Scientia Iranica (2024) 31(20), 1906-1915

Sharif University of Technology
Scientia Iranica
Transactions B: Mechanical Engineering
https://scientiairanica.sharif.edu

A mixed pseudo-spectral FFT-FE method for asymmetric nonlinear heat transfer of a functionally graded hollow cylinder

Mehdi Dehghan [a,*], Amin Moosaie [b], and Mohammad Zamani Nejad [b]

a. Faculty of Mechanical Engineering, Malek Ashtar University of Technology, Shahin Shahr, Iran.
b. Department of Mechanical Engineering, Yasouj University, P.O. Box 75914-353, Yasouj, Iran.

* Corresponding author: m_dehghan@mut-es.ac.ir *(M. Dehghan)*

Received 8 April 2022; received in revised form 27 February 2023; accepted 19 June 2023

Keywords
Nonlinear thermal analysis;
Temperature–dependent material properties;
FG hollow cylinder;
Pseudo–spectral FFT–FE method.

Abstract

In this article, a novel spectral method based on the integral transform and Finite Element (FE) method is introduced for nonlinear thermal analysis of a hollow cylinder under asymmetric boundary excitations. The material properties are temperature-dependent and vary in terms of spatial coordinates. This dependency makes the problem to be nonlinear. The intended nonlinear heat conduction equation is discretized using FEs in the radial direction. Fast Fourier transform (FFT) technique with the uniform distribution of the harmonics in the circumferential direction, is used to discretize the periodic domain and boundary conditions. The use of the FFT algorithm is accompanied by a significant save in computational times and efforts. In such problems, the pseudo-spectral technique, as an evolved model of the spectral method, is utilized whenever the material properties vary in terms of the periodic variables or there exists a nonlinear term. The convolution sum technique is appropriately used to transform the nonlinear terms in the Fourier space. Thermal boundary conditions at the inner surface of the cylinder are considered in asymmetrical form. In compliance with the other analytical and numerical solutions, the present mixed-method benefits from the fast rate of convergence and high accuracy.

1. Introduction

Initially, Functionally Graded Materials (FGMs) are introduced as temperature resistance in the aerospace industries [1]. Gradual properties in such materials are reached by varying the volume fractions of constituting materials in terms of the desirable spatial variables. For example, the composition of a Functionally Graded (FG) plate is formed by varying material properties between two free surfaces of the plate, from ceramic towards the metal, gradually. Low thermal conductivity and resistance to high temperatures of ceramic materials make them an attractive choice to be used as thermal barriers. On the other hand, the flexibility of the metallic constituent of the FGM blend prohibits brittle fracture due to excessive thermal stresses.

In FGMs it is often observed that a highly conductive alloy is integrated with a low conductive ceramic. So, heat conduction analyses of these materials are extremely concerned by many scholars. Up to now, a set of advanced numerical solutions based on the Finite Element Method (FEM) is investigated for heat conduction problems [2-4]. Annasabi and Erchiqui [2] solved the nonlinear heat equation in terms of the Kirchhoff transformation, $\vartheta(T)$. The coefficient of heat conduction has been represented by a piecewise function. Each function is assumed to be described by a B-spline polynomial function. Nonlinear governing equations have been analyzed by using the FE approach. In a challenging problem, an enriched FEM, where the basis functions are augmented with a summation of exponential functions, has been executed for nonlinear transient heat transfer in FGMs [3]. In a similar work, a novel Polynomial Element Differential Method (PEDM) was presented by Zhou et al. [4] for solving two-dimensional nonlinear transient heat conduction problems. New shape functions with respect to isoparametric coordinates were derived to conduct the polynomial elements with an internal node. The finite difference scheme was used for discretizing the transient term. Newton iterative method was then conducted in nonlinearity governing equations.

From the physical aspect, either homogeneous or heterogeneous (e.g. FG) material properties are temperature-dependent in general. This will be prominent when high temperature changes are attained [5]. That is, the temperature dependence of material properties can be

To cite this article:
M. Dehghan, A. Moosaie, and M. Zamani Nejad"A mixed pseudo–spectral FFT–FE method for asymmetric nonlinear heat transfer of a functionally graded hollow cylinder", *Scientia Iranica* (2024), **31**(20), pp. 1906–1915. https://doi.org/10.24200/sci.2023.60237.6682

ignored in the lower temperature gradients. It causes the governing equations to become nonlinear and analysis to become time consuming and complicated. Further complications arise, especially for heat conduction analysis of a hollow cylinder, as the temperature boundary conditions vary in terms of the circumferential spatial variable (asymmetric boundary conditions).

On the authors' knowledge, pseudo-spectral nonlinear analysis of heat conduction equation for an FG hollow cylinder under asymmetric excitations is scarce. Sladek et al. [6] could analyze transient heat conduction of the heterogeneous FG cylinder by using the local boundary integral-based computational method. An algorithm containing the transfinite element method is implemented by Azadi and Shariyat [7] to show that how temperature-dependent material properties affect the heat transfer of an FG cylinder. An approximate solution based on the regular perturbation method is proposed by Moosaie [8] to investigate steady heat transfer for an FG cylinder with temperature-dependent material properties. Shojaeefard and Najibi [9] considered highly nonlinear governing equations of two-directional FG cylinders with temperature-dependent materials.

Xu et al. [10] proposed a hybrid method combining the Galerkin Free Element Method (GFREM) and the Local Radial Point Interpolation Method (LRPIM) for solving steady and transient heat conduction problems with temperature-dependent thermophysical properties in heterogeneous media. In simulation of heat conduction with temperature-dependent physical properties and boundary conditions, the general FEM instead of the conventional FEM is introduced by Yao et al. [11]. Numerical examples show that general FEM yields results with higher accuracy and stability than conventional FEM in simulation of transient heat conduction with temperature-dependent physical properties. In addition to the above-mentioned studies, a hybrid analytical/numerical approach using a mathematical framework of the Distribution Transfer Function Method (DTFM) is introduced for investigating transient heat conduction through the composite hollow cylinder structures [12]. A Localized Method of Fundamental Solution (LMFS) has been introduced to solve the nonlinear heat conduction problem by Wang et al. [13]. For this reason, Kirchhoff transforms have been used combined with an innovative, non-traditional, Fictitious Time Integration Method (FTIM).

Spectral methods are generally classified as numerical-based discretization techniques which are applicable to solve the PDEs. Especially in shells of revolution, the Fourier transform technique recognizes as a vigorous tool to decrease the computational costs. Fast Fourier Transform (FFT) gives numerous advantages such as available computer implementation and fast convergence for solving this type of problem. Numerical methods may be inherently accompanied by some deficiencies in modeling of geometry, discretization, and satisfaction of boundary conditions. Semi-analytical and mixed methods seem to be appropriate ways to alleviate these shortcomings. In this article, the FFT technique and the FE method through the Fourier spectral method are combined to achieve more advantages in the nonlinear thermal analysis of a thick hollow cylinder. In shells of revolution analysis, the periodic field-variable (as for example, in the circumferential direction) is approximated using a set of trigonometric functions (i.e. Fourier series). To the best of the authors' knowledge, the nonlinear thermal analysis of an FG thick hollow cylinder using numerical hybrid FFT-FE technique is scarce. One can be referred to the numerical framework base on the FFT transform and FEM proposed for thermal analysis of disk brakes [14-16]. Currently, a novel combined FFT-*p*FE method based on the FFT and *p*-version FEM has been introduced by Dehghan et al. [17] to thermo-electro-elastic analysis of a thick hollow cylinder.

