Computing the population mean on the use of auxiliary information under ranked set sampling

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Abstract
In this manuscript, a generalized class of estimators has been developed for estimating a finite population means in ranked set sampling scheme. The expressions for bias and mean square error (MSE) of the proposed class of estimators have been derived up to the first order of approximation. Some estimators are shown to be a member of the proposed class. The proposed class of estimators has been compared through the MSE criterion over the other existing member estimators of the proposed class of estimators. The theoretical conditions are obtained under which the proposed class of estimators has performed better. Efficiency comparisons, empirical study, and simulation study also delineate the soundness of our proposed generalized class of the estimators under ranked set sampling (RSS).

Keywords: Auxiliary variable; Mean square error; Monte-Carlo simulation; Ratio estimator; Product estimator; Ranked set sampling.

Mathematics Subject Classification: 62D05.

1. Introduction
To reduce the sampling error many researchers attempted to use additional information (highly correlated with the character under study) which is known as auxiliary (auxiliary) information. This information is available for each unit and may be known well in advance. If it is not readily available for each unit of the population or information on it may be collected through past surveys. The study character, consider $Y$ may be the field in agriculture survey and auxiliary character $X$ may be area under cultivation, $Y$ may be the income of households.
and $X$ the number of earning member in household, $Y$ may be the number of patients is being treated in the hospital and $X$ may be the number of doctors available in the hospital and so on auxiliary information can be stated. It is to be mentioned that Cochran [1] was the pioneer in using auxiliary information at the estimation stage. He envisages the ratio estimator for estimating the population mean or total of a variate under investigation. The ratio and product method of estimation are well-known methods for estimating the population means of a study variable using of auxiliary information. When the correlation between the study variate and auxiliary variate is positive the ratio estimator can be employed quite effectively. If the correlation between study variate and auxiliary variate is negative (high) the product estimator envisaged by Robson [2] and rediscovered by Murthy [3] is used. Keeping this fact in view and also owing to the stronger intuitive appeal survey statisticians are more inclined towards the use of ratio and the product estimators in practice. The estimators unbiased, ratio and product are as respectively

$$
\bar{y}_n = \frac{1}{n} \sum_{i=1}^{n} y_i = t_0 ,
$$

(1)

$$
\bar{y}_R = \frac{\bar{y}}{\bar{x}} = t_R ,
$$

(2)

$$
\bar{y}_P = \frac{\bar{y}}{\bar{x}} = t_P ,
$$

(3)

where $\bar{y}$ and $\bar{x}$ are the sample means of the study variable $Y$ and the auxiliary variable $X$, respectively. Also, $\bar{X}$ is the mean of the auxiliary variable $X$.

$MSE$ of the corresponding estimators in Eqs. (1) to (3) are respectively as

$$
MSE(t_0) = f \left( S_y^2 \right) = Var(t_0) ,
$$

(4)

$$
MSE(t_R) = f \left( S_y^2 + R^2 S_x^2 - 2RS_{yx} \right) ,
$$

(5)

$$
MSE(t_P) = f \left( S_y^2 + R^2 S_x^2 + 2RS_{yx} \right) ,
$$

(6)

where,

$$
f = \left( 1 - \frac{1}{N} \right) , \quad S_y^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})^2 , \quad S_x^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (x_i - \bar{X})^2 ,
$$

$$
S_{yx} = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y}) (x_i - \bar{X}) \quad \text{and} \quad R = \frac{\bar{Y}}{\bar{X}} .
$$

Researchers are always keen to better enhancement of their results, for this, they develop new methods or techniques. In the finding of a better substitute for simple random sampling, McIntyre [4] has propounded a ranked set sampling (RSS) technique. Which is far better than
simple random sampling and economically efficient too. Nowadays, RSS has been widely applied in the field of medical sciences, biology, agriculture, environmental science and many fields of statistics. When the measurements are quite cumbersome and extravagant, then ranking of the variables comparable to others sampling scheme be analogously easy and cost-effective. Ranking will be perfect, if rank of the observations within each set do tally with the numeric order of veiled X values, otherwise, it is imperfect. In the explanation of RSS, many grantrors like Takahasi and Wakimoto [5], Dell and Clutter [6], Stokes [7], Samawi and Muttlak [8], Al-Saleh and Al-Omari [9], Bouza [10], Wolfe [11], Al-Omari et. al. [12], Ai-Omari [13], Mandowara and Mehta [14], Al-Omari and Gupta [15], Pal and Singh [16], Vishwakarma et al. [17], Jeelani et al. [18], Noor et al. [19], Al-Omari and Haq [20], Saini and Kumar [21], and Singh and Vishwakarma [22] have contributed through different estimation procedures and techniques in the various fields of RSS for the estimation of population parameters. Ahmed et al. [23] has given the predictive estimation of population mean using ranked set sampling, under which shown that when natural (usual) unbiased, ratio and regression estimators are used as predictors give the corresponding predictive estimators same as natural unbiased, ratio and regression estimators under the RSS. Mehta et al. [24] introduced a general procedure for estimating finite population mean using ranked set sampling. Koyuncu and Al-Omari [25] developed generalized robust-regression-type estimators under different ranked set sampling. Vishwakarma and Singh [26, 27] Computed the effect of measurement errors on ranked set sampling estimators of the population mean and given some applications to solar energy data.

In the procedure of ranked set sample, we have l bivariate random samples of size l from a population of size N, and are ranked within each sample concerning for ancillary variable X associated with Y. In the RSS procedure, we take the first smallest unit of the first data set size l and specify it for first measurement unit and scraping the rest of units. Similarly, we take the second smallest observation of the second data set size l and specify it for second observation and scraping the rest. Proceeding this way, total l bivariate units for up to lth term are counted, and after k cycle of these procedure, total n = kl bivariate RSS units are treated as SRS data too (used for calculation in SRS). In the extraction of RSS data, there are total kl^2 units, but only n = kl units are counted for actual computation. \(\left( X_{j(i)}, Y_{j(i)}; j = 1, 2, 3, ..., k; i = 1, 2, 3, ..., l \right)\) be the paired bivariate quantified sets of the ith units in the jth cycle. We have unbiased estimator on the basis of study variable under RSS along with MSE as
\[ T_0 = \bar{y}_{[n]} = \frac{1}{n} \sum_{i=1}^{n} y_{i}, \quad \text{(7)} \]

\[ \text{MSE}(T_0) = \bar{y}^2 \left( f C_y^2 - W_y^2 \right) = \text{Var}(T_0). \quad \text{(8)} \]

