Reduced-Order Approximation of Bilinear Systems Using a New Hybrid Method based on Balanced Truncation and Iterative Rational Krylov Algorithms

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Abstract: In this work, a hybrid approach is proposed for the reduced order approximation of the bilinear system by combining the Balanced Truncation (BT) and Bilinear Iterative Rational Krylov Algorithm (BIRKA). Bilinear BT (BBT) has low accuracy but guarantees stability, while BIRKA convergence suffers from sensitivity to initial choice of reduced-order system. To start, the proposed approach minimizes the Integral Square Error (ISE) index to specify the order of the reduced bilinear approximation. To assurance BIRKA convergence, two approaches, BBT and Linear BT (LBT), are applied to prepare the initial guess of the reduced-order approximation. Although BBT prepare a good stable initial guess for BIRKA, solving the generalized Lyapunov equations to find the solution is very computationally expensive. The initial guess is provided by LBT through solving the Lyapunov equations, which decreases computational complexity. Furthermore, the eigenvalues are replaced by the condition number in BIRKA to decrease complexity. To verify the efficiency of the proposed approach, three bilinear test systems are being examined. Finally, the performance of the proposed approach is compared with several classical approaches. The finding indicate that the convergence probability of BIRKA increases. Also, the time for the determining the Model Order Reduction (MOR) decreases.

Keywords: Model order reduction; Bilinear system; Balanced truncation; Iterative rational Krylov algorithm; Monte Carlo simulation.

1. Introduction

Bilinear systems are a significant type of nonlinear systems that connects linear and nonlinear systems. In the literature, bilinear systems are extensively utilized to model many engineering and real-world systems, such as power systems [1], heat transfer [2], and electrical circuits [3]. Bilinear systems are used to approximate weakly nonlinear systems through Carleman bilinearization, which is one of its most important applications [4, 5]. Although, the approximation of nonlinear systems by bilinearization methods frequently results in a high-order model. Therefore, Model Order Reduction (MOR) of bilinear systems has been studied by scientists for analysis and control purposes. Linear MOR techniques, such as Proper Orthogonal Decomposition (POD) [6], BT, Krylov subspace methods [7, 8], and swarm intelligence-based methods [9], are the foundation of most bilinear MOR methods.

First, Hsu et al. used the BT method for MOR of bilinear systems [10]. In [10], the BT method applied earlier to linear systems was extended to bilinear systems. The main drawback of this method was the enormous computational load for computing the Gramians of controllability and the Gramians of observability. After that, many researchers focused on MOR of bilinear systems based on the BT method and improved it [11, 12]. However, the Bilinear BT (BBT) method still has disadvantages, such as high computational cost and relatively low accuracy.

The Krylov subspace was originally used for MOR of linear systems, but [13] extended its application to bilinear systems. Next, MOR of a Multi-Input Multi-Output (MIMO) bilinear system using the Krylov subspace was presented in [14, 15, 16]. There are several MOR methods based on projection techniques that have been proposed for bilinear systems [17, 18, 19]. In [17], a reduced bilinear system is formed to match the desired number of moments of multivariable transfer functions associated with the kernels of the Volterra series representation of the original system through one-sided projection. The efficiency and accuracy of the achieved reduced-order bilinear approximation based on a one-sided projection approach was increased by the two-sided projection method, which theoretically allows double the number of interpolated derivatives of the first two transfer functions [18]. In [19], the problem of full-state approximation by MOR for stochastic and bilinear systems is studied. It was proved that when dominant subspaces were identified based on reachability Gramians, the reduced-order bilinear approximation can be find using Galerkin projection.

In [20], the H_2-optimal MOR problem for the bilinear system was presented. In this method, the necessary conditions for H_2 optimality were achieved and then, the gradient flow technique was applied to optimize the H_2 error. Benner and Breiten [21] have proposed two algorithms for minimizing the H_2-error norm of MOR of bilinear systems. These algorithms are based on generalized Sylvester equations and the BIRKA. In [22], the new Volterra series interpolation framework was proposed and then the optimality conditions for H_2 were determined. Moreover, MOR was introduced for bilinear systems by computing controllability and observability Gramians at pre-determined frequency intervals to minimize the H_2 norm of the error [23]. The Truncated BIRKA (TBIRKA) is an improvement of the BIRKA method that uses the truncated Gramians to minimize the H_2 norm of the error system in order to decrease the computational volume of the BIRKA [24]. The stability analysis of the BIRKA and TBIRKA algorithms and the exactness of the reduced-order approximations obtained by these algorithms with three inexact solvers were studied [25-26]. The BIRKA demonstrated excellent performance in MOR of bilinear systems; however, there's no assurance of the algorithm's convergence [27].

In this paper, Monte Carlo simulations are used to show that by randomly choosing the starting point for the BIRKA, the probability of convergence of the bilinear system is low. Motivated by this sensitivity analysis and in order to guarantee the convergence, a new method is proposed for MOR of the bilinear system based on a combination of BT and BIRKA. First, the order of the reduced bilinear approximation is specified by minimizing the integral square error index based on the eigenvalues of the system matrix. Then, in order to increase the probability of convergence of BIRKA, two approaches are proposed to obtain a proper starting point of the reduced-order system for BIRKA. These approaches include the BBT and Linear BT (LBT) methods. The reduced-order system obtained with BBT has low accuracy but is stable and provides a good initial guess for BIRKA. This choice has been studied to determine whether it can lead to convergence of the BIRKA method. However, the solution of the generalized Lyapunov equations involves a high computational cost. It is noted that although decreasing the computational volume is very important in MOR, still achieving lower order models with appropriate accuracy in approximating the original system, especially in order reduction of high order controllers and high order systems that need to be run multiple times, are more important.

On the other hand, LBT provides a suitable starting point but proposes solving the Lyapunov equations for a solution that requires less computation. Using the reduced-order approximation achieved in the previous stage as the initial guess, BIRKA is implemented to achieve the final reduced-order bilinear system. Instead of eigenvalues, the condition number concept is utilized to further decrease the complexity of BIRKA. To demonstrate its capability, the proposed hybrid method have been used to approximate three test systems. These test systems are the model of Burgers' equation and the transmission line circuit model. Finally, the achieved results are compared with some famous MOR methods such as BBT, Bilinear POD (BPOD), and BIRKA to illustrate the proposed method's performance and capability. Moreover, the convergence rate and the order reduction duration time are evaluated.

The major contributions of this paper with regard to the relevant literature can be outlined as follows:

- Analyzing the properties and order reduction of the bilinear system model.
- Introducing a hybrid order reduction approach for bilinear systems using the combination of BBT and BIRKA to take advantages of both approaches.
- Studying of convergence conditions for BIRKA and the impact of the proposed method on the convergence of BIRKA.
- Increasing the chance of convergence of BIRKA by providing a suitable initial guess.
- Decreasing computational complexity or order reduction time by using condition number instead of eigenvalue vector and LBT instead of BBT.

This paper is divided into the following sections: Section 2 introduces MOR of bilinear systems as a general problem. Sections 3 and 4 present the basics of the BBT and BIRKA methods, respectively. Section 5 describes the proposed MOR method for bilinear systems. In section 6, the proposed MOR method reduces three high-order bilinear systems. The results show that the proposed MOR method outperforms other MOR methods such as BBT, BIRKA, and BPOD. Finally, section 7 concludes the paper.

2. MOR of Bilinear Systems

Below is a description of the Single-Input, Single-Output (SISO) bilinear system:

$$\zeta:\begin{cases}\dot{x}(t)=Ax(t)+Nx(t)u(t)+Bu(t)\\ y(t)=Cx(t)\end{cases} \tag{1}$$

where $A,N \in R^{n\times n}, B, C^T \in R^n$ are the matrices of the bilinear system, $x(t)\in R^n$ is the state vector, and $u(t)\in R$ and $y(t)\in R$ are the input and output of the bilinear system, respectively. Also, *n* represents the order of the original bilinear system.

Suppose that the Eq. (1) is of high order. MOR aims to create a system where both the original bilinear system and the reduced-order approximation have similar responses such that $y(t)$ almost equals $y_r(t)$ for all allowable inputs. Also, the reduced-order approximation has same form to Eq. (1). The representation shown below is for the reduced-order bilinear approximation:

$$\zeta_r:\begin{cases}\dot{x}_r(t)=A_r x_r(t)+N_r x_r(t)u(t)+B_r u(t)\\ y_r(t)=C_r x_r(t)\end{cases} \tag{2}$$

The unknown matrices $A_r, N_r \in R^{r\times r}, B_r, C_r^T \in R^r$ must be determined to achieve the reduced-order bilinear approximation. Also, the state vector is represented by $x_r(t)\in R^r$, while $u(t), y_r(t)\in R$ correspond to the input and output of reduced-order approximation of bilinear system, respectively. Also, the order of the reduced-order bilinear approximation is *r*, and it's smaller than *n*. If the original bilinear system is stable, then the reduced-order bilinear approximation must also be stable.

3. Balanced Truncation For Bilinear Systems

The Gramians of controllability for the bilinear system of Eq. (1) is given by [28]:

$$P = \sum_{i=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P_i P_i^T dt_1 \cdots dt_i \tag{3}$$

where

$$P_1(t_1) = e^{At_1} B,\ P_i(t_1, \cdots, t_i) = e^{At_i} N P_{i-1} \tag{4}$$

Also, the Gramians of observability for the bilinear system of Eq. (1) is provided as follows:

$$Q = \sum_{i=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} Q_i^T Q_i dt_1 \cdots dt_i \tag{5}$$

where

$$Q_1(t_1) = C e^{At_1},\ Q_i(t_1, \cdots, t_i) = Q_{i-1} N e^{At_i} \tag{6}$$

Theorem 1 [20]. Consider the bilinear system of Eq. (1) with a stable matrix *A*. If the controllability Gramian *P* of the system is defined as in Eq. (3) exists, then the Gramian *P* satisfies the generalized Lyapunov equation, given by

$$AP + PA^T + NPN^T + BB^T = 0 \tag{7}$$

Theorem 1 can be extended to apply to the observability Gramian, leading to the conclusion that solving the following generalized Lyapunov equation yields the observability Gramian:

$$A^T Q + AQ + N^T QN + C^T C = 0 \tag{8}$$

The generalized Lyapunov equations of Eq. (7) and Eq. (8) can be solved by the following iterative method [29].

