Angle-monotonicity of theta-graphs for points in convex position

Davood Bakhshesh\(^1\) and Mohammad Farshi\(^2\)

\(^1\) Department of Computer Science, University of Bojnord, Bojnord, Iran.

\(^2\) Combinatorial and Geometric Algorithms Lab., Department of Computer Science, Yazd University, Yazd, P.O. Box 89195-741, Iran.

Abstract. For $0 < \gamma < 180^\circ$, a geometric path $P = (p_1, \ldots, p_n)$ is called angle-monotone with width $\gamma$ from $p_1$ to $p_n$ if there exists a closed wedge of angle $\gamma$ such that every directed edge $\overrightarrow{p_ip_{i+1}}$ of $P$ lies inside the wedge whose apex is $p_i$. A geometric graph $G$ is called angle-monotone with width $\gamma$ if for any two vertices $p$ and $q$ in $G$, there exists an angle-monotone path with width $\gamma$ from $p$ to $q$. In this paper, we show that for any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph $\Theta_{4k+i}$ on a set of points in convex position is angle-monotone with width $90^\circ + \frac{\theta}{4}$, where $\theta = \frac{360^\circ}{4k+i}$. Moreover, we present two sets of points in the plane, one in convex position and the other in non-convex position, to show that for every $0 < \gamma < 180^\circ$, the graph $\Theta_4$ is not angle-monotone with width $\gamma$.

Keywords: t-spanner · Angle-monotone path · Theta-graph · Stretch factor · Convex position.

1 Introduction

Let $S$ be a set of points in the plane. For two points $p, q \in S$, the Euclidean distance between $p$ and $q$ is denoted by $|pq|$. A geometric graph $G = (S, E)$ is a weighted graph such that any edge $(x, y)$ of $G$ is a straight-line segment between $x$ and $y$ and the weight of $(x, y)$ is $|xy|$. The length of a path $P = (p_1, p_2, \ldots, p_r)$ between $p_1$ and $p_r$ in $G$ is denoted by $|P|$, and it is defined as $|P| = \sum_{i=1}^{r-1} |p_ip_{i+1}|$. For any two points $p, q \in S$, the stretch factor (or dilation) between $p$ and $q$ in a geometric graph $G$ is the ratio of the length of a shortest path between $p$ and $q$ in $G$ over $|pq|$. The stretch factor of a geometric graph $G$ is the maximum stretch factor between all pairs of vertices of $G$.

Let $t > 1$ be a real number. A geometric graph $G$ is called a $t$-spanner if the stretch factor of $G$ is at most $t$. In computational geometry, constructing the
Let \( \theta > 0 \) be a real number. In [11], Dehkordi et al., introduced \( \theta \)-paths. Let \( W^\theta_p \) be a \( 90^\circ \) closed wedge delimited by the rays starting at \( p \) with the slopes \( \theta - 45^\circ \) and \( \theta + 45^\circ \). A path \((p_1, p_2, \ldots, p_n)\) is called a \( \theta \)-path if for every integer \( i \) with \( 1 \leq i \leq n - 1 \), the vector \( \overrightarrow{p_ip_{i+1}} \) lies in the wedge \( W^\theta_p \). Using the concept of \( \theta \)-paths, Bonichon et al. [12] introduced angle-monotone graphs. A geometric graph \( G = (S, E) \) is called angle-monotone if for any two points \( u, v \in S \), there is a real number \( \theta > 0 \) such that \( G \) contains a \( \theta \)-path between \( u \) and \( v \). Bonichon et al. [12] generalized the concept of angle-monotone graphs to angle-monotone graphs with width \( \gamma \). Let \( 0 < \gamma < 180^\circ \). A geometric path \( P = (p_1, \ldots, p_n) \) is called angle-monotone with width \( \gamma \) from \( p_1 \) to \( p_n \) if for some closed wedge of angle \( \gamma \), every vector \( \overrightarrow{p_ip_{i+1}} \) lies in the wedge whose apex is \( p_i \) (see Fig. 1).

A geometric graph \( G \) is called angle-monotone with width \( \gamma \) if for any vertex \( p \) of \( G \), there is an angle-monotone path with width \( \gamma \) from \( p \) to all other vertices of \( G \). It is remarkable that if a path is angle-monotone with width \( \gamma \) from \( x \) to \( y \), then the path is also angle-monotone with width \( \gamma \) from \( y \) to \( x \).

