An economic order quantity model for two deteriorating items with mutually complementary price and time dependent demand

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October 9, 2023

Abstract

We present an inventory model to determine the optimal selling price and cycle time for two mutually complementary commodities that are subject to deterioration. Each commodity's demand is influenced by its own selling price, the selling price of the complementary product, and the passage of time. Numerical examples and sensitivity analysis results are presented to demonstrate the usefulness of the inventory model. We conducted sensitivity analysis on the impacts of the changes in key parameters of the model on the decision variables and the objective (profitability) of the inventory system. We observed that as the deterioration rate of either item increases, the model proposes shorter replenishment cycle length, which reduces the profit. Our model's novelty is the inclusion of mutual (two-way) complementarity in the Economic Order Quantity (EOQ) model, where both items are deteriorating and have time-dependent demands.

Keywords— Inventory management; Economic Order Quantity; Complementary items; Deteriorating items; Time-dependent demand.

1 Introduction

The classic Economic Order Quantity (EOQ) model of Harris [1] made many simplifying assumptions, three of which include, the independence of items managed from each other, that the items do not deteriorate, and that the demand is constant over time. While these assumptions simplify the modelling procedure and enhance the tractability of the solution, many systems behave differently in reality.

The model presented in this paper is important because of its pervasiveness in many business environments where the assumptions of the classic EOQ model breakdown, like the retail chain for consumables such as food items and electronics, where the level of sale attainable by the seller is heavily influenced by the volume driven by the business in the low margin environment with short product life span, continuous decline in product value, changing demand levels, multi-way influence of products on each other's demands, and price sensitivity of demand.

Retailers in such environments may adopt a strategy whereby some products that are termed loss-leaders have their prices discounted to bring many customers to their store, but due to the impact of these loss-leaders on the demand for other products, the customers buy more of some other products that bring in sufficient sale and revenue to offset the discount on the loss leader and generate more profit for the store. Operators of a such retail chain would be interested in answering several questions such as:

What should be the optimal selling price for the items to maximise the profit?
What should be the optimal replenishment cycle length for which profit is maximised?
What is the corresponding optimal order quantity for each item managed?

The model formulated in this study addresses these questions in the context of two deteriorating items which are mutual complements of each other, with shared ordering cost through joint purchase from a common supplier or supply chain milk run. Moreover, the items have both time and price dependent demands. The solution of the model assists the retailers of such items in their decision-making on replenishment planning and pricing policies.

Complementary items are items that are sold or consumed together to realise some full utility by customers, hence, these items are said to experience joint demand [2]. Karaöz et al. [3] suggest that though the complementarity of items can be implicitly assumed to be part of a constant term or another parameter, it is important to explicitly define the complementary of the items in the model to understand its effect on the pricing and profitability. This vital suggestion is pursued in this research and is an especially important consideration in the ever-growing technology industry that is known for producing and supplying complementary items such as printers and ink cartridges, computer hardware and software, cell phones and sim cards, etc. as complementary duals.

1.1 Purpose

The goal of this paper is to develop and propose an inventory model that seeks to find optimal selling price and cycle time that maximize profit for mutually complementary products, where the two complementary products have a common ordering cost and are deteriorating in nature.

1.2 Model's Applicability

In today's market, demand for some items is seldom constant. There are business decisions that may influence the demand for an item such as placement and display of the item on the shelf, promotions, time passage, selling price, and interrelationship of an item with other items such as substitutable and complementary relationships. In this paper selling price, time progression, and complementarity of items are factored into the model so that an inventory policy for managing inventory characterised by these aspects is developed.

Furthermore, in several markets, the inventory level of items does not deplete as a result of demand only. The inventory level of certain items may also be reduced due to deterioration or decay. The deterioration of inventory items has caught the attention of many inventory managers and researchers. As a result, deterioration has been incorporated into inventory models by researchers such as Ghare and Schrader [4], Kumar et al. [5], and Aliyu and Sani [6] to name a few. Deterioration of items is also incorporated into the model developed in this paper.

Moreover, many items are not independently replenished, hence, their shipment cost may be shared. This is due to the novel design of logistics processes like transport milk runs, cross docking and supply mixing amongst others. Such initiatives reduce the overall ordering cost, consequently changing the balance between ordering and holding costs, and hence, the optimal order quantity and replenishment interval. These are real situations, especially in the retail environment, and the model presented in this paper addresses all these challenges and opportunities together in an integrated manner.

2 Literature Review

This section presents a brief review of the pertinent literature starting from models of dependent demand, which includes price dependence, stock level dependence and time dependence of demand, after which deteriorating inventory systems are considered, and finally, models of shared fixed costs. There are, of course, many models that fit into more than one of these categories, for instance models that have both price dependence and product deterioration. We would present such models based on the characteristics we consider quite prominent in this particular article.

The classic lot sizing model assumes that the demand for a product is constant, which may not always be the case. The reality is that the demand for a product may be time-dependent [7], inventory-dependent or price-dependent [8]. EOQ models with price-dependent demand often involve complementary price dependent items and substitute items, with researchers such as Edalatpour and Mirzapour [9], Rajesh and Vinod [10] and Taleizadeh et al. [11] to name a few, have incorporated such price dependencies in their models. Maity and Maiti [12] developed an optimal production inventory policy for complementary and substitute products that are subject to deterioration, with the demand that is stock-dependent. Karaöz et al. [3] presented a finite single-item EOQ model for a product with a demand that is sensitive to time, its selling price, and the prices of the complementary and substitute products. Under Vendor Managed Inventory (VMI), Hemmati et al. [13] presented an inventory model for complementary items with stock-dependent demand. Their paper suggested an integrated two-stage model, which considers one supplier who is the manufacturer of the two complementary items, and one buyer who stocks the items in the warehouse to satisfy demand. Mokhtari [2] presented an EOQ model for joint complementary and substitutable items with the objective of minimizing the inventory cost. Edalatpour and Mirzapour [9] investigated a simultaneous pricing model for substitute and complementary products under nonlinear holding cost. Their model was aimed at finding optimal values of replenishment cycle time and the products' selling prices when demand is given by a price-sensitive function. Taleizadeh et al. [11] developed an economic lot-sizing inventory model for complementary and substitutable products that are deteriorating in nature. The model investigates the best pricing and inventory strategy for the items. Under asymmetric substitution, Rajesh and Vinod [10] analysed the impact of substitution cost and joint replenishment on inventory decisions under joint substitutable and complementary items. The model was aimed at determining optimal values for order quantity, total cost, and case based extreme rates of substitution. Under spectral risk measure, Yanhai and Jinwen [14] presented an EOQ model for establishing an optimal ordering policy for complementary components considering partial backordering and emergency replenishment. Poormoaied [15] developed an economic lot-sizing inventory model to investigate inventory decisions under periodic review for two complementary products with joint Poisson arrival. The researcher explored the influence of the interrelated demand phenomenon on optimal base stock levels as well as on the period length of the review policy.

