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Two new numerical approaches for the fractional distribution of the model of a system of lakes via modified hat and quasi-hat functions

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Abstract. In this article, two numerical approaches are presented to solve a system of three fractional differential equations that express the pollution of lakes. In our recent study, a new class of hat functions, called QHFs, are constructed. The proposed approaches utilize Modified Hat Functions (MHFs) and QHFs. Fractional-order operational of MHFs and QHFs are used to build algorithms that transform the main problem into a system of six equations with six unknowns and three equations with three unknowns, respectively. Absolute errors of obtained approximate solutions and convergence analysis of the utilized approach will be studied. Finally, three examples are provided to illustrate the capabilities of these algorithms. The pollution monitoring results are reported in some tables and figures for different values of α.

1. Introduction

Water pollution causes very destructive effects on the environment. Researchers use mathematical modeling to monitor pollution and help plan ways to protect the environment. Biazar et al. studied and solved the mathematical model of lake pollution for the first time [1]. This model consists of a system of three lakes connected to each other by some channels. Mathematical modeling of this problem leads to a system of three fractional differential equations. There are several numerical methods for solving such systems of differential equations (see Refs. [2–5]). The model has been examined by many researchers using different approaches [6–8]. Khader et al. used the operational matrix method based on the shifted Chebyshev polynomials to solve the water pollution model [9]. Prakasha and Veeresha inspected the model of pollution for a system of lakes using the q-homotopy analysis transform method [10]. The Haar wavelet collocation method has been suggested for solving a novel model for the contamination of a system of three artificial lakes in [11]. Ghosh et al. established a new iterative method to solve the model of the amount of pollution in lakes connected with some rivers [12]. Shiri and Baleanu

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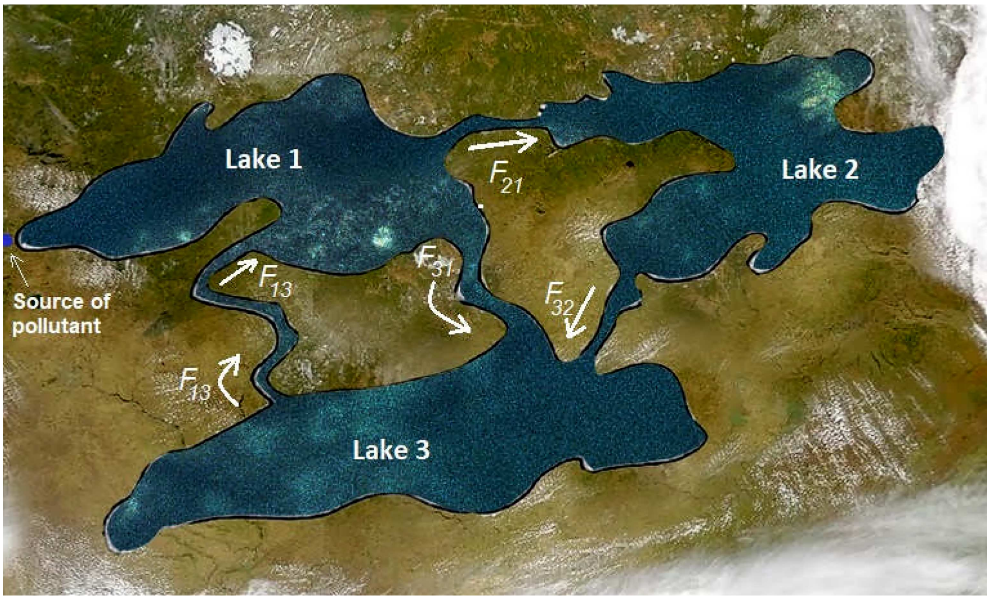


Figure 1. Aerial map of the arrangement of lakes with interconnecting channels.

presented a high-order numerical method to solve the pollution model [13]. Also, Yönet et al. applied a Taylor series to solve the pollution system [14]. Figure 1 shows the three lakes with interconnected canals. A pollutant enters the first lake from the indicated source, named by $p(t)$. Suppose $u_i(t)$ and V_i express the amount of pollutant and the water volume in the lake i for $t \geq 0$, $i = 1, 2, 3$, respectively. Modeling the dynamic behavior of pollution distribution in a system of lakes can be stated as follows:

$$
\begin{aligned}
&{}_0^C D_t^\alpha u_1(t) = p(t) + \frac{F_{13}}{V_3} u_3(t) - \frac{F_{31}}{V_1} u_1(t) - \frac{F_{21}}{V_1} u_1(t),\\
&{}_0^C D_t^\alpha u_2(t) = \frac{F_{21}}{V_1} u_1(t) - \frac{F_{32}}{V_2} u_2(t),\\
&{}_0^C D_t^\alpha u_3(t) = \frac{F_{31}}{V_1} u_1(t) + \frac{F_{32}}{V_2} u_2(t) - \frac{F_{13}}{V_3} u_3(t),\\
&\qquad 0 < \alpha \leq 1, \quad \mathrm{t} \in [0, T]. \qquad (1)
\end{aligned}
$$

Subject to the initial conditions $u_1(0) = \lambda_1$, $u_2(0) = \lambda_2$, and $u_3(0) = \lambda_3$. Here $u_1(t)$, $u_2(t)$, and $u_3(t)$ are unknown functions, λ_1, λ_2, λ_3, F_{13}, F_{31}, F_{21}, F_{32}, V_1, V_2, and V_3 are the appropriate parameters. The operator ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative [15]. Table 1 shows the meaning of parameters and variables for the pollution problem in the system of lakes. The volume of water in each lake is assumed to be constant and the flow into each lake must balance the outflow. The present study discusses some of the properties of Riemann-Liouville integral operators based on the Modified Hat Functions (MHFs) and Quasi-Hat Functions (QHFs) to solve a fractional distribution model of lakes for the first time. In Section 2, some characteristics and basic definitions of the fractional calculus are explained. Section 3 is dedicated to introducing the operational matrices of MHFs and QHFs. Section 4 studies the absolute error of approximation of a function by a truncated series of MHFs and QHFs. Section 5 presents two numerical algorithms for problem 1. The principal problem will be reduced to several simple linear algebraic equations by applying the operational matrix methods of MHFs and QHFs. The convergence analysis of the proposed schemes is discussed in Section 6. As evidence of the validity and accuracy of the utilized approach, three numerical examples are provided in Section 7, and a conclusion and discussion are provided in Section 8.

2. Basic concepts and definitions

Here, several definitions and properties are explained that will be used in this manuscript. In this research the Riemann-Liouville integral operator of the α-th order (I_t^α), and the Caputo fractional differential operator of order α (${}_0^C D_t^\alpha$) will be used. They are well addressed in [15]. The Riemann-Liouville integral and the Caputo fractional derivative operators satisfies the following properties:

$$
\begin{aligned}
&I_t^\alpha(I_t^\beta u(t)) = I_t^\beta(I_t^\alpha u(t)) = I_t^{\alpha+\beta} u(t),\\
&I_t^\alpha({}_0^C D_t^\alpha u(t)) = u(t) - \sum_{i=0}^{n-1} u^{(i)}(0) \frac{t^i}{i!},\\
&\qquad n-1 < \alpha \leq n, \quad t > 0. \qquad (2)
\end{aligned}
$$

2.1. Recalling of MHFs

Hat functions are defined on a closed interval $[0, T]$

Table 1. Nomenclature section of the paper.