First, the bilinear term is removed from Eq. (7). Consequently, the generalized Lyapunov equation is transformed into the subsequent Lyapunov equation:

$$A\hat{P}_1 + \hat{P}_1 A^T + BB^T = 0 \tag{9}$$

An initial solution for the generalized Lyapunov equation is obtained by solving the Lyapunov equation of Eq. (9). Then, the following iterative formula is used to determine the controllability Gramian in each iteration:

$$A\hat{P}_i + \hat{P}_i A^T + N\hat{P}_{i-1}N^T + BB^T = 0,\ i = 2,3,\cdots \tag{10}$$

The Gramian of controllability is finally specified in the following manner:

$$P = \lim_{i \to \infty} \hat{P}_i \tag{11}$$

The Gramian of observability can also be determined, similar to the Gramian of controllability. After computing the controllability and observability Gramians, similar to the standard BT method, the following stages can be performed to obtain the reduced-bilinear approximation:

Stage 1. Determine low-rank approximation of Gramians: $P \approx RR^T$ and $Q \approx SS^T$;

Stage 2. Compute SVD of $S^T R$ as follows:

$$S^T R = U\Sigma V = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T \tag{12}$$

The Σ_1 contains the *r* largest singular values of $S^T R$.

Stage 3. Construct the transformation matrices T_1 and T_2 as follows:

$$T_1 = SU_1\Sigma_1^{-\frac{1}{2}} \tag{13}$$

$$T_2 = RV_1\Sigma_1^{-\frac{1}{2}} \tag{14}$$

Stage 3. Multiplying the transformation matrices to the system of Eq. (1) to determine the reduced-order bilinear approximation:

$$A_r = T_2^T A T_1 \ , N_r = T_2^T N T_1 \ , B_r = T_2^T B \ , C_r = CT \tag{15}$$

4. Interpolation-Based H_2-Optimal Model Reduction of Bilinear System

A different approach for MOR is to computing models that satisfy H_2 optimality conditions.

4.1. H_2 Norm For Bilinear Systems

The H_2 norm of the original bilinear system of Eq. (1) is described as follows:

$$\|\zeta\|_{H_2}^2 = \sum_{k=1}^{\infty} \sup_{x_1>0,\cdots x_k>0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \|G_k\left(x_1+iy_1,\cdots,x_k+iy_k\right)\|_F^2 \, dy_1 \cdots dy_k \tag{16}$$

Where G_k is the transfer function representation of Eq. (1). In [20], it was proved that the H_2 norm of the bilinear system based on Gramians of Eq. (1) could be calculated as follows:

$$\|\zeta\|_{H_2}^2 = trace\left(CPC^T\right) = trace\left(B^T QB\right) \tag{17}$$

As mentioned in section 2, both the original bilinear system and the reduced-order bilinear approximation have almost same responses. In other words, when the following error index is minimized, the original bilinear system and the reduced-order bilinear system responses are equivalent:

$$E = \|y\left(t\right) - y_r\left(t\right)\|_{H_2}^2 \tag{18}$$

To avoid the time-consuming simulation of the original bilinear system, the error criterion below is used as an alternative to minimizing Eq. (18) [30]:

$$E = \|\zeta_{err}\|_{H_2}^2 = \|\zeta - \zeta_r\|_{H_2}^2 \tag{19}$$

To create ζ_{err}, following formulation of the error-system is constructed:

$$\zeta_{err} : \begin{cases} \dot{x}_{err}\left(t\right) = A_{err}x_{err}\left(t\right) + \left(N_{err}x_{err}\left(t\right) + B_{err}\right)u\left(t\right) \\ y_{err}\left(t\right) = C_{err}x_{err}\left(t\right) \end{cases} \tag{20}$$

Where

$$A_{err} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, N_{err} = \begin{bmatrix} N & 0 \\ 0 & N_r \end{bmatrix}, B_{err} = \begin{bmatrix} B \\ B_r \end{bmatrix}, C_{err} = \begin{bmatrix} C & -C_r \end{bmatrix}, x_{err} = \begin{bmatrix} x \\ x_r \end{bmatrix} \tag{21}$$

According to Eq. (17), the H_2 norm of the error-system can be determined as follows:

$$E = \|\zeta_{err}\|_{H_2}^2 = tr\left(C_{err} P_{err} C_{err}^T\right) = tr\left(B_{err}^T Q_{err} B_{err}\right) \tag{22}$$

Where P_{err} and Q_{err} are the Gramians of the error-system, respectively.

The H_2 optimality conditions of a bilinear system are determined by the following theorem [25]:

Theorem 2. Let ζ and ζ_r be the original system (1) and a reduced-order system (2), respectively. Then, the H_2-norm of the error system (20) can be given by

$$\begin{aligned} E &= vec\left(I_2\right)^T \left(\begin{bmatrix} C & -\tilde{C} \end{bmatrix} \otimes \begin{bmatrix} C & -C_r \end{bmatrix}\right) \times \\ &\left(-\begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes \begin{bmatrix} I_n & 0 \\ 0 & I_r \end{bmatrix} - \begin{bmatrix} I_n & 0 \\ 0 & I_r \end{bmatrix} \otimes \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} - \begin{bmatrix} N & 0 \\ 0 & \tilde{N} \end{bmatrix} \otimes \begin{bmatrix} N & 0 \\ 0 & N_r \end{bmatrix}\right)^{-1} \times \\ &\left(\begin{bmatrix} B \\ \tilde{B} \end{bmatrix} \otimes \begin{bmatrix} B \\ B_r \end{bmatrix}\right) vec\left(I_2\right) \end{aligned} \tag{23}$$

Where $\otimes$ denotes the Kronecker product, *vec* is vectorized operator, *I* is identity matrix, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{N}$ are the initial guesses for the reduced-order bilinear system. Also, $R\Lambda R^{-1}$ is the spectral decomposition of A_r, $\tilde{B} = R^{-1}B_r$, $\tilde{C} = CR$ and $\tilde{N} = R^{-1}N_r R$.

Theorem 3. [31] Let ζ be a bilinear system of order *n*. Let ζ_r be a H_2−optimal approximation of order *r*. Then, ζ_r satisfies the following multi-point Volterra series interpolation conditions:

$$\begin{aligned} &\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\phi_{l_1,l_2,\cdots,l_k} G_k\left(-\lambda_{l_1},-\lambda_{l_2},\cdots,-\lambda_{l_k}\right) = \\ &\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\phi_{l_1,l_2,\cdots,l_k} G_{r_k}\left(-\lambda_{l_1},-\lambda_{l_2},\cdots,-\lambda_{l_k}\right) \text{and} \\ &\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\phi_{l_1,l_2,\cdots,l_k} \left(\sum_{j=1}^{k}\frac{\partial}{\partial s_j} G_k\left(-\lambda_{l_1},-\lambda_{l_2},\cdots,-\lambda_{l_k}\right)\right) = \end{aligned} \tag{24}$$

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\phi_{l_1,l_2,\cdots,l_k}\left(\sum_{j=1}^{k}\frac{\partial}{\partial s_j}G_{rk}\left(-\lambda_{l_1},-\lambda_{l_2},\cdots,-\lambda_{l_k}\right)\right)$$

Where $\phi_{l_1,l_2,\cdots,l_k}$ and $\lambda_{l_1},\lambda_{l_2},\cdots,\lambda_{l_k}$ are residues and poles of the transfer function G_{rk} associated with ζ_r, respectively. It is impossible to obtain the poles and residues of the reduced H_2 model because this model is unknown. However, in [21], a new algorithm was proposed, which ensures the H_2 optimality conditions of theorem 3 are satisfied, provided that the algorithm is converging. This algorithm is called BIRKA. It should be noted that the BIRKA obtains the locally H_2 reduced model [27].

5. Proposed MOR Method

Consider the bilinear SISO system indicated by Eq. (1). It is assumed that this system is stable. The aim is to get a reduced-order bilinear approximation that has the identical structure as the original bilinear system in Eq. (2). For this purpose, a hybrid method based on the BBT method and the H_2-optimal model reduction is proposed. As mentioned earlier, the BBT method has some drawbacks, such as low accuracy and high computational cost, while the BBT ensures stability. On the other hand, the convergence of BIRKA is a major problem, while the reduced-order approximation achieved by BIRKA is accurate. Therefore, by combining these two methods, the advantages of both methods can be used and their disadvantages can be eliminated.

The desired order for the reduced-bilinear approximation of Eq. (1) is determined in the first step. To do this, the number of modes with the largest energy should be identified. Evaluating high-energy modes can be achieved through the use of dominant poles and Hankel singular values methods. The number of high energy modes is equal to the order of the reduced-model. Taking into account the real part of the eigenvalues of Eq. (1) is the first step in choosing an initial guess as an order for the reduced bilinear approximation. This approach determines the reduced-model order by finding the number of eigenvalues with the closest real part to the origin. The Choice of the initial order should be conservative. A bilinear MOR method such as BPOD [32] is used to decrease the one by one the initial order. A major challenge in the POD method is the selection of the snapshot data. To create a POD basis, snapshot data is typically collected from one or several runs of high-fidelity numerical simulations of the original system. The POD basis from snapshots can be generated more efficiently using a new partitioned method introduced in [33], which takes advantage of parallelism for computation.

Until the ISE index significantly increases, this procedure would be iteratively repeated. Therefore, the lowest order with negligible error is the most appropriate. It can be noted that to avoid the solution of the original systems in the ISE criterion, the response of the original system can be replaced by the response of the approximated system with the conservative order determined by the Hankel singular value.

The second step is to determine a suitable starting point for the BIRKA. It has already been mentioned, and will be shown in the next section using simulations that the convergence of BIRKA is sensitive to the initial guess for the reduced-order approximation. New conditions for the convergence of BIRKA are also addressed. Therefore, it is crucial for BIRKA to provide an appropriate initial guess of the reduced-order approximation. To this end, two method are proposed to determine the proper initial guess of the reduced-order approximation.

