Scientia Iranica D (2023) 30(6), 2116–2123

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions D: Computer Science & Engineering and Electrical Engineering
http://scientiairanica.sharif.edu

Angle-monotonicity of theta-graphs for points in convex position

D. Bakhshesh[a,*] and M. Farshi[b]

a. *Department of Computer Science, University of Bojnord, Bojnord, Iran.*
b. *Combinatorial and Geometric Algorithms Lab., Department of Computer Science, Yazd University, Yazd, P.O. Box 89195-741, Iran.*

Received 26 August 2022; received in revised form 18 May 2023; accepted 17 July 2023

KEYWORDS
t-spanner;
Angle-monotone path;
Theta-graph;
Stretch factor;
Convex position.

Abstract. For $0 < \gamma < 180°$, a geometric path $P = (p_1, \dots, p_n)$ is called angle-monotone with width γ from p_1 to p_n if there exists a closed wedge of angle γ such that every directed edge $\overrightarrow{p_i p_{i+1}}$ of P lies inside the wedge whose apex is p_i. A geometric graph G is called angle-monotone with width γ if for any two vertices p and q in G, there exists an angle-monotone path with width γ from p to q. In this paper, we show that for any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph Θ_{4k+i} on a set of points in convex position is angle-monotone with width $90° + \frac{i\theta}{4}$, where $\theta = \frac{360°}{4k+i}$. Moreover, we present two sets of points in the plane, one in convex position and the other in non-convex position, to show that for every $0 < \gamma < 180°$, the graph Θ_4 is not angle-monotone with width γ. Furthermore, we improve the stretch factor of graphs $\Theta_4, \Theta_5, \Theta_7, \Theta_9, \Theta_{11}, Y_4$, and Y_5, when the points are in convex position. Finally, we provide a lower bound of 3.66 for Y_4 that solves an open problem.

1. Introduction

Let S be a set of points in the plane. For two points $p, q \in S$, the Euclidean distance between p and q is denoted by $|pq|$. A *geometric* graph $G = (S, E)$ is a weighted graph such that any edge (x, y) of G is a straight-line segment between x and y and the weight of (x, y) is $|xy|$. The length of a path $P = (p_1, p_2, \dots, p_r)$ between p_1 and p_r in G is denoted by $|P|$, and it is defined as $|P| = \sum_{i=1}^{r-1} |p_i p_{i+1}|$. For any two points $p, q \in S$, the *stretch factor* (or *dilation*) between p and q in a geometric graph G is the ratio of the length of a shortest path between p and q in G over $|pq|$. The stretch factor of a geometric graph G is the maximum stretch factor between all pairs of vertices of G.

Let $t > 1$ be a real number. A geometric graph G is called a *t-spanner* if the stretch factor of G is at most t. In computational geometry, constructing the geometric graphs with low stretch factor, small number of edges (small size) and low weight is an important problem. We refer the reader to the book [1] and the papers [2–10] to study t-spanners and their algorithms.

Let $\theta > 0$ be a real number. In [11], Dehkordi et al., introduced *θ-paths*. Let W_p^θ be a 90° closed wedge delimited by the rays starting at p with the slopes $\theta - 45°$ and $\theta + 45°$. A path $(p_1, p_2, \dots, p_n)$ is called a *θ-path* if for every integer i with $1 \leq i \leq n-1$, the vector $\overrightarrow{p_i p_{i+1}}$ lies in the wedge $W_{p_i}^\theta$. Using the concept of θ-paths, Bonichon et al. [12] introduced *angle-monotone graphs*. A geometric graph $G = (S, E)$ is called *angle-monotone* if for any two points $u, v \in S$, there is a real number $\theta > 0$ such that G contains a θ-path between u and v. Bonichon et al. [12] generalized the concept

*. *Corresponding author. Tel.: +98 58 32201000; Fax: +98 58-32284605*
E-mail addresses: d.bakhshesh@ub.ac.ir (D. Bakhshesh); mfarshi@yazd.ac.ir (M. Farshi)

doi: 10.24200/sci.2023.61034.7110

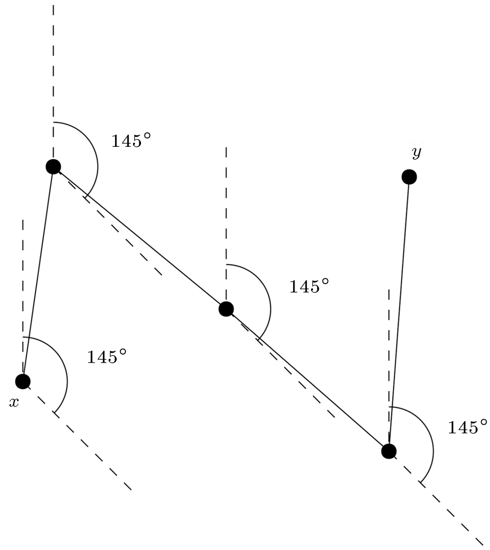


Figure 1. An angle-monotone path between x and y with width $\gamma = 145°$.

of angle-monotone graphs to angle-monotone graphs with width γ. Let $0 < \gamma < 180°$. A geometric path $P = (p_1, \ldots, p_n)$ is called *angle-monotone with width* γ from p_1 to p_n if for some closed wedge of angle γ, every vector $\overrightarrow{p_i p_{i+1}}$ lies in the wedge whose apex is p_i (see Figure 1).

A geometric graph G is called *angle-monotone with width* γ if for any vertex p of G, there is an angle-monotone path with width γ from p to all other vertices of G. It is remarkable that if a path is angle-monotone with width γ from x to y, then the path is also angle-monotone with width γ from y to x.

