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Computing the population mean on the use of auxiliary information under ranked set sampling

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Abstract. In this manuscript, a generalized class of estimators has been developed for estimating finite population means in a Ranked Set Sampling (RSS) scheme. The expressions for bias and Mean Square Error (MSE) of the proposed class of estimators have been derived up to the first order of approximation. Some estimators are shown to be a member of the proposed class. The proposed class of estimators has been compared through the MSE criterion over the other existing member estimators of the proposed class of estimators. The theoretical conditions are obtained under which the proposed class of estimators has performed better. Efficiency comparisons, empirical studies, and simulation studies also delineate the soundness of our proposed generalized class of estimators under RSS.

1. Introduction

To reduce the sampling error, many researchers attempted to use additional information (highly correlated with the character under study), which is known as auxiliary information. This information is available for each unit and may be known well in advance. If it is not readily available for each population unit, information on it may be collected through past surveys. The study character, consider Y may be the field in agriculture survey and auxiliary character X may be the area under cultivation, Y may be the income of households and X the number of earning members in the household, Y may be the number of patients is being treated in the hospital, and X may be the number of doctors available in the hospital and so on auxiliary information can be stated. It is to be mentioned that Cochran [1] was the pioneer in using auxiliary information at the estimation stage. He envisages the ratio estimator for estimating the population mean or total of a variate under investigation. The ratio and product estimation methods are well-known methods for estimating the population means of a study variable using auxiliary information. When the correlation between the study variate and auxiliary variate is positive, the ratio estimator can be employed quite effectively. If the correlation between the study variate and auxiliary variate is negative (high), the product estimator envisaged by Robson [2] and rediscovered by Murthy [3] is used. Keeping this fact in view and also owing to the stronger intuitive appeal, survey statisticians are more inclined towards the use of ratio and product estimators in practice. The estimator's unbiased, ratio, and product are as follows, respectively:

$$\bar{y}_n = \frac{1}{n}\sum_{i=1}^{n} y_i = t_0, \tag{1}$$

$$\bar{y}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) = t_R, \tag{2}$$

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$$\bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right) = t_P, \tag{3}$$

where $\bar{y}$ and $\bar{x}$ are the sample means of the study variable Y and the auxiliary variable X, respectively. Also, $\bar{X}$ is the mean of the auxiliary variable X.

Mean Square Error (MSE) of the corresponding estimators in Eqs. (1) to (3) are, respectively, as:

$$MSE\left(t_0\right) = f\left(S_y^2\right) = Var\left(t_0\right), \tag{4}$$

$$MSE\left(t_R\right) = f\left(S_y^2 + R^2 S_x^2 + 2RS_{yx}\right), \tag{5}$$

$$MSE\left(t_P\right) = f\left(S_y^2 + R^2 S_x^2 + 2RS_{yx}\right), \tag{6}$$

where:

$$f = \left(\frac{1}{n} - \frac{1}{N}\right),$$

$$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^2,$$

$$S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^2,$$

$$S_{xy} = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(y_{hi} - \bar{Y}\right)\left(x_{hi} - \bar{X}\right), \quad \text{and}$$

$$R = \frac{\bar{Y}}{\bar{X}}.$$

Researchers are always keen to enhance their results; for this, they develop new methods or techniques. In finding a better substitute for simple random sampling, McIntyre [4] has propounded a Ranked Set Sampling (RSS) technique, which is far better than simple random sampling and economically efficient, too. Nowadays, RSS has been widely applied in medical sciences, biology, agriculture, environmental science, and many fields of statistics. When the measurements are cumbersome and extravagant, ranking the variables comparable to other sampling schemes is analogously easy and cost-effective. Ranking will be perfect if the rank of the observations within each set does tally with the numeric order of veiled X values; otherwise, it is imperfect. In the explanation of RSS, many grantors like Takahasi and Wakimoto [5], Dell and Clutter [6], Stokes [7], Samawi and Muttlak [8], Al-Saleh and Al-Omari [9], Bouza [10], Wolfe [11], Al-Omari et al. [12], Ai-Omari [13], Mandowara and Mehta [14], Al-Omari and Gupta [15], Pal and Singh [16], Vishwakarma et al. [17], Jeelani et al. [18], Noor Ul Amin et al. [19], Al-Omari and Haq [20], Saini and Kumar [21], and Singh and Vishwakarma [22] have contributed through different estimation procedures and techniquesin the various fields of RSS for the estimation of population parameters. Ahmed et al. [23] have given the predictive estimation of population mean using RSS, which shows that when natural (usual) unbiased, ratio and regression estimators are used as predictors give the corresponding predictive estimators same as natural unbiased, ratio and regression estimators under the RSS. Mehta et al. [24] introduced a general procedure for estimating finite population mean using RSS. Koyuncu and Al-Omari [25] developed generalized robust-regression-type estimators under different RSS. Vishwakarma and Singh [26,27] computed the effect of measurement errors on RSS estimators of the population mean and gave some applications to solar energy data.

In the procedure of ranked set sample, we have l bivariate random samples of size l from a population of size N, and are ranked within each sample concerning for ancillary variable X associated with Y. In the RSS procedure, we take the first smallest unit of the first data set size l, specify it for the first measurement unit, and scrap the rest of the units. Similarly, we take the second smallest observation of the second data set size l, specify it for the second observation, and scrap the rest. Proceeding this way, total l bivariate units for up to lth term are counted, and after k cycles of this procedure, total $n = kl$ bivariate RSS units are treated as Simple Random Sampling (SRS) data, too (used for calculation in SRS). In the extraction of RSS data, there are total kl^2 units, but only $n = kl$ units are counted for actual computation. $\left(X_{j(i)}, Y_{j[i]}\,;\, j = 1,2,3,...,k;\ i = 1,2,3,...,l\right)$ are the paired bivariate quantified sets of the ith units in the jth cycle. We have an unbiased estimator on the basis of study variable under RSS along with MSE as:

$$T_0 = \bar{y}_{[n]} = \frac{1}{n}\sum_{i=1}^{n} y_{[i]}, \tag{7}$$

$$MSE\left(T_0\right) = \bar{Y}^2\left(f\,C_y^2 - W_y^2\right) = Var\left(T_0\right). \tag{8}$$

Employing the concept of auxiliary variables, Samawi and Muttlak [8] and Bouza [28] have propounded ratio and product estimators for population mean using RSS as follows, respectively:

$$\bar{y}_{[R]} = \bar{y}_{[n]}\left(\frac{\bar{X}}{\bar{x}_{(n)}}\right) = T_{[R]}, \tag{9}$$

$$\bar{y}_{[P]} = \bar{y}_{[n]}\left(\frac{\bar{x}_{(n)}}{\bar{X}}\right) = T_{[P]}, \tag{10}$$

where:

$$\bar{y}_{[n]} = \frac{1}{n}\sum_{i=1}^{n} y_{[i]}, \qquad \bar{x}_{(n)} = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}, \quad \text{and}$$

$$\hat{R}_{RSS} = \frac{\bar{Y}_{RSS}}{\bar{X}_{RSS}} = \frac{\bar{y}_{[n]}}{\bar{x}_{(n)}}.$$

To obtain Bias and MSE, we have a ratio estimator from Eq. (9) as:

$$T_{[R]} = \bar{y}_{[R]} = \frac{\bar{y}_{[n]}}{\bar{x}_{(n)}}\bar{X} = H\left(\bar{x}_{(n)}, \bar{y}_{[n]}\right). \quad (11)$$

Using Taylor's series and from Eq. (11), we have:

$$T_{[R]} = H\left(\bar{X}, \bar{Y}\right) + H_0\left(\bar{y}_{[n]} - \bar{Y}\right) + H_1\left(\bar{x}_{(n)} - \bar{X}\right)$$
$$+ H_2\left(\bar{x}_{(n)} - \bar{X}\right)^2 + H_3\left(\bar{y}_{[n]} - \bar{Y}\right)^2$$
$$+ H_4\left(\bar{y}_{[n]} - \bar{Y}\right)\left(\bar{x}_{(n)} - \bar{X}\right) + ..., \quad (12)$$

Eq. (13) is shown in Box I, and such that it satisfies the following conditions:

(i) The function $H\left(\bar{x}_{(n)}, \bar{y}_{[n]}\right)$ is continuous and bounded in D (dimension);

(ii) The first and second-order partial derivatives exist and are continuous and bounded in D.