Parameters and variables	Explanation	Unit
$u_1(t)$	The amount of the pollutant in Lake 1 at any time t	ppm
$u_2(t)$	The amount of the pollutant in Lake 2 at any time t	ppm
$u_3(t)$	The amount of the pollutant in Lake 3 at any time t	ppm
$p(t)$	The rate at which the pollutant enters the first lake per unit of time t	ppm
V_1	The volume of water in Lake 1	m^3
V_2	The volume of water in Lake 2	m^3
V_3	The volume of water in the Lake 3	m^3
F_{13}	The flow rate from Lake 3 into Lake 1	m^3/year
F_{21}	The flow rate from Lake 1 into Lake 2	m^3/year
F_{31}	The flow rate from Lake 1 into Lake 3	m^3/year
F_{32}	The flow rate from Lake 3 into Lake 2	m^3/year
t	Time	year
T	Period of time	year
λ_1	The initial amount of the pollutants in Lake 1	ppm
λ_2	The initial amount of the pollutants in Lake 2	ppm
λ_3	The initial amount of the pollutants in Lake 3	ppm

and have shapes similar to hats [16,17]. The interval is segregated into n number of sub-intervals of equal length, as $[ih,(i+1)h]$, $i=0,1,2,\cdots,n-1$, where $h=\frac{T}{n}$, and $n\geq 2$, $n=2\mathrm{K}$, $K\in\mathbb{N}$. MHFs are defined as follows [16]:

$$\psi_0(t)=\begin{cases}\frac{1}{2h^2}(t-h)(t-2h), & 0\leq t\leq 2h,\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

if i is odd, and $1\leq i\leq n-1$;

$$\psi_i(t)=\begin{cases}\frac{-1}{h^2}(t-(i-1)h)(t-(i+1)h), & (i-1)h\leq t\leq (i+1)h,\\ 0 & \text{otherwise}\end{cases} \tag{4}$$

if i is even, and $2\leq i\leq n-2$;

$$\psi_i(t)=\begin{cases}\frac{1}{2h^2}(t-(i-1)h)(t-(i-2)h), & (i-2)h\leq t\leq ih,\\ \frac{1}{2h^2}(t-(i+1)h)(t-(i+2)h), & ih\leq t\leq (i+2)h,\\ 0 & \text{otherwise}\end{cases} \tag{5}$$

and at the last point

$$\psi_n(t)=\begin{cases}\frac{1}{2h^2}(t-(T-h))(t-(T-2h)), & T-2h\leq t\leq T,\\ 0 & \text{otherwise}\end{cases} \tag{6}$$

2.2. Definition of QHFs

The concept of QHFs on a closed interval $[0,T]$ is derived based on the idea of the hat functions [18]. The domain of the QHFs is the same as those of the hat functions that were introduced right now. QHFs are defined as follows for i even, and $0\leq i\leq n$;

$$\phi_i(t)=\begin{cases}\frac{1}{2h^2}(t-(i+1)h)(t-(i+2)h), & ih\leq t<(i+2)h,\\ 0 & \text{otherwise}\end{cases} \tag{7}$$

when i is odd, and $1\leq i\leq n-1$;

$$\phi_i(t)=\begin{cases}-\frac{1}{2h^2}(t-(i-1)h)(t-(i+2)h), & (i-1)h\leq t<(i+1)\ h,\\ 0 & \text{otherwise,}\end{cases} \tag{8}$$

where in $n \geq 2$ is an even positive integer, $h = \frac{T}{n}$. The MATLAB package is used to plot QHFs on the interval $[0, 1]$ for $n = 4$ (Figure 2).

The following properties can be achieved by using the MHFs and QHFs definitions, respectively [16,18];

$$\psi_i(jh) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad \sum_{i=0}^{n} \psi_i(t) = 1 \tag{9}$$

and

$$\phi_i(jh) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad \sum_{i=0}^{n} \phi_i(t) = 1 \tag{10}$$

2.3. MHFs & QHFs expansion

An arbitrary function $u(t)$, can be approximated by a linear combination of MHFs or QHFs as the following, respectively:

$$u(t) \simeq u_n(t) = \sum_{i=0}^{n} a_i \psi_i(t) = \mathrm{A}^T \Psi(t), \tag{11}$$

$$u(t) \simeq u_n(t) = \sum_{i=0}^{n} \overline{a}_i \phi_i(t) = \overline{\mathrm{A}}^T \Phi(t), \tag{12}$$

so that:

$$\Psi(t) = [\psi_0(t), \psi_1(t), ..., \psi_n(t)]^T,$$

$$\Phi(t) = [\phi_0(t), \phi_1(t), ..., \phi_n(t)]^T, \tag{13}$$

and

$$\mathrm{A} = [a_0, a_1, ..., a_n]^T,$$

$$\overline{\mathrm{A}} = [\overline{a}_0, \overline{a}_1, ..., \overline{a}_n]^T, \tag{14}$$

where

$$a_i = \overline{a}_i = u(ih), \qquad i = 0, ..., n. \tag{15}$$

3. Operational matrices of MHFs & QHFs

The purpose of this section is to present the fractional-order integral operational matrices based on the MHFs and QHFs.

3.1. Fractional order operational matrix ofintegration

Let us to state the following theorems:

Theorem 1. *Suppose* $\Psi(t)$ *is given by Eq. (13) and* $\alpha > 0$, *then:*

$$I_t^{\alpha} \Psi(t) \simeq P^{\alpha} \Psi(t), \tag{16}$$

where P^{α} is the $(n+1) \times (n+1)$ operational matrix of the Riemann-Liouville integral, illustrated as follows:

$$P^{(\alpha)} = \frac{h^{\alpha}}{2\Gamma(\alpha+3)} \begin{pmatrix} 0 & \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \cdots & \zeta_{n-1} & \zeta_n \\ 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \cdots & \eta_{n-1} & \eta_n \\ 0 & 0 & 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_{n-3} & \beta_{n-2} \\ 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \cdots & \eta_{n-3} & \eta_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \beta_1 & \beta_2 & \cdots & \beta_{n-5} & \beta_{n-4} \\ 0 & 0 & 0 & 0 & 0 & \eta_1 & \eta_2 & \cdots & \eta_{n-5} & \eta_{n-4} \\ 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \beta_1 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \eta_1 & \eta_2 \end{pmatrix} \tag{17}$$

where

$$\zeta_1 = \alpha\,(2\alpha + 3)\,,$$

$$\zeta_k = (k^{\alpha+1}(2k - 3\alpha - 6) + 2k^{\alpha}(\alpha+1)(\alpha+2) + (k-2)^{\alpha+1}(2 - 2k - \alpha)),$$

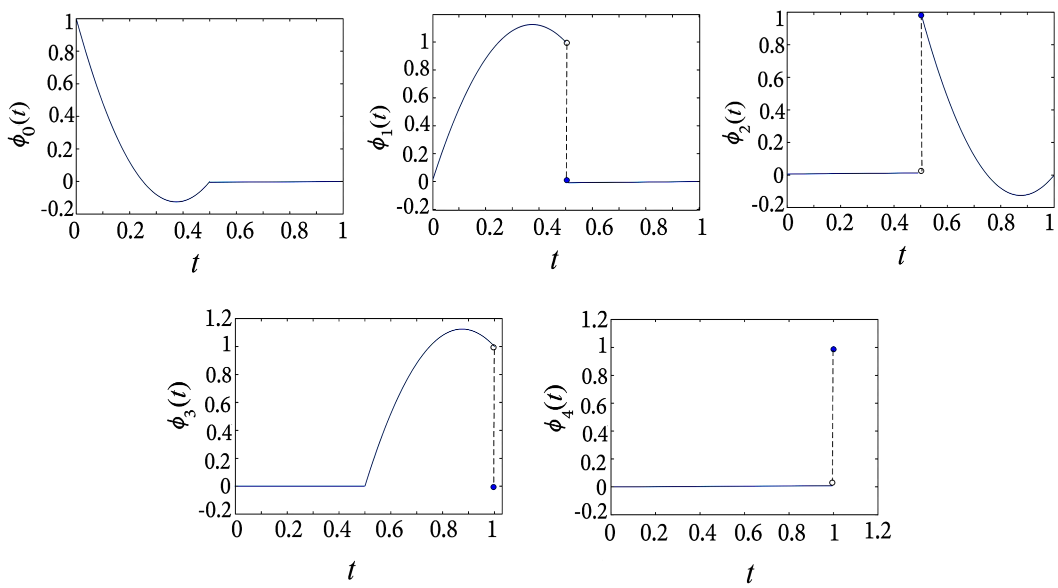


Figure 2. Plots of the QHFs; up to $n = 4$; $T = 1$.

$$k = 2, 3, ..., n,$$

$$\beta_1 = 4(\alpha + 1),$$

$$\beta_k = 4\left((k-2)^{\alpha+1}(k+\alpha) + k^{\alpha+1}(2+\alpha-k)\right),$$

$$k = 2, 3, ..., n,$$

$$\eta_k = -\alpha\delta_{1k} + (2)^{\alpha+1}(2-\alpha)\delta_{2k} + ((3)^{\alpha+1}(4-\alpha) - 6(2+\alpha))\delta_{3k},$$

$$k = 1, 2, 3,$$

$$\eta_k = (k-4)^{\alpha+1}(6-2k-\alpha) - 6(k-2)^{\alpha+1}(2+\alpha) + (k)^{\alpha+1}(2k-2-\alpha),$$

$$k = 4, ..., n, \tag{18}$$

where δ_{ij} is the Kronecker delta.