In [11], Dehkordi et al., showed that any Gabriel triangulation is an angle-monotone graph with width \( 90^\circ \). In [13], Lubiw and Mondal showed that for any set of points in the plane, there is an angle-monotone graph with width \( 90^\circ \) with a subquadratic size. Furthermore, they showed that for any angle \( \beta \) with \( 0 < \beta < 45^\circ \), and for any set of points in the plane, there is an angle-monotone graph with width \( (90^\circ + \beta) \) of size \( O(\frac{n}{\beta}) \). In [14], Bakhshesh and Farshi presented a point set in the plane that its Delaunay triangulation is not angle-monotone with width less than \( 140^\circ \). In [15], Bakhshesh and Farshi proved that the minimum value of an angle \( \gamma \) that for any set of points in the plane there is a plane angle-monotone graph with width \( \gamma \) is equal to \( 120^\circ \).

One of the most popular graphs in computational geometry is \( \theta \)-graphs which was introduced by Clarkson [16] and independently by Keil [17]. Informally, for every point set \( S \) in the plane and an integer \( m \geq 2 \), the \( \theta \)-graph \( \Theta_m \) is constructed by partitioning the plane into \( m \) cones at each point \( p \in S \), and joining the closest point to \( p \) at each cone (in the next section, closest will be defined). Bonichon et al. [12] proved that for any set of points in the plane, half-\( \Theta_6 \)-graph, a plane subgraph of \( \Theta_6 \), whose edges are obtained by selecting every other cone i.e. alternate cones- is angle-monotone with width \( 120^\circ \). In [11], Dehkordi et al. prove that for every set of \( n \) points in the plane that are in convex position, there exists an angle-monotone graph (angle-monotone graph with width \( 90^\circ \)) with \( O(n \log n) \) edges. To the best of our knowledge, it is unknown if the \( \theta \)-graphs except \( \Theta_6 \) are angle-monotone with a constant width.

In this paper, we show that for any set of points in convex position, and any integer \( k \geq 1 \) and any \( i \in \{2, 3, 4, 5\} \), the \( \theta \)-graph \( \Theta_{4k+i} \) is angle-monotone with width \( 90^\circ + \frac{i\theta}{4} \), where \( \theta = \frac{360^\circ}{4k+i} \). Moreover, we present two sets of points in the plane, one in convex position and the other in non-convex position, to show that for every \( 0 < \gamma < 180^\circ \), the graph \( \Theta_4 \) is not angle-monotone with width \( \gamma \).
2 Preliminaries

Let \( m \geq 3 \) be an integer, and let \( \theta = \frac{2\pi}{m} \) be a real number. For any integer \( i \) with \( 0 \leq i < m \) and a point \( p \) in the plane, let \( R_i^p \) be the ray emanating from \( p \) making the angle \( \theta \times i \) with the positive \( x \)-axis (the angles are considered in counter-clockwise). Let \( C_i^p \) be the cone which is constructed by the rays \( R_i^p \) and \( R_{i+1}^p \). Note that we assume that \( R_{m}^p = R_0^p \). For a point \( r \) and a cone \( C_i^p \), we say \( C_i^p \) contains \( r \) (or, \( r \in C_i^p \)) if \( r \) lies strictly between \( R_i^p \) and \( R_{i+1}^p \), or lies on \( R_{i+1}^p \). If \( r \) lies on \( R_i^p \), then \( r \notin C_i^p \). For a point set \( S \), the theta-graph \( \Theta_m \) is constructed as follows. For each point \( p \in S \), we partition the plane into \( m \) cones \( C_0^p, C_1^p, \ldots, C_{m-1}^p \) (see Fig. 2). Then, for each cone \( C_i^p \) containing at least one point of \( S \) other than \( p \), let \( r_i \in C_i^p \) be a point such that \( |pr_i| \) is minimum where \( r_i \) is the perpendicular projection of \( r_i \) onto the bisector of \( C_i^p \). Then, we add the edge \((p, r_i)\) to the graph. We assume that a pair \((a, b)\) is a directed edge. We call the point \( r \) the closest point to \( p \) in \( C_i^p \). For a point \( q \in C_i^p \), the canonical triangle \( T_{pq} \) is the isosceles triangle which is constructed by the rays of \( C_i^p \) and the line through \( q \) perpendicular to the bisector of \( C_i^p \). For more details on theta-graphs, see [1].

Let \( S \) be a set of \( n \geq 3 \) points in the plane that are in convex position. In the following, when we use the notation \( G \), we mean one of the graphs \( \Theta_{4k+2}, \Theta_{4k+3}, \Theta_{4k+4} \) and \( \Theta_{4k+5} \). Throughout the paper, we assume that \( p \) and \( q \) are two distinct points in \( S \) and suppose, without loss of generality, that \( q \in C_0^p \). Let \( W_O \) be the wedge with apex at the origin \( O \) that is the union of all cones \( C_t^O \) with \( \left\lfloor \frac{m-1}{4} \right\rfloor \leq t \leq \left\lfloor \frac{m-2}{4} \right\rfloor \). Let \( W'_O \) be the reflection of \( W_O \) with respect to the point \( O \). Now, let \( U_O \) be a wedge with apex at the origin \( O \) such that \( U_O = W'_O \cup C_0^O \) (see Fig. 3).