Therefore, from Eq. (12) and Appendix A, we have:

$$T_{[R]} \cong H\left(\bar{X}, \bar{Y}\right) + H_0\bar{Y}e_0 + H_1\bar{X}e_1 + H_2\bar{X}^2e_1^2$$
$$+ H_3\bar{Y}^2e_0^2 + H_4\bar{Y}\bar{X}e_0e_1 + O\left(e_i\right). \quad (14)$$

Taking expectation on both sides of Eq. (14) and from Appendix A, we get:

$$E\left(T_{[R]}\right) \cong H\left(\bar{X}, \bar{Y}\right) + H_2\bar{X}^2\left(fC_x^2 - W_{x(i)}^2\right)$$
$$+ H_3\bar{Y}^2\left(fC_y^2 - W_{y[i]}^2\right)$$
$$+ H_4\bar{Y}\bar{X}\left(fC_{yx} - W_{yx(i)}\right),$$
$$\cong \bar{Y} + \left(\frac{\bar{Y}}{\bar{X}^2}\right)\bar{X}^2\left(fC_x^2 - W_{x(i)}^2\right)$$
$$+ (0)\,\bar{Y}^2\left(fC_y^2 - W_{y[i]}^2\right) + \left(-\frac{1}{\bar{X}}\right)$$
$$\bar{Y}\bar{X}\left(fC_{yx} - W_{yx(i)}\right), E\left(T_{[R]} - \bar{Y}\right)$$
$$\cong \bar{Y}\left(fC_x^2 - W_{x(i)}^2\right) - \bar{Y}\left(fC_{yx} - W_{yx(i)}\right).$$

Hence,

$$Bias\left(T_{[R]}\right) = \bar{Y}\left[f\left(C_x^2 - C_{yx}\right) - \left(W_{x(i)}^2 - W_{yx(i)}\right)\right]. \quad (15)$$

From Eq. (14) we have:

$$\left(t_{[R]} - \bar{Y}\right) \cong H\left(\bar{X}, \bar{Y}\right) + H_0\bar{Y}e_0 + H_1\bar{X}e_1$$
$$+ O\left(e_i\right) - \bar{Y},$$
$$\left(t_{[R]} - \bar{Y}\right) \cong \bar{Y}e_0 - R\bar{X}e_1 + O\left(e_i\right). \quad (16)$$

Now, squaring and taking the expectation of both sides of Eq. (16) and from Appendix A, we get:

$$E\left(T_{[R]} - \bar{Y}\right)^2 \cong E\left(\bar{Y}e_0\right)^2 + R^2E\left(\bar{X}e_1\right)^2$$
$$-2RE\left(\bar{X}e_1\right)\left(\bar{Y}e_0\right),$$

$$MSE\left(T_{[R]}\right) \cong E\left(\bar{Y}e_0\right)^2 + R^2E\left(\bar{X}e_1\right)^2$$
$$-2RE\left(\bar{X}e_1\right)\left(\bar{Y}e_0\right),$$
$$\cong \bar{Y}^2E(e_0)^2 + R^2\bar{X}^2E(e_1)^2 - 2R\bar{X}\bar{Y}E\left(e_1e_0\right),$$
$$\cong f\left(\bar{Y}^2C_y^2 + R^2\bar{X}^2C_x^2 - 2R\bar{X}\bar{Y}\rho_{yx}C_xC_y\right)$$
$$-\left(\bar{Y}^2w_{y[i]}^2 + R^2\bar{X}^2w_{x(i)}^2 - 2R\bar{X}\bar{Y}w_{yx(i)}\right),$$
$$\cong f\left(S_y^2 + R^2S_x^2 - 2RS_{yx}\right)$$
$$-\frac{1}{l^2r}\left(\sum_{i=1}^{l}\tau_{y[i]}^2 + R^2\sum_{i=1}^{l}\tau_{x(i)}^2 - 2R\sum_{i=1}^{l}\tau_{yx(i)}\right).$$

Hence,

$$MSE\left(T_{[R]}\right) = f\left(S_y^2 + R^2S_x^2 - 2RS_{yx}\right)$$
$$-\left(T_y^2 + R^2T_x^2 - 2RT_{yx}\right). \quad (17)$$

$$\left.\begin{aligned}
&H\left(\bar{X}, \bar{Y}\right) = \tfrac{\bar{Y}}{\bar{X}}\bar{X} = \bar{Y}, \quad H_0\left(\bar{X}, \bar{Y}\right) = \frac{\partial\left(H\left(\bar{x}_{(n)}, \bar{y}_{[n]}\right)\right)}{\partial\bar{y}_{[n]}} = 1,\\
&H_1\left(\bar{X}, \bar{Y}\right) = \frac{\partial\left(H\left(\bar{x}_{(n)}, \bar{y}_{[n]}\right)\right)}{\partial\bar{x}_{(n)}} = -\frac{\bar{X}}{\bar{x}_{(n)}^2}\bar{y}_{[n]}\bigg|_{\left(\bar{X}, \bar{Y}\right)} = -\frac{\bar{Y}}{\bar{X}} = -R,\\
&H_2\left(\bar{X}, \bar{Y}\right) = \frac{1}{2}\frac{\partial^2\left(H\left(\bar{x}_{(n)}, \bar{y}_{[n]}\right)\right)}{\partial\bar{x}_{(n)}^2} = \frac{\bar{X}}{\bar{x}_{(n)}^3}\bar{y}_{[n]}\bigg|_{\left(\bar{X}, \bar{Y}\right)} = \frac{\bar{Y}}{\bar{X}^2},\\
&H_3\left(\bar{X}, \bar{Y}\right) = \frac{1}{2}\frac{\partial^2\left(H\left(\bar{x}_{(n)}, \bar{y}_{[n]}\right)\right)}{\partial\bar{y}_{[n]}^2} = 0,\\
&H_4\left(\bar{X}, \bar{Y}\right) = \frac{\partial^2\left(H\left(\bar{x}_{(n)}, \bar{y}_{[n]}\right)\right)}{\partial\bar{x}_{(n)}\partial\bar{y}_{[n]}} = -\frac{\bar{X}}{\bar{x}_{(n)}^2}\bigg|_{\left(\bar{X}, \bar{Y}\right)} = -\frac{1}{\bar{X}}.
\end{aligned}\right\} \quad (13)$$

Box I

Similarly, we have Bias and MSE of product estimator as:

$$Bias\left(T_{[P]}\right)=f\bar{Y}\left[\left(\rho_{yx}C_xC_y-w_{yx(i)}\right)\right], \tag{18}$$

$$MSE\left(T_{[P]}\right)=f\left(S_y^2+R^2S_x^2+2RS_{yx}\right) - \left(T_y^2+R^2T_x^2+2RT_{yx}\right). \tag{19}$$

Also, the MSE of the product estimator can be obtained by putting $R=-R$ in Eq. (19).

In the context of auxiliary variables, Mandowara and Mehta [14] have also given some ratio and product-type estimators under RSS, which are as follows:

$$\bar{y}_{rss[mm1]}=\bar{Y}_{RSS}\left(\frac{\bar{X}+C_x}{\bar{X}_{RSS}+C_x}\right)=\hat{T}_{[1]}, \tag{20}$$

$$\bar{y}_{rss[mm2]}=\bar{Y}_{RSS}\left(\frac{\bar{X}+\beta_2(x)}{\bar{X}_{RSS}+\beta_2(x)}\right)=\hat{T}_{[2]}, \tag{21}$$

$$\bar{y}_{rss[mm3]}=\bar{Y}_{RSS}\left(\frac{\beta_2(x)\bar{X}+C_x}{\beta_2(x)\bar{X}_{RSS}+C_x}\right)=\hat{T}_{[3]}, \tag{22}$$

$$\bar{y}_{rss[mm4]}=\bar{Y}_{RSS}\left(\frac{C_x\bar{X}+\beta_2(x)}{C_x\bar{X}_{RSS}+\beta_2(x)}\right)=\hat{T}_{[4]}, \tag{23}$$

$$\bar{y}_{rss[mm5]}=\bar{Y}_{RSS}\left(\frac{C_x\bar{X}_{RSS}+\beta_2(x)}{C_x\bar{X}+\beta_2(x)}\right)=\hat{T}_{[5]}, \tag{24}$$

and their Biases and MSEs correspondingly are as:

$$Bias\left(\bar{y}_{rss[mm1]}\right)=\bar{Y}\left[f\left(C_1^2C_x^2-C_1\rho_{yx}C_yC_x\right) - \left(C_1^2W_x^2-C_1W_{yx}\right)\right], \tag{25}$$