Employing the concept of auxiliary variables, Samawi and Muttlak [8] and Bouza [28] have propounded ratio and product estimators for population mean using RSS as respectively

\[ \bar{y}_{[\alpha]} = \bar{y}_{[n]} \left( \frac{X}{\bar{X}(\alpha)} \right) = T_{[\alpha]}, \quad \text{(9)} \]

\[ \bar{y}_{[P]} = \bar{y}_{[n]} \left( \frac{X}{\bar{X}} \right) = T_{[P]}, \quad \text{(10)} \]

where \( \bar{y}_{[n]} = \frac{1}{n} \sum_{i=1}^{n} y_{[i]}, \quad \bar{x}_{(\alpha)} = \frac{1}{n} \sum_{i=1}^{n} x_{(i)} \) and \( \hat{R}_{\text{RSS}} = \frac{\bar{y}_{\text{RSS}}}{\bar{x}_{(n)}}. \)

To obtaining \textit{Bias} and \textit{MSE}, we have ratio estimator from Eq. (9) as

\[ T_{[\alpha]} = \bar{y}_{[\alpha]} = \frac{\bar{y}_{[n]}}{\bar{x}_{(n)}} \bar{X} = H \left( \bar{x}_{(n)}, \bar{y}_{[n]} \right), \quad \text{(11)} \]

Using Taylor’s series and from Eq. (11), we have

\[ T_{[\alpha]} = H \left( \bar{X}, \bar{Y} \right) + H_0 \left( \bar{x}_{(n)} - \bar{X} \right) + H_1 \left( \bar{x}_{(n)} - \bar{X} \right) + H_2 \left( \bar{x}_{(n)} - \bar{X} \right)^2 + H_3 \left( \bar{y}_{[n]} - \bar{Y} \right)^2 + H_4 \left( \bar{y}_{[n]} - \bar{Y} \right) \left( \bar{x}_{(n)} - \bar{X} \right) + \ldots, \quad \text{(12)} \]

where,

\[ H \left( \bar{X}, \bar{Y} \right) = \frac{\bar{Y}}{\bar{X}}, \quad H_0 \left( \bar{X}, \bar{Y} \right) = \left. \frac{\partial H \left( \bar{x}_{(n)}, \bar{y}_{[n]} \right)}{\partial \bar{y}_{[n]} \bar{x}_{(n)}} \right|_{\bar{X}, \bar{Y}} = \frac{\bar{X}}{\bar{y}_{[n]}}, \quad \text{and} \quad \frac{\bar{Y}}{\bar{X}} = -R, \]

\[ H_1 \left( \bar{X}, \bar{Y} \right) = \left. \frac{\partial^2 H \left( \bar{x}_{(n)}, \bar{y}_{[n]} \right)}{\partial \bar{y}_{[n]}^2 \bar{x}_{(n)}} \right|_{\bar{X}, \bar{Y}} = \frac{\bar{X}}{\bar{y}_{[n]}^2}, \]

\[ H_2 \left( \bar{X}, \bar{Y} \right) = \left. \frac{\partial^2 H \left( \bar{x}_{(n)}, \bar{y}_{[n]} \right)}{\partial \bar{y}_{[n]}^2 \bar{x}_{(n)}^2} \right|_{\bar{X}, \bar{Y}} = \frac{\bar{X}}{\bar{y}_{[n]}^2}, \]

\[ H_3 \left( \bar{X}, \bar{Y} \right) = \left. \frac{\partial^2 H \left( \bar{x}_{(n)}, \bar{y}_{[n]} \right)}{\partial \bar{y}_{[n]}^2 \bar{x}_{(n)}} \right|_{\bar{X}, \bar{Y}} = 0, \]

\[ H_4 \left( \bar{X}, \bar{Y} \right) = \left. \frac{\partial^2 H \left( \bar{x}_{(n)}, \bar{y}_{[n]} \right)}{\partial \bar{y}_{[n]} \partial \bar{x}_{(n)}^2} \right|_{\bar{X}, \bar{Y}} = \frac{1}{\bar{X}}, \quad \text{and} \quad \frac{\bar{Y}}{\bar{X}} = -1 \]

and such that it satisfies the following conditions
(i) The function \( H(\bar{x}, \bar{y}) \) is continuous and bounded in \( D \) (dimension).

(ii) The first and second order partial derivatives exist and are continuous and bounded in \( D \).

Therefore, from Eq. (12) and appendix (i), we have

\[
T_{\{R\}} \approx H(\bar{x}, \bar{y}) + H_0 \bar{e}_0 + H_1 \bar{x}e_i + H_2 \bar{x}^2 e_i^2 + H_3 \bar{y}^2 e_i^2 + H_4 \bar{y}X e_i e_i + O(e_i).
\]

(14)

Taking expectation on the both side of (14) and from appendix (i), we get

\[
E(T_{\{R\}}) \approx \bar{y} + H_2 \bar{x}^2 \left( fC_x - W_{sl(i)} \right) + H_2 \bar{y}^2 \left( fC_y - W_{yl(i)} \right) + H_4 \bar{y} \bar{x} e_i e_i + O(e_i).
\]

(15)

From Eq. (14) we have

\[
(t_{\{R\}} - \bar{y}) \equiv H(\bar{x}, \bar{y}) + H_0 \bar{e}_0 + H_1 \bar{x}e_i + O(e_i) - \bar{y},
\]

(16)

Now squaring and taking expectation of both side of Eq. (16) and from appendix (i), we get

\[
E(T_{\{R\}}) \approx H(\bar{x}, \bar{y}) + H_0 \bar{e}_0 + H_1 \bar{x}e_i + O(e_i) - \bar{y},
\]

(17)

Similarly, we have bias and MSE of product estimator as

bias\( (T_{\{R\}}) = \bar{y} f \left( C_x - C_{xy} \right) - \left( W_{sl(i)} - W_{yl(i)} \right) \]

(18)