Deteriorating products inventory management is another area that has enjoyed significant research. It is very common to assume that a product depletes through its demand or sales only, but in practice, a product may also deplete through decay or deterioration. Deterioration is often encountered in inventory items such as electronic components e.g., cell batteries and printed circuit boards, or food items. Ghare and Schrader [4] produced a classic work on this phenomenon. Their seminal model was built on the premise of constant demand and exponentially deteriorating items. Since the introduction of their model, the topic of deterioration has received attention from many other researchers. Kumar et al. [5] studied an inventory model for a deteriorating item under trade credit. They formulated an inventory model that is subject to the conditions in which the demand is selling price-dependent, and the holding cost is parabolic time varying. Shukla et al. [7] developed an economic lot-sizing model for deteriorating items with exponential demand rates and permissible shortages which are partially backlogged. Considering a single warehouse system, Tripathi and Mishra [16] presented an EOQ model for deteriorating items with a demand that is stock dependent. In their model, shortages are allowed and are fully backlogged. Chang et al. [17] dealt with an inventory model which has stock and price dependent demand for items that are deteriorating and are subject to limited shelf space. Jaggi and Mittal [18] developed an economic lot-sizing inventory model for deteriorating items subject to imperfect quality. Maragatham and Palani [19] formulated an inventory model for deteriorating items with a demand rate that is a function of the selling price, holding and ordering cost as well as the passage of time, where the deterioration rate in the model is also a function of time. Mishra [20] established and EOQ model for two deteriorating items that are substitutable. The model considers stockouts where partial substitution occurs when one of the items is out of stock, and moreover, the demand for items was considered deterministic and constant. Aliyu and Sani [6] investigated a pricing model for deteriorating items under generalised exponentially increasing demand with constant holding cost and constant deterioration rate. Amiri et al. [21] developed an inventory model for deteriorating items using Evidence Reasoning Algorithm (ERA) and imprecise inventory costs. The model was used to determine the optimal profit and the number of replenishment cycles together with the order quantity in each cycle. Mashud [22] considered an EOQ model for deteriorating items with different types of demand and fully backlogged shortages. Rajesh and Vinod [23] investigated the impact of deterioration and the cost of replacement on the best inventory choices for a system of two substitutable goods, where one item is made up of two complementary components. Al-Salami et al. [24] presented a study where an efficient Genetic Algorithm (GA) based inventory control model was created to reach optimal cost and reorder levels of food-related deteriorating products. Feng et al. [25] developed and inventory model for perishable goods where demand curve is dependent on unit price, display stock and the expiry date. Adak and Mahapatra [26] analyzed the reliability's impact on an inventory system that incorporates stochastic deterioration, variable demand and holding costs. For an overview of the more recent studies on deteriorating inventory, Perez and Torres [27] conducted a comprehensive review of deteriorating items, where a structural content analysis of 317 selected peer-reviewed research articles that were published from 2001 to 2018 was performed.

There are not much paper that have presented models where products share their fixed inventory costs (i.e. ordering and/or set up costs), but two such models have been reported here. Adetunji et al. [28] developed an EOQ model for returned multi-type containers with shared ordering cost. This problem involves the repositioning of containers from ports having surplus containers to ports in need of containers, or temporary storage in container depots. It assumed that many containers may be moved by a single mode of transport (like rail), and also shared storage at the port, hence, the need to manage not only the number of containers present in the ports, but also the mix of these containers. They showed that a significant cost saving benefit may be realized due to the sharing of the substantial fixed cost of such repositioning problem, and the conditions under which such savings may be attained. Adetunji et al. [29] also presented a case where there are multiple containers sharing repair facility, the storage space for these containers is limited, and there is the need to balance both the level and the mix of containers made available for shipping and showed the benefit of the shared cost of the repair centre under the space constraint.

In concluding this literature review section and highlighting the contribution of this paper, Table 1 gives a summary of reviewed research papers covering the related topics. The first and second columns on the table show that while most research work on complementary items has been conducted jointly with substitute items, there are, however, opportunities to explore the implications of other characteristics of the system in addition to complementariness of the items. As such, this paper will focus on complementary products, specifically mutual complements. As seen in the table, there is no work that has been produced on mutually complementary items.

This study, in particular, will focus on mutually complementary items that are both subject to deterioration, where replenishment costs can be shared, as has been noted by other researchers like Edalatpour and Mirzapour [9] that in the recent market, most customers prefer buying their day-to-day items all at once and from the same place. This behaviour has been referred to as holus-bolus buying i.e., buying all at once. This behaviour has sparked further interest of many researchers in this area, and hence, the increase in research work on joint or bundle pricing, which may also readily incorporate a shared ordering or replenishment cost.

Contextualising the work presented here relative to the reviewed research works, the model presented in this paper can be said to be an extension of the Karaöz et al. [3] model, and some other models where different operating characteristics of the inventory environment are considered in each such model. The unique combination of the environment of this particular problem in comparison to the other models are presented in Table 1.

(Insert Table 1)

The rest of the paper is organised as follows. Section 3 covers the model definition, together with notations and assumptions adopted in the model. Section 4 presents model development. Section 5 presents numerical and sensitivity analysis, and finally, the conclusion is presented in Section 6.

3 Problem definition

Complementary products are goods that are sold together, and most researchers, for example, Chen and Nalebuf [30] and Karaöz et al. [3] have formulated their model such that one of the products is considered a primary product and the other as its complement. In such cases, it means one product is considered more essential to the joint use but not vice versa. In this paper, however, an inventory model, and hence policy, is developed for complementary products that are equally essential to the joint use with each other. In such a case, the consumer values the complementarity of both products, that is, if a customer purchases product A they will also likely purchase product B and vice versa. Typically, there is often no alternative to replace either product in joint use. The inventory model proposed in this paper seeks optimal values of the selling prices and the cycle length that maximise profit for the mutually complementary products. The products are also subject to deterioration, and the demand is considered to be an exponential function of some key parameters including the product's selling price, the complement product's selling price, and the passage of time. Moreover, the products may be jointly ordered (e.g. through joint shipment), and the ordering cost of the products is defined by a fixed and some variable components.