3 Angle-monotonicity of theta-graphs

In this section, we show that for any integer \( k \geq 1 \) and any \( i \in \{2, 3, 4, 5\} \), the theta-graph \( \Theta_{4k+i} \) is angle-monotone with width \( 90^\circ + \frac{i\theta}{4} \). To this end, we show that there is an angle-monotone path between \( p \) and \( q \) in \( G \) with width \( 90^\circ + \frac{i\theta}{4} \). Let \( P = (p = v_0, v_1, \ldots, v_l) \) be the directed path in \( G \) such that \( v_{i+1} \in C_0^p \) is the closest to \( v_i \), and \( v_l \) is the last vertex of the path \( P \) that lies in \( T_{pq} \). Let \( \bar{P} \) be the directed path which is obtained by reversing the direction of all edges of \( P \). If \( v_l = q \), then obviously \( P \) is an angle-monotone path from \( p \) to \( q \) with width \( \theta \). Then, we are done. Now, in what follows, we assume that \( v_l \neq q \). Suppose, without loss of generality, that \( q \) is below \( P \cup C_{0}^q \) (see Fig. 4). Let \( Q = (q = a_0, a_1, \ldots, a_g = v_l) \) be the path constructed by the algorithm \( \Theta\text{-Walk}(q, v_l) \) (see Algorithm 3.1). The path \( Q \) is a path between \( q \) and \( v_l \) in \( G \) such that for any \( a_i \) there exists a cone \( C_j^{a_i} \) such that \( v_l \in C_j^{a_i} \) and \((a_i, a_{i+1})\) is an edge of \( G \).

3.1 The graphs \( \Theta_{4k+2} \) and \( \Theta_{4k+4} \)

We first prove the following lemma.
Theorem 1. For any set $S$ of points in the plane that are in convex position and for any integer $k \geq 1$, the graph $G = \Theta_{4k+2}$ is angle-monotone with width $90° + \frac{\theta}{2}$. 

Algorithm 3.1: $\Theta$-WALK($a, b$) (see [1])

output: A path between $a$ and $b$ in theta-graphs
1. $a_0 = a$;
2. $i := 0$;
3. while $a_i \neq b$ do
   4. $s := \text{an integer such that } b \in C_s^a$;
   5. $a_{i+1} := \text{a point of } C_s^a \cap S\{a_i\} \text{ such that } (a_i, a_{i+1}) \text{ is an edge of } \Theta_k$;
   6. $i := i + 1$;
4. return the path $(a_0, a_1, \ldots, a_i)$;

Lemma 1. If $G = \Theta_{4k+2}$, then every edge $(a_i, a_{i+1})$ of the path $Q$ lies in the wedge $W_a$.

Proof. Let $\ell_1$ be the horizontal line passing through $v_1$, and $\ell_2$ be the line passing through $v_1$ that forms an angle $\theta$ with the positive $x$-axis. Let $c_1$ and $c_2$ be the intersection of $\ell_1$ and $\ell_2$ with the sides of the triangle $T_{pq}$ which are incident to $p$ (see Fig. 5). Based on the construction of the path $P$, the vertex $v_{i-1}$ lies in the quadrilateral $pc_1v_i\ell_2$. Let $j$ be an integer such that $q \in C_j^p$. Since we assume that $q$ is below $P \cup C_j^p$, we have $3k + 2 \leq j \leq 4k + 1$. Since $q \in C_j^p$, we have $v_i \in C_j^{p-(2k+1)}$. Consider the triangle $T_{qv_i}$. Let $x$ and $y$ be the two other vertices of $T_{qv_i}$ as depicted in Fig. 5. Let $d_1 \neq v_i$ be the intersection of $\ell_1$ and $T_{qv_i}$, and let $d_2 \neq v_i$ be the intersection of $\ell_2$ and $T_{qv_i}$. It is notable that it is possible that the segment $xy$ completely lies on the line $\ell_2$. In this case, we assume that $d_2 = y$. Now, if any vertex $u$ of the path $Q$ lies in the triangle $\triangle v_1yd_2$, since $v_{i-1}$ lies in the quadrilateral $pc_1v_i\ell_2$, the triangle $qvvd_{i-1}$ contains the vertex $v_i$ that contradicts the convexity of the points. Hence, no vertices of $Q$ lie in the $\triangle v_1yd_2$. By the similar reasons, no vertices of $Q$ lie in the triangle $\triangle qvd_1v_i$. Since $C_j^p \cup T_{pq}$ does not contain any point of $S$, the path $Q$ completely lies in the triangle $\triangle qvd_1v_i$. Then, for any edge $(a_i, a_{i+1})$ of $Q$, there is an integer $t$ with $j - (2k + 1) \leq t \leq 2k$ such that $a_{i+1} \in C_t^{a_i}$. Since $3k + 2 \leq j \leq 4k + 1$, clearly $(a_i, a_{i+1})$ lies in the wedge $W_{a_i}$. \]