In [11], Dehkordi et al., showed that any Gabriel triangulation is an angle-monotone graph with width 90°. In [13], Lubiw and Mondal showed that for any set of points in the plane, there is an angle-monotone graph with width 90° with a subquadratic size. Furthermore, they showed that for any angle β with $0 < \beta < 45°$, and for any set of points in the plane, there is an angle-monotone graph with width $(90° + \beta)$ of size $O(\frac{n}{\beta})$. In [14], Bakhshesh and Farshi presented a point set in the plane that its Delauany triangulation is not angle-monotone with width less than 140°. In [15], Bakhshesh and Farshi proved that the minimum value of an angle γ that for any set of points in the plane there is a plane angle-monotone graph with width γ is equal to 120°.

One of the most popular graphs in computational geometry is *theta-graphs* which was introduced by Clarkson [16] and independently by Keil [17]. Informally, for every point set S in the plane and an integer $m \geq 2$, the theta-graph Θ_m is constructed by partitioning the plane into m cones at each point $p \in S$, and joining the *closest* point to p at each cone (in the next section, closest will be defined). Bonichon et al. [12] proved that for any set of points in the plane,

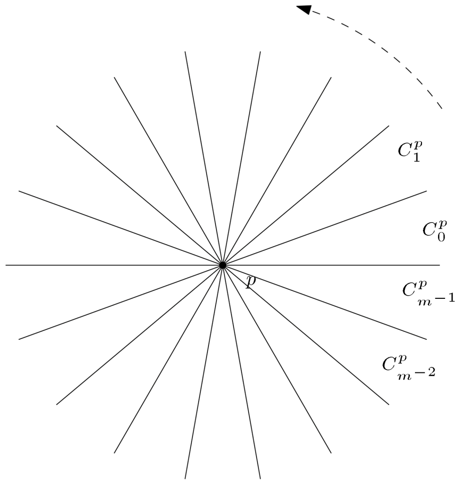


Figure 2. Partition the plane into $m = 18$ cones with apex at p.

half-Θ_6*-graph*, a plane subgraph of Θ_6, whose edges are obtained by selecting every other cone i.e. alternate cones- is angle-monotone with width 120°. In [11] Dehkordi et al. prove that for every set of n points in the plane that are in convex position, there exists an angle-monotone graph (angle-monotone graph with width 90°) with $O(n \log n)$ edges. To the best of our knowledge, it is unknown if the theta-graphs except Θ_6 are angle-monotone with a constant width.

In this paper, we show that for any set of points in convex position, and any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph Θ_{4k+i} is angle-monotone with width $90° + \frac{i\theta}{4}$, where $\theta = \frac{360°}{4k+i}$. Moreover, we present two sets of points in the plane, one in convex position and the other in non-convex position, to show that for every $0 < \gamma < 180°$, the graph Θ_4 is not angle-monotone with width γ.

2. Preliminaries

Let $m \geq 3$ be an integer, and let $\theta = \frac{2\pi}{m}$ be a real number. For any integer i with $0 \leq i < m$ and a point p in the plane, let $\mathcal{R}_i^p$ be the ray emanating from p making the angle $\theta \times i = 2\pi i/m$ with the positive x-axis (the angles are considered in counter-clockwise). Let C_i^p be the cone which is constructed by the rays $\mathcal{R}_i^p$ and $\mathcal{R}_{i+1}^p$. Note that we assume that $\mathcal{R}_m^p = \mathcal{R}_0^p$. For a point r and a cone C_i^p, we say C_i^p contains r (or, $r \in C_i^p$) if r lies strictly between $\mathcal{R}_i^p$ and $\mathcal{R}_{i+1}^p$, or lies on $\mathcal{R}_{i+1}^p$. If r lies on $\mathcal{R}_i^p$, then $r \notin C_i^p$. For a point set S, the theta-graph Θ_m is constructed as follows. For each point $p \in S$, we partition the plane into m cones $C_0^p, C_1^p, \ldots, C_{m-1}^p$ (see Figure 2). Then, for each cone C_i^p containing at least one point of S other than p, let $r_i \in C_i^p$ be a point such that $|pr_i'|$ is minimum where r_i' is the perpendicular projection of r_i onto the bisector of C_i^p. Then, we add the edge (p, r_i) to the graph. We assume that a pair (a, b) is a directed edge. We call the

```
output: A path between a and b in theta-graphs
a_0 = a;
i := 0;
while a_i ≠ b do
    s := an integer such that b ∈ C_s^{a_i};
    a_{i+1} := a point of C_s^{a_i} ∩ S\{a_i} such that (a_i.a_{i+1}) is an edge of Θ_k;
    i := i + 1;
end
return the path (a_0, a_1, ..., a_i);
```

Algorithm 1. Θ-Walk (a, b) (see [1]).

point r the *closest* point to p in C_i^p. For a point $q \in C_i^p$, the *canonical triangle* T_{pq} is the isosceles triangle which is constructed by the rays of C_i^p and the line through q perpendicular to the bisector of C_i^p. For more details on theta-graphs, see [1].

Let S be a set of $n \geq 3$ points in the plane in a convex position. In the following, when we use the notation G, we mean one of the graphs Θ_{4k+2}, Θ_{4k+3}, Θ_{4k+4}, and Θ_{4k+5}. Throughout the paper, we assume that p and q are two distinct points in S and suppose, without loss of generality, that $q \in C_0^p$. Let $\mathcal{W}_O$ be the wedge with apex at the origin O that is the union of all cones C_t^O with $\lceil \frac{m-1}{4} \rceil \leq t \leq \lceil \frac{m-2}{2} \rceil$. Let $\mathcal{W}'_O$ be the reflection of $\mathcal{W}_O$ with respect to the point O. Now, let $\mathcal{U}_O$ be a wedge with apex at the origin O such that $\mathcal{U}_O = \mathcal{W}'_O \cup C_0^O$ (see Figure 3).