MSE\( (T_{\{R\}}) = \bar{y}^2 f S_x^2 + R^2 S_x^2 - 2RS_{xy} - \frac{1}{I^2} \left( \sum_{i=1}^{I} T_{s,i}^2 + R^2 \sum_{i=1}^{I} T_{s,i}^2 - 2R \sum_{i=1}^{I} T_{s,i} \right).
\]

(19)

Also, MSE of the product estimator can be obtained by putting \( R = - R \) in Eq. (19).
In the context of auxiliary variable, Mandowara and Mehta [14] have also given some ratio and product-type estimators under RSS, are as:

\[
\bar{y}_{rss[\text{nn1}]} = \bar{y}_{\text{RSS}} \left( \frac{\bar{X} + C_x}{\bar{X}_{\text{RSS}} + C_x} \right) = \hat{T}_1, \quad (20)
\]

\[
\bar{y}_{rss[\text{nn2}]} = \bar{y}_{\text{RSS}} \left( \frac{\bar{X} + \beta_2(x)}{\beta_2(x)\bar{X}_{\text{RSS}} + \beta_2(x)} \right) = \hat{T}_2, \quad (21)
\]

\[
\bar{y}_{rss[\text{nn3}]} = \bar{y}_{\text{RSS}} \left( \frac{\beta_2(x)\bar{X} + C_x}{\beta_2(x)\bar{X}_{\text{RSS}} + C_x} \right) = \hat{T}_3, \quad (22)
\]

\[
\bar{y}_{rss[\text{nn4}]} = \bar{y}_{\text{RSS}} \left( \frac{C_x\bar{X} + \beta_2(x)}{C_x\bar{X}_{\text{RSS}} + \beta_2(x)} \right) = \hat{T}_4, \quad (23)
\]

\[
\bar{y}_{rss[\text{nn5}]} = \bar{y}_{\text{RSS}} \left( \frac{C_x\bar{X}_{\text{RSS}} + \beta_2(x)}{C_x\bar{X} + \beta_2(x)} \right) = \hat{T}_5, \quad (24)
\]

and their Biases and MSEs correspondingly are as

\[
\text{Bias}\left(\bar{y}_{rss[\text{nn1}]}\right) = \bar{y} \left[ f \left( C_1^2C_x^2 - C_1\rho_{yx}C_yC_x \right) - \left( C_1^2W_x^2 - C_1W_{yx} \right) \right], \quad (25)
\]

\[
\text{MSE}\left(\bar{y}_{rss[\text{nn1}]}\right) = \left[ f \left( S_y^2 + R_y^2S_x^2 - 2R_yS_{yx} \right) - \left( T_y^2 + R_y^2T_x^2 - 2R_yT_{yx} \right) \right], \quad (26)
\]

\[
\text{Bias}\left(\bar{y}_{rss[\text{nn2}]}\right) = \bar{y} \left[ f \left( C_2^2C_x^2 - C_2\rho_{yx}C_yC_x \right) - \left( C_2^2W_x^2 - C_2W_{yx} \right) \right], \quad (27)
\]

\[
\text{MSE}\left(\bar{y}_{rss[\text{nn2}]}\right) = \left[ f \left( S_y^2 + R_y^2S_x^2 - 2R_yS_{yx} \right) - \left( T_y^2 + R_y^2T_x^2 - 2R_yT_{yx} \right) \right], \quad (28)
\]

\[
\text{Bias}\left(\bar{y}_{rss[\text{nn3}]}\right) = \bar{y} \left[ f \left( C_3^2C_x^2 - C_3\rho_{yx}C_yC_x \right) - \left( C_3^2W_x^2 - C_3W_{yx} \right) \right], \quad (29)
\]

\[
\text{MSE}\left(\bar{y}_{rss[\text{nn3}]}\right) = \left[ f \left( S_y^2 + R_y^2S_x^2 - 2R_yS_{yx} \right) - \left( T_y^2 + R_y^2T_x^2 - 2R_yT_{yx} \right) \right], \quad (30)
\]

\[
\text{Bias}\left(\bar{y}_{rss[\text{nn4}]}\right) = \bar{y} \left[ f \left( C_4^2C_x^2 - C_4\rho_{yx}C_yC_x \right) - \left( C_4^2W_x^2 - C_4W_{yx} \right) \right], \quad (31)
\]

\[
\text{MSE}\left(\bar{y}_{rss[\text{nn4}]}\right) = \left[ f \left( S_y^2 + R_y^2S_x^2 - 2R_yS_{yx} \right) - \left( T_y^2 + R_y^2T_x^2 - 2R_yT_{yx} \right) \right], \quad (32)
\]

\[
\text{Bias}\left(\bar{y}_{rss[\text{nn5}]}\right) = \bar{y} \left[ f \left( C_5\rho_{yx}C_yC_x \right) - \left( C_5W_{yx} \right) \right], \quad (33)
\]

\[
\text{MSE}\left(\bar{y}_{rss[\text{nn5}]}\right) = \left[ f \left( S_y^2 + R_y^2S_x^2 + 2R_yS_{yx} \right) - \left( T_y^2 + R_y^2T_x^2 + 2R_yT_{yx} \right) \right], \quad (34)
\]
2. Proposed generalized class of estimators under ranked set sampling

Motivated by Upadhyaya and Singh [29], Sisodia and Dwivedi [30], we have suggested a class of estimators under RSS as

\[ T = \bar{Y}_{RSS} \left( \frac{A\bar{X} + B}{A\bar{X}_{RSS} + B} \right)^{\alpha} \]  

(35)

For various values of \( A, B \) and \( \alpha \) we can get a class ratio and product type estimators. Where, \( \bar{Y}_{RSS} \) and \( \bar{X}_{RSS} \) are earlier mentioned in section 1 and \( A, B \) can be \( C_x \) (coefficient of variation), \( \sigma_x \) (standard deviation), \( \beta_1(x) \) (skewness), \( \beta_2(x) \) (kurtosis) and \( \rho \) (correlation coefficient).