3.1 Notations

The notations and decision variables used in developing the inventory model are presented in Table 2.
(Insert Table 2)

3.2 Assumptions

The following are the main assumptions made in formulating the mathematical model:

- The demand for each of the mutually complementary products, that is, products 1 and 2, is dependent on the product's selling price and the selling price of the complementary product.
- Demand for each of the products exponentially decreases with time.

- Each product's holding cost is linear and independent of the other's.
- The products can be jointly ordered with the ordering cost consisting of a shared fixed portion and a portion that is proportional to the number of products jointly ordered.
- The lead time of the two products can be synchronized by ordering the relevant quantities and such that they both run out and attain the zero-inventory property at the same time.
- Shortages are not allowed for the two products.
- The deterioration rate for each product is defined by a constant exponential function.

4 Model development

This study considers an inventory policy where two deteriorating complementary products are jointly ordered and delivered to the retailer with order quantities , at unit cost for the product, and the batch ordering cost, per order. The system maintains a zero-inventory property, meaning the previous batch runs out when the new batch is just arriving. Therefore, for every cycle, products 1 and 2 start with the quantities and respectively. The quantity ordered for each product gradually decreases due to demand and deterioration to zero at the end of the cycle at time . The inventory level for the items during the cycle is graphically represented in Figure 1.

(Insert Figure 1)

The demand functions for the products are denoted by the following equations:

$$D_1 = A_1 e^{-a_1 P_1 - b_1 P_2 - \beta_1 t} \tag{1}$$

and

$$D_2 = A_2 e^{-a_2 P_1 - b_2 P_2 - \beta_2 t} \tag{2}$$

The change in inventory level for products 1 and 2 at any given time t is governed by the following differential equations

$$\frac{dI_1(t)}{dt} + \theta_1 I_1(t) = -A_1 e^{-a_1 P_1 - b_1 P_2 - \beta_1 t} \qquad 0 \leq t \leq T \tag{3}$$

$$\frac{dI_2(t)}{dt} + \theta_2 I_2(t) = -A_2 e^{-a_2 P_1 - b_2 P_2 - \beta_2 t} \qquad 0 \leq t \leq T \tag{4}$$

The boundary conditions are given by the following equations

$$I_1(0) = Q_1, I_2(0) = Q_2 \quad \text{and} \quad I_1(T) = 0, I_2(T) = 0 \tag{5}$$

To derive equations for the instantaneous inventory levels, $I_1(t)$ and $I_2(t)$, for products 1 and 2, consider (3), for product 1, the equation is of the form

$$\frac{dI_1(t)}{dt} + f(t)y = g(t) \quad \text{with integrating factor} \quad I.F = e^{ace10 \int f(t)} = e^{\theta_1 t} \tag{6}$$

the general solution is

$$I_1(t) e^{\theta_1 t} = -A_1 e^{-a_1 P_1 - b_1 P_2} \int e^{t(\theta_1 - \beta_1)} dt \tag{7}$$

Integrating Equation (7) to get the inventory level function for product 1, results in

$$I_1(t) e^{\theta_1 t} = -\frac{A_1 e^{-a_1 P_1 - b_1 P_2}}{\theta_1 - \beta_1} e^{t(\theta_1 - \beta_1)} + C \tag{8}$$

Using boundary condition $I_1(T) = 0$ to solve for C in Equation (8) and simplifying the equation results in the inventory level function for product 1, which is

$$I_1(t) = \frac{A_1 e^{-a_1 P_1 - b_1 P_2 - \beta_1 t}}{\theta_1 - \beta_1} \left[e^{(\theta_1 - \beta_1)(T-t)} - 1 \right] \tag{9}$$

Similarly, inventory level function $I_2(t)$ can be obtained for product 2 as

$$I_2(t) = \frac{A_2 e^{-a_2 P_1 - b_2 P_2 - \beta_2 t}}{\theta_2 - \beta_2} \left[e^{(\theta_2 - \beta_2)(T-t)} - 1 \right] \tag{10}$$

Substituting the initial conditions from Equations (5) into (9) and (10) respectively results in the following maximum inventory level functions for products 1 and 2:

$$I_1(0) = Q_1 = \frac{A_1 e^{-a_1 P_1 - b_1 P_2}}{\theta_1 - \beta_1} \left[e^{(\theta_1 - \beta_1)T} - 1 \right] \tag{11}$$

$$I_2(0) = Q_2 = \frac{A_2 e^{-a_2 P_1 - b_2 P_2}}{\theta_2 - \beta_2} \left[e^{(\theta_2 - \beta_2)T} - 1 \right] \tag{12}$$

For simplicity let

$$E = A_1 e^{-a_1 P_1 - b_1 P_2} \quad \text{and} \quad F = A_2 e^{-a_2 P_1 - b_2 P_2} \tag{13}$$

Holding cost (HC)
The total holding cost per cycle is given by the following function

$$\begin{aligned} HC = HC_1 + HC_2 &= h_1 \int_0^T I_1(t)dt + h_2 \int_0^T I_2(t)dt \\ &= h_1 \frac{E e^{(\theta_1 - \beta_1)T)}}{\theta_1 - \beta_1} \left[\frac{1 - e^{-\theta_1 T}}{\theta_1} \right] - h_1 \frac{E}{\theta_1 - \beta_1} \left[\frac{1 - e^{-\beta_1 T}}{\beta_1} \right] + \\ & h_2 \frac{F e^{(\theta_2 - \beta_2)T)}}{\theta_2 - \beta_2} \left[\frac{1 - e^{-\theta_2 T}}{\theta_2} \right] - h_2 \frac{F}{\theta_2 - \beta_2} \left[\frac{1 - e^{-\beta_2 T}}{\beta_2} \right] \end{aligned} \tag{14}$$

Ordering cost (OC)
The model assumes one order is placed per cycle. The total ordering cost for the two items consists of the shared ordering cost given by the following equation:

$$OC = k_0 + \sum_{n=1}^{m} k_n, \quad \text{where} \quad m = 2 \quad \text{in this case without any loss of generality} \tag{15}$$

Revenue (TR) **and Purchase cost** (PC)
Revenue for each product is obtained by multiplying the order quantity for the specific product by its selling price, $Q_n P_n$, while, the purchase cost is similarly obtained by multiplying the unit purchase price with the order quantity, $Q_n c_n$.