In the first approach, the BBT is used to the original bilinear system of Eq. (1). The reduced-order system obtained by BBT is relatively accurate, but its computational volume is high since it must solve the generalized Lyapunov equations to find its solution.

In the second approach, it is possible to use the LBT method instead of the BBT reduced model, since the initial guess for BIRKA is an approximate system. In this case, the bilinear term is eliminated. The advantage of this method over the previous method is the reduction in computational volume. That is, instead of solving the generalized Lyapunov equation, the Lyapunov equation is solved. Therefore, the computational complexity is decreased significantly.

Discussion: Determining the controllability and observability Gramians leads to an increase in the computational volume in BT method. In practice, low rank approximate Gramians can be used to decrease the volume of calculations. However, these Gramians do not guarantee the stability of the system. To solve this problem, the modified frequency limited BT method can be applied, which, in addition to using low-rank approximate Gramians to decrease the volume of computing, also makes the reduced-order approximation asymptotically stable under some mild conditions [34].

According to the characteristics of the reduced-order approximation achieved by BBT and LBT methods, a new suitable initial guess for the BIRKA algorithm is provided, which improve the convergence of BIRKA.

After determining the starting point in the previous stage, BIRKA is used to find the reduced-order bilinear approximation. In BIRKA, instead of the eigenvalues of A_r , the condition number of A_r is used to decrease the time of order reduction. The condition number is determined by dividing the largest singular value by the smallest singular value.

Thus, the proposed method can be summarized as below:

Step 1: In a conservative manner, consider the initial order based on the real part of the eigenvalues of Eq. (1). Decreasing each order sequentially and applying BPOD to each is done until a significant increase in the ISE index is observed. An appropriate order is the lowest with negligible error.

Step 2: Determine the initial guess of the reduced-order approximation for the BIRKA using one of the following proposed methods:

First approach: Apply the BBT method to reduce the original bilinear system as an initial reduced-order approximation in BIRKA.

Second approach: Apply the LBT method to reduce the original bilinear system as an initial reduced-order model in BIRKA. In this approach, the bilinear term is considered as a zero matrix with compatible size.

Step 3: Apply the BIRKA with the initial guess of step 2 to find the reduced-order approximation. To decrease the computational complexity of BIRKA, utilize condition numbers in place of eigenvalues.

Step 4: Simulate the obtained reduced-order approximation and examine its properties.

In the following section, simulations are utilized to exhibit the performance and properties of the proposed method.

The pseudocode of the proposed method is outlined in Table 1.

5.1. Convergence of proposed method

In [35] it is shown that BIRKA is convergent if the following relation holds:

$$\left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \left(\tilde{N}^T \otimes N \right) \right\|_2 < 1 \qquad (25)$$

Since the dimension of $\left(-I_r \otimes A - \Lambda \otimes I \right) \in \mathbb{R}^{m \times m}$ is large, it might not be feasible to directly calculate the inverse matrix due to memory limitations. Therefore, the Kronecker product calculation should be avoided. For this purpose, the estimation of Eq. (25) is introduced [35]:

$$\left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \left(\tilde{N}^T \otimes N \right) \right\|_2 \qquad (26)$$

$$\leq \left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \right\|_2 \left\| \left(\tilde{N}^T \otimes N \right) \right\|_2$$

$$\leq \left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \right\|_2 \left\| \left(\tilde{N}^T \right) \right\|_2 \left\| \left(N \right) \right\|_2$$

Therefore, if $\left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \right\|_2 \left\| \left(\tilde{N}^T \right) \right\|_2 \left\| \left(N \right) \right\|_2 < 1$, BIRKA is usable.

The following Lemma has been used to calculate $\left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \right\|_2$ without explicit inversion of the matrix [35].

Lemma 1. *For a normal matrix* M:

$$\left\| M^{-1} \right\|_2 = \frac{1}{min_{i=1\ldots n} \left| \lambda_i \left(M \right) \right|} \tag{27}$$

By using Lemma 1, the following proposition has been obtained for calculation of $\left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \right\|_2$.

Proposition 1. For $A \in \mathbb{R}^{n \times n}$, symmetric, $\Lambda = diag\left(\Lambda_1, \cdots, \Lambda_r \right)$:

$$\left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \right\|_2 = \frac{1}{\theta} \tag{28}$$

where

$$\theta = min_{k=1\ldots r} \left| \lambda_{min} \left(-A - \Lambda_k I \right) \right| \; for \; \Lambda_k \in \mathbb{R} \tag{29}$$

Eq. (28) shows that with increasing θ, $\left\| \left(-I_r \otimes A - \Lambda \otimes I \right)^{-1} \right\|_2$ decreases. Therefore, it should determine the upper and lower bounds of θ.

$$\theta = min_{k=1,\ldots,r} \left| \lambda_{min} \left(-A - \Lambda_k I \right) \right| = min_{k=1,\ldots,r} \left| \lambda_{max} \left(A + \Lambda_k I \right) \right| \tag{30}$$

$$\theta = min_{k=1,\ldots,r} \left| \lambda_{max} \left(A + \Lambda_k I \right) \right| \leq \sum_{i=1}^{n} \lambda_i \left(A + \Lambda_k I \right) = tr\left(A + \Lambda_k I \right) \tag{31}$$

$$\theta = min_{k=1,\ldots,r} \left| \lambda_{max} \left(A + \Lambda_k I \right) \right| \leq tr\left(A \right) + tr\left(\Lambda_k I \right) \tag{32}$$

Theorem 4. [36] $\rho(A) \geq \frac{1}{m}\left|tr(A)\right|$.

In theorem 4, m is rank of A and $\rho(A)$ is spectral radius of an arbitrary real matrix A which defined as follows:

$$\rho(A) \overset{\Delta}{=} max\left\{\left|\lambda\right|\right\} \tag{33}$$

Based on theorem 4 and the spectral radius of a matrix, the lower bound of θ is determined as follows:

$$\begin{aligned}\theta = min_{k=1,\ldots,r}\left|\lambda_{max}\left(A+\Lambda_k I\right)\right| &= min_{k=1,\ldots,r}\left|\rho\left(A+\Lambda_k I\right)\right| \\ \geq \frac{1}{m}\left|tr\left(A+\Lambda_k I\right)\right| &= \frac{1}{m}\left|tr(A)+tr\left(\Lambda_k I\right)\right|\end{aligned} \tag{34}$$

Therefore, it can conclude that θ will be in the following range.

$$\frac{1}{m}\left|tr(A)+tr(\Lambda_k I)\right| \leq \theta \leq \left(tr(A)+tr\left(\Lambda_k I\right)\right) \tag{35}$$

Theorem 5. [20] The Volterra series of bilinear system uniformly convergence on the interval $[0,\infty)$ for all bounded input if (i) A is stable, (ii) $\left\|N\right\| < \alpha / Z\beta$, where α and β are two positive scalar such that $\left\|exp(At)\right\| \leq \beta exp(-\alpha t)\quad t \geq 0$ holds and $Z \geq \left\|u(t)\right\|$.

The BIBO stability of the bilinear system in Eq. (1) can be considered by theorem 5, which requires that A is stable and N is bounded.

Because the reduced-order bilinear approximation achieved by the BBT is stable, it can conclude that $\left\|N_r\right\|$ is sufficiently bounded. Also, if $\left(\left|tr(A)+tr(\Lambda_k I)\right|\right) > m$, then $\left\|\left(-I_r \otimes A - \Lambda \otimes I\right)^{-1}\right\|_2 < 1$. On the other hand, due to stability reasons, $\Lambda_k < 0$. According to the BBT concept, the elements of Λ_k are the most important modes. Since the dimension of Λ is very small compared to the original system, and the elements of Λ are the most important eigenvalues of the original system, it can conclude that

$\left(\left|tr\left(A\right)+tr\left(\Lambda_k I\right)\right|\right)>m$ holds. Therefore, by choosing the initial reduced-order approximation by the BBT method, Eq. (26) is held, and the BIRKA is converged.

6. Simulation Results

In this section, three test systems are considered as high-order bilinear systems that should be approximated by the proposed method. The first test system is the Burgers' equation which is widely used in numerous fields of applied mathematics, such as fluid mechanics [37], heat transfer [38], and traffic flow problems [39]. The second test system is a nonlinear transmission line circuit.

6.1. Test system 1: Burgers' Equation

The one-dimensional viscid Burgers' equation is presented as follows [40]:

$$\begin{aligned}
&\frac{\partial w}{\partial t}+w\,\frac{\partial w}{\partial x}=\frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x}\right)\ \ for\ \left(x,t\right)\in\left(0,L\right)\times\left(0,T\right)\\
&w\left(x,0\right)=p\left(x\right)\quad \text{for}\,x\in\left(0,L\right)\\
&w\left(0,t\right)=u\left(t\right)\quad \text{for}\,t\in\left(0,T\right)\\
&w\left(L,t\right)=q\left(x\right)\quad \text{for}\,t\in\left(0,T\right)
\end{aligned}\tag{36}$$

where w is the spatial coordinate, t is the temporal coordinate, $w\left(x,t\right)$ is the speed of the fluid at the indicated spatial and temporal coordinates. Also, $\nu>0$ is the viscosity of the fluid. The viscosity of the fluid is a physical property that remains constant.

Applying an equidistant step size, a spatial discretization is applied to Eq. (36). Then, the spatial discretization of Burgers' equations of Eq. (36) has been approximated as a nonlinear state-space control system as follows:

$$
\frac{d}{dt}\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_i \\ \vdots \\ w_k \end{bmatrix} = \begin{bmatrix} -\frac{w_1 w_2}{2h} + \frac{\nu}{h^2}\left(w_2 - 2w_1\right) \\ -\frac{w_2}{2h}\left(w_3 - w_1\right) + \frac{\nu}{h^2}\left(w_3 - 2w_2 + w_1\right) \\ \vdots \\ -\frac{w_i}{2h}\left(w_{i+1} - w_{i-1}\right) + \frac{\nu}{h^2}\left(w_{i+1} - 2w_i + w_{i-1}\right) \\ \vdots \\ -\frac{w_N w_{N-1}}{2h} + \frac{\nu}{h^2}\left(-2w_k + w_{k-1}\right) \end{bmatrix} + \begin{bmatrix} -\frac{w_1}{2h} + \frac{\nu}{h^2} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \qquad (37)
$$

Where k is the number of interior points of the interval $\left(0, L\right)$, and h is step size considered as $h = \frac{L}{N+1}$.