$$MSE\left(\bar{y}_{rss[mm1]}\right)=\left[f\left(S_y^2+R_1^2S_x^2-2R_1S_{yx}\right) - \left(T_y^2+R_1^2T_x^2-2R_1T_{yx}\right)\right], \tag{26}$$

$$Bias\left(\bar{y}_{rss[mm2]}\right)=\bar{Y}\left[f\left(C_2^2C_x^2-C_2\rho_{yx}C_yC_x\right) - \left(C_2^2W_x^2-C_2W_{yx}\right)\right], \tag{27}$$

$$MSE\left(\bar{y}_{rss[mm2]}\right)=\left[f\left(S_y^2+R_2^2S_x^2-2R_2S_{yx}\right) - \left(T_y^2+R_2^2T_x^2-2R_2T_{yx}\right)\right], \tag{28}$$

$$Bias\left(\bar{y}_{rss[mm3]}\right)=\bar{Y}\left[f\left(C_3^2C_x^2-C_3\rho_{yx}C_yC_x\right) - \left(C_3^2W_x^2-C_3W_{yx}\right)\right], \tag{29}$$

$$MSE\left(\bar{y}_{rss[mm3]}\right)=\left[f\left(S_y^2+R_3^2S_x^2-2R_3S_{yx}\right) - \left(T_y^2+R_3^2T_x^2-2R_3T_{yx}\right)\right], \tag{30}$$

$$Bias\left(\bar{y}_{rss[mm4]}\right)=\bar{Y}\left[f\left(C_4^2C_x^2-C_4\rho_{yx}C_yC_x\right) - \left(C_4^2W_x^2-C_4W_{yx}\right)\right], \tag{31}$$

$$MSE\left(\bar{y}_{rss[mm4]}\right)=\left[f\left(S_y^2+{R_4}^2S_x^2-2R_4S_{yx}\right) - \left(T_y^2+R_4^2T_x^2-2R_4T_{yx}\right)\right], \tag{32}$$

$$Bias\left(\bar{y}_{rss[mm5]}\right)=\bar{Y}\left[f\left(C_4\rho_{yx}C_yC_x\right) - \left(C_4W_{yx(i)}\right)\right], \tag{33}$$

$$MSE\left(\bar{y}_{rss[mm5]}\right)=\left[f\left(S_y^2+{R_4}^2S_x^2+2R_4S_{yx}\right) - \left(T_y^2+R_4^2T_x^2+2R_4T_{yx}\right)\right]. \tag{34}$$

2. Proposed generalized class of estimators under RSS

Motivated by Upadhyaya and Singh [29], Sisodia and Dwivedi [30], we have suggested a class of estimators under RSS as:

$$T=\bar{Y}_{RSS}\left(\frac{A\bar{X}+B}{A\bar{X}_{RSS}+B}\right)^{\alpha}. \tag{35}$$

For various values of A, B, and $C\alpha$, we can get a class ratio and product type estimators, where $\bar{Y}_{RSS}$ and $\bar{X}_{RSS}$ are earlier mentioned in Section 1, and A and B can be C_x (coefficient of variation), σ_x (standard deviation), $\beta_1(x)$ (skewness), $\beta_2(x)$ (kurtosis) and ρ (correlation coefficient).

To obtain Bias and MSE, we have proposed an estimator as:

$$T=\bar{Y}_{RSS}\left(\frac{A\bar{X}+B}{A\bar{X}_{RSS}+B}\right)^{\alpha}=\bar{y}_{[n]}\left(\frac{A\bar{X}+B}{A\bar{x}_{(n)}+B}\right)^{\alpha} = H^*\left(\bar{x}_{(n)},\bar{y}_{[n]}\right). \tag{36}$$

Using Taylor's series on Eq. (36), we have:

$$\left.\begin{aligned}
&H^*\left(\bar{X},\bar{Y}\right)=\bar{Y},\ H_0^*\left(\bar{X},\bar{Y}\right)=\tfrac{\partial\left(H^*\left(\bar{x}_{(n)},\bar{y}_{[n]}\right)\right)}{\partial\bar{y}_{[n]}}=1,\\
&H_1^*\left(\bar{X},\bar{Y}\right)=\tfrac{\partial\left(H^*\left(\bar{x}_{(n)},\bar{y}_{[n]}\right)\right)}{\partial\bar{x}_{(n)}}=-\tfrac{(A\bar{X}+B)^{\alpha}\bar{y}_{[n]}A\alpha}{(A\bar{x}_{(n)}+B)^{\alpha+1}}\Big|_{(\bar{X},\bar{Y})}=-\tfrac{A\alpha\bar{Y}}{A\bar{X}+B}=-R_T,\\
&H_2^*\left(\bar{X},\bar{Y}\right)=\tfrac{1}{2}\tfrac{\partial^2\left(H^*\left(\bar{x}_{(n)},\bar{y}_{[n]}\right)\right)}{\partial\bar{x}_{(n)}^2}=-\tfrac{1}{2}\tfrac{(A\bar{X}+B)^{\alpha}\bar{y}_{[n]}A\alpha(\alpha+1)}{(A\bar{x}_{(n)}+B)^{\alpha+2}}\Big|_{(\bar{X},\bar{Y})}=\tfrac{A^2\alpha(\alpha+1)\bar{Y}}{\left(A\bar{X}+B\right)^2},\\
&H_3^*\left(\bar{X},\bar{Y}\right)=\tfrac{1}{2}\tfrac{\partial^2\left(H^*\left(\bar{x}_{(n)},\bar{y}_{[n]}\right)\right)}{\partial\bar{y}_{[n]}^2}=0,\\
&H_4^*\left(\bar{X},\bar{Y}\right)=\tfrac{\partial^2\left(H^*\left(\bar{x}_{(n)},\bar{y}_{[n]}\right)\right)}{\partial\bar{x}_{(n)}\partial\bar{y}_{[n]}}=-\tfrac{(A\bar{X}+B)^{\alpha}A\alpha}{(A\bar{x}_{(n)}+B)^{\alpha+1}}\Big|_{(\bar{X},\bar{Y})}=-\tfrac{A\alpha}{A\bar{X}+B}.
\end{aligned}\right\}\quad(38)$$

Box II

$$T=H^*\left(\bar{X},\bar{Y}\right)+H_0^*\left(\bar{y}_{[n]}-\bar{Y}\right)+H_1^*\left(\bar{x}_{[n]}-\bar{X}\right)$$
$$+H_2^*\left(\bar{x}_{[n]}-\bar{X}\right)^2+H_3^*\left(\bar{y}_{[n]}-\bar{Y}\right)^2$$
$$+H_4^*\left(\bar{y}_{[n]}-\bar{Y}\right)\left(\bar{x}_{[n]}-\bar{X}\right)+....,\quad(37)$$

Eq. (38) is shown in Box II, and such that it satisfies the following conditions:

(i) The function $H^*\left(\bar{x}_{(n)},\bar{y}_{[n]}\right)$ is continuous and bounded in D (Dimension);

(ii) The first and second-order partial derivatives exist and are continuous and bounded in D.

Therefore, from Eq. (37) and Appendix A, we have:

$$T\cong H^*\left(\bar{X},\bar{Y}\right)+H_0^*\bar{Y}e_0+H_1^*\bar{X}e_1+H_2^*\bar{X}^2e_1^2$$
$$+H_3^*\bar{Y}^2e_0^2+H_4^*\bar{Y}\bar{X}e_0e_1+O\left(e_i\right).\quad(39)$$

Taking expectation on both sides of Eq. (39) and from Appendix A, we get:

$$E\left(T\right)\cong H^*\left(\bar{X},\bar{Y}\right)+H_2^*\bar{X}^2\left(fC_x^2-W_{x(i)}^2\right)$$
$$+H_3^*\bar{Y}^2\left(fC_y^2-W_{y[i]}^2\right)$$
$$+H_4^*\bar{Y}\bar{X}\left(fC_{yx}-W_{yx(i)}\right).$$

$$\cong\bar{Y}+\left(\frac{A^2\alpha\left(\alpha+1\right)\bar{Y}}{2\left(A\bar{X}+B\right)^2}\right)\bar{X}^2\left(fC_x^2-W_{x(i)}^2\right)$$
$$+\left(0\right)\bar{Y}^2\left(fC_y^2-W_{y[i]}^2\right)$$
$$+\left(-\frac{A\alpha}{A\bar{X}+B}\right)\bar{Y}\bar{X}\left(fC_{yx}-W_{yx(i)}\right),$$

$$E\left(T-\bar{Y}\right)\cong\bar{Y}\left(\frac{\alpha\left(\alpha+1\right)}{2}C_0^2\left(fC_x^2-W_{x(i)}^2\right)\right)$$
$$-\bar{Y}\left(\alpha C_0\left(fC_{yx}-W_{yx(i)}\right)\right).$$