Now, we have the following lemma.

Lemma 2. If $G = \Theta_{4k+2}$, then every edge $(x, y)$ of the path $P \cup \tilde{Q}$ lies in the wedge $U_x$.

Proof. By Lemma 1, every edge $(a, b)$ of $Q$ lies in the wedge $W_a$. Therefore, every edge $(b, a)$ of $\tilde{Q}$ lies in the wedge $W_b$. On the other hand, every edge $(v_i, v_{i+1})$ of $P$ lies in the cone $C_0^a$. Since $U_0 = W_0 \cup C_0^a$, every edge $(x, y)$ of the path $P \cup \tilde{Q}$ lies in the wedge $U_x$. \]
Consider the points $p$ and $q$. By Lemma 2, every edge $(x, y)$ of the path $P \cup Q$ lies in the wedge $\mathcal{U}_x$. Therefore, the path $P \cup Q$ is an angle-monotone path from $p$ to $q$ in $G$ with width $k\theta + \theta$. Note that for $G = \Theta_{4k+2}$, the angle of the wedge $\mathcal{U}_x$ is $k\theta + \theta$. Since $\theta = \frac{360}{4k+2}$, we have $k\theta + \theta = 90^\circ - \frac{\theta}{2} + \theta = 90^\circ + \frac{\theta}{2}$. Hence, $P \cup Q$ is an angle-monotone path with width $90^\circ + \frac{\theta}{2}$. This completes the proof. \[\square\]

Similar to the proof of Theorem 1, for $G = \Theta_{4k+4}$ with $k \geq 1$, we can prove that the path $P \cup Q$ is an angle-monotone path from $p$ to $q$ with width $(k+1)\theta + \theta = 90^\circ + \theta$. Note that for $G = \Theta_{4k+4}$, the angle of the wedge $\mathcal{U}_x$ is $(k+1)\theta + \theta$. Hence, we have the following theorem.

**Theorem 2.** For any set $S$ of points in the plane that are in convex position and for any integer $k \geq 1$, the graph $G = \Theta_{4k+4}$ is angle-monotone with width $90^\circ + \theta$.

In [12], Bonichon et al., showed that any angle-monotone graph with width $\gamma < 180^\circ$ is a $t$-spanner with $t = 1/\cos \frac{\gamma}{2}$. Hence, we have the following result.

**Corollary 1.** For any set of points in the plane that are in convex position and for any integer $k \geq 1$, the graphs $\Theta_{4k+2}$ and $\Theta_{4k+4}$ have the stretch factor at most $1/\cos \left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ and $1/\cos \left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, respectively.

### 3.2 The graphs $\Theta_{4k+3}$ and $\Theta_{4k+5}$

We first assume that $G = \Theta_{4k+3}$. Here, we present an algorithm that finds an angle-monotone path $P$ between $p$ and $q$ in $G$ with a constant width. The algorithm is as follows. It first finds the path $P = (p = v_0, \ldots, v_l)$ which was introduced earlier. If $v_l = q$, then clearly $P = P$ is an angle-monotone path with width $\theta$, and we are done. Now, in the following we assume that $v_l \neq q$. Let $a$ be the topmost vertex of the triangle $T_{pq}$ and let $b \neq p$ be the other vertex of $T_{pq}$. Let $m$ be the midpoint of $ab$. The algorithm considers the following cases.