As a typical choice, the number of interior points and the fluid viscosity for approximating of Burgers' equation with a nonlinear control system are considered to be 30 and 3, respectively, resulting in a nonlinear system of order 30. The complexity of high-order nonlinear control systems makes their analysis and design challenging. Hence, Carleman bilinearization is applied to Eq. (37) to achieve a bilinear form of Burgers' equations. In this case, with the choice mentioned earlier of the number of interior points, the order of the bilinear approximation is $k^2 + k = 930$. As seen, the bilinear model acquired from Carleman bilinearization is high-order and requires order reduction.

6.1.1. Sensitivity Analysis of BIRKA

Although BIRKA is one of the powerful approaches for MOR of bilinear systems, in some cases, BIRKA does not roughly converge to the desired answer. This section shows that proper selection of the initial guess for BIRKA can significantly affect the convergence. To this end, the bilinear model of Burgers' equation is reduced for several initial guesses using BIRKA. Fig. 1 shows the BIRKA results for the three different randomly selected initial guesses. Also, Fig. 2 presents the absolute error for three different initial guesses of BIRKA. In this analysis, the input of the bilinear Burgers' equation is $u(t) = \exp(-t)$. According to Figs. 1 and 2, it is clear that the convergence and accuracy of the BIRKA depend on the initial guess.

For further analysis, Monte Carlo simulations have been used to evaluate the success rate of the BT method instead of the random values of the initial value. The simulation was performed for 500 runs. The mean of generalized Sylvester equation solutions is shown in Fig. 3. In this figure, two bounds of responses are distinguished. The lower bound corresponds to the convergent responses, while the upper bound is dedicated to the non-converging ones. Based on the Monte Carlo simulation, the algorithm's success rate is 41%. It means, in MOR of the bilinear system of Burgers’ model by the BIRKA, there is a 41% chance of selecting the initial values that will lead to the convergence of the BIRKA.

6.1.2. Approximation of Test system 1

In the subsequent steps, the bilinear model of Burgers' would be reduced using the proposed method:

Step 1: In this step, the suitable order of the reduced bilinear model is specified. To do this, it should specify an initial order. It should be emphasized that the system's crucial modes are those whose real eigenvalues are near the origin. These are some effective eigenvalues of the system: -7.41, -66.69, -185.25, -363.09, and -600.21. As a pessimistic choice, the initial order for the reduced-order approximation was 10. Then, the reduced-order bilinear approximation with orders 10, 9, …, 1 are determined by the bilinear proper orthogonal decomposition method. Then, for each reduced-order system with a different order, the ISE criteria are calculated. Fig. 4 shows the ISE index values that change with the order of the system. The results show that order 2 is acceptable.

Step 2: The initial guess for the reduced-order system is determined by both approaches proposed in the previous section, based on the obtained order of the reduced bilinear model.

In the first approach, the BBT method is used to order reduction of the bilinear form of Burgers' equation as follows:

$$\dot{x}_{r1}(t)=\begin{bmatrix}-7.4672 & -2.7663\\ -1.3793 & -77.7916\end{bmatrix}x_{r1}+\begin{bmatrix}8.6100\times10^{-6} & 2.2408\times10^{-5}\\ 9.4358\times10^{-5} & 2.4557\times10^{-4}\end{bmatrix}x_{r1}u+\begin{bmatrix}3.0383\times10^{4}\\ 3.3297\times10^{5}\end{bmatrix}u \tag{38}$$

$$y_{r1}(t)=\begin{bmatrix}4.4136\times10^{-4} & -2.46\times10^{-4}\end{bmatrix}x_{r1}$$

In the second approach, the bilinear term of Burgers' model is considered as zero. Also, the bilinear term of the initial guess of the reduced-system is considered as a zero matrix. Then, the standard LBT is applied to the linear system of Burgers' equation as follows:

$$\dot{x}_{r1}(t)=\begin{bmatrix}-3.909 & 11.06\\ -11.06 & -34.23\end{bmatrix}x_{r1}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}x_{r1}u+\begin{bmatrix}2.134\\ 2.534\end{bmatrix}u$$

$$y_{r1}(t)=\begin{bmatrix}2.134 & -2.534\end{bmatrix}x_{r1} \tag{39}$$

Step 3: The reduced-order approximation acquired by any proposed approaches is applied to BIRKA as a starting point. Then, the BIRKA is used to obtain the reduced-order approximation of the bilinear system of Burgers' equation as follows:

$$\dot{x}_{r2}(t)=\begin{bmatrix}-40.9143 & 14.2829\\ -12.2869 & -2.6459\end{bmatrix}x_{r2}+\begin{bmatrix}0.0292 & 0.0164\\ 0.0033 & 0.0019\end{bmatrix}x_{r2}u+\begin{bmatrix}-904.2622\\ -103.6879\end{bmatrix}u$$

$$y_{r2}(t)=\begin{bmatrix}-0.0020 & 0.0398\end{bmatrix}x_{r2} \tag{40}$$

It can be noted that, because the main basis of the MOR method is the BIRKA and only the initial value of the reduced-order model changes, the obtained model is the same in both approaches. The only element that changes is the order reduction time and the convergence rate of the method.

For further analysis, the convergence conditions have been checked for this test system numerically. In the Burgers' test system, the trace of A is -4.8339×10^{9}, the norm of the bilinear term is 931, and $m = m = 2\times930 = 1860$. Also, using the BBT as an initial guess for the BIRKA, the trace of Λ is -85.2587, and the norm of the reduced-order bilinear term is 2.6417×10^{-4}. It can be shown that the proposed convergence conditions, i.e., $\left(\left|tr\left(A\right)+tr\left(\Lambda_k I\right)\right|\right) > m$ and sufficiently bounded of $\|N\|$ and $\|N\|_r$, are validated.

Fig. 5 depicts the response of the obtained reduced bilinear approximation to input $u\left(t\right)=exp\left(-t\right)$, which has been compared with several famous MOR methods such as BBT, BIRKA, and BPOD methods. Furthermore, the time evaluation of the absolute error is also depicted in Fig. 6. Compared to other methods, the proposed method's results satisfactorily match the original system's response, with a smaller error than the others.

In addition, the response's significant characteristics, such as order reduction time, peak value, steady-state value, and ISE index, are compared for quantitative and numerical evaluation between the obtained results and the mentioned MOR approaches. These characteristics are investigated in Table 2. Figs. 5-6 and Table 2 confirm that the reduced-order bilinear approximation is the most accurate approximation among the other approaches. Hence, the proposed approximation maintains almost all the features of the original bilinear system, while its convergence is significantly faster than that of BIRKA. Besides, according to Table 2, a combination of LBT+BIRKA is used in the proposed method. The total time for order reduction has been significantly reduced, while the properties of the reduced model have remained the same. It is expected that the time of order reduction for BBT+BIRKA will be relatively long because of the need to solve generalized Lyapunov equations. On the other hand, LBT+BIRKA has a faster convergence than the BBT-BIRKA, since the Lyapunov equations are solved instead of the generalized Lyapunov equations.

Another enhancement in this paper is the utilization of the condition number instead of eigenvalues in BIRKA in the proposed approach. This change decreases the computational cost, as reported in Table 3. In this table, the computational properties of the proposed method have been brought into more detail. As can be seen in BBT+ BIRKA, the total elapsed time is higher due to the need to solve the generalized Lyapunov equation. On the other hand, employing LBT in conjunction with the condition number instead of the eigenvalues in the BIRKA reduces the total order reduction time and its number of iterations to 26.5274 and 22, respectively.

For further investigation, the input of the Burgers' equation is changed to $u(t)=e^{-1.5t}\sin(10\mathrm{t})\cos(t)$. Figs. 7 and 8 illustrate an evaluation between the responses of the reduced-order bilinear model and the classical approaches. It can be observed that the proposed approximations are closer to the original model of the Burgers’ equation than other methods when the input is changed.

It can be deduced from the results that BIRKA's convergence rate improved from 41% to 100%. In other words, when the initial guess in the BIRKA algorithm comes from the BT methods, the convergence probability is 59% higher than when the initial guess is chosen randomly. On the other hand, Table 2 illustrates that the simulation time increases notably in cases where the BIRKA algorithm fails to converge. Thus, the proposed method decreases the order reduction time as an index of computational complexity compared to BIRKA when it does not converge.

6.2. Test System 2: Nonlinear Transmission Line Circuit

The nonlinear transmission line circuit is another standard test system used to evaluate MOR methods [17]. The following is a presentation to the state-space form of the nonlinear transmission line circuit:

$$\dot{x}(t)=\begin{bmatrix} -g(x_1)-g(x_1-x_2) \\ -g(x_1-x_2)-g(x_2-x_3) \\ \vdots \\ -g(x_{k-1}-x_k)-g(x_k-x_{k+1}) \\ \vdots \\ -g(x_{v-1}-x_v) \end{bmatrix}+\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}u(t) \tag{41}$$

$$y(t)=\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}x(t)$$

where $x \in R^{v\times 1}$ is the state variables, $f \in R^{v}$ is nonlinear state evolution function, $b \in R^{v\times 1}$ and $c \in R^{1\times v}$ are input and output, respectively. For the transmission line circuit, v is equal to 20, which represents the number of nonlinear resistors. Also, the voltage and current of each resistor depend on each other as follows:

$$g(x)=exp(x)+x-1 \tag{42}$$

The nonlinear transmission line model is converted into the bilinear model using the Carleman bilinearization method. The acquired bilinear model has an order of $v^2+v=420$. The bilinear model obtained through Carleman bilinearization has a high order and requires order reduction.