Hence,

$$Bias\left(T\right)\cong\bar{Y}\left[\left(\frac{\alpha\left(\alpha+1\right)}{2}C_0^2\left(f\,C_x^2-W_{x(i)}^2\right)\right)\right.$$
$$\left.-\left(\alpha C_0\left(f\,C_{yx}-W_{yx(i)}\right)\right)\right].\quad(40)$$

From Eq. (39) we have:

$$\left(T-\bar{Y}\right)\cong\left(H^*\left(\bar{X},\bar{Y}\right)+H_0^*\bar{Y}e_0+H_1^*\bar{X}e_1+O\left(e_i\right)-\bar{Y}\right),$$
$$\left(T-\bar{Y}\right)\cong\left(\bar{Y}e_0-R_T\bar{X}e_1+O\left(e_i\right)\right).\quad(41)$$

Squaring and taking expectation on both sides of Eq. (41) and from Appendix A, we get:

$$E\left(\hat{T}-\bar{Y}\right)^2\cong\bar{Y}^2E\left(e_0^2\right)-2R_T\bar{X}\bar{Y}E\left(e_0e_1\right)$$
$$+R_T^2\bar{X}^2E\left(e_1^2\right),$$

$$MSE\left(\hat{T}\right)\cong f\left(\bar{Y}^2C_y^2+R_T^2\bar{X}^2C_x^2\right.$$
$$\left.-2R_T\bar{X}\bar{Y}\rho_{yx}C_yC_x\right)-\left(\bar{Y}^2W_y^2\right.$$
$$\left.+R_T^2\bar{X}^2W_x^2-2R_T\bar{X}\bar{Y}W_{yx}\right),$$

$$\cong f\left(S_y^2+R_T^2S_x^2-2R_TS_{yx}\right)$$
$$-\frac{1}{l^2r}\left(\sum_{i=1}^{l}\tau_{y[i]}^2+R_T^2\sum_{i=1}^{l}\tau_{x(i)}^2-2R_T\sum_{i=1}^{l}\tau_{yx(i)}\right).$$

$$MSE(\hat{T})\cong\left[f\left(S_y^2+{R_T}^2S_x^2-2R_TS_{yx}\right)\right.$$
$$\left.-\left(T_y^2+R_T^2T_x^2-2R_TT_{yx}\right)\right].\quad(42)$$

The optimum value of α to minimize the can easily be found by equating its derivative to zero. i.e.:

$$\frac{\partial}{\partial\alpha}\left(MSE(\hat{T})\right)=0 \text{ and } \frac{\partial^2}{\partial\alpha^2}\left(MSE(\hat{T})\right)>0,$$

$$\frac{\partial}{\partial\alpha}\Big(f\left(S_y^2+R_T^2S_x^2-2R_TS_{yx}\right) - \left(T_y^2+R_T^2T_x^2-2R_TT_{yx}\right)\Big)=0,$$

$$\frac{\partial}{\partial\alpha}\left(fS_y^2-T_y^2\right)+\frac{\partial}{\partial\alpha}\left(fR_T^2S_x^2-R_T^2T_x^2\right) -2\frac{\partial}{\partial\alpha}\left(fR_TS_{yx}-R_TT_{yx}\right)=0,$$

i.e.: $0+\left(fS_x^2-T_x^2\right)\frac{\partial}{\partial\alpha}\left(R_T^2\right)-2\left(fS_{yx}-T_{yx}\right)\frac{\partial}{\partial\alpha}\left(R_T\right)=0,$

$$\left(fS_x^2-T_x^2\right)\left(\frac{A\bar{Y}}{A\bar{X}+B}\right)^2(2\alpha) -2\left(fS_{yx}-T_{yx}\right)\left(\frac{A\bar{Y}}{A\bar{X}+B}\right)=0, \tag{43}$$

where:

$$\alpha=\frac{\left(fS_{yx}-T_{yx}\right)}{\left(fS_x^2-T_x^2\right)\left(\frac{A\bar{Y}}{A\bar{X}+B}\right)}=\frac{\left(fS_{yx}-T_{yx}\right)}{\left(fS_x^2-T_x^2\right)}\left(\frac{A\bar{X}+B}{A\bar{Y}}\right).$$

We denote the value of α as α^*, therefore $\alpha^*=\frac{\left(fS_{yx}-T_{yx}\right)}{\left(fS_x^2-T_x^2\right)}\left(\frac{A\bar{X}+B}{A\bar{Y}}\right)$, and by differentiating Eq. (43) with respect to α another time, we have $\frac{\partial^2}{\partial\alpha^2}\left(MSE(\hat{T})\right)=2\left(fS_x^2-T_x^2\right)\left(\frac{A\bar{Y}}{A\bar{X}+B}\right)^2>0$, which proves optimality.

Therefore, we replace α with α^* in the expression of MSE in Eq. (43) and we obtain the minimum MSE of the proposed estimator as follows:

$$MSE_{\min}(\hat{T})=\Big[f(S_y^2+{R_T^*}^2S_x^2-2R_T^*S_{yx}) - \left(T_y^2+R_T^{*2}T_x^2-2R_T^*T_{yx}\right)\Big]=\hat{T}_{[opt]}, \tag{44}$$

where $R_T^*=\left(\frac{A\alpha^*\bar{Y}}{A\bar{X}+B}\right)=\frac{fS_{yx}-T_{yx}}{fS_x^2-T_x^2}$.

3. Particular cases of the proposed class of estimators

Through Eq. (33) in the Section 2, we have:

$$T=\bar{Y}_{RSS}\left(\frac{A\bar{X}+B}{A\bar{X}_{RSS}+B}\right)^\alpha. \tag{45}$$

Now, for different values of A, B, and α we can get a class of estimators, shown in Table 1. Hence, we obtain a class of ratio and product estimators under RSS using standard deviation σ_x, coefficient of variation C_x, coefficient of skewness $\beta_1(x)$, coefficient of kurtosis $\beta_2(x)$, auxiliary variable X, and coefficient of correlation ρ of both study and auxiliary variables. Also, we can obtain the expression of Biases and MSEs of the above class of estimators $\hat{T}_i$ for $i=0,R,P,1,2,3,...,13$ and $\alpha=-1,1$ as:

$$Bias\left(T_i\right)=\bar{Y}\Bigg[\left(\frac{\alpha\left(\alpha+1\right)}{2}C_i^2\left(fC_x^2-W_{x(i)}^2\right)\right) - \left(\alpha C_i\left(fC_{yx}-W_{yx(i)}\right)\right)\Bigg], \tag{46}$$

Table 1. Class of ratio and product estimators.