To obtaining \( Bias \) and \( MSE \), we have proposed estimator as

\[ T = \bar{Y}_{RSS} \left( \frac{A\bar{X} + B}{A\bar{X}_{RSS} + B} \right)^{\alpha} = \bar{Y}_{[n]} \left( \frac{A\bar{X} + B}{A\bar{X}_{[n]} + B} \right)^{\alpha} = H^* (\bar{X}_{(n)}, \bar{Y}_{[n]}) \]  

(36)

Using Taylor’s series on (36), we have

\[ T = H^* (\bar{X}, \bar{Y}) + H^*_0 (\bar{Y}_{[n]} - \bar{Y}) + H^*_1 (\bar{X}_{[n]} - \bar{X}) + H^*_2 (\bar{X}_{[n]} - \bar{X})^2 + H^*_3 (\bar{Y}_{[n]} - \bar{Y})^2 + H^*_4 (\bar{Y}_{[n]} - \bar{Y})(\bar{X}_{[n]} - \bar{X}) + .... \]  

(37)

where,

\[ H^* (\bar{X}, \bar{Y}) = \bar{Y}, \quad H^*_0 (\bar{X}, \bar{Y}) = \frac{\partial(H^*(\bar{X}_{(n)}, \bar{Y}_{[n]}))}{\partial \bar{Y}_{[n]}}, \quad H^*_1 (\bar{X}, \bar{Y}) = \frac{\partial(H^*(\bar{X}_{(n)}, \bar{Y}_{[n]}))}{\partial \bar{X}_{[n]}}, \quad H^*_2 (\bar{X}, \bar{Y}) = \frac{1}{2} \frac{\partial^2(H^*(\bar{X}_{(n)}, \bar{Y}_{[n]}))}{\partial \bar{X}_{[n]}^2}, \quad H^*_3 (\bar{X}, \bar{Y}) = \frac{1}{2} \frac{\partial^2(H^*(\bar{X}_{(n)}, \bar{Y}_{[n]}))}{\partial \bar{Y}_{[n]}^2}, \quad H^*_4 (\bar{X}, \bar{Y}) = \frac{1}{2} \frac{\partial^2(H^*(\bar{X}_{(n)}, \bar{Y}_{[n]}))}{\partial \bar{X}_{[n]} \partial \bar{Y}_{[n]}} \]

and such that it satisfies the following conditions

i) The function \( H^*(\bar{X}_{(n)}, \bar{Y}_{[n]}) \) is continuous and bounded in \( D \) (Dimension).
ii) The first and second order partial derivatives exist and are continuous and bounded in $D$.

Therefore from Eq. (37) and appendix (i), we have

$$ T \equiv H' \left( \bar{X}, \bar{Y} \right) + H'_0 \bar{Y} e_i + H'_1 \bar{X} e_i + H'_2 \bar{X}^2 e_i^2 + H'_3 \bar{Y}^2 e_i + H'_4 \bar{X} \bar{Y} e_i e_i + O (e_i) . $$

(39)

Taking expectation on the both side of Eq. (39) and from appendix (i), we get

$$ E(T) \equiv H' \left( \bar{X}, \bar{Y} \right) + H'_0 \bar{Y} e_0 + H'_1 \bar{X} e_0 + H'_2 \bar{X}^2 e_0^2 + H'_3 \bar{Y}^2 e_0 + H'_4 \bar{X} \bar{Y} e_0 e_0 + O (e_0) , $$

$$ \equiv \bar{Y} + \frac{A^2 \alpha (\alpha + 1) \bar{Y}^2}{2 (A \bar{X} + B)^2} \bar{X}^2 (fC_x^2 - W_{x(i)}) + (0) \bar{Y}^2 (fC_y^2 - W_{y[i]}) + \left( - \frac{A \alpha}{A \bar{X} + B} \right) \bar{Y} (fC_{xy} - W_{xy(i)}) , $$

$$ E(T - \bar{Y}) \equiv \bar{Y} \left( \frac{\alpha (\alpha + 1)}{2} C_0 (fC_x^2 - W_{x(i)}) \right) - \bar{Y} \left( \alpha C_0 (fC_{xy} - W_{xy(i)}) \right) . $$

(40)

Hence,

$$ Bias(T) \equiv \bar{Y} \left[ \frac{\alpha (\alpha + 1)}{2} C_0 (fC_x^2 - W_{x(i)}) \right] - \bar{Y} \left( \alpha C_0 (fC_{xy} - W_{xy(i)}) \right) . $$

From (39) we have

$$ (T - \bar{Y}) \equiv \left( H' \left( \bar{X}, \bar{Y} \right) + H'_0 \bar{Y} e_0 + H'_1 \bar{X} e_0 + O (e_0) \right) - \bar{Y} , $$

$$ (T - \bar{Y}) \equiv \left( \bar{Y} e_0 - R_x \bar{X} e_0 + O (e_0) \right) . $$

(41)

Squaring and taking expectation on both side of Eq. (41) and from appendix (i), we get

$$ E \left( \hat{T} - \bar{Y} \right)^2 \equiv \bar{Y}^2 E(e_i^2) - 2 R_x \bar{X} \bar{Y} E(e_i e_i) + R_x^2 \bar{X}^2 E(e_i^2) , $$

$$ MSE(\hat{T}) \equiv f \left( \bar{Y}^2 C_y^2 + R_y^2 \bar{X}^2 C_x^2 - 2 R_y \bar{X} \bar{Y} \rho_{xy} C_x C_y \right) - \left( \bar{Y}^2 W_y^2 + R_y^2 \bar{X}^2 W_x^2 - 2 R_y \bar{X} \bar{Y} W_{xy} \right) , $$

$$ \equiv f \left( S_y^2 + R_y^2 S_x^2 - 2 R_y S_{xy} \right) - \frac{1}{2 r} f \left( \sum_{j=1}^2 \tau_{y(j)}^2 + \sum_{i=1}^2 \sum_{j=1}^2 \tau_{x(i)}^2 - 2 R_y \sum_{i=1}^2 \tau_{xy(i)} \right) , $$

$$ MSE(\hat{T}) \equiv f \left( S_y^2 + R_y^2 S_x^2 - 2 R_y S_{xy} \right) - \left( T_y^2 + R_y^2 T_x^2 - 2 R_y T_{xy} \right) . $$

(42)