In this article, the following steps are considered. At first, a brief history of the pseudo-spectral method and its advantages in the nonlinear solution of the partial differential equations are explained. The convolution sum, as an auxiliary technique, is introduced to remedy difficulties that arise from the nonlinear terms in the Fourier space. In the next step, nonlinear heat conduction equation of the hollow cylinder considering the temperature-dependent heat conductivity is discretized using the proposed mixed FFT-FE method. It reduces the computational efforts of 2-D Partial Differential Governing Equations (PDGE) including boundary conditions into 1-D PDGE for a long cylinder. Finally, numerical results correspond to the asymmetric excitation are presented and the proposed method is validated.

2. Pseudo-spectral methods beyond transform techniques

In the spectral methods, the appropriate selection of the trial and test functions is most influential on the accuracy of the results. Basically, trial functions are considered as a combination of the base functions and are used to approximate the field variables of the problem. Meanwhile, weight functions are assumed to satisfy the governing equations and related boundary conditions. In particular cases, e.g. in the Galerkin method, these functions are named as approximate and weight functions, respectively. The choice of the bases (trial) functions is the main distinction between the early spectral method and the traditional numerical methods such as FEs and finite differences. In this section, we deal with a specific class of spectral methods known as the Fourier spectral method. Generally, the trigonometric functions are used in this approach as the trial and test functions. for satisfying the governing equations as well as the boundary conditions, the Galerkin approach can be employed. In the Galerkin spectral method, when the nonlinear terms appear or the coefficients of the differential equation are varying with respect to the periodic variable, the implementation of some *convolution concepts* is unavoidable. Products of periodic functions in the Fourier space are the most important reason for this implementation. For example, consider $\alpha(x)$ and $\beta(x)$ to be periodic functions, such that we have:

$$\xi(x) = \alpha(x)\beta(x). \tag{1}$$

Using an infinite series expansion, the convolution sum can be defined as the following:

$$\hat{\xi}_k = \sum_{m+n=k} \hat{\alpha}_m \hat{\beta}_n, \tag{2}$$

where

$$\alpha(x) = \sum_{m=-\infty}^{\infty} \hat{\alpha}_m e^{imx}, \quad \beta(x) = \sum_{n=-\infty}^{\infty} \hat{\beta}_n e^{inx} \tag{3}$$

and

$$\hat{\xi}_k = \frac{1}{2\pi}\int_0^{2\pi} \xi(x) e^{-ikx} dx. \tag{4}$$

Here, the field variables, $\alpha(x)$ and $\beta(x)$, are considered as truncated Fourier series whose finite wave numbers are $\leq$

$\frac{N}{2}$. It is assumed that $\xi \in S_{2N}$, whereas, related trigonometric polynomials belong to S_N. The trigonometric polynomial space, S_N, is considered as:

$$S_N = \text{Span}\left\{e^{ikx} \,\middle|\, -\frac{N}{2} \leq k \leq \frac{N}{2} - 1\right\}. \tag{5}$$

$\hat{\xi}_k$ are only admissible for the interval $|x| \leq \frac{N}{2}$. In the Fourier space, for product of Eq. (1) in the finite wave numbers and prescribed order, $\frac{N}{2}$, we have:

$$\hat{\xi}_k = \sum_{\substack{m+n=k \\ |m|,|n| \leq \frac{N}{2}}} \hat{\alpha}_m \hat{\beta}_n, |k| \leq \frac{N}{2}. \tag{6}$$

Direct calculation of Eq. (6) yields operations of $O(N^2)$ [18]. These operations in three dimensions are of $O(N^4)$. This is an expensive computational effort. Especially, when one can consider that, for the finite-difference method, these amounts of operations are $O(N)$ and $O(N^3)$ in one and three dimensions, respectively. Using an integral transform method, e.g. the Fourier spectral method, these operations reduce to $O(N\log_2 N)$ in one dimension and $O(N^3\log_2 N)$ in three dimensions. For the first time, this technique has been introduced by Orszag [19] and Eliasen et al. [20]. It can be considered as the most important effort done to gain spectral convergence, especially for large-scale problems.

The approach used in the Fourier space, S_N, for evaluating Eq. (6) for α and β has the following steps. One should transform $\hat{\alpha}_m$ and $\hat{\beta}_n$ into the real space using Discrete Fourier Transform (DFT). Then, the multiplication is done similarly to Eq. (1) in the real space. Afterward, the so-obtained result is transformed into the frequency space. For a better demonstration, the following discrete transforms are considered:

$$\alpha_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{\alpha}_k e^{ikx_j}, \quad \beta_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{\beta}_k e^{ikx_j}, \qquad j = 0,1,\dots,N-1, \tag{7}$$

and we define:

$$\xi_j = \alpha_j \beta_j, \qquad j = 0,1,\dots.,N-1, \tag{8}$$

and

$$\tilde{\xi}_k = \frac{1}{N}\sum_{j=0}^{N-1} \xi_j e^{-ikx_j}, k = -\frac{N}{2},\dots,\frac{N}{2} - 1 \tag{9}$$

where

$$x_j = \frac{2\pi j}{N}.$$

$\tilde{\xi}_k$ are discrete Fourier coefficients related to function *S*. Now, using the orthogonality relation given below:

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{-ipx_j} = \begin{cases} 1 & \text{if } p = Nm, m = 0, \pm 1, \pm 2, \dots \\ 0 & \text{other wise} \end{cases} \tag{10}$$

The discrete Fourier coefficients of Eq. (9), $\tilde{\xi}_k$, lead to:

$$\tilde{\xi}_k = \sum_{m+n=k} \hat{\alpha}_m \hat{\beta}_n + \sum_{m+n=k\pm N} \hat{\alpha}_m \hat{\beta}_n = \hat{\xi}_k + \sum_{m+n=k\pm N} \hat{\alpha}_m \hat{\beta}_n. \tag{11}$$

The second term in the right-hand side of the above equation is known as the *aliasing error*. This error sometimes, especially using spectral Galerkin methods, causes inappropriate approximation of the differential equation. This approach has been introduced by Orszag in 1971 [21] as the *pseudo-spectral method*. It can be easily extended the pseudo-spectral method for computing convolution sum in the three-dimensional space. There are two distinct methods for eliminating error of Eq. (11), the *padding technique* and the *phase shifts* [22].