Proof. For the poof and more details see Ref. [16].

Theorem 2. *Let $\Phi(t)$ be given by Eq. (13) and $\alpha > 0$, then*:

$$I_t^\alpha \Phi(t) \simeq Q^\alpha \Phi(t). \tag{19}$$

So that Q^α is the $(n+1)\times(n+1)$ operational matrix of order α, which is resulted by defining the Riemann-Liouville integral, and is generally represented as follows:

$$Q^{(\alpha)} = \frac{h^\alpha}{2\Gamma(\alpha+3)} \begin{pmatrix} 0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \cdots & \rho_{n-1} & \rho_n \\ 0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \cdots & \sigma_{n-1} & \sigma_n \\ 0 & 0 & 0 & \rho_1 & \rho_2 & \cdots & \rho_{n-3} & \rho_{n-2} \\ 0 & 0 & 0 & \sigma_1 & \sigma_2 & \cdots & \sigma_{n-3} & \sigma_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \rho_1 & \rho_2 \\ 0 & 0 & 0 & 0 & 0 & & \sigma_1 & \sigma_2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{20}$$

where

$$\rho_1 = \alpha(2\alpha + 3),$$

$$\rho_k = (k^{\alpha+1}(2k - 3\alpha - 6) + 2k^\alpha(\alpha+1)(\alpha+2) + (k-2)^{\alpha+1}(2-2k-\alpha)), \qquad k = 2, 3, ..., n,$$

$$\sigma_1 = 3\alpha + 4,$$

$$\sigma_k = (k-2)^{\alpha+1}(2k+\alpha-2) - 2(k-2)^\alpha(2+\alpha)(1+\alpha) - (k)^{\alpha+1}(2k-6-3\alpha),$$

$$k = 2, 3, ..., n. \tag{21}$$

Proof. For the poof and more details see Ref. [18]

If we approximate a function with MHFs and QHFs, we can use Eqs. (16) and (19), to obtain the following result:

$$I_t^\alpha u(t) \simeq I_t^\alpha\left(\sum_{i=0}^n a_i\psi_i(t)\right) \simeq I_t^\alpha\left(\mathrm{A}^T\Psi(t)\right) \simeq \mathrm{A}^T P^\alpha \Psi(t), \tag{22}$$

$$I_t^\alpha u(t) \simeq I_t^\alpha\left(\sum_{i=0}^n \overline{a}_i\phi_i(t)\right) \simeq I_t^\alpha\left(\overline{\mathrm{A}}^T\Phi(t)\right) \simeq \overline{\mathrm{A}}^T Q^\alpha \Phi(t). \tag{23}$$

4. Error analysis

In this section, QHFs and MHFs are proposed to estimate the absolute errors for approximating an arbitrary function.

According to Ref. [18], the absolute error when approximating an arbitrary function using QHFs (12), can be written as:

$$|u(t) - u_n(t)| \simeq \frac{h}{2}\left|(j-k)(2+(-1)^k+k-j)u'(kh)\right|, \tag{24}$$

wherein $j = t/h$, $t \in (kh, (k+1)h)$, $k = 0, 1, 2, ..., n$.

An analysis is performed to approximate an arbitrary function with MHFs (11), as follows [16]:

$$u_n(t) \simeq u(kh) + (t-kh)u'(kh) + \frac{(t-kh)^2}{2}u''(kh). \tag{25}$$

Also, the absolute error at the points $t \in (kh, (k+1)h)$ is as follows:

$$|u(t) - u_n(t)| \simeq \left|u(t) - u(kh) - (t-kh)u'(kh) - \frac{(t-kh)^2}{2}u''(kh)\right|. \tag{26}$$

For $t \in (kh, (k+1)h)$, $k = 0, 1, 2, ..., n$, $h \to 0$, obtain:

$$|u(t) - u_n(t)| \simeq \frac{1}{2}\left|(jh-kh)^2 u''(kh)\right| = \frac{h^2}{2}\left|(j-k)^2 u''(kh)\right|. \tag{27}$$

where $j = t/h$. The absolute error of QHFs and MHFs for $j \to k$, $k = 0, ..., n$, results:

$$|u(kh) - u_n(kh)| \simeq 0, \tag{28}$$

thus $\forall k$, while $h \to 0$ or $n \to \infty$, result in:

$$|u(t) - u_n(t)| \to 0. \tag{29}$$

5. Numerical process

In this section, two numerical approaches are presented for the solution of Eq. (1). As well recalling, the definition of Riemann-Liouville integral operator of order α is as follows [15].

$$I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau. \tag{30}$$

We apply Eq. (30) to both sides of the original equation Eq.(1). Then, using the properties of the Riemann-Liouville integral operator and the Caputo fractional differential operator, and the initial conditions of Eq. (1), we obtain:

$$\begin{aligned} u_1(t) &= \lambda_1 + I_t^\alpha p(t) + \frac{F_{13}}{V_3} I_t^\alpha u_3(t) - \frac{F_{31}}{V_1} I_t^\alpha u_1(t) \\ &\quad - \frac{F_{21}}{V_1} I_t^\alpha u_1(t), \\ u_2(t) &= \lambda_2 + \frac{F_{21}}{V_1} I_t^\alpha u_1(t) \\ &\quad - \frac{F_{32}}{V_2} I_t^\alpha u_2(t), \\ u_3(t) &= \lambda_3 + \frac{F_{31}}{V_1} I_t^\alpha u_1(t) + \frac{F_{32}}{V_2} I_t^\alpha u_2(t) \\ &\quad - \frac{F_{13}}{V_3} I_t^\alpha u_3(t). \end{aligned} \tag{31}$$

5.1. *Description of numerical algorithm based on MHFs*

As described in Section 2, we can approximate all the functions in Eq. (31) with MHFs (11), as follows:

$$u_v(t) \simeq \sum_{i=0}^n a_{vi} \psi_i(t) = A_v{}^T \Psi(t), \quad v = 1,2,3, \tag{32}$$

$$\lambda_v \simeq \lambda_v \sum_{i=0}^n \psi_i(t) = \lambda_v E^T \Psi(t), \quad v = 1,2,3, \tag{33}$$

$$p(t) \simeq \sum_{i=0}^n p(ih) \psi_i(t) = G^T \Psi(t), \tag{34}$$

so that:

$$A_v = [a_{v0}, a_{v1}, ..., a_{vn}]^T, \quad v = 1,2,3, \tag{35}$$

$$E = [1,1,...,1]^T, \tag{36}$$

$$G = [p(0), p(h), ..., p(nh)]^T. \tag{37}$$

Combining Eqs. (16), (22) and substitution Eqs. (32–34) into Eqs. (31) results in:

$$\begin{aligned} A_1^T \Psi(t) &= \lambda_1 E^T \Psi(t) + G^T P^\alpha \Psi(t) \\ &\quad + \frac{F_{13}}{V_3} A_3{}^T P^\alpha \Psi(t) - \frac{F_{31}}{V_1} A_1{}^T P^\alpha \Psi(t) \\ &\quad - \frac{F_{21}}{V_1} A_1{}^T P^\alpha \Psi(t), \\ A_2^T \Psi(t) &= \lambda_2 E^T \Psi(t) + \frac{F_{21}}{V_1} A_1{}^T P^\alpha \Psi(t) \\ &\quad - \frac{F_{32}}{V_2} A_2{}^T P^\alpha \Psi(t), \\ A_3{}^T \Psi(t) &= \lambda_3 E^T \Psi(t) + \frac{F_{31}}{V_1} A_1{}^T P^\alpha \Psi(t) \\ &\quad + \frac{F_{32}}{V_2} A_2{}^T P^\alpha \Psi(t) - \frac{F_{13}}{V_3} A_3{}^T P^\alpha \Psi(t). \end{aligned} \tag{38}$$