The optimum value of $\alpha$ to minimize the $MSE(\hat{T})$ can easily be found by equating its derivative to zero. i.e.,

$$ \frac{\partial}{\partial \alpha} \left( MSE(\hat{T}) \right) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial \alpha^2} \left( MSE(\hat{T}) \right) > 0 , $$

$$ \frac{\partial}{\partial \alpha} \left( f \left( S_y^2 + R_y^2 S_x^2 - 2 R_y S_{xy} \right) - \left( T_y^2 + R_y^2 T_x^2 - 2 R_y T_{xy} \right) \right) = 0 , $$

$$ \frac{\partial}{\partial \alpha} \left( fS_y^2 - T_y^2 \right) + \frac{\partial}{\partial \alpha} \left( fR_y^2 S_x^2 - R_y^2 T_x^2 \right) - 2 \frac{\partial}{\partial \alpha} \left( fR_y S_{xy} - R_y T_{xy} \right) = 0 , $$

$$ \frac{\partial}{\partial \alpha} \left( fS_y^2 - T_y^2 \right) + \frac{\partial}{\partial \alpha} \left( fR_y^2 S_x^2 - R_y^2 T_x^2 \right) - 2 \frac{\partial}{\partial \alpha} \left( fR_y S_{xy} - R_y T_{xy} \right) = 0 , $$

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i.e. 
\[0 + \left(f_{S_x} - T_x^2\right) \frac{\partial}{\partial \alpha} \left(R_t^2\right) - 2 \left(f_{S_{xy}} - T_{xy}\right) \frac{\partial}{\partial \alpha} \left(R_t\right) = 0,\]

\[(f_{S_x} - T_x^2) \left(\frac{A\bar{Y}}{AX + B}\right)^2 \left(2\alpha\right) - 2 \left(f_{S_{xy}} - T_{xy}\right) \left(\frac{A\bar{Y}}{AX + B}\right) = 0,\] (43)

where, 
\[\alpha = \frac{\left(f_{S_{xy}} - T_{xy}\right)}{\left(f_{S_x} - T_x^2\right) \left(\frac{A\bar{Y}}{AX + B}\right) \left(\frac{A\bar{Y}}{AX + B}\right)}.\]

We denote the value of \(\alpha\) as \(\alpha'\), therefore \(\alpha' = \frac{\left(f_{S_{xy}} - T_{xy}\right)}{\left(f_{S_x} - T_x^2\right) \left(\frac{A\bar{Y}}{AX + B}\right)}\) again differentiating Eq. (43) with respect to \(\alpha\), we have
\[
\frac{\partial^2}{\partial \alpha^2} \left(MSE(\hat{T})\right) = 2 \left(f_{S_x} - T_x^2\right) \left(\frac{A\bar{Y}}{AX + B}\right)^2 > 0,
\]
which proves optimality.

Therefore, we replace \(\alpha\) with \(\alpha'\) in the expression of \(MSE\) in Eq. (43) and we obtain minimum \(MSE\) of the proposed estimator as follows:
\[
MSE_{\min}(\hat{T}) = \left[f \left(S_y^2 + R_t^2 S_x^2 - 2R_t S_{xy}\right) - \left(T_y^2 + R_t^2 T_x^2 - 2R_t T_{xy}\right)\right] = \hat{T}_{[\text{opt}]},
\] (44)

where, \(R_t' = \frac{A\alpha' \bar{Y}}{AX + B} = \frac{f_{S_{xy}} - T_{xy}}{f_{S_x} - T_x^2}\).

3. Particular cases of proposed class of estimators

Through Eq. (33) in the previous section 2, we have
\[
T = \bar{Y}_{RSS} \left(\frac{AX + B}{AX_{RSS} + B}\right)^{\alpha}.\] (45)

Now for different values of \(A, B\) and \(\alpha\) we can get a class of estimators, shown in table 1:

Hence, we obtain a class of ratio and product estimators under RSS using standard deviation \(\sigma_x\), coefficient of variation \(C_r\), coefficient of skewness \(\beta_1(x)\), and coefficient of kurtosis \(\beta_2(x)\) auxiliary variable \(X\) and coefficient of correlation \(\rho\) of both study and auxiliary variables. Also, we can obtain the expression of \(Biases\) and \(MSEs\) of above class of estimators \(\hat{T_i}\) for \(i = 0, R, P, 1, 2, 3 \ldots 13\) and \(\alpha = -1, 1\) as

\[
\text{Bias}(T_i) = \bar{Y} \left[\frac{\alpha(\alpha + 1)}{2} C_i^2 \left(f_{C_x}^2 - W_{x(i)}^2\right) - \alpha C_i \left(f_{C_{xy}} - W_{xy(i)}\right)\right],
\] (46)

\[
MSE(\hat{T}_i) = \left[f \left(S_y^2 + R_t^2 S_x^2 - 2R_t S_{xy}\right) - \left(T_y^2 + R_t^2 T_x^2 + 2R_t T_{xy}\right)\right].
\] (47)
4. Efficiency comparisons

Case 1: comparison with ratio estimator from Eq. (17) and Eq. (42)

\[ \text{MSE}(\hat{T}) < \text{MSE}(T_{r}) , \text{ if } \]
\[ f \left( R^2_y S^2_{xy} - 2 R_y S_{xy} \right) - \left( R^2_y T^2_x - 2 R_y T_{xy} \right) < f \left( R^2_y S^2_{xy} - 2 R S_{xy} \right) - \left( R^2_y T^2_x - 2 R T_{xy} \right) , \]
\[ R^2_y \left( f S^2_y - T^2_x \right) - 2 R_y \left( f S_{xy} - T_{xy} \right) < R^2 \left( f S^2_y - T^2_x \right) - 2 R \left( f S_{xy} - T_{xy} \right) , \]
\[ \left( f S^2_y - T^2_x \right) \left( R^2_y - R^2 \right) < 2 \left( f S_{xy} - T_{xy} \right) (R_y - R) , \]
\[ \frac{2 \left( f S_{xy} - T_{xy} \right)}{f S^2_y - T^2_x} - R < R_y < R . \] (48)

Case 2: comparison with product estimator from Eq. (19) and Eq. (42)

\[ \text{MSE}(\hat{T}) < \text{MSE}(T_{p}) , \text{ if } \]
\[ f \left( R^2_y S^2_{xy} - 2 R_y S_{xy} \right) - \left( R^2_y T^2_x - 2 R_y T_{xy} \right) < f \left( R^2_y S^2_{xy} + 2 R S_{xy} \right) - \left( R^2_y T^2_x + 2 R_y T_{xy} \right) , \]
\[ R^2 \left( f S^2_y - T^2_x \right) - 2 R \left( f S_{xy} - T_{xy} \right) < R^2 \left( f S^2_y - T^2_x \right) + 2 R \left( f S_{xy} - T_{xy} \right) , \]
\[ \left( f S^2_y - T^2_x \right) \left( R^2_y - R^2 \right) < 2 \left( f S_{xy} - T_{xy} \right) (R_y + R) , \]
\[ - R < R_y < \frac{2 \left( f S_{xy} - T_{xy} \right)}{f S^2_y - T^2_x} + R . \] (49)

The above efficiency comparison clearly shows that the proposed generalized class of estimators of the population mean under ranked set sampling is more efficient than the ratio and product estimators of the population mean under ranked set sampling. Also, the class of different ratio and product estimators will be more efficient than to correspondingly natural ratio estimator and product estimators in RSS if both cases 1 and 2 are satisfied with the conditions. Empirical study and simulation study will also clearly illustrate the efficiency.