	A	**B**	**α**	**Estimators**
$\hat{T}_{[0]}$	0	0	0	$\bar{Y}_{RSS}$
$\hat{T}_{[R]}$	1	0	1	$\bar{Y}_{RSS}\left(\frac{\bar{X}}{\bar{X}_{RSS}}\right)$
$\hat{T}_{[P]}$	1	0	−1	$\bar{Y}_{RSS}\left(\frac{\bar{X}_{RSS}}{\bar{X}}\right)$
$\hat{T}_{[1]}$	1	C_x	1	$\bar{Y}_{RSS}\left(\frac{\bar{X}+C_x}{\bar{X}_{RSS}+C_x}\right)$
$\hat{T}_{[2]}$	1	$\beta_2(x)$	1	$\bar{Y}_{RSS}\left(\frac{\bar{X}+\beta_2(x)}{\bar{X}_{RSS}+\beta_2(x)}\right)$
$\hat{T}_{[3]}$	$\beta_2(x)$	C_x	1	$\bar{Y}_{RSS}\left(\frac{\beta_2(x)\bar{X}+C_x}{\beta_2(x)\bar{X}_{RSS}+C_x}\right)$
$\hat{T}_{[4]}$	C_x	$\beta_2(x)$	1	$\bar{Y}_{RSS}\left(\frac{C_x\bar{X}+\beta_2(x)}{C_x\bar{X}_{RSS}+\beta_2(x)}\right)$
$\hat{T}_{[5]}$	1	ρ	1	$\bar{Y}_{RSS}\left(\frac{\bar{X}+\rho}{\bar{X}_{RSS}+\rho}\right)$
$\hat{T}_{[6]}$	1	σ_x	−1	$\bar{Y}_{RSS}\left(\frac{\bar{X}_{RSS}+\sigma_x}{\bar{X}+\sigma_x}\right)$
$\hat{T}_{[7]}$	$\beta_1(x)$	σ_x	−1	$\bar{Y}_{RSS}\left(\frac{\beta_1(x)\bar{X}_{RSS}+\sigma_x}{\beta_1(x)\bar{X}+\sigma_x}\right)$
$\hat{T}_{[8]}$	$\beta_2(x)$	σ_x	−1	$\bar{Y}_{RSS}\left(\frac{\beta_2(x)\bar{X}_{RSS}+\sigma_x}{\beta_2(x)\bar{X}+\sigma_x}\right)$
$\hat{T}_{[9]}$	1	C_x	−1	$\bar{Y}_{RSS}\left(\frac{\bar{X}_{RSS}+C_x}{\bar{X}+C_x}\right)$
$\hat{T}_{[10]}$	1	$\beta_2(x)$	−1	$\bar{Y}_{RSS}\left(\frac{\bar{X}_{RSS}+\beta_2(x)}{\bar{X}+\beta_2(x)}\right)$
$\hat{T}_{[11]}$	$\beta_2(x)$	C_x	−1	$\bar{Y}_{RSS}\left(\frac{\beta_2(x)\bar{X}_{RSS}+C_x}{\beta_2(x)\bar{X}+C_x}\right)$
$\hat{T}_{[12]}$	C_x	$\beta_2(x)$	−1	$\bar{Y}_{RSS}\left(\frac{C_x\bar{X}_{RSS}+\beta_2(x)}{C_x\bar{X}+\beta_2(x)}\right)$
$\hat{T}_{[13]}$	1	ρ	−1	$\bar{Y}_{RSS}\left(\frac{\bar{X}_{RSS}+\rho}{\bar{X}+\rho}\right)$

$$MSE\left(\hat{T}_i\right) = \left[f\left(S_y^2 + R_i^2 S_x^2 - 2R_i S_{yx}\right) - \left(T_y^2 + R_i^2 T_x^2 + 2R_i T_{yx}\right)\right]. \qquad (47)$$

4. Efficiency comparisons

To show the performance of the proposed estimators, we have compared theoretically in two cases:

- **Case 1:** Comparison with ratio estimator from Eqs. (17) and (42):

$$MSE(\hat{T}) < MSE(T_{[R]}), \quad \text{if:}$$

$$f\left(R_T^2 S_x^2 - 2R_T S_{yx}\right) - \left(R_T^2 T_x^2 - 2R_T T_{yx}\right) < f\left(R^2 S_x^2 - 2RS_{yx}\right) - \left(R^2 T_x^2 - 2RT_{yx}\right),$$

$$R_T^2\left(f\,S_x^2 - T_x^2\right) - 2R_T\left(f\,S_{yx} - T_{yx}\right) < R^2\left(f\,S_x^2 - T_x^2\right) - 2R\left(f\,S_{yx} - T_{yx}\right),$$

$$\left(f\,S_x^2 - T_x^2\right)\left({R_T}^2 - R^2\right) < 2\left(f\,S_{yx} - T_{yx}\right)\left(R_T - R\right),$$

$$\frac{2\left(f\,S_{yx} - T_{yx}\right)}{\left(f\,S_x^2 - T_x^2\right)} - R < R_T < R. \qquad (48)$$

- **Case 2:** Comparison with product estimator from Eqs. (19) and (42):

$$MSE(\hat{T}) < MSE(T_{[P]}), \quad \text{if:}$$

$$f\left(R_T^2 S_x^2 - 2R_T S_{yx}\right) - \left(R_T^2 T_x^2 - 2R_T T_{yx}\right) < f\left(R^2 S_x^2 + 2RS_{yx}\right) - \left(R_T^2 T_x^2 + 2R_T T_{yx}\right),$$

$$R_T^2\left(f\,S_x^2 - T_x^2\right) - 2R_T\left(f\,S_{yx} - T_{yx}\right) < R^2\left(f\,S_x^2 - T_x^2\right) + 2R\left(f\,S_{yx} - T_{yx}\right),$$

$$\left(f\,S_x^2 - T_x^2\right)\left({R_T}^2 - R^2\right) < 2\left(f\,S_{yx} - T_{yx}\right)\left(R_T + R\right),$$

$$-R < R_T < \frac{2\left(f\,S_{yx} - T_{yx}\right)}{\left(f\,S_x^2 - T_x^2\right)} + R. \qquad (49)$$

The above efficiency comparison clearly shows that the proposed generalized class of estimators of the population mean under RSS is more efficient than the ratio and product estimators of the population mean under RSS. Also, the class of different ratio and product estimators will be more efficient than the correspondingly natural ratio estimator and product estimators in RSS if both Cases 1 and 2 are satisfied with the conditions. Empirical study and simulation study will also clearly illustrate the efficiency.

5. Empirical study

Numerically, to explore the properties of the proposed generalized class of the estimators over member estimators of the proposed class of estimators for mean in RSS and over unbiased estimator. We are compiling two natural data sets from Singh [31;1111-1113]:

- **Data set-I:** Concerns the agricultural loans [in thousand dollars ($000)] in different states of the USA in 1997 of all outstanding operating banks. y: Farm loans (real estate), x: Farm loans (non-real estate). The required values for the estimation of means are as:

$N = 50, \quad X = 43908.0, \quad Y = 27771.730,$

$\bar{X} = 878.160, \quad \bar{Y} = 555.430,$

$S_x^2 = 1176526, \quad S_y^2 = 342021.5,$

$C_x^2 = 1.52560, \quad C_y^2 = 1.10860,$

$S_{xy} = 509910.410, \quad \beta_1(x) = 2.5914,$

$\beta_2(x) = 4.6171, \quad \rho_{xy} = 0.8038.$

Data set-II: Concerns the hypothetical situation of a small village having only 30 old persons (aged over 50 years). y: Duration of the sleep (in minutes), x: Age in years. The required values for the estimation of means are as:

$N = 30, \quad X = 2018.0, \quad Y = 11526.0,$

$\bar{X} = 67.2670, \quad \bar{Y} = 384.20,$

$S_x^2 = 85.2370, \quad S_y^2 = 3582.580,$

$C_x^2 = 0.01880, \quad C_y^2 = 0.02430,$

$S_{xy} = -472.6070, \quad \beta_1(x) = 0.1982,$

$\beta_2(x) = 3.7316, \quad \rho_{xy} = -0.8552.$

We have taken 25 RSS samples from both the population data sets. In Data set-I, we have taken set size $l = 4$ with $k = 3$ replications, so that $n = lk = 12$. Similarly, in Data set-II, we have taken set size $l = 3$ with $k = 3$ replications, so that $n = lk = 9$. Further, for this 25 RSS sample data from both population data sets, we have calculated MSEs (rounded to 0) of the

Table 2. MSE of the ratio types estimators under RSS and t_0.

Population 1, $N = 50$, $\rho_{xy} = 0.8038$										7
	$MSE(t_0) = 28502$						$MSE(t_R) = 13972$			
Samples	T_x^2	T_y^2	T_{yx}	$\hat{T}_{[R]}$	$\hat{T}_{[1]}$	$\hat{T}_{[2]}$	$\hat{T}_{[3]}$	$\hat{T}_{[4]}$	$\hat{T}_{[5]}$	$\hat{T}_{[opt]}$
1	62424	15987	30598	11718	11648	11699	11714	11661	11706	8543
2	48927	8748	11861	654	652	653	654	652	653	650
3	26863	9019	15193	13426	13310	13395	13420	13332	13406	9014
4	21693	14955	17363	12301	12149	12260	12293	12178	12275	5275
5	29251	10525	16318	12387	12273	12356	12380	12294	12367	8017
6	83106	15782	35677	10073	10056	10069	10072	10059	10070	9609
7	53017	1012	22236	10769	10715	10755	10766	10725	10760	9268
8	24200	5451	11187	12990	12889	12963	12985	12908	12973	9779
9	40657	12535	21385	12222	12122	12195	12216	12141	12205	8202
10	38625	9313	16091	9561	9488	9541	9557	9501	9548	7457
11	31944	6107	13437	12083	12000	12061	12078	12015	12068	9623
12	36527	10419	13017	5406	5344	5389	5402	5355	5395	3959
13	72105	18408	34414	10251	10196	10236	10248	10207	10242	7577
14	30509	10624	17078	12745	12632	12715	12739	12653	12725	8313
15	74493	27343	45122	13906	13790	13875	13899	13812	13886	864
16	30734	9869	16268	12386	12278	12357	12379	12298	12367	8415
17	67055	13063	29246	11078	11036	11067	11076	11044	11071	9775
18	60603	19769	29675	7496	7425	7477	7492	7438	7484	4344
19	37511	11311	20253	13274	13168	13245	13268	13188	13255	9020
20	60893	12205	26437	10848	10799	10834	10845	10808	10839	9357
21	23460	9287	9229	6974	6883	6949	6969	6900	6958	4379
22	33326	15582	18815	8857	8744	8827	8851	8765	8837	4256
23	53109	16761	28830	12434	12337	12408	12429	12355	12417	7587
24	59025	3457	12803	3097	3130	3105	3098	3124	3102	2453
25	40336	9777	19515	12744	12655	12720	12738	12672	12728	9575

estimators for lighting the properties of estimators, and results are shown in Tables 2 and 3, respectively.