3. Angle-monotonicity of theta-graphs

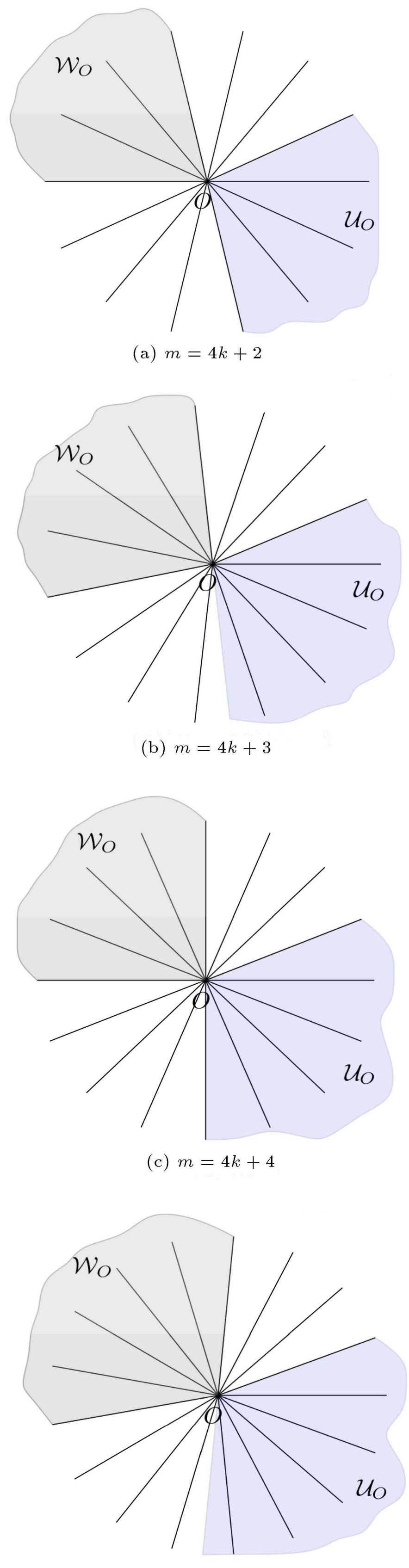
In this section, we show that for any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph Θ_{4k+i} is angle-monotone with width $90° + \frac{i\theta}{4}$. To this end, we show that there is an angle-monotone path between p and q in G with width $90° + \frac{i\theta}{4}$. Let $P = (p = v_0, v_1, \ldots, v_l)$ be the directed path in G such that $v_{i+1} \in C_0^{v_i}$ is the closest point to v_i, and v_l is the last vertex of the path P that lies in T_{pq}. Let $\ddot{P}$ be the directed path, which is obtained by reversing the direction of all edges of P. If $v_l = q$, then obviously P is an angle-monotone path from p to q with width θ. Then, we are done. Now, in what follows, we assume that $v_l \neq q$. Suppose, without loss of generality, that q is below $P \cup C_0^{v_l}$ (see Figure 4). Let $Q = (q = a_0, a_1, \ldots, a_g = v_l)$ be the path constructed by the algorithm Θ-Walk(q, v_l) (see Algorithm 1). The path Q is a path between q and v_l in G such that for any a_i there exists a cone $C_j^{a_i}$ such that $v_l \in C_j^{a_i}$ and (a_i, a_{i+1}) is an edge of G.

3.1. The graphs Θ_{4k+2} and Θ_{4k+4}

We first prove the following lemma.

Lemma 1. *If $G = \Theta_{4k+2}$, then every edge (a_i, a_{i+1}) of the path Q lies in the wedge $\mathcal{W}_{a_i}$.*

Proof. Let ℓ_1 be the horizontal line passing through v_l, and ℓ_2 be the line passing through v_l, forming an angle θ with the positive x-axis. Let c_1 and c_2 be the



(a) $m = 4k + 2$

(b) $m = 4k + 3$

(c) $m = 4k + 4$

(d) $m = 4k + 5$

Figure 3. The wedges $\mathcal{W}_O$ and $\mathcal{U}_O$ for the different values of m.

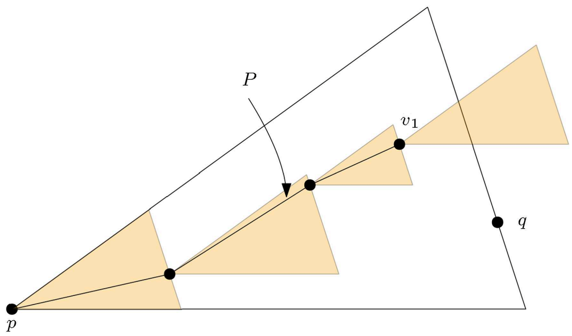


Figure 4. The point q and $P \cup C_0^{v_l}$.