- **Case 1:** $q$ lies on the segment $am$. Now, let $Q = (q = a_0, \ldots, v_l)$ be the path constructed by the algorithm $\Theta$-Walk$(q, v_l)$. Then, the algorithm outputs the path $P = P \cup Q$.
- **Case 2:** $q$ lies on the segment $bm$. Let $P' = (q = u_0, \ldots, u_s)$ be the path in $G$ such that $u_{i+1} \in C_{2k+1}^{u_i}$ and $u_{i+1}$ is the closest point to $u_i$, and $u_s$ is the last vertex of the path $P'$ that lies in $T_{qp}$. Let $b'$ be the topmost vertex of the triangle $T_{qp}$ and let $a'$ be the bottommost vertex of $T_{qp}$. Let $m'$ be the midpoint of $a'b'$. Since $q \in C_0^b$, it is easy to see that $p$ lies on the segment $a'm'$. Now, there are two cases:
  - **(I):** $P$ and $P'$ have a common vertex $w$. The algorithm outputs the path $R$ which is formed by the portion of $P$ from $v_0$ to $w$ followed by the portion of $P'$ from $w$ to $q$.
  - **(II):** $P$ and $P'$ do not have any common vertex. Now, consider two following cases: (a) there is a vertex $q \neq q$ of the path $P'$ below the path $P$. (b) all vertices of $P'$ are above the path $P$. For the case (a), let
be the last vertex of $P'$ below the path $P$ and let $Q'$ be the constructed path by the algorithm $\Theta$-Walk($p, u_h$). Then, the algorithm outputs path $P = P' \cup Q'$. For the case (b), first the path $Q = \Theta$-Walk($q, v_l$) is constructed. Then, the algorithm outputs the path $P = P \cup Q$.

For more details, see Algorithm 3.2.

\begin{algorithm}[H]
\caption{ANGLE-MONOTONE-PATH-$\Theta_{4k+3}(p, q)$}
\begin{algorithmic}[1]
\State \textbf{output}: An angle-monotone path between $p$ and $q$ in $\Theta_{4k+3}$
\State $P := \emptyset$; \Comment{Initialize $P$ as empty}
\State Compute the path $P = (p = v_0, \ldots, v_l)$; \Comment{Compute path between $p$ and $q$}
\If{$v_l \neq q$}
\If{$q$ lies on the segment $am$}
\State $Q := \Theta$-Walk($q, v_l$); \Comment{Compute $Q$}
\State $P := P \cup Q$; \Comment{Combine $P$ and $Q$}
\EndIf
\Else
\State Compute the path $P' = (q = u_0, \ldots, u_s)$; \Comment{Compute $P'$}
\If{$P$ and $P'$ have a common vertex $w$}
\State $R :=$ the path which is formed by the portion of $P$ from $v_0$ to $w$ followed by the portion of $P'$ from $w$ to $q$; \Comment{Compute $R$}
\State $P := R$; \Comment{Replace $P$ with $R$}
\EndIf
\Else
\If{there is a vertex $g \neq q$ of the path $P'$ below the path $P$}
\State $u_h :=$ the last vertex of $P'$ below $P$; \Comment{Find last vertex below $P$}
\State $Q' := \Theta$-Walk($p, u_h$); \Comment{Compute $Q'$}
\State $P := P' \cup Q'$; \Comment{Combine $P'$ and $Q'$}
\EndIf
\Else
\State $Q := \Theta$-Walk($q, v_l$); \Comment{Compute $Q$}
\State $P := P \cup Q$; \Comment{Combine $P$ and $Q$}
\EndIf
\EndIf
\EndIf
\State \textbf{return} $P$.
\end{algorithmic}
\end{algorithm}

In the following, we show that the path $P$ returned by Algorithm 3.2 is an angle-monotone path between $p$ and $q$ with width $90^\circ + \frac{3\theta}{4}$. We first prove the following lemma.

Lemma 3. If $q$ lies on the segment $am$, then every edge $(a_i, a_{i+1})$ of the path $Q = (q = a_0, \ldots, v_l)$ lies in the wedge $W_{a_i}$.

\textbf{Proof.} Let $j$ be an integer such that $v_l \in C_j$. Since we assumed that $q$ is below $P \cup C_0$, we have $k + 1 \leq j \leq 2k + 1$. Consider the triangle $T_{qv_l}$. Let $x$ and $y$ be the two other vertices of $T_{qv_l}$ as depicted in Fig. 6(a). It is notable that the line passing through $p$ and $m$ is parallel to the line passing through $q$ and $y$. Then, since $q$ lies on the segment $am$, the point $p$ is below the line passing through $q$.
and $y$. Hence, because of the convexity of the points, no points of $Q$ lie in the triangle $\triangle qy$. Consider the lines $l_1$ and $l_2$, and the points $d_1$ and $d_2$ as defined in the proof of Lemma 1. By the reasons similar to the proof of Lemma 1, we can prove that the path $Q$ completely lies in the triangle $\triangle qy$. Then, for any edge $(a_i, a_{i+1})$ of $Q$, there is an integer $t$ with $j \leq t \leq 2k+1$ such that $a_{i+1} \in C_t^w$. Clearly, this shows that $(a_i, a_{i+1})$ lies in the wedge $W_{a_i}$.

Now, we prove the following lemma.