6.2.1. Sensitivity Analysis of BIRKA

In this subsection, to reduce the order of the high-order bilinear model of the transmission line circuit, the BIRKA method is used. Similar to the proposed procedure in test system 1, the order of the reduced approximation is considered as 3. It has been observed that the BIRKA for MOR of the bilinear transmission line system has a convergence rate of 82% when the initial guesses of the reduced-model for the BIRKA are randomly chosen, and the simulation is repeated 50 times. Therefore, an increase in the convergence rate of BIRKA has been achieved by implementing the proposed method.

6.2.2. Approximation of the Test system 2

In this subsection, the proposed approach is applied to the bilinear model of the transmission line circuit. The reduced-order bilinear approximation obtained by the proposed approach is given by

$$
\begin{aligned}
\dot{x}_r(t) &= \begin{bmatrix} -106.3575 & 36.1771 & 3.2816 \\ 75.7145 & -91.0158 & 36.8621 \\ -24.5642 & 45.0254 & -52.8670 \end{bmatrix} x_r + \\
&\quad \begin{bmatrix} -0.6687 & -1.4783 & 0.6278 \\ 0.2579 & 0.5673 & -0.2630 \\ -0.0530 & -0.2633 & -0.2414 \end{bmatrix} x_r u + \begin{bmatrix} -0.2396 \\ -1.0467 \\ 0.1906 \end{bmatrix} u \qquad (43) \\
y_r(t) &= \begin{bmatrix} -0.2967 & -0.5946 & 0.3389 \end{bmatrix} x_r
\end{aligned}
$$

Similar to test system 1, the achieved reduced-order approximation is compared with some well-known MOR methods such as BPOD, BBT, and BIRKA. In Fig. 9, responses of reduced-order bilinear approximations to input $u(t) = e^{-1.5t}\sin(10\mathrm{t})\cos(t)$ are depicted. Also, Fig. 10 illustrates the absolute error as it varies with time.

Table 4 presents a comparison of characteristics of responses, including final value, ISE index, peak value, and order reduction time.

The proposed methods and BIRKA results are similar to the high-order bilinear model of the transmission line circuit, as shown in Figs 9 and 10 and Table 4.

It can be concluded that the probability of BIRKA convergence rate increases from 82% to 100% success by the proposed methods. In other words, the probability of convergence of BIRKA by the proposed method has increased by 18% compared to the original BIRKA.

Again, similar to test system 1, the convergence conditions have been numerically evaluated for this system. In this example, the trace of A is -6.5559×10^{4}, the norm of the bilinear term is 2, and $m = rn = 2\times420 = 840$. Also, using the BBT as an initial guess for the BIRKA, the trace of Λ is -4.0769, and the norm of the reduced-order bilinear term is 0.2409. It can be shown that the proposed convergence conditions, i.e., $\left(\left|tr\left(A\right)+tr\left(\Lambda_k I\right)\right|\right) > m$ and sufficiently bounded of $\|N\|$ and $\|N\|_r$ are validated.

It can be noted that because the proposed methods are based on BIRKA, the reduced-order model determined by BBT+BIRKA and LBT+BIRKA are identical. The convergence and success rates of the algorithm are the only distinguishing factors between the methods. In Fig. 11, the norm of the solution of the generalized Sylvester equations for the proposed methods and 50 times runs of BIRKA are demonstrated.

It is seen that the proposed methods lead to a fast and reliable determination of the generalized Sylvester equations rather than random initial guesses.

Finally, the impact of the condition number concept instead of eigenvalues in BIRKA for test system 2 is presented in Table 5.

Similar to test system 1, it can be seen that condition number increases the convergence rate of the BIRKA. Also, the proposed methods led to the convergence of the BIRKA.

Remark: It should be noted that sometimes the MOR methods such as BT, BIRKA, and BPOD may be longer than the simulation time of the original system. However, in order reduction of controllers and reduced-order approximation of the high-order system which requires several runs, the accuracy of the lowest-order model is more important. It should be also noted that the MOR is commonly an offline procedure and once the reduced-order approximation is obtained, the simulation time is drastically reduced.

6.3. Test System 3

To illustrate the effect of condition number instead eigenvalues on BIRKA, the following bilinear system is considered [41]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ N_1 & 0 \end{bmatrix} x(t) u(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & C_2 \end{bmatrix} x(t) \end{aligned} \tag{44}$$

Where $A_1 \in R^{100\times100}$, $A_2 \in R^{100\times100}$, $B_1 \in R^{100\times1}$ and $C_2 \in R^{1\times100}$

$$\begin{aligned} A_1 &= \begin{bmatrix} -10 & 2 & & & \\ 7 & -10 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & & 7 & -10 \end{bmatrix}, A_2 = \begin{bmatrix} -5 & 2 & & & \\ 2 & -5 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & & 2 & -5 \end{bmatrix} \\ N_1 &= \begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T, C_2 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \end{aligned} \tag{45}$$

To fair comparison, the initial guess for both methods should be identical. The reduced-order bilinear approximation as a starting point for the BIRKA which determined by both methods is:

$$\begin{aligned} \dot{x}_r(t) = &\begin{bmatrix} -1.0124 & -4.12\times10^{-15} \\ -4.13\times10^{-15} & -1.0505 \end{bmatrix} x_r + \begin{bmatrix} 2.16\times10^{-13} & 2.0052 \\ -2.34\times10^{-26} & -2.16\times10^{-13} \end{bmatrix} x_r u + \\ &\begin{bmatrix} -1.08\times10^{-12} \\ -10.0004 \end{bmatrix} u \end{aligned} \tag{46}$$

$$y_r(t) = \begin{bmatrix} -9.9336 & 1.07\times10^{-12} \end{bmatrix} x_r$$

Simulation times for achieving the reduced-order bilinear model for original BIRKA and BIRKA equipped with condition number are 0.5596 and 0.2868, respectively. Therefore, it can be concluded that when the condition number is used instead of eigenvalues, the simulation time is decreased.

It should be noted that the responses of reduced-order bilinear system for test system 3 for original BIRKA and BIRKA equipped with condition number are identical.

For further analysis, responses of high-order bilinear test system 3 and their reduced-order approximation which determined by proposed method and some classical methods such as BT and BPOD to input $u(t) = \exp(-t)$ is shown in Fig. 12. Similar to the previous examples, the time evolution of the absolute error is illustrated in Fig. 13. Table 6 provides a comparison of some response features for numerical evaluation. Because the reduced-order bilinear approximation achieved by both methods is the same, the presented results are presented only for one of the methods.

According to Figs. 12-13 and Table 6 it can be confirmed that the proposed method is highest match to the bilinear test system 3.

7. Conclusions

A new approach for MOR of bilinear systems has been proposed in the paper, which is a hybrid approach combining BT and BIRKA. First, the ISE index and BPOD are applied to determine the appropriate order for the reduced-order approximation. Since BIRKA is very sensitive to the initial guess of the reduced-order approximation, two approaches have been proposed to determine the suitable initial guess for BIRKA. These are the BBT method and the LBT method. The initial guess of the reduced-order approximation proposed by these approaches ensures the convergence of BIRKA. It has been studied and confirmed that the algorithm converges when the BBT method's reduced-order approximation is used as the initial guess for BIRKA. As a result, the convergence rate of the proposed method compared to BIRKA has increased. In addition, by employing the LBT+ BIRKA, the overall reduction time has been decreased drastically. As a further improvement, the eigenvalues in BIRKA has been replaced by condition number. As a result, the time of order reduction decreased and the convergence rate increased significantly again. Three bilinear test systems were approximated and evaluated with well-known MOR approaches such as BBT, BIRKA, and BPOD to verify the efficiency and capability of the proposed approach. The results indicate a significant improvement in the BIRKA's convergence rate and probability of success.

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Biographies

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Nooshin Bigdeli was born in 1978 in Iran, and completed her Ph.D. degree in Electrical Engineering majoring in Control at Sharif University of Technology, Tehran, Iran in 2007. She is currently professor of Control Engineering Department of Imam Khomeini International University, Qazvin, Iran. Her research interests include control systems, intelligent systems, chaos control, model predictive control as well as model order reduction in high order systems.

Figure and Table Captions

Fig. 1. Comparison of responses of the bilinear model of Burgers' equation for three different initial guesses for BIRKA.

Fig. 2. Time evolution of absolute error of different initial guesses for Burgers' equation approximations by BIRKA.

Fig. 3. Mean of generalized Sylvester equation solutions for 500 independent runs.

Fig. 4. Varying the ISE index according to order of reduced bilinear model of Burgers' equation

Fig. 5. Comparison of responses of the bilinear model of Burgers' equation and their reduced-order model approximations.

Fig. 6. Time evolution of absolute error of various methods for Burgers' equation approximations

Fig. 7. Comparison of responses of the bilinear system of Burgers' equation and their reduced-order models approximations for another input

Fig. 8. Time evolution of absolute error of various methods for Burgers' equation approximations for another input

Fig. 9. Comparison of responses of the bilinear model of transmission line circuit and their reduced-order model approximations

Fig. 10. Time evolution of absolute error of various methods for bilinear transmission line circuit approximations

Fig. 11. Convergence of norm of solution of generalized Sylvester equation for BBT+BIRKA, LBT+BIRKA and 50 times BIRKA runs with random initial reduced-order system for test system 2

Fig. 12. Comparison of response of the bilinear test system 3 and their reduced-order model approximations

Fig. 13. Time evolution of absolute error of various methods for test system approximations

Table 1. Pseudocode of the proposed method

Table 2. Comparison of methods for Burgers' equation approximations.