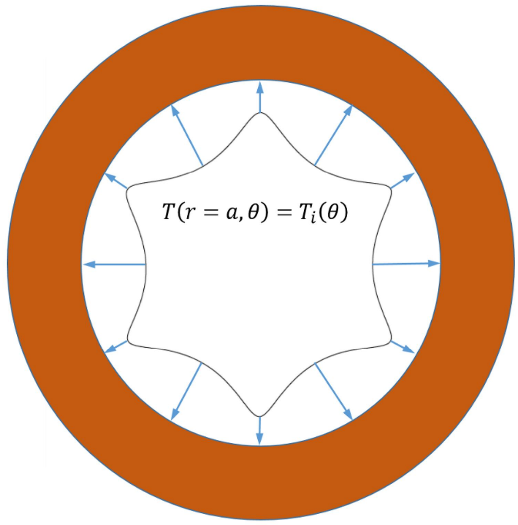


Figure 1. Asymmetric distribution of temperature field.

3. Nonlinear heat conduction formulation

3.1. Governing equation

In the previous section, a brief review of the pseudo-spectral method as well as the convolution sum technique has been presented. Now, a novel pseudo-spectral method as a combination of the FFT and FE method is utilized to investigate the nonlinear heat conduction phenomenon for an FG hollow cylinder with temperature-dependent material properties exposed to an asymmetric thermal excitation, as schematically depicted in Figure 1. Such asymmetric thermal excitation can be seen in the fluid flow within pipes subjected to thermal asymmetrical boundary conditions which take place in many real industrial situations such as those related to solar thermal devices, aerial pipelines subjected to external temperatures, etc. Other examples and attempts to solve this problem can be seen on [23-26]. Firstly, a graded model is introduced to indicate that how the material properties vary with respect to the thickness direction. The coefficient of heat conduction for a cylinder whose material properties are graded in the radial direction can be defined as $\lambda = \lambda(T, r)$, where T is the temperature field and *r* refers to the radial coordinate of the cylinder.

Micromechanics models of FGMs consisting of the Voigt model (in accordance with the mixed models of volume fraction), self-consistent method and Mori–Tanaka scheme have been introduced in the past few decades. The Mori–Tanaka model is applicable to a discontinuous particulate phase [27], whereas the Voigt model is more simple and more convenient in approximating FG material properties. Volume fraction is a spatial function in thickness or radius direction $(V_m(r), V_c(r))$. V_c and V_m are the ceramic and metal volume fractions and they can be expressed by

$V_c + V_m = 1$. The effective coefficient of thermal conductivity is considered to be in the following generalized form [28]:

$$\lambda_{eff}(r,T) = [1 - V_m(r)]\lambda_c(T) + \lambda_m(T)V_m(r) + f[\lambda_c(T), \lambda_m(T), V_m(r)]. \quad (12)$$

It is assumed in Voigt method that $f[\lambda_c, \lambda_m, V_m(r)]$ is negligible. According to the Mori–Tanaka scheme, thermal conductivity can be expressed by Shen [29]:

$$\frac{\lambda_{eff}(r,T) - \lambda_m(T)}{\lambda_c(T) - \lambda_m(T)} = \frac{[1 - V_m(r)]}{1 + V_m(r)\dfrac{\lambda_c(T) - \lambda_m(T)}{3\lambda_m(T)}} \quad (13)$$

In FG structures, the mechanical properties of constituent materials are assumed to be temperature-dependent and can be expressed as a nonlinear function of temperature:

$$P_{\text{eff}} = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + p_3T^3) \quad (14)$$

where P_0, P_{-1}, P_1, P_2 and P_3 are the constant coefficients of the temperature. (noting that $P_{-1}T^{-1}$ usually can be ignored under high temperature circumstances). Substituting Eq. (14) into (12) for thermal conductivity the following general form can be obtain:

$$\lambda_{eff} = \lambda_0(r) + \lambda_1(r)T + \lambda_2(r)T^2 + \lambda_3(r)T^3 + \cdots \quad (15)$$

For better demonstration of the present pseudo-spectral method, a simplified form of the above equation is considered as $\lambda_{\text{eff}} = (\lambda_0 - \lambda_1 T)\left(\frac{r}{r_o}\right)^l$. In which λ_0 and λ_1 are material constants and r_0 is the outer radius of the cylinder.

Here, the steady-state equation of heat conduction in the cylindrical coordinate system is considered for a hollow FG cylinder. Considering the principle of energy conservation as well as the Fourier law of heat conduction along with the general dependency of materials to the spatial variables and temperature field, above mentioned equation can be derived. So, the following field equation can be concluded:

$$\nabla \cdot q = 0, \quad (16)$$

where ∇ and q are the nabla operator and the heat flux vector, respectively. Substituting for heat flux using Fourier law of heat conduction yields:

$$\nabla \cdot (\lambda \nabla T) = 0, \quad (17)$$

and related boundary conditions are considered to be:

$$T(r = r_i) = T_i, T(r = r_o) = T_o, \quad (18)$$

in which r_i and r_o are related to the inner surface and outer surface of the cylinder, respectively.

3.2. FFT-FE discretization

A new model of the Fourier spectral method containing a combination of the integral transform and the FEM along with the pseudo-spectral concept is used for nonlinear analysis. Due to the periodicity of the geometry under consideration, we can gain the ability of the trigonometric approximate functions and Fourier transform to discretize the problem domain. The FFT technique as a vigorous numerical-based algorithm of the Fourier transform is accompanied by some advantages such as high accuracy, ease of implementation, and super-algebraic convergence rate in the approximation of the periodic functions. In this regard, we use the ability of the spectral method, as an appropriate foundation, for combining the different classical and traditional numerical methods, i.e. FFT technique and FEM.

In the Fourier spectral methodology, the partial differential equations are primarily discretized in the transformed Fourier space. Then, the so-obtained results are retransformed into the real space. It should be noted that, in nonlinear differential equations and the cases where the coefficients are varying in terms of the periodic spatial variables, the use of the pseudo-spectral method is obligatory. In the following, we show how one can analyze the nonlinear heat conduction problem under asymmetric thermal excitations by using the proposed pseudo-spectral FEM.

Assuming the temperature filed $T(r, \theta)$ as:

$$\tilde{I} = (0,2\pi), \; I = (0.5,1), \; \Omega = I \times \tilde{I}, \quad (19)$$

we have:

$$\nabla \cdot (\lambda \nabla T) = 0 \quad \text{in } \Omega. \quad (20)$$

All the functions are assumed to have period of 2π in the circumferential direction, θ. Mutually, boundary conditions are considered to be as Eq. (18). Considering a Sobolev space $H^\gamma(Q)(\gamma \geq 0)$ with the corresponding norm of $[H^\gamma(Q)]^n$ and desired infinity domain $Q \subset R^n (n = 1$ or $2)$, the field variable of temperature is defined as bellow:

$$H_Q^1(\Omega) = \{T \in H^1(\Omega) \mid T(0.5, \theta) = T_i, T(1.0, \theta) = T_o, \forall\theta \in \tilde{I}; T(r, \theta) = T(r, \theta + 2\pi) \forall r \in I\}, \quad (21)$$