5. Empirical study

Numerically, to explore the properties of the proposed generalized class of the estimators over member estimators of the proposed class of estimators for mean in RSS and over unbiased estimator. We are compiling two natural data sets from Singh [31; 1111-1113].

Data set-I concerns the agricultural loans [in thousand dollars ($000)] in different states of USA in 1997 of all outstanding operating banks. y: farm loans (real estate), x: farm loans (non-real estate). The required values for the estimation of means are as
Similarly, data set-II concerns the hypothetical situation of a small village having only 30 old persons (age more than 50 years). y: duration of the sleep (in minutes), x: age in years. The required values for the estimation of means are as

\[ N=30, \quad X = 2018.0, \quad Y = 11526.0, \quad \bar{X} = 67.2670, \quad \bar{Y} = 384.20, \quad S_x^2 = 85.2370, \quad S_y^2 = 3582.580, \quad C_x^2 = 0.01880, C_y^2 = 0.02430, \quad S_{xy} = -472.6070, \quad \beta_1(x) = 0.1982, \quad \beta_2(x) = 3.7316, \quad \rho_{xy} = -0.8552. \]

We have taken 25 RSS samples from the both the population data sets. In data set-I, we have taken set size \( n = l = 4 \) with \( k = 3 \) replications, so that \( n = lk = 12 \). Similarly, in data set-II, we have taken set size \( l = 3 \) with \( k = 3 \) replications, so that \( n = lk = 9 \). Further, for this 25 RSS sample data from both population data set, we have calculated MSEs (rounded to 0) of the estimators for lighting the properties of estimators and results are shown in tables 2 and 3 respectively.

Later, we have taken different RSS samples having different set sizes with different replications. We have drawn, total \( n = lk = 6, 8, 10, 9, 12, \) and 15, RSS data from data set-I having set sizes \( l = 3, 4, 5 \) with \( k = 2, 3 \) replications. Similarly, we have drawn, total \( n = lk = 4, 6, 6, \) and 9, RSS data from the data set-II having set sizes \( l = 2, 3 \) with \( k = 2, 3 \) replications. Further, with this RSS data (having different set sizes and replications) from both population data sets, we have calculated relative efficiencies (RE) for more clearance on lighting the properties of the estimators. The results through relative efficiencies calculation are shown in tables 4 and 5.

6. Monte-Carlo simulation
We have carried a Monte-Carlo simulation study for throwing light on the properties of the proposed generalized class of estimators in RSS over the unbiased estimator and extracted estimators from proposed estimators. Monte-Carlo simulation is carried out in R Studio [32] by taking a random bivariate unit from a population with (bivariate normal) means (50.0, 50.0) and covariance matrix \[
\begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix}
\] (\( \rho = -0.9, -0.7, -0.5, 0.5, 0.7, 0.9 \)). We have taken total bivariate RSS samples of sizes \( n = lk = 12, 15, 18, 24, 30, \) and 36 (set sizes \( l = 4, 5, 6 \) with \( k = 3, 6 \) replications ) from the population of size \( N=100 \). We have replicated each simulation
study 5000 times for the estimation of means and MSEs, and the results through relative efficiencies calculation are shown in Tables 6 and 7. The relative efficiency formula is as:

\[
RE(V, t_0) = \frac{MSE(t_0)}{MSE(V)},
\]

where, \( V = t_p, T_{[i]}, \hat{T}_{[i]}, \hat{T}_{oper} \) and \( \hat{T}_{[i]} \) for \( i; 1, 2, 3, ..., 13 \).

7. Results and Discussion

In the empirical study section, Table 2 presents the MSEs of the ratio type estimators under RSS along with MSEs of an unbiased estimator and ratio estimator under SRS. We see, the ratio types member estimators and the proposed estimator under RSS have performed better over unbiased and ratio estimators under SRS. Also, the proposed estimator under RSS has the lowest MSEs in the Table (has shown superiority).

Table 3 exhibits the MSEs of the product type estimators under RSS along with MSEs of an unbiased estimator and product estimator under SRS. We see, the product types member estimators and the proposed estimator under RSS have performed better over unbiased and product estimators under SRS. Also, the proposed estimator under RSS has the lowest MSEs in the Table (has shown superiority).

Similarly, in Tables 4 and 5, our proposed estimator under RSS has shown superiority in terms of relative efficiencies over ratio and product type member estimators under RSS and SRS. Also, the estimators under RSS have increasing relative efficiencies with the increasing values of set sizes and replications.

Alike empirical study section, we compiled a Monte-Carlo simulation study and results are in tables 6 and 7 in terms of relative efficiencies.

Table 6 presents the relative efficiency of the proposed estimator, ratio type member estimators under RSS along with RE of ratio estimator in SRS over mean per unit estimator. We see relative efficiencies of the proposed estimator and member estimators of the proposed estimator in RSS have increased with increasing values of the correlation coefficient, set sizes, and replications.

Table 7 shows the relative efficiency of the proposed estimator, product type member estimators under RSS along with RE of product estimator in SRS over mean per unit estimator. We see relative efficiencies of the proposed estimator and member estimators under RSS have increased with increasing values of the correlation coefficient, set sizes, and replications.
8. Conclusion
In this study, we developed a generalized class of estimators has been developed for estimating a finite population means in ranked set sampling scheme. The simulation study shows that the proposed class of estimators perform better than the other estimators for all the considered cases do and showed findings on a real data example. These results are also similar to the simulation study. Therefore, the proposed methods, in this manuscript, can be considered in many real applications, such as mean estimation in case of missing data, quality control charts for monitoring the process mean, and in acceptance sampling plans. In future studies, the proposed method can be generalized by using different types of ranked set samplings to obtain more efficient results for different cases.