Table 3. Comparison of order reduction time and convergence rate of proposed methods and condition number effect for Burgers' equation approximations

Table 4. Comparison of methods for approximations of test system 2

Table 5. Comparison of order reduction time and convergence rate of proposed methods and condition number effect for approximations of test system 2

Table 6. Comparison of methods for approximations of test system 3

Figures

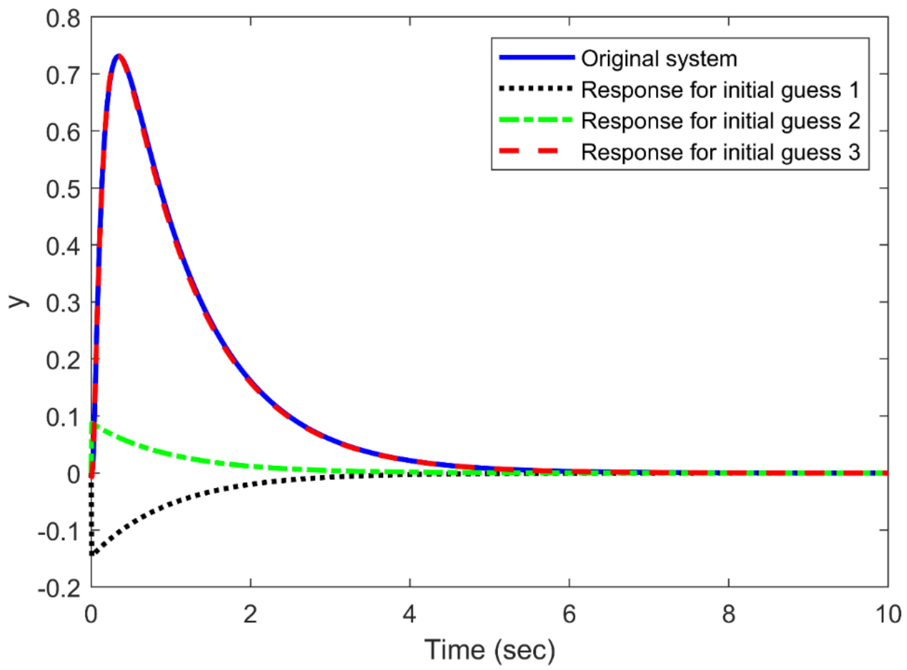


Fig. 5

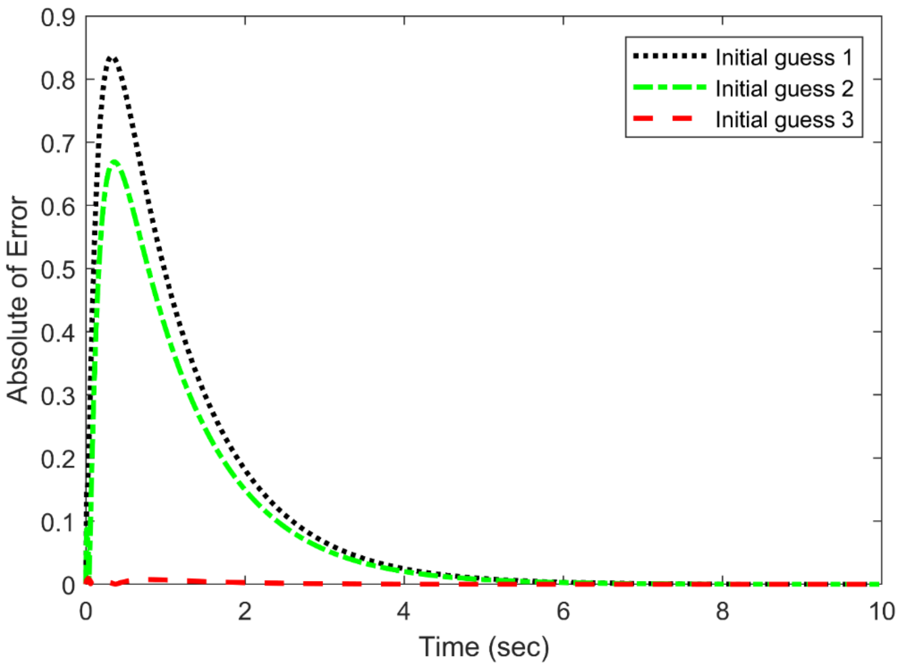

0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Absolute of Error
0
2
4
6
8
10
Time (sec)
Initial guess 1
Initial guess 2
Initial guess 3

Fig. 6

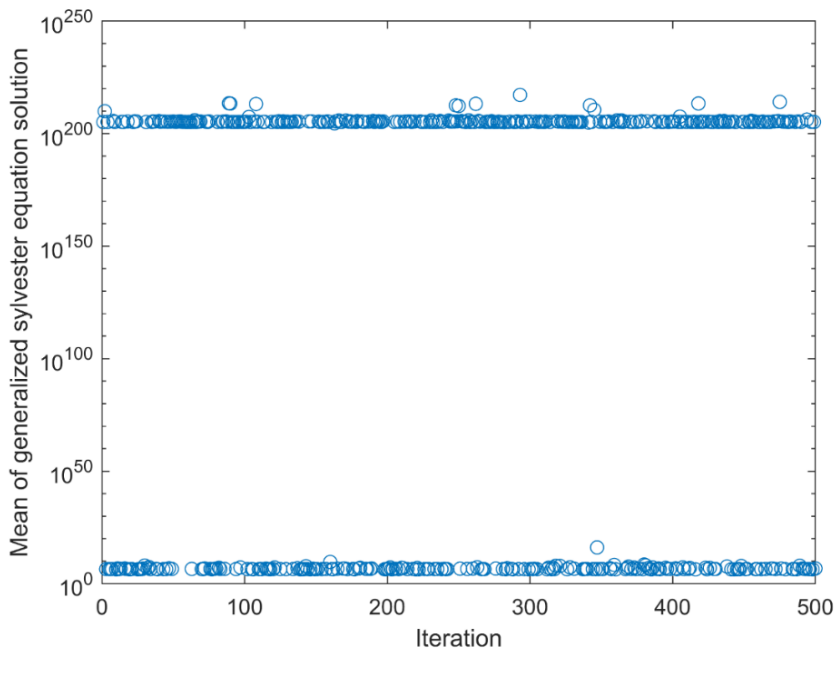
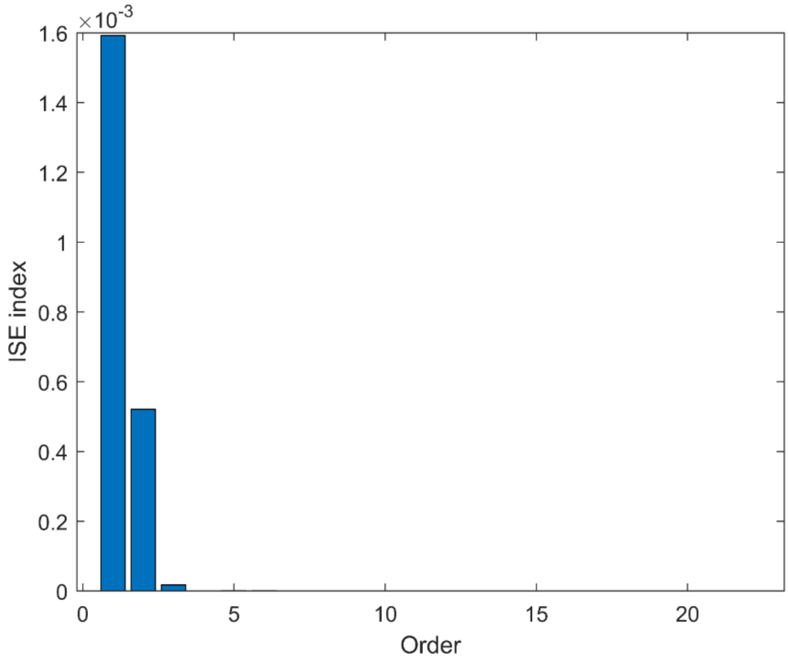
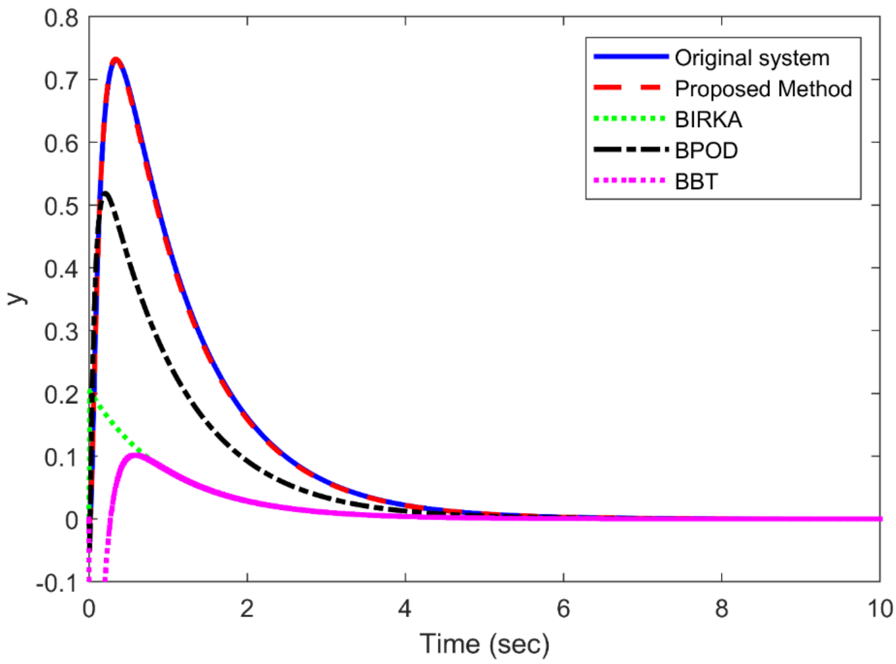

10^{250}
10^{200}
10^{150}
10^{100}
10^{50}
10^{0}
Mean of generalized sylvester equation solution
0
100
200
300
400
500
Iteration

Fig. 7



×10^-3
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
ISE index
0
5
10
15
20
Order

Fig. 8



Original system
Proposed Method
BIRKA
BPOD
BBT
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
y
0
2
4
6
8
10
Time (sec)