Also, we can write:

$$\begin{aligned} & A_1{}^T - \lambda_1 E^T - G^T P^\alpha - \frac{F_{13}}{V_3} A_3{}^T P^\alpha \\ & \quad + \frac{F_{31}}{V_1} A_1{}^T P^\alpha + \frac{F_{21}}{V_1} A_1{}^T P^\alpha = 0, \\ & A_2{}^T - \lambda_2 E^T - \frac{F_{21}}{V_1} A_1{}^T P^\alpha + \frac{F_{32}}{V_2} A_2{}^T P^\alpha = 0, \\ & A_3{}^T - \lambda_3 E^T - \frac{F_{31}}{V_1} A_1{}^T P^\alpha \\ & \quad - \frac{F_{32}}{V_2} A_2{}^T P^\alpha + \frac{F_{13}}{V_3} A_3{}^T P^\alpha = 0. \end{aligned} \tag{39}$$

This system contains $3(n+1)$ gathers with $3(n+1)$ unknown MHFs coefficients. Assume $P^\alpha = [\mu]_{ij}$, $i, j = 0, ..., n$, the following results are obtained from the arrangement of elements in the operational matrix (17):

$$\begin{aligned} & [\mu_{ij}]_{i=0}^n = 0, && j = 0, \\ & [\mu_{ij}]_{i=j+2}^n = 0, && j = 1,3,...,n-1, \\ & [\mu_{ij}]_{i=j+1}^n = 0, && j = 2,4,...,n. \end{aligned} \tag{40}$$

Based on Eq. (15), we get:

$$a_{10} = \lambda_1, \quad a_{20} = \lambda_2, \quad a_{30} = \lambda_3. \tag{41}$$

As a result of the properties of the operational matrix of MHFs, Eqs. (17) and (40), we introduce a recursive system $(k/2)$, $k = 2,4,...,n$. To define system (1), take $P^\alpha = [\mu]_{ij}$, $i = 0,1,k$, $j = 1,k$, and $k = 2$, so Eq. (39) is expressed by Eq. (42) is shown in Box I.

$$system\ (1):\begin{cases} a_{11}-\frac{F_{13}}{V_3}\left[\sum_{i=0}^{2}\mu_{i1}a_{3i}\right]-\left[\sum_{i=0}^{2}p(ih)\mu_{i1}\right]+\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{2}\mu_{i1}a_{1i}\right]-\lambda_1=0,\\ a_{12}-\frac{F_{13}}{V_3}\left[\sum_{i=0}^{2}\mu_{i2}a_{3i}\right]-\left[\sum_{i=0}^{2}p(ih)\mu_{i2}\right]+\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{2}\mu_{i2}a_{1i}\right]-\lambda_1=0,\\ a_{21}-\frac{F_{21}}{V_1}\left[\sum_{i=0}^{2}\mu_{i1}a_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum_{i=0}^{2}\mu_{i1}a_{2i}\right]-\lambda_2=0,\\ a_{22}-\frac{F_{21}}{V_1}\left[\sum_{i=0}^{2}\mu_{i2}a_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum_{i=0}^{2}\mu_{i2}a_{2i}\right]-\lambda_2=0,\\ a_{31}-\frac{F_{31}}{V_1}\left[\sum_{i=0}^{2}\mu_{i1}a_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum_{i=0}^{2}\mu_{i1}a_{2i}\right]+\frac{F_{13}}{V_3}\left[\sum_{i=0}^{2}\mu_{i1}a_{3i}\right]-\lambda_3=0,\\ a_{32}-\frac{F_{31}}{V_1}\left[\sum_{i=0}^{2}\mu_{i2}a_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum_{i=0}^{2}\mu_{i2}a_{2i}\right]+\frac{F_{13}}{V_3}\left[\sum_{i=0}^{2}\mu_{i2}a_{3i}\right]-\lambda_3=0. \end{cases} \quad (42)$$

Box I

$$system\ (2):\begin{cases} a_{13}-\frac{F_{13}}{V_3}\left[\sum_{i=0}^{4}\mu_{i3}a_{3i}\right]-\left[\sum_{i=0}^{4}p(ih)\mu_{i3}\right]+\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{4}\mu_{i3}a_{1i}\right]-\lambda_1=0,\\ a_{14}-\frac{F_{13}}{V_3}\left[\sum_{i=0}^{4}\mu_{i4}a_{3i}\right]-\left[\sum_{i=0}^{4}p(ih)\mu_{i4}\right]+\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{4}\mu_{i4}a_{1i}\right]-\lambda_1=0,\\ a_{23}-\frac{F_{21}}{V_1}\left[\sum_{i=0}^{4}\mu_{i3}a_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum_{i=0}^{4}\mu_{i3}a_{2i}\right]-\lambda_2=0,\\ a_{24}-\frac{F_{21}}{V_1}\left[\sum_{i=0}^{4}\mu_{i4}a_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum_{i=0}^{4}\mu_{i4}a_{2i}\right]-\lambda_2=0,\\ a_{33}-\frac{F_{31}}{V_1}\left[\sum_{i=0}^{4}\mu_{i3}a_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum_{i=0}^{4}\mu_{i3}a_{2i}\right]+\frac{F_{13}}{V_3}\left[\sum_{i=0}^{4}\mu_{i3}a_{3i}\right]-\lambda_3=0,\\ a_{34}-\frac{F_{31}}{V_1}\left[\sum_{i=0}^{4}\mu_{i4}a_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum_{i=0}^{4}\mu_{i4}a_{2i}\right]+\frac{F_{13}}{V_3}\left[\sum_{i=0}^{4}\mu_{i4}a_{3i}\right]-\lambda_3=0, \end{cases} \quad (43)$$

Box II

Solving system (1) which includes 6 equations, coefficients of $\{a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}\}$ can be calculated. Then for $k = 4$, Eq. (43) obtained as shown in Box II, unknown parameters $\{a_{13}, a_{14}, a_{23}, a_{24}, a_{33}, a_{34}\}$ are obtained by solving system (2). Then we continue the successive process

$$\vdots$$

eventually, for $k = n$, result obtained by Eq. (44) is shown in Box III.

By solving system ($\frac{n}{2}$) and finding the six unknown coefficients of this system, all coefficients are completely determined, and we can get the approximate solutions $u_1(t)$, $u_2(t)$, $u_3(t)$, Eq. (32). The proposed approach reduces the original system to $\frac{n}{2}$ systems, which includes six algebraic equations.

5.2. Description of numerical algorithm based on QHFs

To obtain numerical solutions of Eqs. (31) using QHFs (12), we have:

$$u_v(t) \simeq \sum_{i=0}^{n} \overline{a}_{vi}\phi_i(t) = \overline{A}_v{}^T\Phi(t), \qquad v = 1,2,3, \quad (45)$$

$$\lambda_v \simeq \lambda_v \sum_{i=0}^{n} \phi_i(t) = \lambda_v E^T\Phi(t),$$

$$v = 1,2,3, \quad (46)$$

$$p(t) \simeq \sum_{i=0}^{n} p(ih)\phi_i(t) = G^T\Phi(t), \quad (47)$$

wherein:

$$\overline{A}_v = [\overline{a}_{v0}, \overline{a}_{v1}, ..., \overline{a}_{vn}]^T, \qquad v = 1,2,3, \quad (48)$$