**Lemma 4.** If $q$ lies on the segment $bm$, then every edge $(r_i, r_{i+1})$ of the path $R$ lies in the wedge $U_{a_i}$.

**Proof.** According to Algorithm 3.2, the path $R$ is constructed when the paths $P = (v_1, \ldots, v_l)$ and $P' = (u_1, \ldots, u_n)$ have a common vertex. It is clear that for every edge $(v_i, v_{i+1})$ of the path $P$, we have $v_{i+1} \in C_{l+1}^O$, therefore $(v_i, v_{i+1})$ lies in the wedge $U_{v_i}$. On the other hand, for every edge $(u_i, u_{i+1})$ of $P'$, we have $u_{i+1} \in C_{2k+1}^O$. Therefore, $u_i \in C_{4k+1}^O$ or $u_i \in C_{2k+1}^O$. Hence, the edge $(u_{i+1}, u_i)$ lies in the wedge $U_{u_{i+1}}$. This completes the proof.

Let $Y_{\alpha}$ be a wedge with $Y_{\alpha} = \left( \bigcup_{i=3k+2}^{(k-1)\theta} C_{i+1}^O \right) \cup \left( C_{2k+1}^O \right)^{\prime} \left( (C_{2k+1}^O)^{\prime} \right)$ is the reflection of $C_{2k+1}^O$ with respect to the origin $O$). It is clear that the angle of $Y_{\alpha}$ is equal to $(k+1)\theta + \theta/2$. Now, we prove the following lemma.

**Lemma 5.** If $q$ lies on the segment $bm$ and the paths $P$ and $P'$ do not have any common vertex, and there is a vertex $y \neq q$ of the path $P'$ below the path $P$, then every edge $(c_i, c_{i+1})$ of the constructed path $P$ by Algorithm 3.2 lies in the wedge $Y_{c_i}$.

**Proof.** Let $u_h$ be the last vertex of $P'$ below $P$. According Algorithm 3.2, $P = P' \cup Q'$ that $Q'$ is the constructed path by $\Theta$-WALK($p$, $u_h$). It is clear that for every edge $(u_i, u_{i+1})$ of $P'$, we have $u_{i+1} \in C_{2k+1}^O$, and therefore $u_i \in (C_{2k+1}^O)^{\prime}$. Hence, $(u_{i+1}, u_i)$ lies in the wedge $Y_{u_{i+1}}$. Let $Q' = (p = a'_1, a'_2, \ldots, a'_{i+1})$. We claim that every edge $(a'_i, a'_{i+1})$ lies in the wedge $Y_{a'_{i+1}}$. Since $p$ lies on the segment $a'n'$, By the arguments similar to the proof of Lemma 3, the claim is proved. These show that if $(c_i, c_{i+1})$ be an edge of the path $P$, it lies in the wedge $Y_{c_i}$.}

Now, we have the following lemma.

**Lemma 6.** If $q$ lies on the segment $bm$ and the paths $P$ and $P'$ do not have any common vertex, and there is no vertex $y \neq q$ of the path $P'$ below the path $P$, then every edge $(r_i, r_{i+1})$ of the constructed path $P$ by Algorithm 3.2 lies in the wedge $U_{r_i}$.

**Proof.** Let $u_j$ be a vertex of $P'$ above the path $P$. Let $v_i$ be the last vertex of $P$ to the left of $u_j$ (see Fig. 6(b)). Since $p$ is to the left of $u_j$, the vertex $v_i$ always exist. Since there is no vertex $g \neq q$ of the path $P'$ below the path $P$, we have $u_{i-1} = q$. Now, consider the triangle $T_{v_i, u_{i+1}}$. Since $P$ and $P'$ have no common vertex, clearly $u_j \notin T_{v_i, u_{i+1}}$. Hence, if $v_i \neq p$, then the triangle $\triangle pu_{i+1}$ contains
the vertex \( v_i \) which contradicts the convexity of the points. Then, \( v_i = p \). On the other hand, since \( v_j \neq q \), we must have \( v_{i+1} \notin T_{q,s} \), and therefore \( v_i \in C_t^q \) with \( k + 1 \leq t < 2k + 1 \). Now, by the arguments similar to the proof of Lemma 3, we can prove that every edge \((a_i, a_{i+1})\) of the path \( Q \) lies in the wedge \( W_a \). Hence, it is clear that every edge \((r_i, r_{i+1})\) of the path \( P = P \cup \tilde{Q} \) lies in the wedge \( U_r \).