Profit (TP)
Profit per unit time is obtained by dividing the total profit by cycle length, T, which is given by the following function:

$$TP = \frac{TR - PC - [HC + OC]}{T} = \left[(P_1 - c_1)\frac{A_1 e^{-a_1 P_1 - b_1 P_2}}{\theta_1 - \beta_1}\left[e^{(\theta_1-\beta_1)T} - 1\right] + \right.$$
$$(P_2 - c_2)\frac{A_2 e^{-a_2 P_1 - b_2 P_2}}{\theta_2 - \beta_2}\left[e^{(\theta_2-\beta_2)T} - 1\right]$$
$$- \left(h_1 \frac{Ee^{(\theta_1-\beta_1)T}}{\theta_1 - \beta_1}\left[\frac{1 - e^{-\theta_1 T}}{\theta_1}\right] - h_1 \frac{E}{\theta_1 - \beta_1}\left[\frac{1 - e^{-\beta_1 T}}{\beta_1}\right] + \right. \tag{16}$$
$$h_2 \frac{Fe^{(\theta_2-\beta_2)T)}}{\theta_2 - \beta_2}\left[\frac{1 - e^{-\theta_2 T}}{\theta_2}\right] - h_2 \frac{F}{\theta_2 - \beta_2}\left[\frac{1 - e^{-\beta_2 T}}{\beta_2}\right] + k_0 + \sum_{i=1}^{n} k_i)/T$$

Cycle length

The optimal cycle time is derived in this section, but first is the linearisation of the exponential terms contained in Equation (16) by using Maclaurin's expansion for e^x where for simplicity $x = (\theta_1 - \beta_1)T$ which leads to:

$$\begin{aligned} e^{(\theta_1-\beta_1)T} &= \sum_{m=0}^{\infty} \frac{(\theta_1 - \beta_1)^m T^m}{m!} \\ &= 1 + \frac{(\theta_1 - \beta_1)^1 T^1}{1!} + \frac{(\theta_1 - \beta_1)^2 T^2}{2!} + \frac{(\theta_1 - \beta_1)^3 T^3}{3!} \\ &\approx 1 + (\theta_1 - \beta_1)T \end{aligned} \tag{17}$$

All the other exponential terms are approximated as in Equation (17). Substituting these approximations into Equation (16), results in:

$$\begin{aligned} TP = {} & [(P_1 - c_1)\frac{E}{\theta_1 - \beta_1}[1 + (\theta_1 - \beta_1)T - 1] + (P_2 - c_2)\frac{F}{\theta_2 - \beta_2}[1 + (\theta_2 - \beta_2)T - 1] - \\ & (h_1 \frac{E(1 + (E(1 + (\theta_1 - \beta_1)T)}{\theta_1 - \beta_1}[\frac{1 - (1 - \theta_1 T)}{\theta_1}] - h_1 \frac{E}{\theta_1 - \beta_1}[\frac{1 - (1 - \beta_1 T)}{\beta_1}] + \\ & h_2 \frac{F(1 + (F(1 + (\theta_2 - \beta_2)T)}{\theta_2 - \beta_2}[\frac{1 - (1 - \theta_2 T)}{\theta_2}] - h_2 \frac{F}{\theta_2 - \beta_2}[\frac{1 - (1 - \beta_T)}{\beta_2}] + \\ & k_0 + \sum_{i=1}^{n} k_i)]/T \end{aligned} \tag{18}$$

Simplifying Equation (18) results in:

$$TP = \frac{[(P_1 - c_1)ET + (P_2 - c_2)FT - (h_1 ET^2 + h_2 FT^2 + k_0 + \sum_{i=1}^{n} k_i)]}{T} \tag{19}$$

Using the product differentiation rule to find $TP^{'} = (fg)^{'} = fg^{'} + f^{'}g$and then equating $TP^{'} = 0$ the following is obtained.

$$\begin{aligned} TP^{'} = {} & (P_1 - c_1)ET^{-1} + (P_2 - c_2)FT^{-1} - (2h_1 ET + 2h_2 FT)\,T^{-1} - (P_1 - c_1)ET^{-1} \\ & - (P_2 - c_2)FT^{-1} - \left(-h_1 ET^0 - h_2 FT^0 - k_0 T^{-2} - \sum_{i=1}^{n} k_i T^{-2}\right) = 0 \end{aligned} \tag{20}$$

Solving Equation (20) for T to find the optimum T yields:

$$T = \sqrt{\frac{k_0 + \sum_{i=1}^{n} k_i}{h_1 E + h_2 F}} \quad \text{where } E \text{ and } F \text{ have been defined in Equation (11) for simplicity} \tag{21}$$

Proof of optimality
To show that the unit profit function TP is concave, we prove that the Hessian matrix for the profit function Equation (19) is negative (semi)definite.

$$\text{The Hessian matrix for } TP \text{ is given by} H\left(P_1, P_2, T\right) = \begin{pmatrix} \frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1 dP_2} & \frac{d^2TP}{dP_1 dT} \\ \frac{d^2TP}{dP_2 dP_1} & \frac{d^2TP}{dP_2{}^2} & \frac{d^2TP}{dP_2 dT} \\ \frac{d^2TP}{dT dP_1} & \frac{d^2TP}{dT dP_2} & \frac{d^2TP}{dT^2} \end{pmatrix} \tag{22}$$

Second derivatives
Consider Equation (22), the second derivatives of the Hessian matrix are obtained using the profit function given by Equation (19), therefore.

$$\begin{aligned} \frac{d^2TP}{dP_1{}^2} = &\left(P_1 - c_1\right) a_1{}^2 A_1 e^{-a_1 P_1 - b_1 P_2} - 2a_1 A_1 e^{-a_1 P_1 - b_1 P_2} + \\ &a_2{}^2 \left(P_2 - c_2\right) A_2 e^{-a_2 P_1 - b_2 P_2} - T \times \left[h_1 a_1{}^2 A_1 e^{-a_1 P_1 - b_1 P_2} + h_2 a_2{}^2 A_2 e^{-a_2 P_1 - b_2 P_2}\right] \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{d^2TP}{dP_2{}^2} = &\left(P_2 - c_2\right) b_2{}^2 A_2 e^{-a_2 P_1 - b_2 P_2} - 2b_2 A_2 e^{-a_2 P_1 - b_2 P_2} + \\ &b_1{}^2 \left(P_1 - c_1\right) A_1 e^{-a_1 P_1 - b_1 P_2} - T \left[h_1 b_1{}^2 A_1 e^{-a_1 P_1 - b_1 P_2} + h_2 b_2{}^2 A_2 e^{-a_2 P_1 - b_2 P_2}\right] \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{d^2TP}{dP_2 dP_1} = &\left[\left(P_1 - c_1\right)\left(a_1 b_1\right) A_1 e^{-a_1 P_1 - b_1 P_2} - b_1 A_1 e^{-a_1 P_1 - b_1 P_2}\right] \\ &- \left[-a_2 b_2 \left(P_2 - c_2\right) A_2 e^{-a_2 P_1 - b_2 P_2} + a_2 A_2 e^{-a_2 P_1 - b_2 P_2}\right] \\ &- T \left[h_1 a_1 b_1 A_1 e^{-a_1 P_1 - b_1 P_2} + h_2 a_2 b_2 A_2 e^{-a_2 P_1 - b_2 P_2}\right] \end{aligned} \tag{25}$$