And

$$W = \left[H_Q^1(\Omega)\right]^2. \quad (22)$$

For the field variable $T \in W$, the weak form of the Eq. (20) as a bilinear form becomes:

$$(\lambda \nabla T, \nabla w) = 0 \; \forall w \in W. \quad (23)$$

Now, we should consider an appropriate space to approximate the field variable in the non-periodic domain. For this purpose, according to the FE assumptions, the subdomain I is divided into the intervals $I_j = (r_{j-1}, r_j), (1 \leq j \leq J)$ with equal lengths of $h_j = r_j - r_{j-1}$. As shown in Figure 2, the cylinder cross-section is discretized by using N harmonics in the θ-direction and N_r

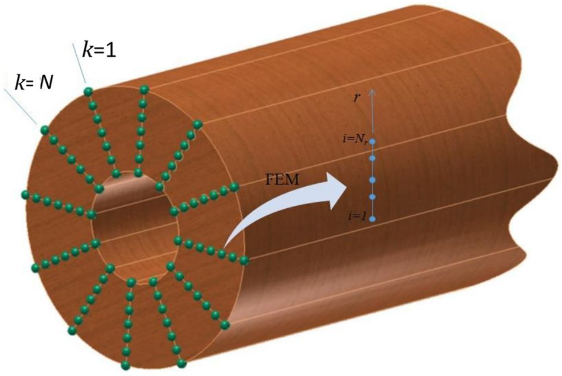


Figure 2. Discretized geometry of the FG hollow cylinder using FFT-FE method.

FE-nodes along the r-direction. If m is assumed to be any non-negative integer and P_m to be a set of polynomials (from the Lagrange family) with degrees equal or less than m, the space of analysis in the radial direction is defined as the following set:

$$L_{m,h} = \left\{\varphi \in H^1(I) | \varphi|_{I_j} \text{ be polynomial of degree } \le \text{m}\right\}. \quad (24)$$

The approximate space regarding periodic domain is introduced by:

$$L_N = \text{Span}\{e^{ik\theta} \mid -N \le k \le N\}, \quad (25)$$

where the interpolation points are introduced via:
$\theta_j = \frac{2\pi j}{2N}(0 \le j \le 2N)$.
Consequently, the approximate space of the problem is defined as below:

$$W = L_{m,h} \otimes L_N. \quad (26)$$

We rewrite Eq. (23) for an FG thick cylinder to extract the weak form of the governing equation. It reads:

$$\int_{r_i}^{r_o} \int_0^{2\pi} \left\{\mathbf{w}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\lambda\frac{\partial T}{\partial \theta}\right)\right]\right\} r dr d\theta = 0, \quad (27)$$

in which $\mathbf{w}$ is the vector of weight functions from the Lagrange family and assumed to be sufficiently smooth and differentiable. In the approximate space of L_N, the following approximation functions are defined for the field variable of temperature and its derivatives:

$$\begin{aligned} T^N &= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{T}_k(r)e^{ik\theta}, \\ \lambda^N &= (\lambda_0 - \lambda_1 T^N)f(r) \\ &= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \left(\lambda_0 - \lambda_1 \hat{T}_k(r)\right) f(r)e^{ik\theta}, \\ \tau^N &= \frac{\partial(T^N)}{\partial\theta} = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} ik\hat{T}_k(r)e^{ik\theta}, \\ \mathcal{H}^N &= \frac{\partial^2(T^N)}{\partial\theta^2} = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} - k^2\hat{T}_k(r)e^{ik\theta}. \end{aligned} \quad (28)$$

Substituting the above approximations into Eq. (27) in the Fourier space, we have:

$$\int_{\Omega^e} \text{w}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{r}{r_o}\right)^l\left[\lambda_0\frac{\partial(T^N)_k^\wedge}{\partial r} - \lambda_1\left(T^N\frac{\partial(T^N)}{\partial r}\right)_k^\wedge\right]\right] + \frac{1}{r^2}\left(\frac{r}{r_o}\right)^l\left[\lambda_0(\mathcal{H}^N)_k^\wedge - \lambda_1((\tau^N)^2)_k^\wedge - \lambda_1(T^N\mathcal{H}^N)_k^\wedge\right]\right\} r dr = 0 \quad (29)$$

For discretizing of the nonlinear terms (e.g. $((\tau^N)^2)_k^\wedge$) in the above equation, the pseudo-spectral method needs to be accomplished as described in Section 2. According to the pseudo-spectral method and by using the convolution sum technique, the nonlinear terms are initially transformed into the real space. Afterward, the multiplication is done and the desired product is then immediately transformed into the Fourier space. The FEM has an important role in the discretization of the transformed governing equations. Accordingly, the following approximation functions of the field variables are defined in the Fourier space:

$$\begin{cases} w_m = \varphi_m(r), \\ T^N(r) = \sum_{m=1}^{N_r} \varphi_m(r)(T^N)_m, \\ \tau^N(r) = \sum_{m=1}^{N_r} \varphi_m(r)(\tau^N)_m, \\ \mathcal{H}^N(r) = \sum_{m=1}^{N_r} \varphi_m(r)(\mathcal{H}^N)_m, \end{cases} \quad (30)$$

where φ_m are linear shape functions from the Lagrange family. Using the integration by part technique and approximation functions of Eq. (30), the weak formulation of Eq. (29) can be derived as below:

$$\begin{aligned} &\int_{r_i}^{r_o} \left\{ r\frac{d\varphi_i}{dr} f(r)\left(\lambda_0 - \lambda_1\left[\sum_{m=1}^{N_r} \varphi_m(T^N)_m\right]^{(s-1)}\right) \left[\sum_{m=1}^{N_r} \frac{d\varphi_m}{dr}(T^N)_m\right]^{(s)} \right. \\ &+\frac{1}{r}\lambda_1 f(r)\varphi_i\left[\sum_{m=1}^{N_r} \varphi_m(\tau^N)_m\right]^{(s-1)}\left[\sum_{m=1}^{N_r} \varphi_m(\tau^N)_m\right]^{(s)} \\ &-\frac{1}{r}f(r)\varphi_i\left(\lambda_0 - \lambda_1\left[\sum_{m=1}^{N_r} \varphi_m(T^N)_m\right]^{(s-1)}\right) \\ &\left.\left[\sum_{m=1}^{N_r} \varphi_m(\mathcal{H}^N)_m\right]^{(s)}\right\} dr - \underbrace{\left[\varphi_i\left(r\lambda^N\frac{\partial T^N}{\partial r}\right)\right]_{r_i}^{r_o}}_{\hat{Q}^N} = 0. \end{aligned} \quad (31)$$

Now, the FEM and the direct iteration technique as efficient methods is used to solve the nonlinear system of governing equations. During the iterative process, values correspond to the temperature field at the s-th iteration can be obtained via:

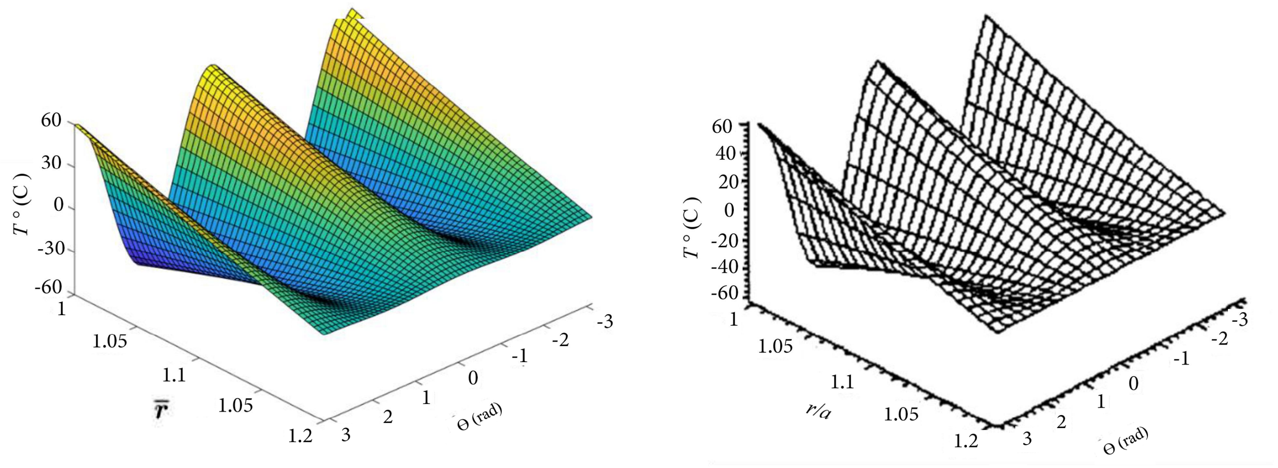
$$\left[K\left(\{T^N\}^{(s-1)}\right)\right]\{T^N\}^{(s)} = \{\hat{Q}^N\} \quad (32)$$

The coefficients matrix K at each iteration can be achieved by substituting the temperature values of the previous time step. Initial values correspond to the field variable, $\{T^N\}^{(0)}$, are considered to be the solution of the linear equation.

4. Numerical demonstrations

In this section, a numerical description of the nonlinear heat transfer analysis of an FG hollow cylinder is presented. In order to verify the solution procedure of the mixed FFT-FE method, a simple linear heat problem is considered with the following boundary conditions:

$$\begin{aligned} &T(a,\theta) = 60\cos(2\theta)^\circ\text{C}, \\ &\sigma_{rr}(a,\theta) = 0, \quad \sigma_{r\theta}(a,\theta) = 0, \\ &u_r(b,\theta) = 0, \quad u_\theta(b,\theta) = 0. \end{aligned} \quad (33)$$



(a) (b)

Figure 3. 2D nonsymmetric distribution of the temperature across the cylinder section: (a) present FFT-FE solution and (b) analytical solution [30].

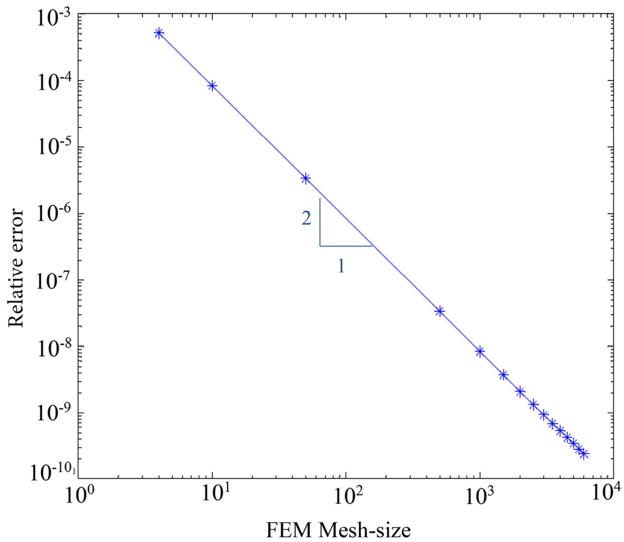


Figure 4. Convergence rate of the FFT-FE method in thickness direction.

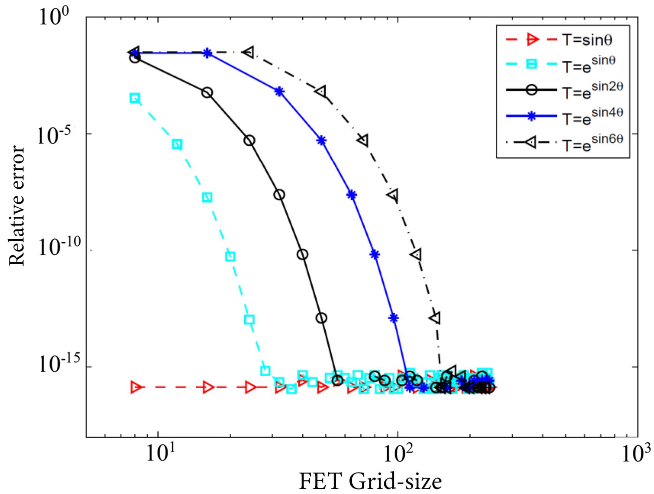


Figure 5. Logarithmic representation of the relative error versus the number of wavenumbers in the FFT technique.

2D distribution of the temperature field is shown in Figure 3. In this figure, the left-hand side pic is belonging to the FFT-FE results. A simple comparison indicates that the obtained results are in good agreement with the exact solution of Ref. [30]. The convergence rate obtained by using linear elements is shown in Figure 4. It is observed from the *log-log* representation that increasing the number of elements in the radial direction is accompanied by a linear convergence behavior. Mutually, fast convergence behavior and high accuracy of the present Fourier spectral method is depicted in Figure 5. Here, some periodic functions are assumed to be as the inner surface boundary conditions of the hollow cylinder. As shown, the rate of convergence varies by varying the complexity of the periodic functions. As expected, for $T(\theta) = \sin\theta$, the desired method can appropriately attain machine precision in finite harmonics. It is obvious from the figure that FFT technique has an exponentially varying convergence behavior.

According to the prescribed flowchart of Figure 6, the nonlinear pseudo-spectral analysis of the FG cylinder is done using both the FFT-FE discretization and direct iteration technique. To assure the accuracy of the suggested mixed method in the nonlinear heat transfer analysis of the hollow cylinder, the obtained results, for different FGM power indexes, are compared with those extracted from the analytical perturbation method [31] and are depicted in Figure 7. The related boundary conditions and material constants are considered as:

$$T_i = 0°\text{C}, \; T_o = 1000°\text{C}, \quad \lambda_0 = 50.16\ \text{W/m°C}, \quad \lambda_1 = 0.0293\ \text{W/m°C}^2. \tag{34}$$

Afterward, two distinct examples considering different temperature intensities on the outer surface of the cylinder are analyzed by means of the proposed FFT-FE method. In this regard, the following asymmetric boundary conditions are assumed:

Example 1:

$$T_i = 0.0°\text{C}, \quad T_o = 500e^{\sin(2\theta)}{}^{\circ}\text{C}, \quad \lambda_0 = 50.16\ \text{W/m°C}, \quad \lambda_1 = 0.0293\ \text{W/m°C}^2. \tag{35}$$