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References


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Biography
Dr. Gajendra Vishwakarma is working as an Associate Professor of statistics in the Department of Mathematics & Computing, Indian Institute of Technology Dhanbad, India with several years of academic as well as industrial research experience in the field of statistics. His research experience covers both applied as well as theoretical provinces. He has published number of research papers in international journal repute. He visited European and Asian (Oceania) countries and doing collaborative research work with them. He has supervised twelve PhD theses so far and many more ongoing. He is an Elected Member of International Statistical Institute Netherlands. Country Representative of ISI Young Statisticians, International Statistical Institute (ISI), Netherlands. Fellow of the Royal Statistical Society UK and Founder member of RSS (Indian Local Group) UK. Associate Fellow of the International Academy of Physical Sciences India. He received international travel awards from DST Govt of India, World Bank Trust Fund for Statistical Capacity Building, USA.
International Biometric Society, USA to attend the scientific events abroad. He has high value ongoing research projects funded by DST, CSIR, ICSSR. He received Young Scientist Award from Center for Advanced Research and Design, India. He has received PROFESSOR P.V. SUKHATME and Smt. Suraj Kali Jain Awards of Indian Society for Medical Statistics, India. SIRE fellowship from DST Govt of India. Recently he has selected as member of National Academy of Sciences India by Department of Science & Technology Government of India. He is editor / associate editor / editorial board member of some reputed journals like Scientia Iranica, BMC Medical Research Methodology, Measurement: Interdisciplinary Research and Perspectives, Revista de Investigacion Operacional etc.

<table>
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<th>$\hat{T}_{[0]}$</th>
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<th>$B$</th>
<th>$\alpha$</th>
<th>$\hat{Y}_{RSS}$</th>
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<td>1</td>
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\[
\hat{T}_{[2]} \quad C_x \quad \beta_2(x) \quad -1 \quad \hat{\varphi}_{RSS} \left( \frac{C_x \bar{X}_{RSS} + \beta_2(x)}{C_x \bar{X} + \beta_2(x)} \right)
\]
\[
\hat{T}_{[3]} \quad 1 \quad \rho \quad -1 \quad \hat{\varphi}_{RSS} \left( \frac{\bar{X}_{RSS} + \rho}{\bar{X} + \rho} \right)
\]

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### Table 3: MSE of the product types estimators under RSS and $t_0$

Population 2, $N=30$, $\rho_{xy} = -0.8552$

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### Table 4: RE of the $\hat{T}_{[i]}$ (ratio) and $\hat{T}_{opt}$ under RSS over unbiased estimator $t_0$

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### Table 5: RE of the $\hat{r}_{[i]}$ (product) and $\hat{T}_{opt}$ under RSS over unbiased estimator $t_0$

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### Table 6: RE of the $\hat{T}_{[i]}$ (ratio) and $\hat{T}_{opt}$ under RSS over unbiased estimator $t_0$

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Table 7: RE of the \( \hat{T}_{[i]} \) (product) and \( \hat{T}_{opt} \) under RSS over unbiased estimator \( t_0 \)

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Appendix

Notations and error terms are as

\[ \begin{align*}
\bar{y}_{[a]} &= \bar{Y}(1 + e_0), \quad \bar{x}_{[a]} = \bar{X}(1 + e_1), \quad E(e_0) = E(e_1) = 0, \quad E(e_0^2) = \left(fC^2 - W^2\right), \quad f = \frac{1}{lk}, \\
E(e_0 e_1) &= \left( fC_{yx} - W_{yx} \right), \quad \tau_{[i]} = \left( \mu_{[i]} - \bar{Y} \right), \quad \tau_{x[i]} = \left( \mu_{x[i]} - \bar{X} \right), \quad \tau_{yx[i]} = \left( \mu_{yx[i]} - \bar{Y} \right)(\mu_{x[i]} - \bar{X}), \\
\delta_y &= \frac{\tau_{y[i]}}{\bar{Y}}, \quad \delta_x = \frac{\tau_{x[i]}}{\bar{X}}, \quad \delta_{yx} = \frac{\tau_{yx[i]}}{\bar{Y} \bar{X}}, \quad W_y = \frac{1}{f^2} \sum_{i=1}^l \delta_{y[i]^2}, \quad W_x = \frac{1}{f^2} \sum_{i=1}^l \delta_{x[i]^2}, \quad W_{yx} = \frac{1}{f^2} \sum_{i=1}^l \delta_{yx[i]^2}, \\
E(e_1^2) &= \left( fC^2_x - W^2_x \right), \quad T_y^2 = \frac{1}{f^2} \sum_{i=1}^l \tau_{y[i]^2}, \quad T_x^2 = \frac{1}{f^2} \sum_{i=1}^l \tau_{x[i]^2}, \quad T_{yx} = \frac{1}{f^2} \sum_{i=1}^l \tau_{yx[i]^2}, \quad R = \frac{\bar{Y}}{\bar{X}}. 
\end{align*} \]

\[ \begin{align*}
C_1 &= \frac{\bar{X}}{\bar{X} + C_x}, \quad C_2 = \frac{\bar{X} + \beta_2(x)}{\bar{X} + \beta_2(x) + C_x}, \quad C_3 = \frac{\beta_1(x)\bar{X}}{C_x \bar{X} + \beta_2(x)} , \quad C_4 = \frac{C_x \bar{Y} + \beta_2(x)}{A\bar{X} + B}, \quad A\bar{Y} = R_f \\
R_1 &= \frac{\bar{Y}}{\bar{X} + C_x}, \quad R_2 = \frac{\bar{Y}}{\bar{X} + \beta_2(x)}, \quad R_3 = \frac{\beta_2(x) \bar{Y}}{C_x \bar{X} + \beta_2(x)}, \quad R_4 = \frac{C_x \bar{Y}}{C_x \bar{X} + \beta_2(x)}, \quad R_5 = -R_4
\end{align*} \]