Based on Lemmas 3, 4, 5 and 6, any path constructed by Algorithm 3.2 is angle-monotone with width \((k+1)\theta + \frac{\theta}{2} \). Since \( \theta = \frac{360^\circ}{2k+1} \), we have \((k+1)\theta + \frac{\theta}{2} = 90^\circ + \frac{3\theta}{4} \). Then, the following theorem holds.

**Theorem 3.** For any set \( S \) of points in the plane that are in convex position and for any integer \( k \geq 1 \), \( \Theta_{4k+3} \) is angle-monotone with width \( 90^\circ + \frac{3\theta}{4} \).

By the arguments similar to the proof of Theorem 3, for \( G = \Theta_{4k+5} \) with \( k \geq 1 \), we can prove that the path \( P \) is an angle-monotone path from \( p \) to \( q \) with width \((k+1)\theta + \frac{\theta}{2} \). Since \( \theta = \frac{360^\circ}{4k+1} \), we have \((k+1)\theta + \frac{\theta}{2} = 90^\circ + \frac{5\theta}{4} \). Then, the following theorem holds.

**Theorem 4.** For any set \( S \) of points in the plane that are in convex position and for any integer \( k \geq 1 \), \( \Theta_{4k+5} \) is angle-monotone with width \( 90^\circ + \frac{5\theta}{4} \).

We close this section with the following result.

**Corollary 2.** For any set of points in the plane that are in convex position, the graphs \( \Theta_{4k+3} \) and \( \Theta_{4k+5} \) with \( k \geq 1 \) have the stretch factor at most \( 1/\cos \left( \frac{\pi}{4} + \frac{3\theta}{8} \right) \) and \( 1/\cos \left( \frac{\pi}{4} + \frac{5\theta}{8} \right) \), respectively.

### 4 Theta-graph \( \Theta_4 \)

In the following, we present two point sets, one in convex position and the other in non-convex position, to show that the graph \( \Theta_4 \) of the point set is not angle-monotone for any width \( \gamma > 0 \). Let \( p_0, p_2, p_3 \) and \( p_5 \) be the vertices of a rectangle with length 2 and width \( 1 + \epsilon \), where \( \epsilon > 0 \) is a small real number (see Fig. 7(a)). Let \( p_1 \) and \( p_4 \) be the midpoints of the segments \( p_0p_2 \) and \( p_3p_5 \), respectively. Now, let \( P = \{p_0, p_1, \ldots, p_5\} \). Consider the theta-graph \( \Theta_4 \) on \( P \). It is not hard to see that the edge set \( E \) of \( \Theta_4 \) is

\[
E = \{(p_0, p_1), (p_1, p_2), (p_2, p_3), (p_3, p_4), (p_4, p_5), (p_5, p_0)\}.
\]

Now, since \( p_0p_2 \) and \( p_3p_5 \) are parallel, it is obvious that for any \( 0 < \gamma < 180^\circ \), any path between \( p_1 \) and \( p_4 \) is not angle-monotone with width \( \gamma \).

Let \( P' = \{p'_0, p'_1, \ldots, p'_5\} \) be a copy of point set \( P \) such that the points of \( P' \) placed below the points of \( P \) as depicted in Fig. 7(b). Let \( V = P \cup P' \). It is easy to see that the edge set \( F \) of the theta-graph \( \Theta_4 \) on the point set \( V \) is

\[
F = E \cup \{(p'_0, p'_1), (p'_1, p'_2), (p'_2, p'_3), (p'_3, p'_4), (p'_4, p'_5), (p'_5, p'_0)\} \cup \{(p'_0, p_5), (p'_1, p_4), (p'_2, p_3)\}.
\]

It is obvious that for any \( 0 < \gamma < 180^\circ \), any path between \( p_1 \) and \( p_4 \) is not angle-monotone with width \( \gamma \). Now, we have the following theorem.

**Theorem 5.** For any angle \( 0 < \gamma < 180^\circ \), the graph \( \Theta_4 \) is not necessarily angle-monotone with width \( \gamma \).
5 Remarks

In Corollaries 1 and 2, we examined the stretch factor of the graphs $\Theta_{4k+2}$, $\Theta_{4k+3}$, $\Theta_{4k+4}$ and $\Theta_{4k+5}$ for the points in convex position. In [18], Bose et al., show that the stretch factor of the graphs $\Theta_{4k+2}$, $\Theta_{4k+3}$, $\Theta_{4k+4}$ and $\Theta_{4k+5}$ are at most $1+2\sin(\theta/2)\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$, $1+2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$ and $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$, respectively.

By comparing the results of Corollaries 1 and 2 with the results of [18], we find that the results of the corollaries do not improve the stretch factors known in [18].