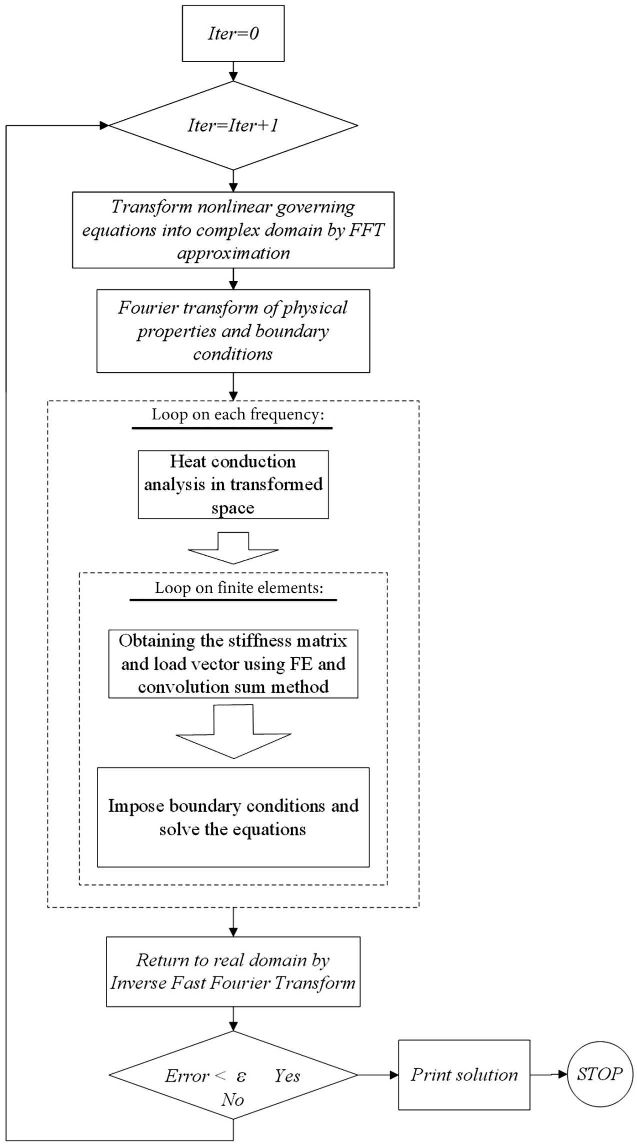


Figure 6. FFT-FE discretization flow diagram of nonlinear heat transfer analysis.

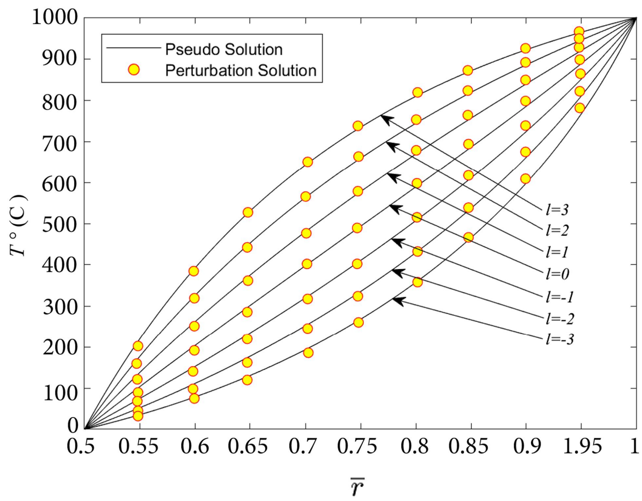


Figure 7. Distribution of the nonlinear temperature field of FG hollow cylinder.

Example 2:

$$T_i = 0.0^{\circ}\mathrm{C}, \qquad T_o = 500e^{2\sin(2\theta)}{}^{\circ}\mathrm{C},$$
$$\lambda_0 = 50.16\ \mathrm{W/m^{\circ}C}, \qquad \lambda_1 = 0.0293\ \mathrm{W/m^{\circ}C^2} \qquad (36)$$

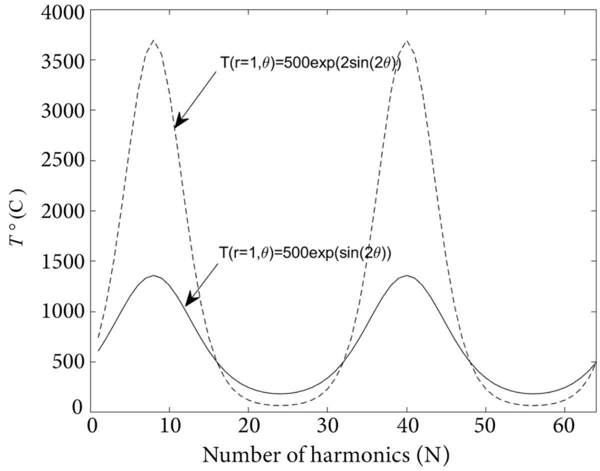


Figure 8. Asymmetric variations of the boundary temperature of the cylinder.

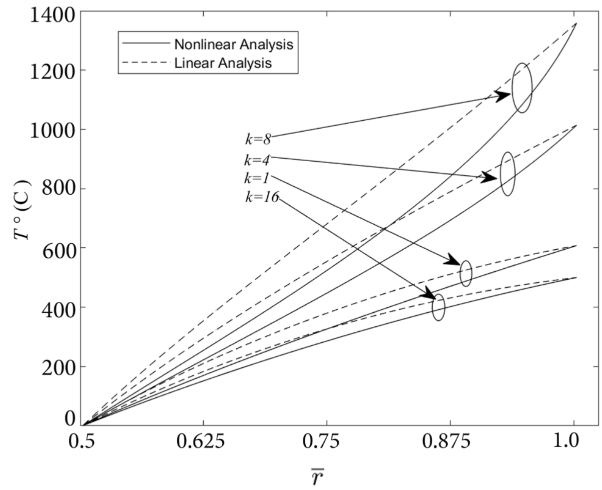


Figure 9. Linear and nonlinear temperature distribution in different harmonics (Example 1).

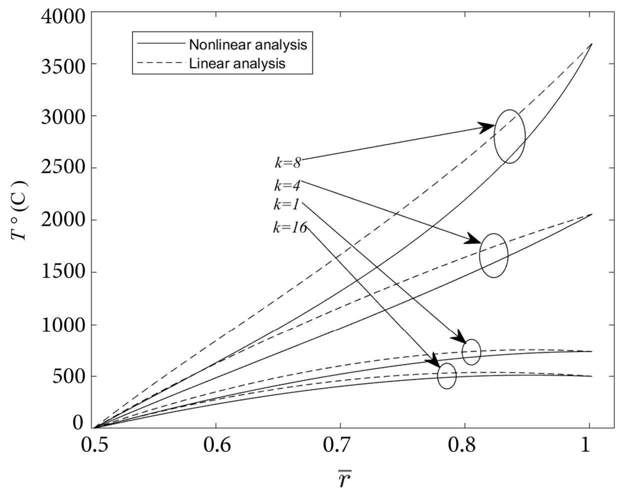


Figure 10. Linear and nonlinear temperature distribution in different harmonics (Example 2).