Equations (23), (24), and (25) are simplified by factorising like terms and substituting the relevant terms with (13), to obtain.

$$\begin{aligned} \frac{d^2TP}{dP_1{}^2} = &\left(P_1 - c_1\right) a_1^2 E - 2a_1 E + a_2^2 \left(P_2 - c_2\right) F - T \left[h_1 a_1^2 E + h_2 a_2^2 F\right] \\ &= a_1 E \left[P_1 - c_1 a_1 - 2 - T h_1 a_1\right] + a_2^2 F \left[P_2 - c_2 - T h_2\right] \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{d^2TP}{dP_2{}^2} = &\left(P_2 - c_2\right) b_2{}^2\ F - 2b_2 F + b_1{}^2 \left(P_1 - c_1\right) E\ - T \left[h_1 b_1{}^2 E + h_2 b_2{}^2 F\ \right] \\ &= b_1{}^2 E \left[\left(P_1 - c_1\right) - T h_1\right] + b_2 F \left[\left(P_2 - c_2\right) b_2 - 2 - T h_2 b_2\right] \end{aligned} \tag{27}$$

$$\begin{aligned}\frac{d^2TP}{dP_2dP_1} &= \left[\left(P_1-c_1\right)\left(a_1b_1\right)E-b_1E\right]-\left[-a_2b_2\left(P_2-c_2\right)F+a_2F\right]- \\ &T\left[h_1a_1b_1E+h_2a_2b_2F\right] \\ &= b_1E\left[\left(P_1-c_1\right)a_1-1-Th_1a_1\right]+a_2F\left[\left(P_2-c_2\right)b_2-1-Th_2b_2\right]\end{aligned} \tag{28}$$

To further simplifying Equations (26) to (28). Let,

$$\varepsilon=\left(P_2-c_2\right)-Th_2 \tag{29}$$

$$\omega=\left(P_1-c_1\right)-Th_1 \tag{30}$$

Now

$$\frac{d^2TP}{dP_1{}^2}=a_1E\left(a_1\omega-2\right)+a_2{}^2F\varepsilon \tag{31}$$

$$\frac{d^2TP}{dP_2{}^2}=b_1{}^2E\omega+b_2F\left(b_2\varepsilon-2\right) \tag{32}$$

$$\frac{d^2TP}{dP_2dP_1}=b_1E(a_1\omega-1)+a_2F(b_2\varepsilon-1) \tag{33}$$

For the derivatives of the profit function with respect to T, the first derivative of TP with respect to T is denoted by

$$\frac{dTP}{dT}=-h_1E-h_2F+T^{-2}\left(k_0+\sum_{i=1}^{n}k_i\right) \tag{34}$$

The 2^{nd} derivatives are:

$$\frac{d^2TP}{dT^2}=-2T^{-3}\left(k_0+\sum_{i=1}^{n}k_i\right) \tag{35}$$

$$\frac{d^2TP}{dTdP_1}=h_1a_1E+h_2a_2F \tag{36}$$

$$\frac{d^2TP}{dTdP_2}=h_1b_1E+h_2b_2F \tag{37}$$

Now, to prove that TP is negative (semi)definite. The determinants need to satisfy the following condition $|H\left(P_1\right)|<0,\ \ |\ H\left(P_2\right)|>0$ and $|H(T)|<0$ where

$$|H\left(P_1\right)|=\frac{d^2TP}{dP_1{}^2},|\ H\left(P_2\right)|=\begin{bmatrix}\frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2}\\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2}\end{bmatrix},|H(T)|=\begin{vmatrix}\frac{d^2TP}{dP_1{}^2} & \frac{d^2TP}{dP_1dP_2} & \frac{d^2TP}{dP_1dT}\\ \frac{d^2TP}{dP_2dP_1} & \frac{d^2TP}{dP_2{}^2} & \frac{d^2TP}{dP_2dT}\\ \frac{d^2TP}{dTdP_1} & \frac{d^2TP}{dTdP_2} & \frac{d^2TP}{dT^2}\end{vmatrix} \tag{38}$$

To derive the conditions for $|H\left(P_1\right)|<0$. Substitute Equation (26) in the relevant determinants in Equation (38)

$$|H\left(P_1\right)|=\frac{d^2TP}{dP_1{}^2}=\left(P_1-c_1\right)a_1{}^2\ E+a_2{}^2\left(P_2-c_2\right)F\ \ -T\left[h_1a_1{}^2E+h_2a_2{}^2F\ \right]-2a_1E \tag{39}$$

It is known that $P_1 > c_1$ and $P_2 > c_2$. Therefore when $(P_1 + P_2) - (c_1 + c_2) \leq T\,(h_1 + h_2)$ then the condition $|H\,(P_1)| < 0$ holds.

To derive the conditions for $|\ H\,(P_2)| > 0$.

The determinant is given by $|\ H\,(P_2)| = \left(\frac{d^2TP}{dP_1{}^2} * \frac{d^2TP}{dP_2{}^2}\right) - \left(\frac{d^2TP}{dP_2dP_1}\right)^2$. Equations (31) to (33) are progressively substituted into$|\ H\,(P_2)|$. Now, the first term is:

$$\begin{aligned}\left(\frac{d^2TP}{dP_1{}^2} * \frac{d^2TP}{dP_2{}^2}\right) &= \left[a_1E\,(\omega a_1 - 2) + a_2{}^2F\varepsilon\ \right] \times \left[b_1{}^2E\omega + b_2F\,(\varepsilon b_2 - 2)\right] \\ &= a_1b_1{}^2E^2\omega\,(\omega a_1 - 2) + a_1b_2FE\,(\omega a_1 - 2)\,(\varepsilon b_2 - 2) + a_2{}^2b_1{}^2EF\varepsilon\omega + \\ & a_2{}^2b_2F^2\varepsilon\,(\varepsilon b_2 - 2)\end{aligned} \tag{40}$$