Later, we took different RSS samples with different set sizes with different replications. We have drawn, total $n = lk = 6$, 8, 10, 9, 12, and 15, RSS data from Data set-I having set sizes $l = 3$, 4, 5 with $k = 2$, 3 replications. Similarly, we have drawn, total $n = lk = 4$, 6, and 9, RSS data from the Data set-II having set sizes $l = 2$, 3 with $k = 2$, 3 replications. Further, with this RSS data (having different set sizes and replications) from both population data sets, we have calculated Relative Efficiencies (RE) for more clearance on lighting the properties of the estimators. The results through relative efficiencies calculation are shown in Tables 4 and 5.

6. Monte-Carlo simulation

We have carried out a Monte-Carlo simulation study to highlight the properties of the proposed generalized class of estimators in RSS over the unbiased estimator and extracted estimators from the proposed estimators. Monte-Carlo simulation is carried out in R Studio [32] by taking a random bivariate unit from a population

Table 3. MSE of the product types estimators under RSS and t_0.

Population 2, $N = 30$, $\rho_{xy} = -0.8552$													
	$MSE(t_0) = 236$								$MSE(t_P) = 63$				
Samples	T_x^2	T_y^2	T_{yx}	$\hat{T}_{[P]}$	$\hat{T}_{[6]}$	$\hat{T}_{[7]}$	$\hat{T}_{[8]}$	$\hat{T}_{[9]}$	$\hat{T}_{[10]}$	$\hat{T}_{[11]}$	$\hat{T}_{[12]}$	$\hat{T}_{[13]}$	$\hat{T}_{[opt]}$
1	2	55	10	49	52	69	49	50	49	49	69	48	48
2	4	188	27	44	41	36	43	42	44	44	36	45	35
3	3	60	12	32	39	64	33	35	32	32	64	31	17
4	3	97	15	34	37	53	35	35	34	34	53	33	30
5	4	227	28	31	24	11	29	27	31	31	11	32	4
6	3	43	8	30	37	64	32	33	30	30	64	29	19
7	3	53	12	45	51	74	46	48	45	45	73	44	36
8	3	82	14	49	51	64	49	49	49	49	64	48	48
9	4	153	23	34	34	40	34	34	34	34	40	34	34
10	5	157	26	31	35	45	32	33	32	32	45	31	10
11	4	143	22	49	48	51	49	49	49	49	51	49	48
12	2	77	11	42	43	57	42	42	42	42	56	42	42
13	4	176	25	43	41	38	42	42	43	43	38	44	38
14	1	137	13	24	19	19	22	21	24	24	19	25	17
15	2	71	8	11	17	40	13	14	12	11	39	11	7
16	2	63	10	62	61	70	61	61	62	62	70	62	61
17	3	143	18	39	37	37	38	38	39	39	37	40	35
18	3	52	11	59	62	80	60	60	59	59	79	59	57
19	4	210	27	35	29	18	33	32	35	35	18	36	15
20	3	91	15	37	42	59	39	40	38	37	59	37	32
21	3	213	26	42	35	20	40	38	42	42	20	43	14
22	3	118	21	62	62	68	62	62	62	62	68	62	62
23	1	79	10	54	52	59	53	53	54	54	59	55	52
24	3	140	20	50	48	49	49	48	50	50	49	50	47
25	4	204	29	52	47	36	50	49	52	52	36	52	29

Table 4. RE of the $\hat{T}_{[i]}$ (ratio) and $\hat{T}_{opt}$ under RSS over unbiased estimator t_0.

		Population 1, $N = 50$, $\rho_{xy} = 0.8038$							
k	m	$\hat{T}_R$	$\hat{T}_{[R]}$	$\hat{T}_{[1]}$	$\hat{T}_{[2]}$	$\hat{T}_{[3]}$	$\hat{T}_{[4]}$	$\hat{T}_{[5]}$	$\hat{T}_{opt}$
2	3	2.04	2.85	2.87	2.85	2.85	2.86	2.85	3.25
3		2.04	3.93	3.96	3.93	3.93	3.95	3.93	4.78
2	4	2.04	2.93	2.95	2.94	2.93	2.95	2.93	4.36
3		2.04	4.87	4.88	4.87	4.87	4.88	4.87	5.06
2	5	2.04	3.57	3.62	3.58	3.57	3.61	3.57	6.12
3		2.04	5.19	5.27	5.21	5.19	5.25	5.20	8.44

Table 5. RE of the $\hat{T}_{[i]}$ (product) and $\hat{T}_{opt}$ under RSS over unbiased estimator t_0.

		Population 2, $N = 30$, $\rho_{xy} = -0.8552$										
k	m	$\hat{T}_P$	$\hat{T}_{[P]}$	$\hat{T}_{[6]}$	$\hat{T}_{[7]}$	$\hat{T}_{[8]}$	$\hat{T}_{[9]}$	$\hat{T}_{[10]}$	$\hat{T}_{[11]}$	$\hat{T}_{[12]}$	$\hat{T}_{[13]}$	$\hat{T}_{opt}$
2	2	3.71	4.60	4.39	3.58	4.55	4.50	4.60	4.60	3.59	4.63	4.74
3		3.71	3.82	3.93	3.68	3.86	3.89	3.82	3.82	3.69	3.79	3.94
2	3	3.71	5.38	5.63	4.74	5.48	5.56	5.38	5.38	4.76	5.32	5.64
3		3.71	6.67	8.02	12.64	7.01	7.33	6.28	6.31	12.58	6.20	12.85

Table 6. RE of the $\hat{T}_{[i]}$ (ratio) and $\hat{T}_{opt}$ under RSS over unbiased estimator t_0.

		$\rho_{xy} = 0.5$							
k	m	$\hat{T}_R$	$\hat{T}_{[R]}$	$\hat{T}_{[1]}$	$\hat{T}_{[2]}$	$\hat{T}_{[3]}$	$\hat{T}_{[4]}$	$\hat{T}_{[5]}$	$\hat{T}_{opt}$
3	4	1.00	1.31	1.37	1.31	1.31	2.12	1.32	2.12
6		1.51	1.71	1.75	1.71	1.71	1.84	1.72	1.91
3	5	1.12	2.00	2.03	2.00	2.00	2.31	2.01	2.31
6		1.20	1.67	1.71	1.67	1.67	2.32	1.68	2.37
3	6	1.19	2.82	2.81	2.82	2.82	2.07	2.82	2.83
6		1.05	2.23	2.22	2.23	2.23	1.76	2.23	2.23
		$\rho_{xy} = 0.7$							
3	4	1.50	1.87	1.88	1.87	1.87	1.64	1.87	1.90
6		1.41	1.70	1.72	1.70	1.70	1.44	1.70	1.75
3	5	1.58	2.77	2.82	2.78	2.77	3.08	2.79	3.14
6		1.40	1.77	1.96	1.77	1.77	2.86	1.80	2.97
3	6	1.83	3.31	3.36	3.31	3.31	3.11	3.33	3.52
6		2.61	2.70	2.75	2.70	2.70	2.65	2.71	2.88
		$\rho_{xy} = 0.9$							
3	4	3.32	4.65	4.68	4.65	4.65	3.94	4.67	4.72
6		4.77	5.74	5.85	5.74	5.74	2.81	5.78	5.86
3	5	4.46	6.09	6.18	6.09	6.09	5.08	6.15	6.42
6		5.17	6.12	6.22	6.12	6.12	3.40	6.16	6.24
3	6	2.92	5.62	6.02	5.63	5.62	8.07	5.77	8.86
6		5.54	11.82	11.97	11.82	11.82	5.00	11.89	11.97

with (bivariate normal) means (50.0, 50.0) and covariance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ ($\rho = -0.9, -0.7, -0.5, 0.5, 0.7, 0.9$). We have taken total bivariate RSS samples of sizes $n = lk = 12$, 15, 18, 24, 30, and 36 (set sizes $l = 4$, 5, 6 with $k = 3$, 6 replications) from the population of size $N = 100$. We have replicated each simulation study 5000 times to estimate means and MSEs, and the results through relative efficiencies calculation are shown in Tables 6 and 7. The relative efficiency formula is as follows:

$$RE(V, t_0) = MSE(t_0)/MSE(V), \tag{50}$$

where $V = t_R, t_P, \hat{T}_{[R]}, \hat{T}_{[P]}, \hat{T}_{opt}$, and $\hat{T}_{[i]}$ for $i = 1, 2, 3, ..., 13$.