A visual view for Circumferential variation of the boundary temperature field of Eqs. (35) and (36) is depicted in Figure 8. Considering these boundary conditions, the desired nonlinear heat equation can be solved by using the pseudo-spectral method. In the following, nonlinear results of the temperature-dependent heat equation in compliance with the linear ones are presented at different harmonics and are depicted in Figures 9 and 10.

2D linear and nonlinear distribution of the temperature field of Example 1 are presented in Figures 11 and 12, respectively. As depicted, the pseudo-spectral method can appropriately solve the nonlinear heat conduction equation of a thick hollow cylinder. By considering the boundary conditions of Eq. (36), the linear and nonlinear numerical results of the heat equation are obtained and presented in Figures 13 and 14.

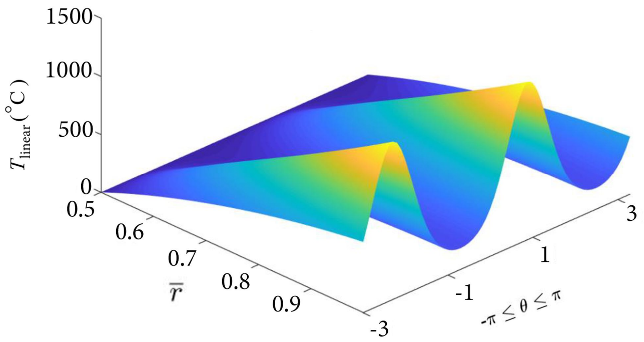


Figure 11. 2D distribution of the temperature field for the linear regime (Example 1).

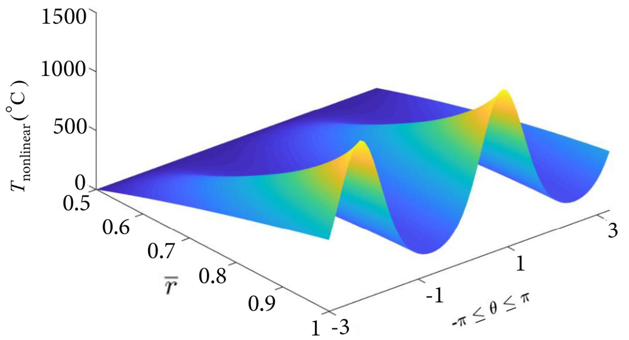


Figure 12. 2D distribution of the temperature field for nonlinear regime (Example 1).

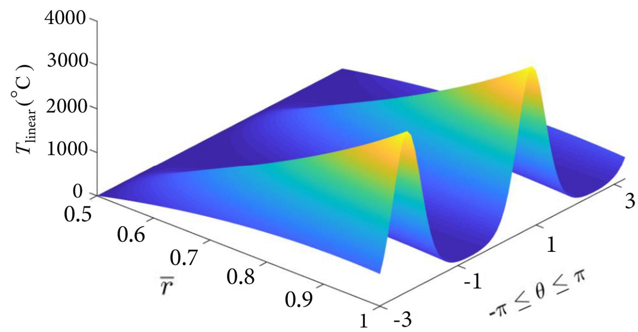
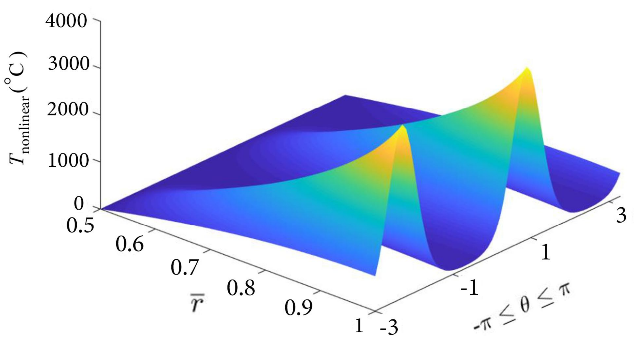


Figure 13. 2D distribution of the temperature field for linear regime (Example 2).



14. 2D distribution of the temperature field for nonlinear regime (Example 2).

5. Conclusion

In this paper, the pseudo-spectral method through a combination of the Discrete Fourier Transform (DFT) and Finite Element Method (FEM) was successfully implemented for nonlinear heat transfer analysis of a hollow Functionally Graded (FG) cylinder while the material properties vary in terms of the temperature field. The Fourier spectral approach, as well as the convolution sum technique, are used to discretize the governing equation and related nonlinear terms. According to the convolution sum technique, the nonlinear terms are transformed into the real space, the multiplication operation is done, then the product of two field variables (or their derivatives) is retransformed into the Fourier space. In the radial and circumferential directions, the FEM method and FFT technique are used to discrete governing equations, respectively. The proposed mixed method benefits low computational cost of the FFT method as well as the ability of the FEM to model the complicated geometries and boundary conditions. Basically, the DFT needs $2N^2$ operations which are done using matrix-vector multiplications, whereas the Fast Fourier Transform (FFT) technique can decrease this amount of operations up to $(5/2)N\log_2 N$. From a numerical point of view, the proposed method is quite efficient. Asymmetric boundary conditions are considered and the so-obtained results were validated using exact solutions. It is concluded from the investigations that the heat conductivity constant λ_1 has a major influence on the numerical convergence of the direct iterative method. The presented hybrid method can be extended to the three-dimensional analysis of homogeneous, composite, and FG thick shells of revolution with temperature-dependent and circumferentially varying material properties.

Nomenclature

α, β	Arbitrary periodic functions
N	Wave numbers
S_N	Trigonometric polynomial space
$\hat{\xi}_k$	Convolution sum
x_j	Variables in periodic domain
T	Temperature field
λ	Coefficient of heat conduction
λ_{eff}	Effective coefficient of thermal conductivity
V_c, V_m	Ceramic and metal volume fractions
P_i	Constant coefficients of the temperature
q	Heat flux vector
∇	Nabla operator
$\tilde{I}$	Periodic domain
I	Radial domain
$H_Q^1(\Omega)$	Sobolev space in which the field variable is defined
$L_{m,h}$	Appropriate space to approximate field variable
T^N	Approximation function related to the temperature field
τ^N	Approximation function related to first derivative of the temperature field
$\mathcal{H}^N$	Approximation function related to second derivative of the temperature field

K Coefficient matrix

References

1. Noda, N. "Thermal stresses in functionally graded materials", *Journal of Thermal Stresses*, **22**(4-5), pp. 477–512 (1999). https://doi.org/10.1080/014957399280841

2. Annasabi, Z. and Erchiqui, F. "Robust Kirchhoff transformation using B-spline for finite element analysis of the non-linear heat conduction", *International Communications in Heat and Mass Transfer*, **120**, 104985 (2021). https://doi.org/10.1016/j.icheatmasstransfer.2020.104985

3. Malek, M., Izem, N., and Seaid, M. "A three-dimensional enriched finite element method for nonlinear transient heat transfer in functionally graded materials", *International Journal of Heat and Mass Transfer*, **155**, 119804 (2020). https://doi.org/10.1016/j.ijheatmasstransfer.2020.119804

