U_1

Table 8: Comparative Analysis with Some Existing Measures

Method	U_1	U_2	U_3	Results
Thao et al.[38]	0.1184	0.1371	0.1359	U_1
Thao et al.[38]	0.5103	0.6123	0.5968	U_1
Le at al.[49]	0.1792	0.2000	0.1917	U_1
Thao[50]	0.2208	0.2375	0.2360	U_1
Wei[11]	0.7273	0.6744	0.6970	U_1
Umar et al.[30]	0.2022	0.2348	0.2326	U_1
Proposed	0.1637	0.2019	0.2512	U_1
Proposed	1.4326	1.7279	2.2817	U_1

Table 9: Clustering results of existing measures

	IHC[42]
Stage 1	{"Sealant"}, {"Floor varnish"}, {"Wall paint"}, {"Carpet"}, {"Chloride flooring"}
Stage 2	{"Sealant"}, {"Floor varnish","Wall paint"}, {"Carpet","Chloride flooring"}
Stage 3	{"Sealant"}, {"Floor varnish","Wall paint","Carpet","Chloride flooring"}
Stage 4	{"Sealant","Floor varnish","Wall paint","Carpet","Chloride flooring"}

Table 10: Clustering results of existing measures

	HPC[17]
Stage 1	{"Sealant"}, {"Floor varnish"}, {"Wall paint"}, {"Carpet"}, {"Chloride flooring"}
Stage 2	{"Sealant"}, {"Floor varnish","Wall paint"}, {"Carpet","Chloride flooring"}
Stage 3	{"Sealant"}, {"Floor varnish","Wall paint","Carpet","Chloride flooring"}
Stage 4	{"Sealant","Floor varnish","Wall paint","Carpet","Chloride flooring"}

Table 11: Characteristic Comparison with the Existing Techniques

Research Articles	**Parametrization Involvement**	**Entropy & Discriminant Measure**	**Assessment Information of Alternatives**
Ganie et al. [25]	$\times$	$\times$	Picture Fuzzy Set
Thao [38]	$\times$	$\times$	Picture Fuzzy Set
Dutta [48]	$\times$	$\times$	Picture Fuzzy Set
Singh [39]	$\times$	$\times$	Picture Fuzzy Set
Umar [30]	$\times$	$\times$	Picture Fuzzy Set
Proposed Methods	✓	✓	Picture Fuzzy Set