The second term

$$\begin{aligned}\left(\frac{d^2TP}{dP_2dP_1}\right)^2 &= (b_1E\,(a_1\omega - 1) + a_2F\,(b_2\varepsilon - 1))^2 \\ &= b_1{}^2E^2(\omega a_1 - 1)^2 + a_2b_1EF\,(\varepsilon b_2 - 1)\,(\omega a_1 - 1) + a_2^2F^2(\varepsilon b_2 - 1)^2\end{aligned} \tag{41}$$

Therefore, considering Equations (40) and (41) it can be deduced that $|H\,(P_2)| > 0$ when $\omega < 0$ and $\varepsilon < 0$. To derive the conditions for $|\ H(T)| < 0,$ the matrix is expanded along a row and the determinant is given by

$$\begin{aligned}|H(T)| &= \frac{d^2TP}{dP_1{}^2}\left(\frac{d^2TP}{dP_2{}^2} \bullet \frac{d^2TP}{dT^2} - \frac{d^2TP}{dTdP_2} \bullet \frac{d^2TP}{dP_2dT}\right) - \\ & \frac{d^2TP}{dP_1dP_2}\left(\frac{d^2TP}{dP_2dP_1} \bullet \frac{d^2TP}{dT^2} - \frac{d^2TP}{dP_2dT} \bullet \frac{d^2TP}{dTdP_1}\right) + \\ & \left(\frac{d^2TP}{dP_1dT}\right)\left(\frac{d^2TP}{dP_2dP_1} \bullet \frac{d^2TP}{dTdP_2} - \frac{d^2TP}{dP_2{}^2} \bullet \frac{d^2TP}{dTdP_1}\right)\end{aligned} \tag{42}$$

The three terms are, thus:

$$\begin{aligned}&\frac{d^2TP}{dP_1{}^2}\left(\frac{d^2TP}{dP_2{}^2} \bullet \frac{d^2TP}{dT^2} - \frac{d^2TP}{dTdP_2} \bullet \frac{d^2TP}{dP_2dT}\right) = \left[a_1E\,(a_1\omega - 2) + a_2{}^2F\varepsilon\right] \bullet \\ &\left[\left[b_1{}^2E\omega + b_2F\,(b_2\varepsilon - 2)\right]\ \times -2T^{-3}\left(k_0 + \sum_{i=1}^{n} k_i\right) - \ (h_1b_1E + h_2b_2F)^2\ \right]\end{aligned} \tag{43}$$

$$\begin{aligned}&\frac{d^2TP}{dP_1dP_2}\left(\frac{d^2TP}{dP_2dP_1} \bullet \frac{d^2TP}{dT^2} - \frac{d^2TP}{dP_2dT} \bullet \frac{d^2TP}{dTdP_1}\right) = -\left[b_1E\,(a_1\omega - 1) + a_2F\,(b_2\varepsilon - 1)\right] \bullet \\ &\left[\left[b_1E\,(a_1\omega - 1) + a_2F\,(b_2\varepsilon - 1)\right] \times \right. \\ &- 2T^{-3}\left(k_0 + \sum_{i=1}^{n} k_i\right) - (h_1b_1E + h_2b_2F)\,(h_1a_1E + h_2a_2F\ )\end{aligned} \tag{44}$$

$$\begin{aligned}&\frac{d^2TP}{dP_1dT}\left(\frac{d^2TP}{dP_2dP_1} \bullet \frac{d^2TP}{dTdP_2} - \frac{d^2TP}{dP_2{}^2} \bullet \frac{d^2TP}{dTdP_1}\right) = \left[h_1a_1E + h_2a_2F\right] \bullet \left[(b_1E\,(a_1\omega - 1) + \right. \\ &a_2F\,(b_2\varepsilon - 1)\ \ (h_1b_1E + h_2b_2F) - \left(b_1{}^2E\omega + b_2F\,(b_2\varepsilon - 2)\right)(h_1a_1E + h_2a_2F)\left.\right]\end{aligned} \tag{45}$$

Combining all terms finally yields:

$$
\begin{aligned}
|H(T)| &= \left[a_1E\left(\omega a_1-2\right)+{a_2}^2F\varepsilon\right]\bullet \\
&\left[-\left[{b_1}^2E\omega+b_2F\left(\varepsilon b_2-2\right)\right]\ \bullet 2T^{-3}\left(k_0+\sum_{i=1}^{n}k_i\right)-\left(h_1b_1E+h_2b_2F\right)^2\right] \\
&+\left(b_1E\left(\omega a_1-1\right)+a_2F\left(\varepsilon b_2-1\right)\ \right)^2\times 2T^{-3}\left(k_0+\sum_{i=1}^{n}k_i\right)+ \\
&2\times\left[h_1a_1E+h_2a_2F\right]\bullet\left[\left(b_1E\left(\omega a_1-1\right)+a_2F(\varepsilon b_2-1)\right)\ \left(h_1b_1E+h_2b_2F\right)\right]- \\
&\left({b_1}^2E\omega+b_2F\left(\varepsilon b_2-2\right)\right)\left(h_1a_1E+h_2a_2F\right)^2<0
\end{aligned}
\tag{46}
$$

Therefore $|H(T)|<0$ when, $\varepsilon<0$ and $\omega<0$.

5 Numerical results

The numerical example and sensitivity analysis of the model developed are discussed next.

5.1 Numerical examples

Consider a retail environment for two deteriorating items that are mutual complements with the parameter values shown in Table 3. The solution to this problem is provided, and the sensitivity analysis of the results considering changes in some of the important input values is performed next.

(Insert Table 3)

Firstly, a test of the optimality condition was done, and it can be seen that the profit function is negative (semi)definite since the following conditions are satisfied: $\varepsilon=(P_2-c_2)-Th_2=(34.1-20.00)-3.05\times5.00=-1.14<0$ and $\omega=(P_1-c_1)-Th_1=(40.12-24)-3.05\times6.00=-2.21<0$.

Next, the problem was solved by substituting the parameter values in Table 3 into the appropriate equations in the model. This was done by implementing the relevant model equations in Excel which yields the results presented in Table 4. Finally, the sensitivity analysis is performed to test the robustness of the model.

(Insert Table 4)

5.2 Sensitivity analysis

Sensitivity analysis was performed by changing the values of each of the parameters a_n, b_n'β_n, θ_n, h_n, and c_n in the model one at a time while keeping values of the other parameters constant as given in Table 3. The changes in the values of each parameter were made in steps of 25 percent decrease and increase from the original values (as given in the table). The impact of the changes on the three decision variables (prices, P_1 and P_2, and the replenishment cycle, T) and the key system outputs (revenue TP and order quantities, Q_1 and Q_1) were tabulated for each change in the parameter value.