$$E = [1, 1, ..., 1]^T,$$

$$G = [p(0), p(h), ..., p(nh)]^T$$

$$system\left(\tfrac{n}{2}\right):\begin{cases} a_{1n-1}-\frac{F_{13}}{V_3}\left[\sum\limits_{i=0}^{n}\mu_{i\,n-1}a_{3i}\right]-\left[\sum\limits_{i=0}^{n}p(ih)\mu_{i\,n-1}\right]+\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum\limits_{i=0}^{n}\mu_{i\,n-1}a_{1i}\right]-\lambda_1=0,\\ a_{1n}-\frac{F_{13}}{V_3}\left[\sum\limits_{i=0}^{n}\mu_{i\,n}a_{3i}\right]-\left[\sum\limits_{i=0}^{n}p(ih)\mu_{i\,n}\right]+\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum\limits_{i=0}^{n}\mu_{i\,n}a_{1i}\right]-\lambda_1=0,\\ a_{2n-1}-\frac{F_{21}}{V_1}\left[\sum\limits_{i=0}^{n}\mu_{i\,n-1}a_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum\limits_{i=0}^{n}\mu_{i\,n-1}a_{2i}\right]-\lambda_2=0,\\ a_{2n}-\frac{F_{21}}{V_1}\left[\sum\limits_{i=0}^{n}\mu_{i\,n}a_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum\limits_{i=0}^{n}\mu_{i\,n}a_{2i}\right]-\lambda_2=0,\\ a_{3n-1}-\frac{F_{31}}{V_1}\left[\sum\limits_{i=0}^{n}\mu_{i\,n-1}a_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum\limits_{i=0}^{n}\mu_{i\,n-1}a_{2i}\right]+\frac{F_{13}}{V_3}\left[\sum\limits_{i=0}^{n}\mu_{i\,n-1}a_{3i}\right]-\lambda_3=0,\\ a_{3n}-\frac{F_{31}}{V_1}\left[\sum\limits_{i=0}^{n}\mu_{i\,n}a_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum\limits_{i=0}^{n}\mu_{i\,n}a_{2i}\right]+\frac{F_{13}}{V_3}\left[\sum\limits_{i=0}^{n}\mu_{i\,n}a_{3i}\right]-\lambda_3=0. \end{cases}\tag{44}$$

Box III

By applying Eqs. (19) and (20) and (45)–(47) in Eqs. (31) the following results are obtained:

$$\overline{A}_1{}^T-\lambda_1E^T-G^TQ^\alpha-\frac{F_{13}}{V_3}\overline{A}_3{}^TQ^\alpha+\frac{F_{31}}{V_1}\overline{A}_1{}^TQ^\alpha+\frac{F_{21}}{V_1}\overline{A}_1{}^TQ^\alpha=0,$$
$$\overline{A}_2{}^T-\lambda_2E^T-\frac{F_{21}}{V_1}\overline{A}_1{}^TQ^\alpha+\frac{F_{32}}{V_2}\overline{A}_2{}^TQ^\alpha=0,$$
$$\overline{A}_3{}^T-\lambda_3E^T-\frac{F_{31}}{V_1}\overline{A}_1{}^TQ^\alpha-\frac{F_{32}}{V_2}\overline{A}_2{}^TQ^\alpha+\frac{F_{13}}{V_3}\overline{A}_3{}^TQ^\alpha=0.\tag{49}$$

The dimension of this system is $3(n+1)\times 3(n+1)$.

Suppose $Q^\alpha=[\theta]_{ij},\ i,j=0,...,n$. As shown in the operational matrix (20), we have:

$$[\theta_{ij}]_{i=0}^{n}=0,\qquad j=0,$$
$$[\theta_{ij}]_{i=0}^{n}=0,\qquad j=n,$$
$$[\theta_{ij}]_{i=j+1}^{n}=0,\qquad j=1,3,...,n-1,$$
$$[\theta_{ij}]_{i=j}^{n}=0,\qquad j=2,4,...,n.\tag{50}$$

Using Eq. (15) and initial values, we get the following results:

$$\overline{a}_{10}=\lambda_1,\quad \overline{a}_{20}=\lambda_2,\quad \overline{a}_{30}=\lambda_3.\tag{51}$$

According to Eqs. (20) and (50), we introduce recursive system $(k/2)$, $k=2,4,...,n$. Thus, the following system is an appropriate representation to write Eq. (49) based on Eq. (50).

system (1):

$$\begin{cases}\overline{a}_{11}-\frac{F_{13}}{V_3}\left[\sum\limits_{i=0}^{1}\theta_{i1}\overline{a}_{3i}\right]-\left[\sum\limits_{i=0}^{1}p(ih)\theta_{i1}\right]+\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum\limits_{i=0}^{1}\theta_{i1}\overline{a}_{1i}\right]-\lambda_1=0,\\ \overline{a}_{21}-\frac{F_{21}}{V_1}\left[\sum\limits_{i=0}^{1}\theta_{i1}\overline{a}_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum\limits_{i=0}^{1}\theta_{i1}\overline{a}_{2i}\right]-\lambda_2=0,\\ \overline{a}_{31}-\frac{F_{31}}{V_1}\left[\sum\limits_{i=0}^{1}\theta_{i1}\overline{a}_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum\limits_{i=0}^{1}\theta_{i1}\overline{a}_{2i}\right]+\frac{F_{13}}{V_3}\left[\sum\limits_{i=0}^{1}\theta_{i1}\overline{a}_{3i}\right]-\lambda_3=0,\end{cases}\tag{52}$$

by solving system (1) which includes 3 equations, coefficients of $\{\overline{a}_{11},\overline{a}_{21},\overline{a}_{31}\}$ can be calculated, then we get $\{\overline{a}_{12},\overline{a}_{22},\overline{a}_{32}\}$, as follows:

$$\overline{a}_{12}=\frac{F_{13}}{V_3}\left[\sum_{i=0}^{1}\theta_{i2}\overline{a}_{3i}\right]+\left[\sum_{i=0}^{1}p(ih)\theta_{i2}\right]-\left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{1}\theta_{i2}\overline{a}_{1i}\right]+\lambda_1,$$
$$\overline{a}_{22}=\frac{F_{21}}{V_1}\left[\sum_{i=0}^{1}\theta_{i2}\overline{a}_{1i}\right]-\frac{F_{32}}{V_2}\left[\sum_{i=0}^{1}\theta_{i2}\overline{a}_{2i}\right]+\lambda_2,$$
$$\overline{a}_{32}=\frac{F_{31}}{V_1}\left[\sum_{i=0}^{1}\theta_{i2}\overline{a}_{1i}\right]+\frac{F_{32}}{V_2}\left[\sum_{i=0}^{1}\theta_{i2}\overline{a}_{2i}\right]-\frac{F_{13}}{V_3}\left[\sum_{i=0}^{1}\theta_{i2}\overline{a}_{3i}\right]+\lambda_3,$$

then, for $k=4$, one has:

system (2):

$$\begin{cases} \overline{a}_{13} - \frac{F_{13}}{V_3}\left[\sum_{i=0}^{3}\theta_{i3}\overline{a}_{3i}\right] - \left[\sum_{i=0}^{3}p(ih)\theta_{i3}\right] \\ \quad + \left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{3}\theta_{i3}\overline{a}_{1i}\right] - \lambda_1 = 0, \\ \overline{a}_{23} - \frac{F_{21}}{V_1}\left[\sum_{i=0}^{3}\theta_{i3}\overline{a}_{1i}\right] + \frac{F_{32}}{V_2}\left[\sum_{i=0}^{3}\theta_{i3}\overline{a}_{2i}\right] \\ \quad -\lambda_2 = 0, \\ \overline{a}_{33} - \frac{F_{31}}{V_1}\left[\sum_{i=0}^{3}\theta_{i3}\overline{a}_{1i}\right] - \frac{F_{32}}{V_2}\left[\sum_{i=0}^{3}\theta_{i3}\overline{a}_{2i}\right] \\ \quad + \frac{F_{13}}{V_3}\left[\sum_{i=0}^{3}\theta_{i3}\overline{a}_{3i}\right] - \lambda_3 = 0, \end{cases} \tag{53}$$

unknown parameters $\{\overline{a}_{13}, \overline{a}_{23}, \overline{a}_{33}\}$ are calculated by solving system (2), then we find $\{\overline{a}_{14}, \overline{a}_{24}, \overline{a}_{34}\}$, as follows:

$$\overline{a}_{14} = \frac{F_{13}}{V_3}\left[\sum_{i=0}^{3}\theta_{i4}\overline{a}_{3i}\right] + \left[\sum_{i=0}^{3}p(ih)\theta_{i4}\right] - \left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{3}\theta_{i4}\overline{a}_{1i}\right] + \lambda_1.$$

$$\overline{a}_{24} = \frac{F_{21}}{V_1}\left[\sum_{i=0}^{3}\theta_{i4}\overline{a}_{1i}\right] - \frac{F_{32}}{V_2}\left[\sum_{i=0}^{3}\theta_{i4}\overline{a}_{2i}\right] + \lambda_2,$$

$$\overline{a}_{34} = \frac{F_{31}}{V_1}\left[\sum_{i=0}^{3}\theta_{i4}\overline{a}_{1i}\right] + \frac{F_{32}}{V_2}\left[\sum_{i=0}^{3}\theta_{i4}\overline{a}_{2i}\right] - \frac{F_{13}}{V_3}\left[\sum_{i=0}^{3}\theta_{i4}\overline{a}_{3i}\right] + \lambda_3.$$