We have another way of finding Bias and MSE of the proposed generalized class of estimator and proposed class of estimator is as

\[ \begin{align*}
T &= \bar{y}_{\text{RSS}} \left( \frac{A\bar{X} + B}{A\bar{X}_{\text{RSS}} + B} \right)^\alpha = \bar{y}_{[a]} \left( \frac{A\bar{X} + B}{A\bar{X}_{[a]} + B} \right)^\alpha. 
\end{align*} \]  

(ii)

Expressing estimator from Eq. (ii) in terms of \( e \)'s and taking approximating terms having up to degree 2, we get

\[ \begin{align*}
T &= \bar{Y}(1 + e_0) \left[ \frac{A\bar{X} + B}{A\bar{X}(1 + e_1) + B} \right]^\alpha, \\
&\approx \bar{Y}(1 + e_0) \left[ \frac{A\bar{X} + B}{(A\bar{X} + B) + A\bar{X}e_0} \right]^\alpha, \\
&\equiv \bar{Y}(1 + e_0) \left[ 1 + C_0 e_1 \right]^\alpha, \quad \text{where} \quad C_0 = A\bar{X} / (A\bar{X} + B), \\
&\equiv \bar{Y} \left[ 1 + e_0 - \alpha C_0 e_1 + \frac{\alpha(\alpha + 1)}{2} C_0^2 e_1^2 - \alpha C_0 e_1 e_0 \right].
\end{align*} \]

Therefore,

\[ \begin{align*}
(T - \bar{Y}) &= \bar{Y} \left[ e_0 - \alpha C_0 e_1 + \frac{\alpha(\alpha + 1)}{2} C_0^2 e_1^2 - \alpha C_0 e_1 e_0 \right]. 
\end{align*} \]  

(iii)

Taking expectation on both side of Eq. (iii) and from appendix Eq. (i), we have

\[ \begin{align*}
E(T - \bar{Y}) &= \bar{Y} E \left[ e_0 - \alpha C_0 e_1 + \frac{\alpha(\alpha + 1)}{2} C_0^2 e_1^2 - \alpha C_0 e_1 e_0 \right].
\end{align*} \]  

21
\[
\begin{align*}
= \frac{\alpha(\alpha+1)}{2} C_0^2 \bar{Y} (e_i^2) - \alpha C_0 \bar{Y} (e_0 e_i), \\
= \frac{\alpha(\alpha+1)}{2} C_0^2 \bar{Y} \left( f C_x^2 - W_x^2 \right) - \alpha C_0 \bar{Y} \left( f \rho_{yx} C_y C_x - W_{yx} \right), \\
= \bar{Y} \left[ f \left( \frac{\alpha(\alpha+1)}{2} C_0^2 C_x^2 - \alpha C_0 \rho_{yx} C_y C_x \right) - \left( \frac{\alpha(\alpha+1)}{2} C_0^2 W_x^2 - \alpha C_0 W_{yx} \right) \right].
\end{align*}
\]

Hence

\[
\text{Bias}(T) = \bar{Y} \left[ f \left( \frac{\alpha(\alpha+1)}{2} C_0^2 C_x^2 - \alpha C_0 \rho_{yx} C_y C_x \right) - \left( \frac{\alpha(\alpha+1)}{2} C_0^2 W_x^2 - \alpha C_0 W_{yx} \right) \right]. \quad (iv)
\]

**Optimality of the Bias**

The optimum bias can be obtained by minimizing bias with respect to \( \alpha \) by taking

\[
\frac{\partial \left( \text{Bias}(T) \right)}{\partial \alpha} = 0,
\]

therefore

\[
\frac{\partial \left( \text{Bias}(T) \right)}{\partial \alpha} = \frac{\partial \left( \bar{Y} \left( \alpha + \frac{1}{2} \right) C_0 \left( f C_x^2 - W_{x(i)}^2 \right) \right) - \bar{Y} \left( C_0 \left( f C_{y(i)} - W_{y(i)} \right) \right)}{\partial \alpha} = 0,
\]

i.e.,

\[
\bar{Y} \left( \alpha + \frac{1}{2} \right) C_0 \left( f C_x^2 - W_{x(i)}^2 \right) - \bar{Y} \left( C_0 \left( f C_{y(i)} - W_{y(i)} \right) \right) = 0,
\]

i.e.,

\[
\alpha_i = \frac{\left( f C_{y(i)} - W_{y(i)} \right)}{\left( f C_x^2 - W_{x(i)}^2 \right)} \times \frac{1}{2}, \quad \text{where, } C_0 = \frac{AX}{AX + B},
\]

replacing \( \alpha \) by \( \alpha_i \) in Eq. (iv), we get

\[
\text{Bias}(T)_{\text{min}} \approx \bar{Y} \left[ f \left( \frac{\alpha_i(\alpha_i+1)}{2} C_0^2 C_x^2 - \alpha_i C_0 \rho_{yx} C_y C_x \right) - \left( \frac{\alpha_i(\alpha_i+1)}{2} C_0^2 W_x^2 - \alpha_i C_0 W_{yx} \right) \right]. \quad (v)
\]

Squaring both side of Eq. (iii) and taking expectation having degree not more 2, we get

\[
E \left( T - \bar{Y} \right)^2 \approx \bar{Y}^2 E \left( e_0 - \alpha C_0 e_i + \frac{\alpha(\alpha+1)}{2} C_0^2 e_i^2 - \alpha \bar{Y} C_0 e_i e_i + O \left( e_i \right) \right)^2,
\]

\[
\approx \bar{Y}^2 \left( E \left( e_0^2 \right) + \alpha^2 C_0^2 E \left( e_i^2 \right) - 2 \alpha C_0 E \left( e_0 e_i \right) \right),
\]

\[
\approx \bar{Y}^2 \left[ f \left( C_y^2 + \alpha^2 C_0^2 C_x^2 - 2 \alpha C_0 C_{yx} \right) - \left( W_x^2 + \alpha^2 C_0^2 W_x^2 - 2 \alpha C_0 W_{yx} \right) \right].
\]

Hence,

\[
\text{MSE}(T) \approx \bar{Y}^2 \left[ f \left( C_y^2 + \alpha^2 C_0^2 C_x^2 - 2 \alpha C_0 C_{yx} \right) - \left( W_x^2 + \alpha^2 C_0^2 W_x^2 - 2 \alpha C_0 W_{yx} \right) \right]. \quad (vi)
\]