The changes in the deterioration rate of product 2 were, however, performed in an asymmetric manner such that optimality conditions are adhered to, using the range of values between -25% and 75% of the change in the parameter. The effect of changes in parameter values for a_1 is shown in Table 5. All other parameters were tested similarly, however, these were not shown in tables but presented together on the same graph so that the sensitivities of the key system properties to these changes can be compared in relative terms. The changes in each of the parameter values were plotted against the profit function and the cycle length, as presented in Figures 2 and 3 respectively.

(Insert Table 5)

Profit Graph
The graph in Figure 2 shows a combined summary of previous tables, the graph shows the effect of parameter changes on profit.

(Insert Figure 2)

Cycle Time graph
The graph in Figure 3 shows the effect of percentage change in parameters on cycle time.

(Insert Figure 3)

The following are the observations from the sensitivity analysis:
As the price coefficient for product 1, a_1, increases, the selling price, P_1, of product 1 decreases while product 2's selling price, P_2, increases. The cycle length Tincreases slightly. The unit profit decreases as seen in Figure 2. Similarly, when a_2 increases, the selling price P_1 for product 1 increases slightly while the selling price P_2 for product 2 decreases marginally. The cycle length increases slightly, and the unit profit decreases as seen in Figures 2 and 3. When b_1 increases, the selling price for product 1, P_1, decreases while the selling price for product 2, P_2, increases marginally. The cycle length also increases slightly while the unit profit decreases as seen in Figures 2 and 3. An increase in b_2 results in an increase in the selling price of P_1 and a decrease in P_2. The cycle length increases while the unit profit decreases drastically as apparent from Figures 2 and 3.

When the selling price coefficients a_n, b_n increase, there is a decrease in TP as seen from the graphs in Figure 2, while on the other hand, the cycle length decreases as seen in Figure 3. P_1 decrease with an increase in β_n for the products. Whereas P_2 decreases with an increase in β_1, P_2 on the other hand, increases with an increase in β_2. Moreover, an increase in unit holding cost, h_n or purchasing cost c_n of the products, results in reduced profit. This is an indication that the retailer should always try to find ways to keep costs low to maximises profit. The model, however, tries to push up the prices for products 1 and 2 in trying to counteract the cost increase.

Finally, from Figure 2, it can be seen that the profit function is more sensitive to higher values of A_1 and A_2. As such, from a management perspective this indicates that having higher values of base demand results in higher profit, inventory managers should take note of this as it is favorable to the model. Conversely, at higher values of c_2 and θ_2, the model yields lower profit. This highlights to inventory managers that the purchasing cost and deterioration rate of product 2 should be closely monitored as higher values of these parameters result in lower profit relative to the other parameters. Furthermore, at lower values of the parameters, the profit is more sensitive to b_1 and a_2 as well as b_2 and θ_1, where b_1 and a_2 result in higher profit whereas b_2 and θ_1 result in lower profit. A highlight to inventory managers from this outcome of the model is the fact that a lower value of deterioration rate for product 1 results in lower profit. From Figure 3, higher values of c_1 and c_2 in the model suggest longer cycle lengths relative to other parameters. This observation indicates that the model is more sensitive to purchasing cost of both products at higher values. Conversely, higher values of the base demand, A_1 and A_2 lower the cycle length as seen in the graph. On the other hand, lower values of h_2 and A_2 increase the cycle length more, relative to other parameters, while lower values of b_1 and c_2 decrease the cycle length less than other parameters.

5.3 Comparisons with existing studies

The model developed in this paper can be said to be an extension of the Karaöz et al [3] model. However, in this study, the model is extended to incorporate the deterioration of items while omitting the substitution of items from the model. As such, the sensitivity analysis of the model developed in this paper is compared with the sensitivity analysis of some parameters of Karaöz et al.[3]'s model as this is a more realistic comparison. This comparison is summarised in Table 6. From the table, the two are comparable in that a change in a common parameter result in similar profit changes in both models.

(Insert Table 6)

6 Conclusion

The inventory level of products does not deplete through demand only. Factors such as deterioration do contribute to inventory depletion. Moreover, demand for items is seldom constant. Factors such as the display of stock, time dynamics, selling price, substitutability, and complementarity of items have a huge influence on the overall demand for an item. Also, there may be cost savings in joint ordering of items. This study's focus has been on developing an optimal inventory policy for mutually complementary items with stocks that deplete due to both demand and deterioration. The demand for each of these items was represented as an exponential function of the product's selling price, the complementary product's selling price, and time. The items could also be ordered together by choosing their joint cycle length, and hence their order quantities, appropriately.

The mathematical model was formulated and solved for the values of the selling price of the two products and the cycle length that maximises the profit. A numerical example was used to demonstrate a practical application of the model and the optimal replenishment time and profit were determined. Sensitivity analysis indicated how changes in certain parameters of the model influence the cycle length, product price, and profitability of the inventory system.

The model can be extended in many ways, for instance, by incorporating substitutable items. This will address some other real-life scenarios, and this may be done together with the complementary products as presented in this model. The model can also be further extended to allow for shortages as this scenario is also quite realistic in many inventory management systems due to opportunities for full and partial backlogging of demand. In addition, the demand for some items may be stimulated by a physical display of the bulk presence of the item, therefore, it may be interesting to extend the model to items that have stock dependent demand.

7 Declarations

Funding- Not applicable.
Conflicts of interest/Competing interests– On behalf of all authors, the corresponding author states that there is no conflict of interest.
Availability of data and material– Can be made available on request.
Code availability- Not applicable.
Authors' contributions- All authors DM, OA, and MS were involved in problem conceptualisation and model development. DM conducted a literature review on the studied topic. MS and OA performed numerical analysis for the developed model. DM was also involved in compiling and writing the manuscript. All authors reviewed and approved the final manuscript.