7. Results and discussion

In the empirical study section, Table 2 presents the MSEs of the ratio type estimators under RSS along with MSEs of an unbiased estimator and ratio estimator under SRS. We see that the ratio types member estimators and the proposed estimator under RSS have performed better over unbiased and ratio estimators under SRS. Also, the proposed estimator under RSS has the lowest MSEs in the table (has shown superiority).

Table 3 exhibits the MSEs of the product type estimators under RSS along with the MSEs of an unbiased estimator and product estimator under SRS. We see the product types member estimators and the proposed estimator under RSS have performed better

Table 7. RE of the $\hat{T}_{[i]}$ (product) and $\hat{T}_{opt}$ under RSS over unbiased estimator t_0.

k	m	$\hat{T}_P$	$\hat{T}_{[P]}$	$\hat{T}_{[6]}$	$\hat{T}_{[7]}$	$\hat{T}_{[8]}$	$\hat{T}_{[9]}$	$\hat{T}_{[10]}$	$\hat{T}_{[11]}$	$\hat{T}_{[12]}$	$\hat{T}_{[13]}$	$\hat{T}_{opt}$
						$\rho_{xy}=-0.5$						
3	4	1.29	1.52	1.54	1.84	1.53	1.57	1.52	1.52	2.01	1.51	2.03
6		1.00	1.35	1.38	1.42	1.36	1.39	1.35	1.35	1.82	1.34	1.82
3	5	1.03	1.54	1.58	2.12	1.56	1.61	1.54	1.54	2.76	1.52	2.81
6		1.31	1.53	1.54	1.55	1.53	1.55	1.53	1.53	1.63	1.53	1.65
3	6	1.00	2.93	2.94	2.99	2.93	2.95	2.93	2.93	2.64	2.92	2.99
6		1.13	1.77	1.79	1.93	1.77	1.82	1.77	1.77	1.86	1.76	2.00
						$\rho_{xy}=-0.7$						
3	4	1.09	1.54	1.60	2.05	1.57	1.62	1.54	1.54	3.00	1.51	3.10
6		1.62	1.69	1.70	1.71	1.69	1.72	1.69	1.69	1.74	1.68	1.85
3	5	1.46	2.13	2.16	2.27	2.14	2.2	2.13	2.13	2.53	2.11	2.61
6		2.48	2.86	2.91	2.95	2.88	2.97	2.86	2.86	2.96	2.82	3.46
3	6	3.15	3.52	3.51	2.35	3.51	3.49	3.52	3.52	2.52	3.53	3.59
6		2.40	3.64	3.65	3.65	3.65	3.66	3.64	3.64	3.02	3.63	3.66
						$\rho_{xy}=-0.9$						
3	4	3.58	3.80	3.88	4.02	3.84	3.96	3.80	3.80	4.06	3.73	4.77
6		5.91	6.79	6.91	7.11	6.84	7.02	6.80	6.79	4.18	6.69	7.26
3	5	2.53	3.38	3.50	5.59	3.45	3.60	3.39	3.39	4.72	3.27	5.61
6		6.25	7.30	7.43	7.38	7.35	7.55	7.30	7.30	5.24	7.19	7.92
3	6	4.60	5.42	5.41	5.16	5.41	5.40	5.42	5.42	3.53	5.42	5.42
6		8.41	9.87	9.83	9.79	9.86	9.72	9.87	9.87	4.05	9.89	9.89

over unbiased and product estimators under SRS. Also, the proposed estimator under RSS has the lowest MSEs in the table (has shown superiority).

Similarly, in Tables 4 and 5, our proposed estimator under RSS has shown superiority in terms of relative efficiencies over ratio and product type member estimators under RSS and SRS. Also, the estimators under RSS have increasing relative efficiencies with the increasing values of set sizes and replications.

Similar to the empirical study section, we compiled a Monte-Carlo simulation study, and the results are in Tables 6 and 7 regarding relative efficiencies.

Table 6 presents the relative efficiency of the proposed estimator, ratio type member estimators under RSS, along with the RE of the ratio estimator in SRS over the mean per unit estimator. We see that the relative efficiencies of the proposed estimator and member estimators of the proposed estimator in RSS have increased with increasing values of the correlation coefficient, set sizes, and replications.

Table 7 shows the relative efficiency of the proposed estimator, product type member estimators under RSS, and the RE of the product estimator in SRS over the mean per unit estimator. We see that the relative efficiencies of the proposed estimator and member estimators under RSS have increased with increasing values of the correlation coefficient, set sizes, and replications.

8. Conclusion

In this study, we developed a generalized class of estimators for estimating finite population means in a ranked set sampling scheme. The simulation study shows that the proposed class of estimators performs better than the others for all the cases and showed findings on a real data example. These results are also similar to the simulation study. Therefore, the proposed methods in this manuscript can be considered in many real applications, such as mean estimation in case of missing data, quality control charts for monitoring the process mean, and acceptance sampling plans. In future studies, the proposed method can be generalized by using different types of ranked set samplings to obtain more efficient results for different cases.

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Appendix A

Notations and error terms obtained by Eq. (A.1) as shown in Box A.I. We have another way of finding Bias and MSE of the proposed generalized class of estimator, and the proposed class of estimator is as:

$$T=\bar{Y}_{RSS}\left(\frac{A\bar{X}+B}{A\bar{X}_{RSS}+B}\right)^{\alpha}=\bar{y}_{[n]}\left(\frac{A\bar{X}+B}{A\bar{x}_{(n)}+B}\right)^{\alpha}. \tag{A.2}$$

Expressing the estimator from Eq. (A.2) in terms of e's and taking approximating terms having up to degree 2, we get:

$$T=\bar{Y}\left(1+e_0\right)\left[\frac{A\bar{X}+B}{A\bar{X}\left(1+e_1\right)+B}\right]^{\alpha},$$

$$\cong \bar{Y}\left(1+e_0\right)\left[\frac{A\bar{X}+B}{(A\bar{X}+B)+A\bar{X}e_1}\right]^{\alpha},$$

$$\cong \bar{Y}\left(1+e_0\right)\left[1+C_0e_1\right]^{-\alpha},$$

$$\text{where,}\quad C_0=A\bar{X}/\left(A\bar{X}+B\right),$$

$$\cong \bar{Y}\left(1+e_0-\alpha C_0e_1+\frac{\alpha\left(\alpha+1\right)}{2}C_0^2e_1^2-\alpha C_0e_1e_0\right).$$

Therefore,

$$\left(T-\bar{Y}\right)=\bar{Y}\left(e_0-\alpha C_0e_1+\frac{\alpha\left(\alpha+1\right)}{2}C_0^2e_1^2 -\alpha C_0e_1e_0\right). \tag{A.3}$$