Then we continue the process

$\vdots$

finally, for $k = n$, result in:

system $(\frac{n}{2})$:

$$\begin{cases} \overline{a}_{1\,n-1} - \frac{F_{13}}{V_3}\left[\sum_{i=0}^{n-1}\theta_{i\,n-1}\overline{a}_{3i}\right] \\ \quad - \left[\sum_{i=0}^{n-1}p(ih)\theta_{i\,n-1}\right] + \left(\frac{F_{31}+F_{21}}{V_1}\right) \\ \quad \left[\sum_{i=0}^{n-1}\theta_{i\,n-1}\overline{a}_{1i}\right] - \lambda_1 = 0, \\ \overline{a}_{2\,n-1} - \frac{F_{21}}{V_1}\left[\sum_{i=0}^{n-1}\theta_{i\,n-1}\overline{a}_{1i}\right] \\ \quad + \frac{F_{32}}{V_2}\left[\sum_{i=0}^{n-1}\theta_{i\,n-1}\overline{a}_{2i}\right] - \lambda_2 = 0, \\ \overline{a}_{3\,n-1} - \frac{F_{31}}{V_1}\left[\sum_{i=0}^{n-1}\theta_{i\,n-1}\overline{a}_{1i}\right] \\ \quad - \frac{F_{32}}{V_2}\left[\sum_{i=0}^{n-1}\theta_{i\,n-1}\overline{a}_{2i}\right] \\ \quad + \frac{F_{13}}{V_3}\left[\sum_{i=0}^{n-1}\theta_{i\,n-1}\overline{a}_{3i}\right] - \lambda_3 = 0, \end{cases} \tag{54}$$

having the coefficients $\{\overline{a}_{1\,n-1}, \overline{a}_{2\,n-1}, \overline{a}_{3\,n-1}\}$ from solving this system, the coefficients $\{\overline{a}_{1\,n}, \overline{a}_{2\,n}, \overline{a}_{3\,n}\}$ are found as follows:

$$\overline{a}_{1n} = \frac{F_{13}}{V_3}\left[\sum_{i=0}^{n-1}\theta_{in}\overline{a}_{3i}\right] + \left[\sum_{i=0}^{n-1}p(ih)\theta_{in}\right] - \left(\frac{F_{31}+F_{21}}{V_1}\right)\left[\sum_{i=0}^{n-1}\theta_{in}\overline{a}_{1i}\right] + \lambda_1,$$

$$\overline{a}_{2n} = \frac{F_{21}}{V_1}\left[\sum_{i=0}^{n-1}\theta_{in}\overline{a}_{1i}\right] - \frac{F_{32}}{V_2}\left[\sum_{i=0}^{n-1}\theta_{in}\overline{a}_{2i}\right] + \lambda_2,$$

$$\overline{a}_{3n} = \frac{F_{31}}{V_1}\left[\sum_{i=0}^{n-1}\theta_{in}\overline{a}_{1i}\right] + \frac{F_{32}}{V_2}\left[\sum_{i=0}^{n-1}\theta_{in}\overline{a}_{2i}\right] - \frac{F_{13}}{V_3}\left[\sum_{i=0}^{n-1}\theta_{in}\overline{a}_{3i}\right] + \lambda_3.$$

Once we determine the three unknown coefficients for system $(\frac{n}{2})$, all the coefficients are determined, and we can get the approximate solutions $u_1(t)$, $u_2(t)$, and $u_3(t)$ (45). In this method, there is a need to solve $\frac{n}{2}$ systems, each of them consists of three linear algebraic equations, that can be done easily by direct methods. The computations are handled using the MATLAB package.

6. Convergence analysis of MHFs and QHFs approachers.

In this section, the convergence of the system (1) is examined based on the approximation of functions using MHFs and QHFs. Consider the system (31), which is equivalent to the system (1). The resulting equations must approximate the following equation:

$$E_{1n}^{\alpha}u(t) = \left| u_1(t) - I_t^{\alpha}p(t) - \frac{F_{13}}{V_3}I_t^{\alpha}u_3(t) + \frac{F_{31}}{V_1}I_t^{\alpha}u_1(t) + \frac{F_{21}}{V_1}I_t^{\alpha}u_1(t) - \lambda_1 \right| \simeq 0,$$

$$E_{2n}^{\alpha}u(t) = \left| u_2(t) - \frac{F_{21}}{V_1}I_t^{\alpha}u_1(t) + \frac{F_{32}}{V_2}I_t^{\alpha}u_2(t) - \lambda_2 \right| \simeq 0,$$

$$E_{3n}^{\alpha}u(t) = \left| u_3(t) - \frac{F_{31}}{V_1}I_t^{\alpha}u_1(t) - \frac{F_{32}}{V_2}I_t^{\alpha}u_2(t) + \frac{F_{13}}{V_3}I_t^{\alpha}u_3(t) - \lambda_3 \right| \simeq 0. \tag{55}$$

Here, two cases can be considered as follows:

1. Utilizing Eq. (27) to approximate the absolute error based on the functions MHFs, results in:

$$e(u_v(t)) \simeq \frac{h^2}{2}\left|(j-k)^2 u_v''(kh)\right|, \quad v=1,2,3. \quad (56)$$

2. As a result of approximating the absolute error using Eq. (24), we obtain:

$$\bar{e}(u_v(t)) \simeq \frac{h}{2}\left|(j-k)(2+(-1)^k+k-j){u'}_v(kh)\right|,$$
$$v=1,2,3, \quad (57)$$

where $j=t/h$, and $t\in(kh,(k+1)h)$, $k=0,1,2,...,n$. For MHFs, utilizing Eqs. (55) and Eq. (56), we get eqations are shown in Box IV, wherein:
$d=\sup\left\{(j-k)^2\right\}$,
$b=\sup_{t,\tau\in[0,T]}\left|\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\right|$, and as $h\to 0$, $E^{\alpha}_{vn}e(t)\to 0$, $v=1,2,3$.

For QHFs, when we take (55) and (57), we get equations are shown in Box V, wherein:
$\bar{d}$=$\sup\left\{(j-k)(2+(-1)^k+k-j)\right\}$,
b=$\sup_{t,\tau\in[0,T]}\left|\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\right|$, and while $h\to 0$, $E^{\alpha}_{vn}(t)\to 0$, $v=1,2,3$.

7. Numerical experiments and discussions

In this section, three different pollution input models are considered to evaluate the effectiveness of the presented methods.

Example 1: Periodic input model
The model considers pollutants enter periodically into Lake 1. Let us consider $p(t)=a\sin(\omega t)+c$, where a and ω denote amplitude and frequency of the variation, respectively, and the average concentration of pollutants is represented by c. We assume Eq. (1) with, a=ω=c=1,

$$\begin{aligned}
{}^{C}_{0}D^{\alpha}_{t}u_1(t) =& 1+\sin(t)+\frac{38}{1180}u_3(t)\\
&-\frac{20}{2900}u_1(t)-\frac{18}{2900}u_1(t),\\
{}^{C}_{0}D^{\alpha}_{t}u_2(t) =& \frac{18}{2900}u_1(t)-\frac{18}{850}u_2(t),\\
{}^{C}_{0}D^{\alpha}_{t}u_3(t) =& \frac{20}{2900}u_1(t)+\frac{18}{850}u_2(t)-\frac{38}{1180}u_3(t),\\
&0<\alpha\leq 1, t\in[0,T], \quad (58)
\end{aligned}$$

with the initial conditions $u_1(0)=0$, $u_2(0)=0$, and $u_3(0)=0$.