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10 Tables and figures

Author(s)	Substitute item	Complementary item	Mutual price dependent	Deterioration	Constant demand? (Type)	Joint replenishment	Shared ordering cost
Maity and Maiti [12]	Yes	Yes	No	Yes	No (Linear)	No	No
Karaöz et al. [3]	Yes	Yes	No	No	No (Natural Exponential)	No	No
Maragatham,and Palani [19]	No	No	No	Yes	No (Exponential)	No	No
Mishra [20]	No	No	No	Yes	Yes	Yes	No
Hemmati et al. [13]	No	Yes	No	No	No (Linear)	Yes	No
Mokhtari [2]	Yes	Yes	No	No	No (Linear)	Yes	No
Edalatpour and Mirzapour [9]	Yes	Yes	No	Yes	No (Linear)	Yes	No
Taleizadeh et al. [11]	Yes	Yes	Yes	Yes	No (Linear)	Yes	No
Aliyu and Sani [6]	No	No	No	Yes	No (Natural exponential)	No	No
Amiri et al. [21]	No	No	No	Yes	Yes	No	No
Mashud [22]	No	No	No	Yes	No (Linear and exponential)	No	No
Rajesh and Vinod [10]	Yes	Yes	No	No	No (Linear)	Yes	No
Yanhai and Jinwen [14]	No	Yes	No	No	No (Probability distribution function)	No	No
Poormoaied [15]	No	Yes	No	No	No (Probability distribution function)	Yes	No
Rajesh and Vinod [23]	Yes	Yes	No	Yes	No (linear)	Yes	No
Salami et al. [24]	No	No	No	Yes	No (linear)	No	No
Feng et al. [25]	No	No	No	Yes	No (linear)	No	No
Adak and Mahapatra [26]	No	No	No	Yes	No (linear)	No	No
This Model	No	Yes	Yes	Yes	No (natural Exponential)	Yes	Yes

Table 1: Article placement from the context of the literature

Notations	Units	Description
n	Dimensionless	An index representing the product, $n = 1, 2$ in this case
A_n	Unit/Time	Base demand (a constant) of product n.
a_n	Constant	Product 1's price coefficient for the demand rate ($a > 0$) of product n.
b_n	Constant	Product 2's price coefficient for the demand rate ($b > 0$) of product n.
c_n	R/Unit	Purchasing cost per unit of product n.
D_n	Unit/Time	Demand for product n.
h_n	R/Unit/Time	Holding cost per unit of product n per unit time.
$I_n(t)$	Unit	Inventory level at time t of product n.
k_0	R/Batch	The constant portion of order cost per order.
k_n	R/Batch	The variable portion of order cost per order for product n.
$_n$	Constant	Deterioration rate of product n, where (0_n1).
$_n$	Constant	Constant governing rate of change with time (decrease or increase) of demand for the nth product.
Decision Variables	**Units**	**Description**
P_n	R	Selling price (R/Unit) of product n.
Q_n	Unit	The order quantity for nth Product.
T	Time	Length of replenishment cycle.

Table 2: Notations and decision variables used in deriving the inventory model.

Parameters	Product 1	Product 2
a_n	0.06	0.02
b_n	0.04	0.08
n	0.40	0.50
h_n	6.00	5.00
c_n	24.00	20.00
A_n	300	450
n	0.28	0.35
e	2.718	-
k_0	500	-
k_n	150	300

Table 3: Numerical values.

P_1	P_2	T	TP	Q_1	Q_2
40.12	34.12	3.05	1234.98	122.56	223.43

Table 4: Model results.

P_1	P_2	T	TP	Q_1	Q_2	a_1	Change in parameter
32.54	27.11	1.42	1293.60	281.25	116.81	0.02	-75
34.99	29.16	1.83	1611.57	258.46	163.13	0.03	-50
39.12	32.60	2.52	1562.80	188.63	194.89	0.05	-25
40.12	34.12	3.05	1234.98	122.56	223.43	0.06	0
35.09	35.75	3.24	1063.04	98.83	234.98	0.08	25
31.63	36.61	3.35	999.85	80.38	246.21	0.09	50
29.04	37.06	3.41	983.42	66.04	256.56	0.11	75

Table 5: The effects of changing $a_1$while keeping other parameters at the original values.

Karaöz et al. [3]'s model.	This Model
An increase in the parameter of β results in an increase in prices together with profit per unit of time.	An increase in the parameter of time β_1 or β_2 makes the profit per unit of time increase as observed in Figure 2.
At higher values of the price coefficients of the products, profit is reduced	Similarly, in this paper at higher values of the price coefficients of the products the unit profit decreases as seen in Figure 2.

Table 6: Comparison of results with existing study

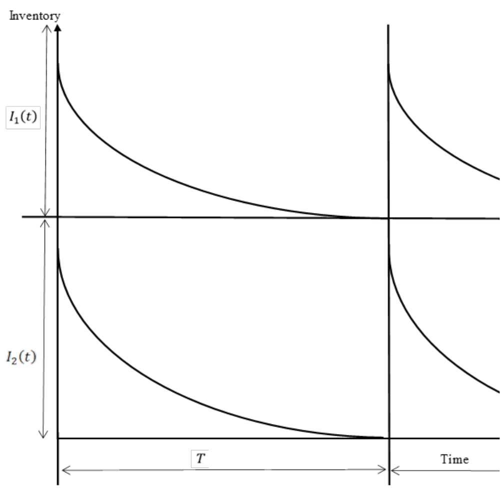

Figure 1: Inventory levels of two deteriorating mutually complementary products with time.

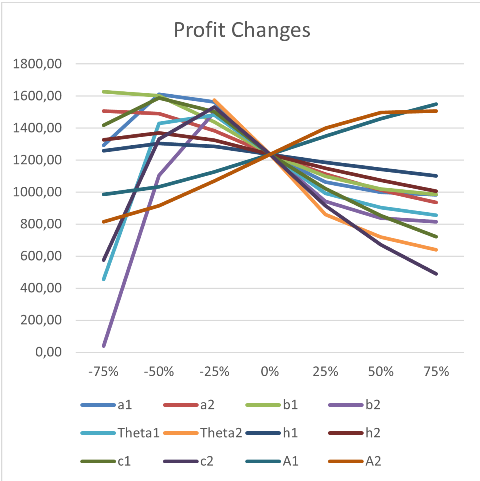

Figure 2: Changes in profit due to parameter changes

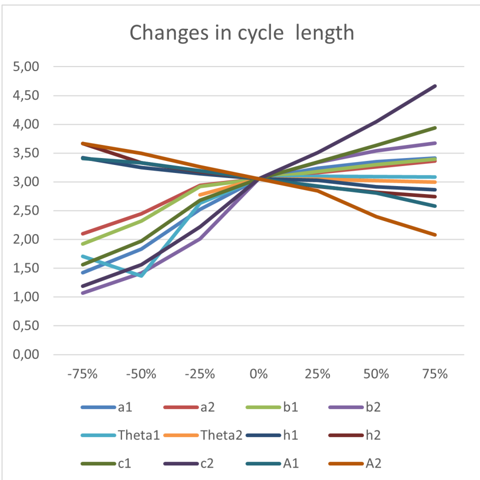

Figure 3: Change in T due to parameter changes.