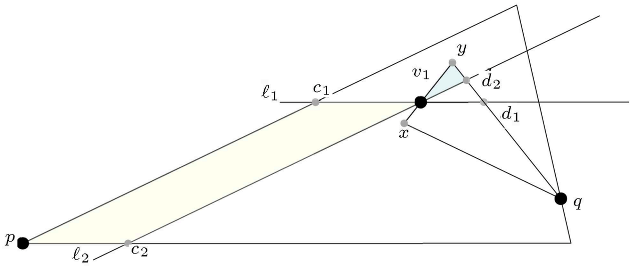


Figure 5. Illustrating the proof of Lemma 1.

intersection of ℓ_1 and ℓ_2 with the sides of the triangle T_{pq} which are incident to p (see Figure 5). Based on the construction of the path P, the vertex v_{l-1} lies in the quadrilateral $pc_1v_lc_2$. Let j be an integer such that $q \in C_j^{v_l}$. Since we assume that q is below $P \cup C_0^{v_l}$, we have $3k+2 \leq j \leq 4k+1$. Since $q \in C_j^{v_l}$, we have $v_l \in C^q_{j-(2k+1)}$. Consider the triangle T_{qv_l}. Let x and y be the two other vertices of T_{qv_l} as depicted in Figure 5. Let $d_1 \neq v_l$ be the intersection of ℓ_1 and T_{qv_l}, and let $d_2 \neq v_l$ be the intersection of ℓ_2 and T_{qv_l}. It is notable that it is possible that the segment xy completely lies on the line ℓ_2. In this case, we assume that $d_2 = y$. Now, if any vertex u of the path Q lies in the triangle $\triangle v_lyd_2$, since v_{l-1} lies in the quadrilateral $pc_1v_lc_2$, the triangle quv_{l-1} contains the vertex v_l that contradicts the convexity of the points. Hence, no vertices of Q lie in the $\triangle v_lyd_2$. For similar reasons, no vertices of Q lie in the triangle $\triangle qv_lp$. Since $C_0^{v_l} \cap T_{pq}$ does not contain any point of S, the path Q completely lies in the triangle $\triangle qd_1v_l$. Then, for any edge (a_i, a_{i+1}) of Q, there is an integer t with $j-(2k+1) \leq t \leq 2k$ such that $a_{i+1} \in C_t^{a_i}$. Since $3k+2 \leq j \leq 4k+1$, clearly (a_i, a_{i+1}) lies in the wedge $\mathcal{W}_{a_i}$. Now, we have the following lemma:

Lemma 2. *If $G = \Theta_{4k+2}$, then every edge (x, y) of the path $P \cup \ddot{Q}$ lies in the wedge $\mathcal{U}_x$.*

Proof. By Lemma 1, every edge (a, b) of Q lies in the wedge $\mathcal{W}_a$. Therefore, every edge (b, a) of $\ddot{Q}$ lies in the wedge $\mathcal{W}'_b$. On the other hand, every edge (v_i, v_{i+1}) of P lies in the cone $C_0^{v_i}$. Since $\mathcal{U}_O = \mathcal{W}'_O \cup C_0^O$, every edge (x, y) of the path $P \cup \ddot{Q}$ lies in the wedge $\mathcal{U}_x$.

Theorem 1. *For any set S of points in the plane that are in convex position and for any integer $k \geq 1$, the graph $G = \Theta_{4k+2}$ is angle-monotone with width $90° + \frac{\theta}{2}$.*

Proof. Consider the points p and q. By Lemma 2, every edge (x, y) of the path $P \cup \ddot{Q}$ lies in the wedge $\mathcal{U}_x$. Therefore, the path $P \cup \ddot{Q}$ is an angle-monotone path from p to q in G with width $k\theta + \theta$. Note that for $G = \Theta_{4k+2}$, the angle of the wedge $\mathcal{U}_x$ is $k\theta + \theta$. Since $\theta = \frac{360°}{4k+2}$, we have $k\theta + \theta = 90° - \frac{\theta}{2} + \theta = 90° + \frac{\theta}{2}$. Hence, $P \cup \ddot{Q}$ is an angle-monotone path with width $90° + \frac{\theta}{2}$. This completes the proof. □

Similar to the proof of Theorem 1, for $G = \Theta_{4k+4}$ with $k \geq 1$, we can prove that the path $P \cup \ddot{Q}$ is an angle-monotone path from p to q with width $(k+1)\theta + \theta = 90° + \theta$. Note that for $G = \Theta_{4k+4}$, the angle of the wedge $\mathcal{U}_x$ is $(k+1)\theta + \theta$. Hence, we have the following theorem.

Theorem 2. *For any set S of points in the plane that are in convex position and for any integer $k \geq 1$, the graph $G = \Theta_{4k+4}$ is angle-monotone with width $90° + \theta$.*

In [12], Bonichon et al. showed that any angle-monotone graph with width $\gamma < 180°$ is a t-spanner with $t = 1/\cos\frac{\gamma}{2}$. Hence, we have the following result.

Corollary 1. *For any set of points in the plane that are in convex position and for any integer $k \geq 1$, the graphs Θ_{4k+2} and Θ_{4k+4} have the stretch factor at most $1/\cos\left(\frac{\pi}{4} + \frac{\theta}{4}\right)$ and $1/\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, respectively.*

3.2. The graphs Θ_{4k+3} and Θ_{4k+5}

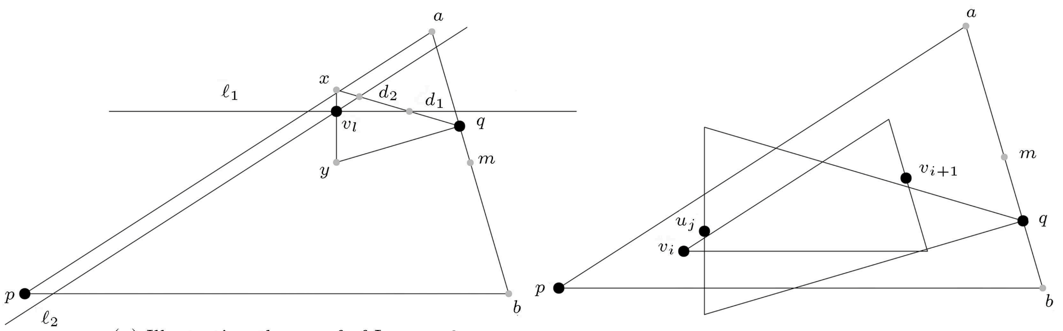
We first assume that $G = \Theta_{4k+3}$. Here, we present an algorithm that finds an angle-monotone path $\mathcal{P}$ between p and q in G with a constant width. The algorithm is as follows. It first finds the path $P = (p = v_0, \ldots, v_l)$ which was introduced earlier. If $v_l = q$, then clearly $\mathcal{P} = P$ is an angle-monotone path with width θ, and we are done. Now, in the following, we assume that $v_l \neq q$. Let a be the topmost vertex of the triangle T_{pq} and let $b \neq p$ be the other vertex of T_{pq}. Let m be the midpoint of ab. The algorithm considers the following cases:

- Case 1: q lies on the segment am. Now, let $Q = (q = a_0, \ldots, v_l)$ be the path constructed by the algorithm Θ-Walk(q, v_l). Then, the algorithm outputs the path $\mathcal{P} = P \cup \ddot{Q}$.
- **Case 2:** q lies on the segment bm. Let $P' = (q = u_0, \ldots, u_s)$ be the path in G such that $u_{i+1} \in C_{2k+1}^{u_i}$ and u_{i+1} is the closest point to u_i, and u_s is the last vertex of the path P' that lies in T_{qp}. Let b' be the topmost vertex of the triangle T_{qp} and let a' be the bottommost vertex of T_{qp}. Let m' be the midpoint

output: An angle-monotone path between p and q in Θ_{4k+3}

$\mathcal{P} := \emptyset$;
Compute the path $P = (p = v_0, \ldots, v_l)$;
if $v_l \neq q$ **then**
  if *q lies on the segment am* **then**
    $Q := \Theta$-Walk(q, v_l);
    $\mathcal{P} := P \cup \ddot{Q}$;
  end
  else
    Compute the path $P' = (q = u_0, \ldots, u_s)$;
    if *P and P' have a common vertex w* **then**
      $R :=$ the path which is formed by the portion of P from v_0 to w followed by the portion of P' from w to q;
      $\mathcal{P} := R$;
    end
    else
      if *there is a vertex $g \neq q$ of the path P' below the path P* **then**
        $u_h :=$ the last vertex of P' below P;
        $Q' := \Theta$-Walk(p, u_h);
        $\mathcal{P} := P' \cup \ddot{Q}'$;
      end
      else
        $Q := \Theta$-Walk(q, v_l);
        $\mathcal{P} = P \cup \ddot{Q}$;
      end
    end
  end
end
else
  $\mathcal{P} := P$;
end
return $\mathcal{P}$;

Algorithm 2. Angle-monotone path between $\Theta_{4k+3}(p,q)$.



(a) Illustrating the proof of Lemma 3 (b) Illustrating the proof of Lemma 6

Figure 6. Illustrating the proofs of Lemmas 3 and 6.

of $a'b'$. Since $q \in C_0^p$, it is easy to see that p lies on the segment $a'm'$. Now, there are two cases:

- **I:** P and P' have a common vertex w. The algorithm outputs the path R, formed by the portion of P from v_0 to w followed by the portion of P' from w to q;
- **II:** P and P' do not have any common vertex. Now, consider two following cases: (a): there is a vertex $g \neq q$ of the path P' below the path P and (b): all vertices of P' are above the path P. For the case (a), let u_h be the last vertex of P' below the path P and let Q' be the constructed path by the algorithm Θ-Walk(p, u_h). Then, the algorithm outputs path $\mathcal{P} = P' \cup \ddot{Q}'$. For the case (b), first the path $Q = \Theta$-Walk(q, v_l) is constructed. Then, the algorithm outputs the path $\mathcal{P} = P \cup \ddot{Q}$.

For more details, see Algorithm 2.

In the following, we show that the path $\mathcal{P}$ returned by Algorithm 2 is an angle-monotone path between p and q with width $90° + \frac{3\theta}{4}$. We first prove the following lemma.

Lemma 3. *If q lies on the segment am, then every edge (a_i, a_{i+1}) of the path $Q = (q = a_0, \ldots, v_l)$ lies in the wedge $\mathcal{W}_{a_i}$.*

Proof. Let j be an integer such that $v_l \in C_j^q$. Since we assumed that q is below $P \cup C_0^{v_l}$, we have $k+1 \leq j \leq 2k+1$. Consider the triangle T_{qv_l}. Let x and y be the two other vertices of T_{qv_l} as depicted in Figure 6(a). It is notable that the line passing through p and m is parallel to the line passing through q and y. Then, since q lies on the segment am, the point p is below the line passing through q and y. Hence, because of the convexity of the points, no points of Q lie in the triangle $\triangle qv_ly$. Consider the lines ℓ_1 and ℓ_2, and the points d_1 and d_2 as defined in the proof of Lemma 1. By reasons similar to the proof of Lemma 1, we can prove that the path Q completely lies in the triangle $\triangle qd_1v_l$. Then, for any edge (a_i, a_{i+1}) of Q, there is an integer t with $j \leq t \leq 2k+1$ such that $a_{i+1} \in C_t^{a_i}$. Clearly, this shows that (a_i, a_{i+1}) lies in the wedge $\mathcal{W}_{a_i}$. $\square$

Now, we prove the following lemma:

Lemma 4. *If q lies on the segment bm, then every edge (r_i, r_{i+1}) of the path R lies in the wedge $\mathcal{U}_{r_i}$.*