Fig. 5

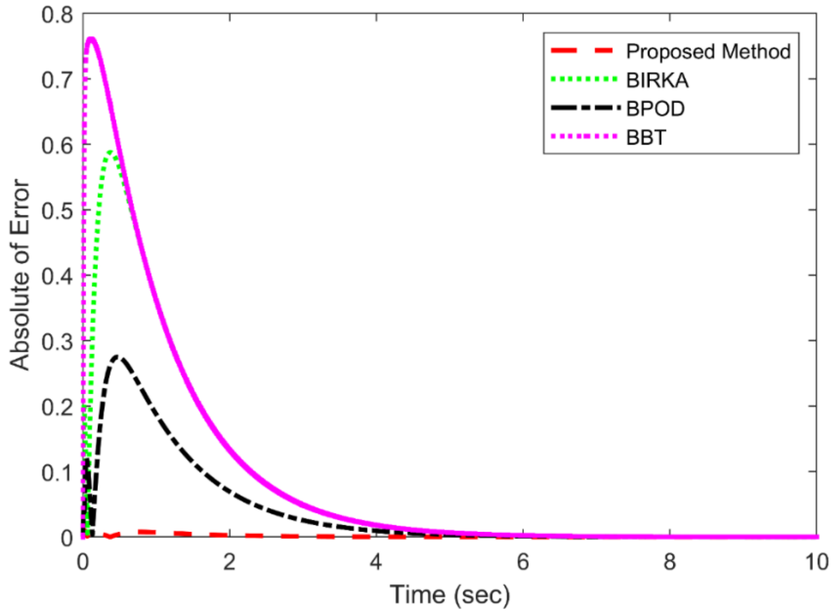

Proposed Method
BIRKA
BPOD
BBT
Absolute of Error
Time (sec)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10

Fig. 6

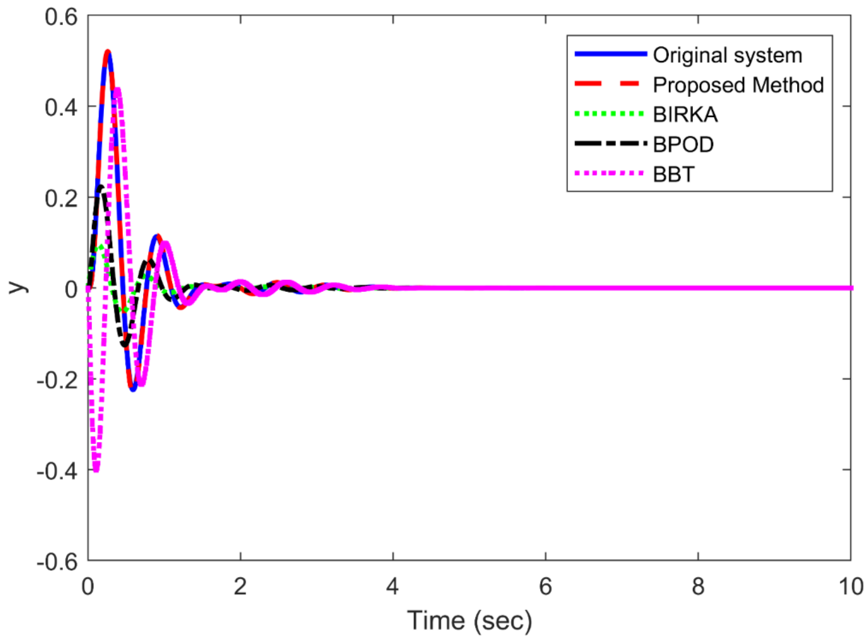

Original system
Proposed Method
BIRKA
BPOD
BBT
y
Time (sec)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
2
4
6
8
10

Fig. 7

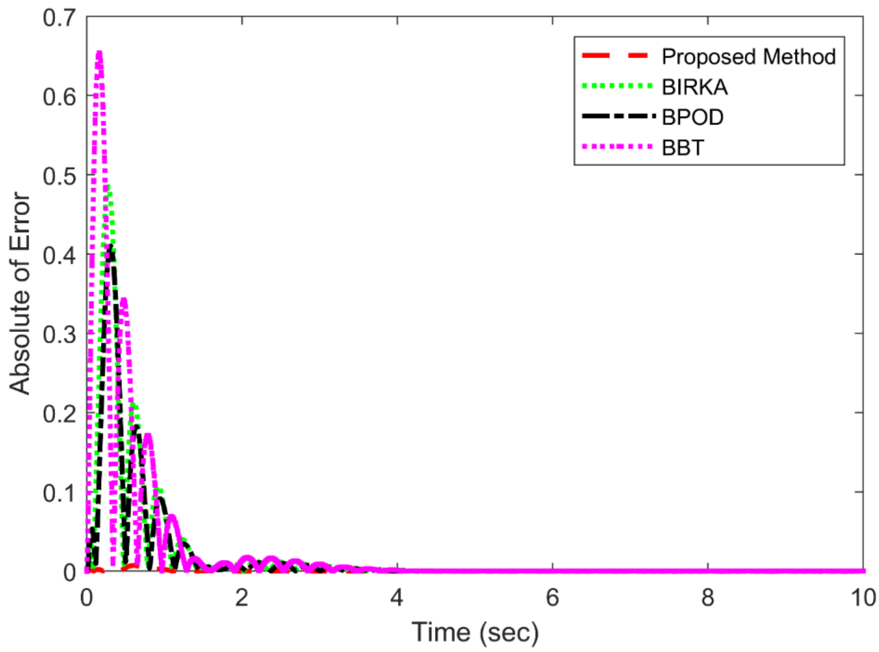

Proposed Method
BIRKA
BPOD
BBT
Absolute of Error
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
Time (sec)

Fig. 8

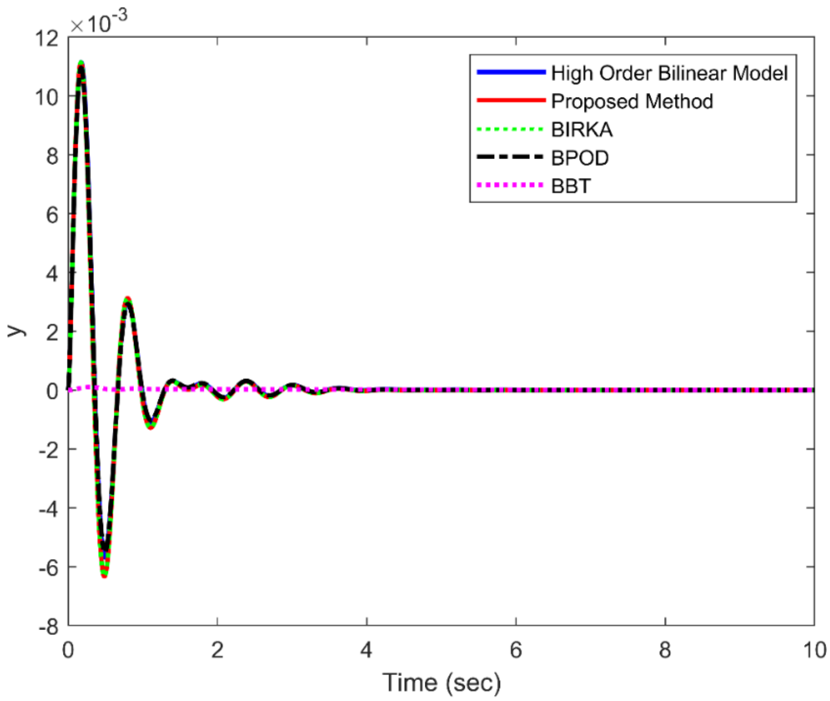
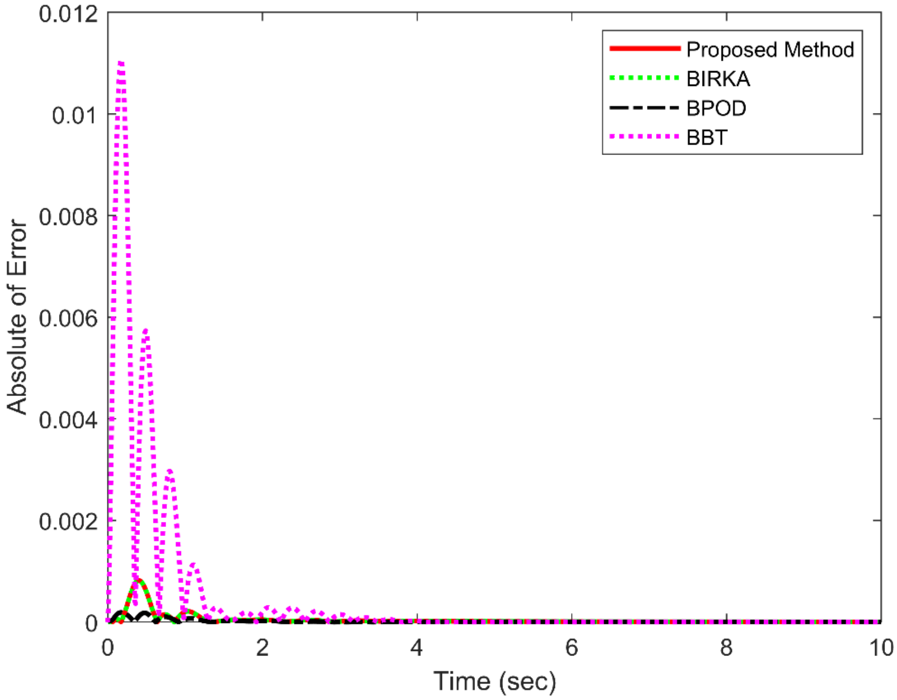
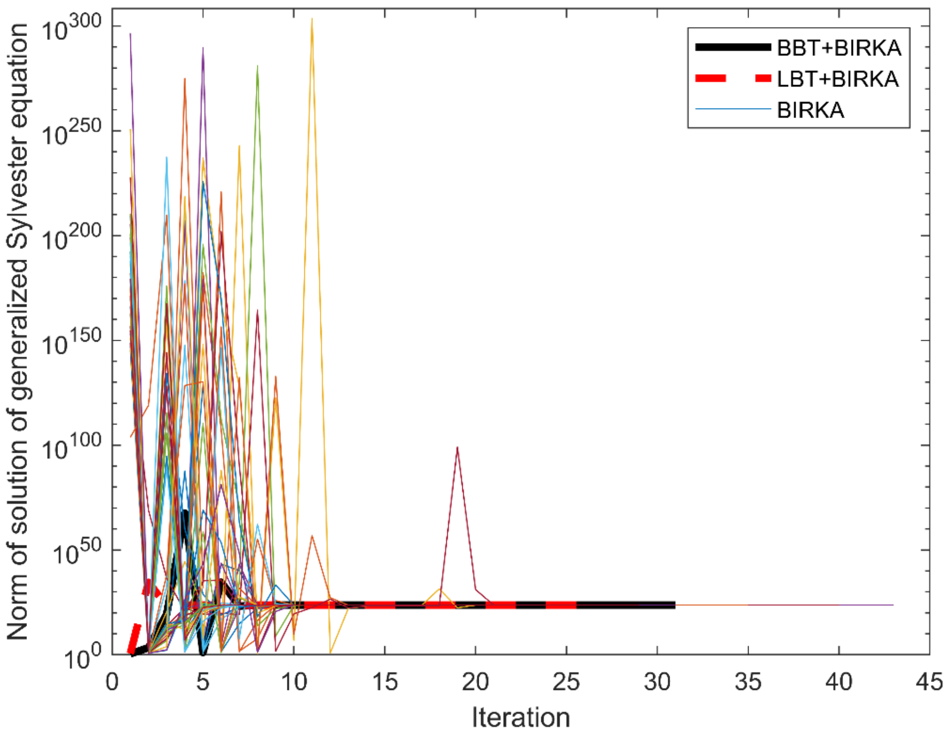