There have been several studies of this problem (58). For the pollution monitoring, results of the

$$\begin{aligned}
E^{\alpha}_{1n}e(t) &\leq \frac{dh^2}{2}\left(|{u_1}''(t)|+b\left(|p''(t)|+\left|\frac{F_{13}}{V_3}{u_3}''(t)\right|+\left|\frac{F_{31}}{V_1}{u_1}''(t)\right|+\left|\frac{F_{21}}{V_1}{u_1}''(t)\right|\right)\right)\leq C_1h^2,\\
E^{\alpha}_{2n}e(t) &\leq \frac{dh^2}{2}\left(|{u_2}''(t)|+b\left(\left|\frac{F_{21}}{V_1}{u_1}''(t)\right|+\left|\frac{F_{32}}{V_2}{u_2}''(t)\right|\right)\right)\leq C_2h^2,\\
E^{\alpha}_{3n}e(t) &\leq \frac{dh^2}{2}\left(|{u_3}''(t)|+b\left(\left|\frac{F_{31}}{V_1}{u_1}''(t)\right|+\left|\frac{F_{32}}{V_2}{u_2}''(t)\right|+\left|\frac{F_{13}}{V_3}{u_3}''(t)\right|\right)\right)\leq C_3h^2,
\end{aligned}$$

Box IV

$$\begin{aligned}
E^{\alpha}_{1n}\bar{e}(t) &\leq \frac{\bar{d}h}{2}\left(|{u_1}'(t)|+b\left(|p'(t)|+\left|\frac{F_{13}}{V_3}{u_3}'(t)\right|+\left|\frac{F_{31}}{V_1}{u_1}'(t)\right|+\left|\frac{F_{21}}{V_1}{u_1}'(t)\right|\right)\right)\leq \overline{C}_1h,\\
E^{\alpha}_{2n}\bar{e}(t) &\leq \frac{\bar{d}h}{2}\left||{u_2}'(t)|+b\left(\left|\frac{F_{21}}{V_1}{u_1}'(t)\right|+\left|\frac{F_{32}}{V_2}{u_2}'(t)\right|\right)\right|\leq \overline{C}_2h,\\
E^{\alpha}_{3n}\bar{e}(t) &\leq \frac{\bar{d}h}{2}\left||{u_3}'(t)|+b\left(\left|\frac{F_{31}}{V_1}{u_1}'(t)\right|+\left|\frac{F_{32}}{V_2}{u_2}'(t)\right|+\left|\frac{F_{13}}{V_3}{u_3}'(t)\right|\right)\right|\leq \overline{C}_3h,
\end{aligned}$$

Box V

Table 2. Numerical results of Example 1 for three lakes, $\alpha = 1$, $n = 32$.

Time (in year)	MHFs			QHFs		
t	Lake 1: $u_1(t)$	Lake 2: $u_2(t)$	Lake 3: $u_3(t)$	Lake 1: $u_1(t)$	Lake 2: $u_2(t)$	Lake 3: $u_3(t)$
0.0	0.00000000	0.00000000	0.00000000	0.0000000	0.0000000	0.0000000
0.125	0.13269583	0.5044×10^{-4}	0.5606×10^{-4}	0.13205645	0.4602×10^{-4}	0.5115×10^{-4}
0.250	0.28064517	0.2094×10^{-3}	0.2329×10^{-3}	0.27937983	0.1996×10^{-3}	0.2220×10^{-3}
0.375	0.44346046	0.4886×10^{-3}	0.5433×10^{-3}	0.44159249	0.4724×10^{-3}	0.5254×10^{-3}
0.500	0.62051910	0.8986×10^{-3}	0.9997×10^{-3}	0.61808134	0.8755×10^{-3}	0.9738×10^{-3}
0.625	0.81097317	0.1450×10^{-2}	0.1613×10^{-2}	0.80800747	0.1418×10^{-2}	0.1578×10^{-2}
0.750	1.01376264	0.2152×10^{-2}	0.2396×10^{-2}	1.01031920	0.2112×10^{-2}	0.2351×10^{-2}
0.875	1.22763196	0.3014×10^{-2}	0.3356×10^{-2}	1.22374664	0.2975×10^{-2}	0.3312×10^{-2}
1.0	1.45114965	0.4044×10^{-2}	0.4504×10^{-2}	1.44690889	0.3994×10^{-2}	0.4449×10^{-2}

Table 3. Numerical results of the proposed and some other methods for the sinusoidal input of Lake 1 with $\alpha = 1$.

Time (in year)	MHFs	QHFs	Bessel polynomials	VPM
t	$n = 10$	$n = 10$	See [7], $n = 10$	See [8]
0.2	0.219654473	0.216408198	0.219654467	–
0.4	0.477756712	0.471445300	0.477756680	–
0.5	0.620522784	0.614912667	–	0.62051
0.6	0.771858751	0.762787012	0.771858670	–
0.8	1.098057381	1.086641630	1.098057233	–
1.0	1.451149882	1.437901320	1.451149651	1.45115

Table 4. Numerical results of the proposed and some other methods for the sinusoidal input of Lake 2 with $\alpha = 1$.

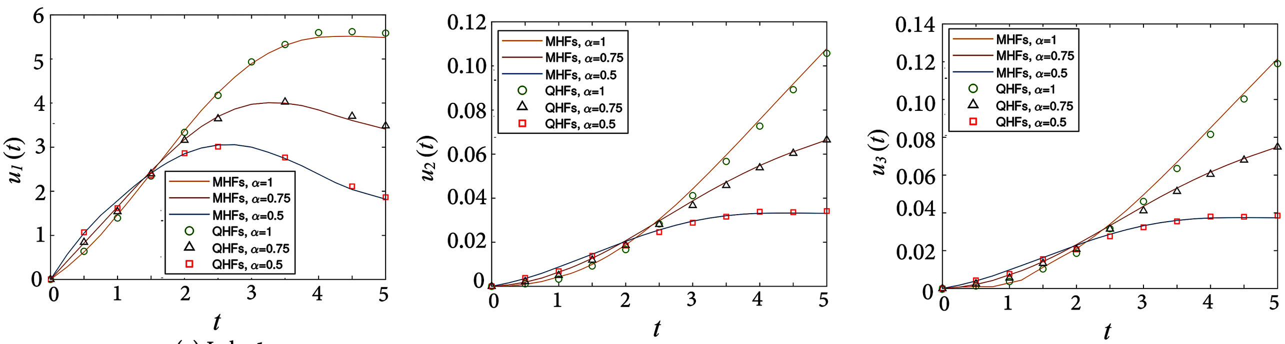
Time (in year)	MHFs	QHFs	Bessel polynomials	VPM
t	$n = 10$	$n = 10$	See [7], $n = 10$	See [8]
0.2	0.1321026×10^{-3}	0.1092542×10^{-3}	0.1320999×10^{-3}	–
0.4	0.5597490×10^{-3}	0.5062570×10^{-3}	0.5597436×10^{-3}	–
0.5	0.8986229×10^{-3}	0.8493274×10^{-3}	–	0.898×10^{-3}
0.6	0.1327956×10^{-2}	0.1236559×10^{-2}	0.1327949×10^{-2}	–
0.8	0.2477604×10^{-2}	0.2341872×10^{-2}	0.2477594×10^{-2}	–
1.0	0.4043740×10^{-2}	0.3858345×10^{-2}	0.4043728×10^{-2}	0.4043×10^{-2}

presented methods and some other methods for one year are shown in Tables 2–5. Similar numerical results are obtained from the two proposed methods in Table 2 clarifying that water pollution in all three lakes rises as time progresses. Figure 3(a) demonstrates the sinusoidal behavior of Lake 1. In this case, in analyzing tables and plots, we find that Lake 2 has low pollution levels compared to Lake 3, and both lakes' pollution levels are increasing exponentially.

Example 2: Exponential input model

In this example, we will consider a system of fractional order differential Eq. (1) with $p(t) = ae^{bt}$ with $a = 1$, $b = 1$, $u_1(0) = 0$, $u_2(0) = 0$, and $u_3(0) = 0$, and other parameters the same as in Example 1.

A numerical analysis is done for this case using MHFs and QHFs with $\alpha = 0.5, 1$, (see Tables 6 and 7). It seems that as alpha decreases from 1 down to 0.5, contamination increases on the time interval $[0, 1]$. The



(a) Lake 1 (b) Lake 2 (c) Lake 3

Figure 3. Distribution of pollution in the three lakes by MHFs and QHFs, for Example 1.

Table 5. Numerical results of the proposed and some other methods for the sinusoidal input of Lake 3 with $\alpha = 1$.