Proof According to Algorithm 2, the path R is constructed when the paths $P = (v_1, \ldots, v_l)$ and $P' = (u_1, \ldots, u_s)$ have a common vertex. It is clear that for every edge (v_i, v_{i+1}) of the path P, we have $v_{i+1} \in C_0^{v_i}$, therefore (v_i, v_{i+1}) lies in the wedge $\mathcal{U}_{v_i}$. On the other hand, for every edge (u_i, u_{i+1}) of P', we have $u_{i+1} \in C_{2k+1}^{u_i}$. Therefore, $u_i \in C_{4k+2}^{u_{i+1}}$ or $u_i \in C_0^{u_{i+1}}$. Hence, the edge (u_{i+1}, u_i) lies in the wedge $\mathcal{U}_{u_{i+1}}$. This completes the proof. $\square$

Let $\mathcal{Y}_O$ be a wedge with $\mathcal{Y}_O = \left(\bigcup_{i=3k+2}^{4k+2} C_i^O\right) \cup \left(C_{2k+1}^O\right)' (\left(C_{2k+1}^O\right)'$ is the reflection of C_{2k+1}^O with respect to the origin O). It is clear that the angle of $\mathcal{Y}_O$ is equal to $(k+1)\theta + \theta/2$. Now, we prove the following lemma:

Lemma 5. *If q lies on the segment bm and the paths P and P' do not have any common vertex, and there is a vertex $g \neq q$ of the path P' below the path P, then every edge (c_i, c_{i+1}) of the constructed path $\mathcal{P}$ by Algorithm 2 lies in the wedge $\mathcal{Y}_{c_i}$.*

Proof. Let u_h be the last vertex of P' below P. According Algorithm 2, $\mathcal{P} = P' \cup \ddot{Q}'$ that Q' is the constructed path by Θ-Walk(p, u_h). It is clear that for every edge (u_i, u_{i+1}) of P', we have $u_{i+1} \in C_{2k+1}^{u_i}$, and therefore $u_i \in \left(C_{2k+1}^{u_{i+1}}\right)'$. Hence, (u_{i+1}, u_i) lies in the wedge $\mathcal{Y}_{u_{i+1}}$. Let $Q' = (p = a_1', a_2', \ldots, a_z' = u_h)$. We claim that every edge (a_i', a_{i+1}') lies in the wedge $\mathcal{Y}_{a_i'}$. Since p lies on the segment $a'm'$, the claim is proved by the arguments similar to the proof of Lemma 3. These show that if (c_i, c_{i+1}) be an edge of the path $\mathcal{P}$, it lies in the wedge $\mathcal{Y}_{c_i}$. $\square$

Now, we have the following lemma:

Lemma 6. *If q lies on the segment bm and the paths P and P' do not have any common vertex, and there is no vertex $g \neq q$ of the path P' below the path P, then every edge (r_i, r_{i+1}) of the constructed path $\mathcal{P}$ by Algorithm 2 lies in the wedge $\mathcal{U}_{r_i}$.*

Proof. Let u_j be a vertex of P' above the path P. Let v_i be the last vertex of P to the left of u_j (see Figure 6(b)). Since p is to the left of u_j, the vertex v_i always exists. Since there is no vertex $g \neq q$ of the path P' below the path P, we have $u_{j-1} = q$. Now, consider the triangle $T_{v_iv_{i+1}}$. Since P and P' have no common vertex, clearly $u_j \notin T_{v_iv_{i+1}}$. Hence, if $v_i \neq p$, then the triangle $\triangle pu_jv_{i+1}$ contains the vertex v_i, which contradicts the convexity of the points. Then, $v_i = p$. On the other hand, since $v_l \neq q$, we must have $v_{i+1} \notin T_{qu_j}$, and therefore $v_l \in C_t^q$ with $k+1 \leq t < 2k+1$. Now, by the arguments similar to the proof of Lemma 3, we can prove that every edge (a_i, a_{i+1}) of the path Q lies in the wedge $\mathcal{W}_{a_i}$. Hence, it is clear that every edge (r_i, r_{i+1}) of the path $\mathcal{P} = P \cup \ddot{Q}$ lies in the wedge $\mathcal{U}_{r_i}$. $\square$

Based on Lemmas 3, 4, 5, and 6, any path constructed by Algorithm 2 is angle-monotone with width $(k+1)\theta + \frac{\theta}{2}$. Since $\theta = \frac{360^\circ}{4k+3}$, we have $(k+1)\theta + \frac{\theta}{2} = 90^\circ + \frac{3\theta}{4}$. Then, the following theorem holds.

Theorem 3. *For any set S of points in the plane that are in convex position and for any integer $k \geq 1$, Θ_{4k+3} is angle-monotone with width $90^\circ + \frac{3\theta}{4}$.*

By the arguments similar to the proof of Theorem 3, for $G = \Theta_{4k+5}$ with $k \geq 1$, we can prove that the path $\mathcal{P}$ is an angle-monotone path from p to q with width $(k+1)\theta + \frac{\theta}{2}$. Since $\theta = \frac{360^\circ}{4k+1}$, we have $(k+1)\theta + \frac{\theta}{2} = 90^\circ + \frac{5\theta}{4}$. Then, the following theorem holds.