×10-3
High Order Bilinear Model
Proposed Method
BIRKA
BPOD
BBT
y
12
10
8
6
4
2
0
-2
-4
-6
-8
0
2
4
6
8
10
Time (sec)

Fig. 9



0.012
0.01
0.008
0.006
0.004
0.002
0
Absolute of Error
0
2
4
6
8
10
Time (sec)
Proposed Method
BIRKA
BPOD
BBT

Fig. 10



10^{300}
10^{250}
10^{200}
10^{150}
10^{100}
10^{50}
10^{0}
Norm of solution of generalized Sylvester equation
0
5
10
15
20
25
30
35
40
45
Iteration
BBT+BIRKA
LBT+BIRKA
BIRKA

Fig. 11

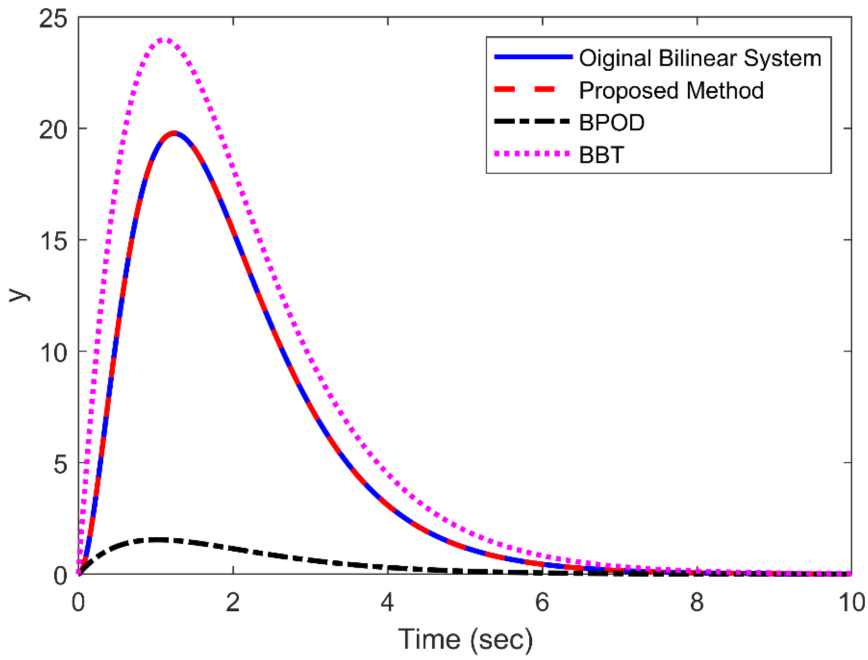


Fig. 12

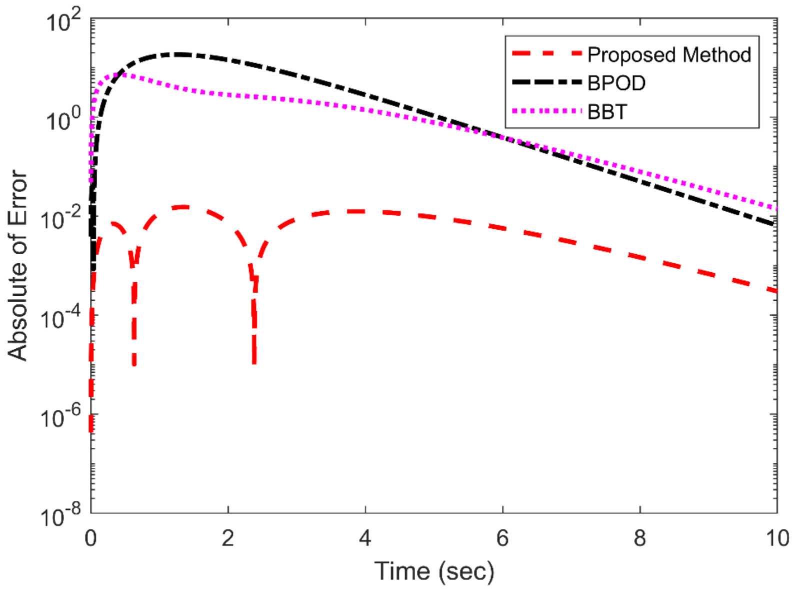


Fig. 13

Tables

Table 1

Input: The system matrices: A, N, B, C, *tol1*, *tol2*
Determine the order of reduced-model, r
Make an initial guess for order of reduced model based on HSV of original system, $\acute{r}$
while error criteria <*tol1* convergence **do**
Obtain reduced-order model by BPOD with order of $\acute{r}$

$\acute{r} = \acute{r} - 1$

achieve the order of reduced-order model, r

end **while**

Make an initial guess of Λ, $\widetilde{N}$, $\tilde{B}$ and $\tilde{C}$ by BT method

Determine low-rank approximation of Gramians

$$P \approx RR^T \text{and } Q \approx SS^T$$

Compute SVD of $S^T R$ as follows:

$$S^T R = U\Sigma V = [U_1 \quad U_2]\begin{bmatrix}\Sigma_1 & 0 \\ 0 & \Sigma_2\end{bmatrix}[V_1 \quad V_2]^T$$

The Σ_1 contains the r largest singular values of $S^T R$

Construct the transformation matrices T_1 and T_2:

$$T_1 = SU_1\Sigma_1^{-\frac{1}{2}}, \qquad T_2 = RV_1\Sigma_1^{-\frac{1}{2}}$$

Determine the reduced-order bilinear model as an initial guess for the BIRKA

$$A_{r1} = T_2^T A T_1 \,, N_{r1} = T_2^T N T_1 \,, B_{r1} = T_2^T B \,, C_{r1} = CT$$

$$A_{r1} =: R\Lambda R^{-1},\ \tilde{B} = R^{-1}B_{r1}, \tilde{C} = C_{r1}R, \widetilde{N} = R^{-1}N_{r1}R$$

while relative change in $\{\kappa_i\} > tol2$ convergence **do**

Solve for V and W:

$V(-\Lambda) + AW + NV\widetilde{N}^T + B\tilde{B}^T = 0$

$W(-\Lambda) + A^T W + N^T W\widetilde{N} + C^T\tilde{C} = 0$

Perform:

$V = orth(V)$ and $W = orth(V)$

Compute the reduced matrices:

$A_r = (W^T V)^{-1}W^T AV,\ N_r = (W^T V)^{-1}W^T NV$

$B_r = (W^T V)^{-1}W^T B, \quad C_r = CV$

Determine the spectral decomposition of $A_r =: R\Lambda R^{-1}$

Define $\tilde{B}$, $\tilde{C}$ and $\widetilde{N}$ are defined as $R^{-1}B_r$, CR and $R^{-1}N_r R$, respectively

end **while**

Output: $\mathrm{A_r}$, $\mathrm{N_r}$, $\mathrm{B_r}$, $\mathrm{C_r}$

Table 2

	Order	Final value	ISE	Order reduction time	Peak
Original system	930	5.42e-05	-	-	0.7310
Proposed Method	**2**	**5.33e-05**	**5.61e-05**	(BBT+ BIRKA) 88.2872 **(LBT+ BIRKA) 28.3370**	**0.7323**
BBT Method	2	9.56e-06	0.4168	84.86	0.6623
BPOD Method	2	3.09e-05	0.0641	8.4804	0.5185
BIRKA	2	9.33e-06	0.2854	180.85	0.2035

Table 3

Method	Total Order Reduction Time (sec)	BIRKA Convergence Rate (iteration)
BBT+ Original BIRKA	88.2872	25
LBT+ Original BIRKA	28.3370	25
BBT+BIRKA with Condition number	87.5815	27
LBT+BIRKA with Condition number	**26.5274**	**22**

Table 4

	Order	Final value	ISE	Order reduction time	Peak
Original system	420	4.10e-06		-	0.0111
Proposed Method	3	5.33e-05	1.53e-07	(BBT+ BIRKA) 515.00 **(LBT+ BIRKA) 458.45**	**0.0111**
BBT Method	3	**3.64e-06**	2.69e-05	6.73	1.60e-04
BPOD Method	3	2.16e-07	**1.31e-08**	31.26	0.0110
BIRKA	3	5.33e-05	1.53e-07	481.04	**0.0111**

Table 5

Method	Total Order Reduction Time (sec)	BIRKA Convergence Rate (iteration)
BBT+ Original BIRKA	515.00	31
LBT+ Original BIRKA	458.97	26
BBT+BIRKA with Condition number	479.47	29
LBT+BIRKA with Condition number	410.55	26

Table 6

	Order	Final value	ISE	Peak
Original system	200	0		19.7680
Proposed Method	2	**7.0206e-04**	**5.6585e-04**	**19.6208**
BBT Method	2	0.0141	59.1178	23.9674
BPOD Method	2	0.0065	546.1963	1.5442