In the following, we indicate whether the bounds on the width presented in Theorems 1, 2, 3 and 4 are tight or not. Consider the graph $\Theta_{4k+2}$. Fig. 8 shows that the upper bound on the width presented in Theorems 1 is tight. We place a vertex $c$ close to the lower corner of $T_{ab}$ that is sufficiently far from the vertex $b$. We also place a vertex $d$ close to the upper corner of $T_{ba}$ that is sufficiently far from the vertex $a$. Now, the graph $\Theta_{4k+2}$ of four points $a, b, c$ and $d$ is as shown in Fig. 8. We can easily see that each of the paths $acb$ and $adb$ are angle-monotone with width $90^\circ + \frac{\theta}{4} - \epsilon$, for some real number $\epsilon > 0$ that only depends on the distance between $c$ ($d$) and the lower corner (upper corner) of $T_{ab}$ ($T_{ba}$). If $\epsilon$ approaches zero, then the width approaches $90^\circ + \frac{\theta}{4}$.

For Theorems 2, 3 and 4, we do not know whether the bounds for the width is tight or not.

6 Conclusion

In this paper, we showed that for any set of points in the plane that are in convex position and for any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph $\Theta_{4k+i}$ is angle-monotone with width $90^\circ + \frac{i\theta}{4}$, where $\theta = \frac{360^\circ}{4k+i}$. Moreover, we presented two sets of points in the plane, one in convex position and the other in non-convex position, to show that for every $0 < \gamma < 180^\circ$, the graph $\Theta_4$ is not angle-monotone with width $\gamma$. Furthermore, we showed that the upper bound on the width presented in Theorems 1 is tight. It is notable that our technique in Section 3.2, does not work for $\Theta_5$ because by the proposed technique, the resulting path $P$ is angle-monotone with width $90^\circ + \frac{5\theta}{4}$. Since for $\Theta_5$, we have $\theta = \frac{2\pi}{5} = 72^\circ$. Then, $90^\circ + \frac{5\theta}{4} = 180^\circ$. We conjecture for any set of points in convex position, $\Theta_5$ is angle-monotone with a constant width. We tried to prove our conjecture but we did not succeed. Finally, we present the following conjecture.

Conjecture 1. For any set of points in the plane that are not convex position, for any integer $k \geq 1$ and any $i \in \{2, 3, 4, 5\}$, the theta-graph $\Theta_{4k+i}$ is angle-monotone with width $90^\circ + \frac{i\theta}{4}$, where $\theta = \frac{360^\circ}{4k+i}$.

References


**Biographies**

**Davood Bakhshesh** is an Assistant Professor at the Department of Computer Science at University of Bojnord in Bojnord, Iran and has been working in University of Bojnord since 2012. He received his BS degree in Computer Science from Vali-e-Asr University of Rafsanjan in 2007, and MS degree in Computer Science from Sharif University of Technology in 2009, and PhD Degree in Computer Science from Yazd University in 2017.

**Mohammad Farshi** is an Associate Professor at the Department of Computer Science at Yazd University in Yazd, Iran, and has been working in the Yazd faculty since 1999. He received his BS in Computer Science from Yazd University

List of captions

– Fig 1: An angle-monotone path between $x$ and $y$ with width $\gamma = 145^\circ$.
– Fig 2: Partition the plane into $m = 18$ cones with apex at $p$.
– Fig 3: The wedges $\mathcal{W}_O$ and $\mathcal{U}_O$ for the different values of $m$.
– Fig 4: The path $P$.
– Fig 5: Illustrating the proof of Lemma 1.
– Fig 6: Illustrating the proofs of Lemma 3 and Lemma 6.
– Fig 7: The point sets $P$ and $V$.
– Fig 8: The lower bound for the width of $\Theta_{4k+2}$.

Figures

\begin{figure}[h]
\centering
\includegraphics[width=.5\textwidth]{figure1}
\caption{An angle-monotone path between $x$ and $y$ with width $\gamma = 145^\circ$.}
\end{figure}
Fig. 2. Partition the plane into $m = 18$ cones with apex at $p$.

Fig. 3. The wedges $W_O$ and $U_O$ for the different values of $m$. 

(a) $m = 4k + 2$

(b) $m = 4k + 3$

(c) $m = 4k + 4$

(d) $m = 4k + 5$
Fig. 4. The path $P$.

Fig. 5. Illustrating the proof of Lemma 1.

(a) Illustrating the proof of Lemma 3.  
(b) Illustrating the proof of Lemma 6.

Fig. 6. Illustrating the proofs of Lemma 3 and Lemma 6.
(a) The point set $P$.

(b) The point set $V = P \cup P'$.

Fig. 7. The point sets $P$ and $V$.

Fig. 8. The lower bound for the width of $\Theta_{4k+2}$.