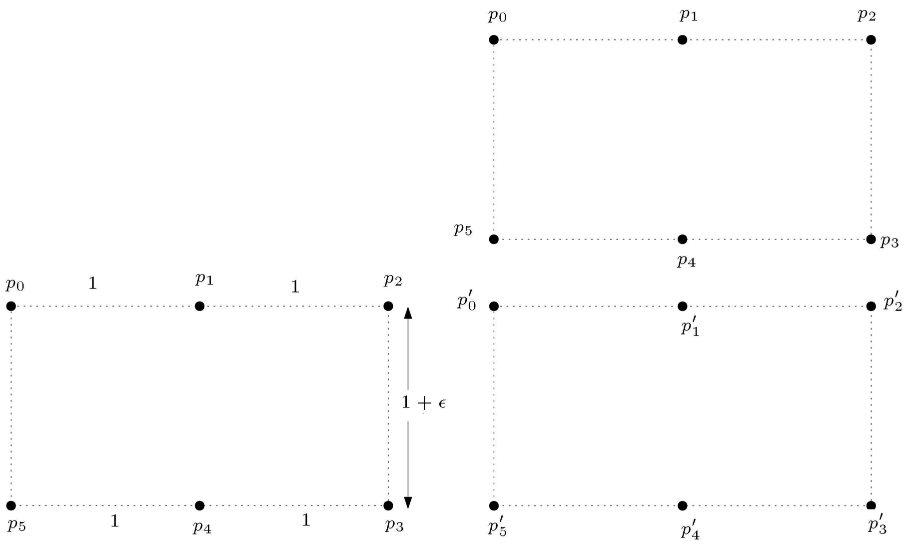
Theorem 4. *For any set S of points in the plane that are in convex position and for any integer $k \geq 1$, Θ_{4k+5} is angle-monotone with width $90^\circ + \frac{5\theta}{4}$.*

We close this section with the following result.

Corollary 2. *For any set of points in the plane that are in convex position, the graphs Θ_{4k+3} and Θ_{4k+5} with $k \geq 1$ have the stretch factor at most $1/\cos\left(\frac{\pi}{4} + \frac{3\theta}{8}\right)$ and $1/\cos\left(\frac{\pi}{4} + \frac{5\theta}{8}\right)$, respectively.*

4. Theta-graph Θ_4

In the following, we present two point sets, one in a convex position and the other in a non-convex position, to show that the graph Θ_4 of the point set is not angle-monotone for any width $\gamma > 0$. Let p_0, p_2, p_3, and p_5 be the vertices of a rectangle with length 2 and width $1+\epsilon$, where $\epsilon > 0$ is a small real number (see Figure 7(a)). Let p_1 and p_4 be the midpoints of the segments p_0p_2 and p_3p_5, respectively.



(a) The point set P. (b) The point set $V = P \cup P'$

Figure 7. The point sets P and V.

Now, let $P = \{p_0, p_1, \ldots, p_5\}$. Consider the theta-graph Θ_4 on P. It is not hard to see that the edge set E of Θ_4 is:

$$E = \{(p_0, p_1), (p_1, p_2), (p_2, p_3), (p_3, p_4), (p_4, p_5), (p_5, p_0)\}.$$

Now, since p_0p_2 and p_3p_5 are parallel, it is obvious that for any $0 < \gamma < 180°$, any path between p_1 and p_4 is not angle-monotone with width γ.

Let $P' = \{p'_0, p'_1, \ldots, p'_5\}$ be a copy of point set P such that the points of P' placed below the points of P as depicted in Figure 7(b). Let $V = P \cup P'$. It is easy to see that the edge set F of the theta-graph Θ_4 on the point set V is:

$$F = E \cup \{(p'_0, p'_1), (p'_1, p'_2), (p'_2, p'_3), (p'_3, p'_4), (p'_4, p'_5), (p'_5, p'_0)\} \cup \{(p'_0, p_5), (p'_1, p_4), (p'_2, p_3)\}.$$

It is obvious that for any $0 < \gamma < 180°$, any path between p_1 and p_4 is not angle-monotone with width γ. Now, we have the following theorem.

Theorem 5. *For any angle $0 < \gamma < 180°$, the graph Θ_4 is not necessarily angle-monotone with width γ.*

5. Remarks

In Corollaries 1 and 2, we examined the stretch factor of the graphs, Θ_{4k+2}, Θ_{4k+3}, Θ_{4k+4}, and Θ_{4k+5} for the points in convex position. In [18], Bose et al. show that the stretch factor of the graphs Θ_{4k+2}, Θ_{4k+3}, Θ_{4k+4} and Θ_{4k+5} are at most: $1 + 2\sin(\theta/2)$, $\cos(\theta/4)/(\cos(\theta/2)\text{-}\sin(3\theta/4))$, $1{+}2\sin(\theta/2)/(\cos(\theta/2)\text{-}\sin(\theta/2))$, and $\cos(\theta/4)/(\cos(\theta/2)\text{-}\sin(3\theta/4))$, respectively.

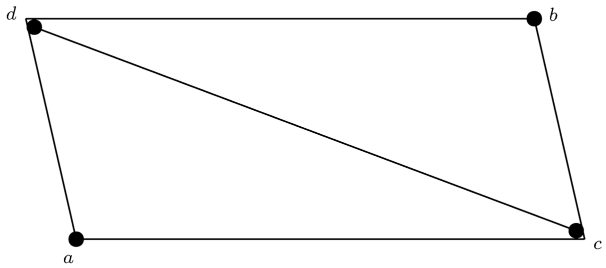


Figure 8. The lower bound for the width of Θ_{4k+2}.

By comparing the results of Corollaries 1 and 2 with the results of [18], we find that the results of the corollaries do not improve the stretch factors known in [18].

In the following, we indicate whether the bounds on the width presented in Theorems 1, 2, 3 and 4 are tight or not. Consider the graph Θ_{4k+2}. Figure 8 shows that the upper bound on the width presented in Theorems 1 is tight. We place a vertex c close to the lower corner of T_{ab} that is sufficiently far from the vertex b. We also place a vertex d close to the upper corner of T_{ba} that is sufficiently far from the vertex a. Now, the graph Θ_{4k+2} of four points a, b, c, and d is as shown in Figure 8. We can easily see that each of the paths acb and adb are angle-monotone with width $90° + \frac{\theta}{2} - \epsilon$, for some real number $\epsilon > 0$ that only depends on the distance between $c(d)$ and the lower corner (upper corner) of T_{ab} (T_{ba}). If ϵ approaches zero, then the width approaches $90° + \frac{\theta}{2}$.