Time (in year)	MHFs	QHFs	Bessel polynomials	VPM
t	$n = 10$	$n = 10$	See [7], $n = 10$	See [8]
0.2	0.1468580×10^{-3}	0.1214503×10^{-3}	0.1468549×10^{-3}	–
0.4	0.6225887×10^{-3}	0.5630246×10^{-3}	0.6225828×10^{-3}	–
0.5	0.9997562×10^{-3}	0.9448416×10^{-3}	–	0.999×10^{-3}
0.6	0.1477775×10^{-2}	0.1375878×10^{-2}	0.1477766×10^{-2}	–
0.8	0.2758474×10^{-2}	0.2606978×10^{-2}	0.2758463×10^{-2}	–
1.0	0.4504327×10^{-2}	0.4297170×10^{-2}	0.4504314×10^{-2}	0.4504×10^{-2}

Table 6. Numerical results of exponential input for three lakes, $\alpha = 0:5$, $n = 10$.

Time (in year)	MHFs			QHFs		
t	Lake 1: $u_1(t)$	Lake 2: $u_2(t)$	Lake 3: $u_3(t)$	Lake 1: $u_1(t)$	Lake 2: $u_2(t)$	Lake 3: $u_3(t)$
0.0	0.00000000	0.00000000	0.00000000	0.0000000	0.0000000	0.0000000
0.1	0.38034144	0.5862×10^{-3}	0.6525×10^{-3}	0.38303751	0.6187×10^{-3}	0.6889×10^{-3}
0.2	0.57477572	0.1346×10^{-2}	0.1500×10^{-2}	0.55180692	0.1114×10^{-2}	0.1240×10^{-2}
0.3	0.75333265	0.2125×10^{-2}	0.2369×10^{-2}	0.75026575	0.2070×10^{-2}	0.2307×10^{-2}
0.4	0.93190143	0.2991×10^{-2}	0.3336×10^{-2}	0.89904555	0.2712×10^{-2}	0.3023×10^{-2}
0.5	1.11720443	0.3944×10^{-2}	0.4400×10^{-2}	1.10960000	0.3834×10^{-2}	0.4277×10^{-2}
0.6	1.31353450	0.4997×10^{-2}	0.5576×10^{-2}	1.26999927	0.4642×10^{-2}	0.5179×10^{-2}
0.7	1.52400197	0.6157×10^{-2}	0.6873×10^{-2}	1.51170788	0.5989×10^{-2}	0.6987×10^{-2}
0.8	1.75156787	0.7438×10^{-2}	0.8305×10^{-2}	1.69561933	0.6987×10^{-2}	0.7800×10^{-2}
0.9	1.99882170	0.8852×10^{-2}	0.9886×10^{-2}	1.98126556	0.8616×10^{-2}	0.9619×10^{-2}
1.0	2.26867203	0.1041×10^{-1}	0.1163×10^{-1}	2.19794126	0.9843×10^{-2}	0.1099×10^{-1}

results show that the pollution in all three lakes exhibit exponential behavior. As expected, the pollution levels in all three lakes are increasing, although Lake 2 appears to be lower pollution.

Example 3: Impulse input model

In this case, the contaminant has been released directly into Lake 1, (input impact method). In this example, we will consider a system of fractional order differential Eq. (1) with $p(t) = 100$ and other parameters the same as in Example 1.

Figure 4 shows the contamination results for different values of α based on MHFs and QHFs. According to Figure 4, the pollution behaviors of the three lakes using the MHFs and QHFs approaches are similar. When $\alpha = 1$, Lake 1 behavior would reflect the effects of a constant concentration of pollution increasing over time. Also, for $\alpha = 1$, Lakes 2 and

Table 7. Numerical results of exponential input for three lakes, $\alpha = 1$, $n = 10$.

Time (in year)	MHFs			QHFs		
t	Lake 1: $u_1(t)$	Lake 2: $u_2(t)$	Lake 3: $u_3(t)$	Lake 1: $u_1(t)$	Lake 2: $u_2(t)$	Lake 3: $u_3(t)$
0.0	0.00000000	0.00000000	0.00000000	0.0000000	0.0000000	0.0000000
0.1	0.10509874	0.3203×10^{-4}	0.3560×10^{-4}	0.10605399	0.3835×10^{-4}	0.4263×10^{-4}
0.2	0.22112299	0.1325×10^{-3}	0.1473×10^{-3}	0.21729710	0.1096×10^{-3}	0.1218×10^{-3}
0.3	0.34654789	0.3084×10^{-3}	0.3429×10^{-3}	0.34920205	0.2907×10^{-3}	0.3234×10^{-3}
0.4	0.49062641	0.5674×10^{-3}	0.5693×10^{-3}	0.48213712	0.5119×10^{-3}	0.5738×10^{-3}
0.5	0.64677541	0.9180×10^{-3}	0.1021×10^{-2}	0.63972225	0.8667×10^{-3}	0.9642×10^{-2}
0.6	0.81922527	0.1021×10^{-2}	0.1524×10^{-2}	0.80504959	0.1270×10^{-2}	0.1413×10^{-2}
0.7	1.00966072	0.1933×10^{-2}	0.2151×10^{-2}	0.99724405	0.1835×10^{-2}	0.2043×10^{-2}
0.8	1.22000692	0.2619×10^{-2}	0.2915×10^{-2}	1.19889530	0.2461×10^{-2}	0.2739×10^{-2}
0.9	1.45232162	0.3441×10^{-2}	0.3831×10^{-2}	1.43336338	0.3284×10^{-2}	0.3656×10^{-2}
1.0	1.70895624	0.4412×10^{-2}	0.4914×10^{-2}	1.67938243	0.4178×10^{-2}	0.4653×10^{-2}

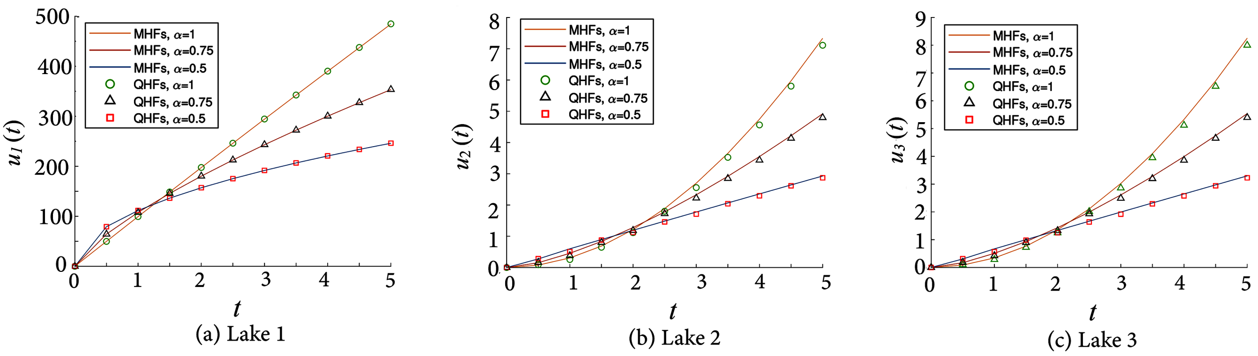


Figure 4. The behavior of pollution in each lake using MHFs and QHFs, for Example 3.

3 exhibit exponential pollution growth.

8. Conclusion

In this paper, two algorithms are implemented to solve a fractional model of pollution in a system of lakes. Applying the Modified Hat Functions (MHFs) and Quasi-Hat Functions (QHFs), algorithms have been developed for monitoring lakes' water pollution for the first time. Analyzing the method's absolute errors and convergence are addressed. A comparison of the numerical results of two proposed algorithms with the results of other numerical approaches confirms both their efficiency and accuracy. This fractional system is solved using MHFs and QHFs with $n/2$ algebraic systems with dimensions of 6×6 and 3×3, respectively. This is one of the benefits of these two algorithms, so the methods proposed are simple and computationally efficient for large values of n. The amount of pollution in Lake 1 is much higher than in Lakes 2 and 3 because it is the principal source of pollutants. Generally, the contamination levels are inversely related to alpha at the beginning, and over time, the pollution levels and alpha values become directly related. A variety of similar problems can be addressed with the proposed methods, especially QHF, and we intend to investigate these issues.

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