Taking expectation on both side of Eq. (A.3) and from appendix Eq. (A.1), we have:

$$E\left(T-\bar{Y}\right)=\bar{Y}E\left(e_0-\alpha C_0e_1+\frac{\alpha\left(\alpha+1\right)}{2}C_0^2{e_1}^2 -\alpha\bar{Y}C_0e_0e_1\right),$$

$$=\frac{\alpha\left(\alpha+1\right)}{2}C_0^2\bar{Y}E\left(e_1^2\right)-\alpha C_0\bar{Y}E\left(e_0e_1\right),$$

$$=\frac{\alpha\left(\alpha+1\right)}{2}C_0^2\bar{Y}\left(fC_x^2-W_x^2\right) -\alpha C_0\bar{Y}\left(f\rho_{yx}C_yC_x-W_{yx}\right),$$

$$\left.\begin{array}{l}
\bar{y}_{[n]}=\bar{Y}\left(1+e_0\right),\bar{x}_{(n)}=\bar{X}\left(1+e_1\right),E\left(e_0\right)=E\left(e_1\right)=0,E\left(e_0^2\right)=\left(fC_y^2-W_y^2\right),f=\frac{1}{lk},\\
E\left(e_0e_1\right)=\left(fC_{yx}-W_{yx}\right),\tau_{y[i]}=\left(\mu_{y[i]}-\bar{Y}\right),\tau_{x(i)}=\left(\mu_{x(i)}-\bar{X}\right),\tau_{yx(i)}=\left(\mu_{y[i]}-\bar{Y}\right)\left(\mu_{x(i)}-\bar{X}\right),\\
\delta_y=\frac{\tau_{y[i]}}{\bar{Y}},\delta_x=\frac{\tau_{x(i)}}{\bar{X}},\delta_{yx}=\frac{\tau_{yx(i)}}{\bar{Y}\bar{X}},W_y^2=\frac{1}{l^2k}\sum_{i=1}^{l}\delta_{y[i]}^2,W_x^2=\frac{1}{l^2k}\sum_{i=1}^{l}\delta_{x(i)}^2,W_{yx}=\frac{1}{l^2k}\sum_{i=1}^{l}\delta_{yx(i)},\\
E\left(e_1^2\right)=\left(fC_x^2-W_x^2\right),T_y^2=\frac{1}{l^2k}\sum_{i=1}^{l}\tau_{y[i]}^2,T_x^2=\frac{1}{l^2k}\sum_{i=1}^{l}\tau_{x(i)}^2,T_{yx}=\frac{1}{l^2k}\sum_{i=1}^{l}\tau_{yx(i)},R=\frac{\bar{Y}}{\bar{X}},\\
C_1=\frac{\bar{X}}{\bar{X}+C_x},C_2=\frac{\bar{X}}{\bar{X}+\beta_2(x)},C_3=\frac{\beta_2(x)\bar{X}}{\beta_2(x)\bar{X}+C_x},C_4=\frac{C_x\bar{X}}{C_x\bar{X}+\beta_2(x)},\frac{A\alpha\bar{Y}}{A\bar{X}+B}=R_T,\\
R_1=\frac{\bar{Y}}{\bar{X}+C_x},R_2=\frac{\bar{Y}}{\bar{X}+\beta_2(x)},R_3=\frac{\beta_2(x)\bar{Y}}{\beta_2(x)\bar{X}+C_x},R_4=\frac{C_x\bar{Y}}{C_x\bar{X}+\beta_2(x)},R_5=-R_4.
\end{array}\right\}. \tag{A.1}$$

Box A.I

$$\frac{\partial\left(Bias\left(T\right)\right)}{\partial\alpha}=\frac{\partial\left(\bar{Y}\left(\frac{\alpha(\alpha+1)}{2}C_0^2\left(fC_x^2-w_{x(i)}^2\right)\right)-\bar{Y}\left(\alpha C_0\left(fC_{yx(i)}-w_{yx(i)}\right)\right)\right)}{\partial\alpha}=0.$$

Box A.II

$$=\bar{Y}\left[f\left(\frac{\alpha\left(\alpha+1\right)}{2}C_0^2C_x^2-\alpha C_0\rho_{yx}C_yC_x\right)\right.$$
$$\left.-\left(\frac{\alpha\left(\alpha+1\right)}{2}C_0^2W_x^2-\alpha C_0W_{yx}\right)\right].$$

Hence:

$$Bias\left(T\right)=\bar{Y}\left[f\left(\frac{\alpha\left(\alpha+1\right)}{2}C_0^2C_x^2\right.\right.$$
$$\left.-\alpha C_0\rho_{yx}C_yC_x\right)-\left(\frac{\alpha\left(\alpha+1\right)}{2}C_0^2W_x^2\right.$$
$$\left.\left.-\alpha C_0W_{yx}\right)\right]. \quad \text{(A.4)}$$

Optimality of the Bias

The optimum Bias can be obtained by minimizing Bias with respect to α by taking $\frac{\partial(Bias(T))}{\partial\alpha}=0$, therefore the equation also can be obtained as shown in Box A.II.

$$\text{i.e.,}\quad \bar{Y}\left(\left(\alpha+\frac{1}{2}\right)C_0^2\left(fC_x^2-w_{x(i)}^2\right)\right)$$
$$-\bar{Y}\left(C_0\left(fC_{yx(i)}-w_{yx(i)}\right)\right)=0,$$

$$\text{i.e.,}\quad \alpha_1=\frac{\left(fC_{yx(i)}-w_{yx(i)}\right)}{C_0\left(fC_x^2-w_{x(i)}^2\right)}-\frac{1}{2},\quad \text{where}$$

$$C_0=\frac{A\bar{X}}{A\bar{X}+B},$$

replacing α by α_1 in Eq. (A.4), we get:

$$Bias(T)_{\min}\approx\bar{Y}\left[f\left(\frac{\alpha_1\left(\alpha_1+1\right)}{2}C_0^2C_x^2\right.\right.$$
$$\left.-\alpha_1C_0\rho_{yx}C_yC_x\right)-\left(\frac{\alpha_1\left(\alpha_1+1\right)}{2}C_0^2W_x^2\right.$$
$$\left.\left.-\alpha_1C_0W_{yx}\right)\right]. \quad \text{(A.5)}$$

Squaring both sides of Eq. (A.3) and taking its expectation of having a degree of not more than 2, we get:

$$E\left(T-\bar{Y}\right)^2\cong\bar{Y}^2E\left(e_0-\alpha C_0e_1+\frac{\alpha\left(\alpha+1\right)}{2}C_0^2e_1^2\right.$$
$$\left.-\alpha\bar{Y}C_0e_0e_1+O\left(e_i\right)\right)^2,$$
$$\cong\bar{Y}^2\left(E\left(e_0^2\right)+\alpha^2C_0^2E\left(e_1^2\right)-2\alpha C_0E\left(e_0e_1\right)\right),$$
$$\cong\bar{Y}^2\left[f\left(C_y^2+\alpha^2C_0^2C_x^2-2\alpha C_0C_{yx}\right)\right.$$
$$\left.-\left(W_y^2+\alpha^2C_0^2W_x^2-2\alpha C_0W_{yx}\right)\right].$$

Hence,

$$MSE\left(T\right)\cong\bar{Y}^2\left[f\left(C_y^2+\alpha^2C_0^2C_x^2-2\alpha C_0C_{yx}\right)\right.$$
$$\left.-\left(W_y^2+\alpha^2C_0^2W_x^2-2\alpha C_0W_{yx}\right)\right]. \quad \text{(A.6)}$$

Biography

Gajendra K. Vishwakarma is working as an Associate Professor of statistics in the Department of Mathematics & Computing, Indian Institute of Technology Dhanbad, India, with several years of academic and industrial research experience in statistics. His research experience covers both applied as well as theoretical provinces. He has published a number of research papers in reputed international journals. He visited European and Asian (Oceania) countries and did collaborative research work with them. He has supervised twelve PhD theses so far and many more ongoing. He is an Elected Member of the International Statistical Institute Netherlands. Country Representative of ISI Young Statisticians, International Statistical Institute (ISI), Netherlands. Fellow of the Royal Statistical Society UK and Founder member of RSS (Indian Local Group) UK. Associate Fellow of the International Academy of Physical Sciences India. He received international travel awards from the DST Govt of India, the World Bank Trust Fund for Statistical Capacity Building of the USA, and the International Biometric Society of the USA to attend scientific events abroad. He has high-value ongoing research projects funded by DST, CSIR, and ICSSR. He received the Young Scientist Award from the Center for Advanced Research and Design, India. He has received Professor P.V. SUKHATME and Smt. Suraj Kali Jain Awards of Indian Society for Medical Statistics and SIRE fellowship from DST Govt of India. Recently, he has been selected as a member of the National Academy of Sciences India by the Department of Science & Technology Government of India. He is an Editor/Associate Editor/Editorial Board member of some reputed journals like Scientia Iranica, BMC Medical Research Methodology, Measurement: Interdisciplinary Research and Perspectives, Revista de Investigacion Operacional, etc.