For Theorems 2, 3, and 4, we do not know whether the bounds for the width are tight or not.

6. Conclusion

In this paper, we showed that for any set of points in the plane that are in convex position and for any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph Θ_{4k+i} is angle-monotone with width $90^\circ + \frac{i\theta}{4}$, where $\theta = \frac{360^\circ}{4k+i}$. Moreover, we presented two sets of points in the plane, one in a convex position and the other in a non-convex position, to show that for every $0 < \gamma < 180^\circ$, the graph Θ_4 is not angle-monotone with width γ. Furthermore, we showed that the upper bound on the width presented in Theorem 1 is tight. It is notable that our technique in Section 3.2 does not work for Θ_5 because, by the proposed technique, the resulting path $\mathcal{P}$ is angle-monotone with width $90^\circ + \frac{5\theta}{4}$. Since for Θ_5, we have $\theta = \frac{2\pi}{5} \equiv 72^\circ$. Then, $90^\circ + \frac{5\theta}{4} = 180^\circ$. We conjecture that for any set of points in a convex position, Θ_5 is angle-monotone with a constant width. We tried to prove our conjecture, but we did not succeed. Finally, we present the following conjecture.

Conjecture 1. For any set of points in the plane that are not convex position, for any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph Θ_{4k+i} is angle-monotone with width $90^\circ + \frac{i\theta}{4}$, where $\theta = \frac{360^\circ}{4k+i}$.

References

1. Narasimhan, G. and Smid, M., *Geometric Spanner Networks*, Cambridge University Press (2007).
2. Bakhshesh, D. and Farshi, M. "A lower bound on the stretch factor of Yao graph Y_4", *Scientia Iranica*, **29**(6), pp. 3244–3248 (2022).
3. Abam, M.A. and Seraji, M.J.R. "Geodesic spanners for points in $\mathbb{R}^3$ amid axis-parallel boxes", *Inform Process Lett.*, **166**, 106063 (2021).
4. Abam, M.A. and Borouny, M.S. "Local geometric spanners", *Algorithmica*, **83**, pp. 3629–3648 (2021).
5. Akitaya, H.A., Biniaz, A., and Bose, P. "On the spanning and routing ratios of the directed Θ_6-graph", *Comp. Geom-Theor. App.*, **105**(106), 101881 (2022).
6. De Carufel, J.L., Bose, P., Paradis, F., et al. "Local routing in WSPD-based spanners", *Journal of Computational Geometry*, **12**, pp. 1–34 (2021).
7. van Renssen, A. and Wong, G. "Bounded-degree spanners in the presence of polygonal obstacle", *Theor. Comput. Sci.*, **854**, pp. 159–173 (2021).
8. Bakhshesh, D. and Farshi, M. "(Weakly) self-approaching geometric graphs and spanners", *Comp. Geom-Theor. App.*, **78**, pp. 20–36 (2019).
9. Bakhshesh, D. and Farshi, M. "A degree 3 plane 5.19-spanner for points in convex position", *Scientia Iranica*, **28**, pp. 3324–3331 (2021).
10. Iranfar, B. and Farshi, M. "On the expected weight of the theta graph on uncertain points", *Journal of Algorithms and Computation*, **52**(1), pp. 163–174 (2020).
11. Dehkordi, H.R., Frati, F., and Gudmundsson, J. "Increasing-chord graphs on point sets", *Journal of Graph Algorithms and Applications*, **19**(2), pp. 761–778 (2015).
12. Bonichon, N., Bose, P., Carmi, P., et al. "Gabriel triangulations and angle-monotone graphs: Local routing and recognition", In *Proceedings of the 24th International Symposium on Graph drawing (GD 2016)*, pp. 519–531 (2016).
13. Lubiw, A. and Mondal, D. "Construction and local routing for angle-monotone graphs", *Journal of Graph Algorithms and Applications*, **23**(2), pp. 345–369 (2019).
14. Bakhshesh, D. and Farshi, M. "Angle-monotonicity of Delaunay triangulation", *Comp. Geom-Theor. App.*, **94**(10), 1711 (2021).
15. Bakhshesh, D. and Farshi, M. "On the plane angle-monotone graphs", *Comp. Geom-Theor. App.*, **100**(10), 1818 (2022).
16. Clarkson, K. "Approximation algorithms for shortest path motion planning", In *Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing*, STOC'87, pp. 56–65, New York, NY, USA,. ACM (1987).
17. Keil, J.M., *Approximating the Complete Euclidean Graph*, R. Karlsson and A. Lingas, Editors, SWAT *88*, pp. 208–213, Berlin, Heidelberg (1988).
18. Bose, P., D. Carufel, J.-L., Morin, P., et al. "Towards tight bounds on theta-graphs: More is not always better", *Theor. Comput. Sci.*, **616**, pp. 70–93 (2016).

Biographies

Davood Bakhshesh is an Assistant Professor at the Department of Computer Science at the University of Bojnord in Bojnord, Iran, and has been working at the University of Bojnord since 2012. He received his BSc degree in Computer Science from Vali-e-Asr University of Rafsanjan in 2007, an MSc degree in Computer Science from Sharif University of Technology in 2009, and a PhD degree in Computer Science from Yazd University in 2017.

Mohammad Farshi is an Associate Professor at the Department of Computer Science at Yazd University in Yazd, Iran, and has been working in the Yazd faculty since 1999. He received his BSc in Computer Science from Yazd University in 1996, MSc in Pure Mathematics from Shiraz University in 1999, and PhD in Computer Science from Eindhoven University of Technology (TU/e) in 2008.