4. Zhou, L., Lv, J., Cui, M., et al. "A polygonal element differential method for solving two-dimensional transient nonlinear heat conduction problems", *Engineering Analysis with Boundary Elements*, **146**, pp. 448–459 (2023). https://doi.org/10.1016/j.enganabound.2022.10.015

5. Noda, N. "Thermal stresses in materials with temperature-dependent properties", *Applied Mechanics Reviews*, **44**(9), pp. 383-397 (1991). https://doi.org/10.1115/1.3119511

6. Sladek, J., Sladek, V., and Zhang, C. "Transient heat conduction analysis in functionally graded materials by the meshless local boundary integral equation method", *Computational Materials Science*, **28**(3-4), pp. 494–504 (2003). https://doi.org/10.1016/j.commatsci.2003.08.006

7. Azadi, M. and Shariyat, M. "Nonlinear transient transfinite element thermal analysis of thick-walled FGM cylinders with temperature-dependent material properties", *Meccanica*, **45**(3), pp. 305-318 (2010). https://doi.org/10.1007/s11012-009-9249-4

8. Moosaie, A. "Axisymmetric steady temperature field in FGM cylindrical shells with temperature-dependent heat conductivity and arbitrary linear boundary conditions", *Archives of Mechanics*, **67**(3), pp. 233–251 (2015).

9. Shojaeefard, M.H. and Najibi, A. "Nonlinear transient heat conduction analysis of hollow thick temperature-dependent 2D-FGM cylinders with finite length using numerical method", *Journal of Mechanical Science and Technology*, **28**(9), pp. 3825–3835 (2014). https://doi.org/10.1007/s12206-014-0846-3

10. Xu, B.-B., Gao, X.-W., and Cui, M. "An efficient and accurate hybrid weak-form meshless method for transient nonlinear heterogeneous heat conduction problems", *Engineering with Computers*, pp. 1–16 (2020). https://doi.org/10.1007/s00366-020-01050-7

11. Yao, X., Wang, Y., and Leng, J. "A general finite element method: Extension of variational analysis for nonlinear heat conduction with temperature-dependent properties and boundary conditions, and its implementation as local refinement", *Computers & Mathematics with Applications,* **100**, pp. 11–29 (2021). https://doi.org/10.1016/j.camwa.2021.08.024

12. de Andrade, G.S., de Lemos, M.J., and Colombo, D. "A new hybrid analytical/numerical method for transient heat conduction in composite hollow cylinders applied to plug and abandonment of oil wells", *International Journal of Thermal Sciences*, **168**, 106981 (2021). https://doi.org/10.1016/j.ijthermalsci.2021.106981

13. Wang, F., Liu, Y.-C., and Zheng, H. "A localized method of fundamental solution for numerical simulation of nonlinear heat conduction", *Mathematics*, **10**(5), p. 773 (2022). https://doi.org/10.3390/math10050773

14. Cho, C. and Ahn, S. "Transient thermoelastic analysis of disk brake using the fast Fourier transform and finite element method", *Journal of Thermal Stresses*, **25**(3), pp. 215–243 (2002). https://doi.org/10.1080/014957302317262288

15. Floquet, A. and Dubourg, M. "Nonaxisymmetric effects for three-dimensional analysis of a brake", *Journal of Tribology*, **116**(3), pp. 401–407 (1994). DOI: 10.1115/1.2928852

16. Floquet, A. and Dubourg, M.-C. "Realistic braking operation simulation of ventilated disk brakes", *Journal of Tribology*, **118**(3), pp. 466–472 (1996). https://doi.org/10.1115/1.2831559

17. Dehghan, M., Moosaie, A., and Z Nejad, M. "Hybrid integral transform and *p*-version FEM for thermo-mechanical analysis of a functionally graded piezoelectric hollow cylinder subjected to asymmetric loads", *Scientia Iranica*, **28**(4), pp. 2229–2249 (2021). DOI: 10.24200/sci.2021.56289.4646

18. Orszag, S.A. "Spectral methods for problems in complex geometrics", *Numerical Methods for Partial Differential Equations: Elsevier*, pp. 273–305 (1979). https://doi.org/10.1016/B978-0-12-546050-7.50014-9

19. Orszag, S.A. "Numerical methods for the simulation of turbulence", *The Physics of Fluids*, **12**(12), pp. II-250–II-257 (1969). https://doi.org/10.1063/1.1692445

20. Eliasen, E., Machenhauer, B., and Rasmussen, E. "On a numerical method for integration of the hydrodynamical equations with a spectral representation of the horizontal fields", *Kobenhavns Universitet, Institut for Teoretisk Meteorologi*, (1970).

21. Orszag, S.A. "Numerical simulation of incompressible flows within simple boundaries. I. Galerkin (spectral) representations", *Studies in Applied Mathematics*, **50**(4), pp. 293–327 (1971). https://doi.org/10.1002/sapm1971504293

22. Patterson Jr, G. and Orszag, S.A. "Spectral calculations of isotropic turbulence: Efficient removal of aliasing interactions", *The Physics of Fluids*, **14**(11), pp. 2538–2541 (1971). https://doi.org/10.1063/1.1693365

23. Arutyunyan, R. "Three-dimensional simulation of heat and mass transfer processes during SHF heating of oil flow in a pipeline", *Journal of Engineering Physics and Thermophysics,* **94**(3), pp. 745–752 (2021). https://doi.org/10.1007/s10891-021-02352-3

24. Dehdarinejad, M., Behbahani-Nejad, M., and Hajidavalloo, E. "Experimental and numerical investigation of heat loss in transition modes of buried hot fuel oil pipeline", *Proceedings of the Institution of Mechanical Engineers*, *Part C: Journal of Mechanical Engineering Science*, **235**(22), pp. 6567–6576 (2021). https://doi.org/10.1177/09544062211000794

25. Na, T., Baodong, C., Limin, H., et al. "Research on the overall heat transfer coefficients of crude oil pipeline and product pipeline laid in one ditch", *Petroleum Science and Technology*, **30**(3), pp. 247–255 (2012). https://doi.org/10.1080/10916466.2010.551821

26. Stepanov, A. and Egorova, G. "Simulation of heat transfer of heating-system and water pipelines under northern conditions", *Journal of Engineering Physics and Thermophysics*, **5**(89), pp. 1284–1288 (2016). DOI: 10.1007/s10891-016-1493-6

27. Shen, H.-S. and Wang, Z.-X. "Assessment of Voigt and Mori–Tanaka models for vibration analysis of functionally graded plates", *Composite Structures*, **94**(7), pp. 2197–2208 (2012). https://doi.org/10.1016/j.compstruct.2012.02.018

28. Zhang, H., Kan, W., and Hu, X. "Green's function approach to the nonlinear transient heat transfer analysis of functionally graded materials", *International Journal of Thermal Sciences*, **71**, pp. 292–301 (2013). https://doi.org/10.1016/j.ijthermalsci.2013.04.025

29. Shen, H.-S. "Nonlinear vibration of shear deformable FGM cylindrical shells surrounded by an elastic medium", *Composite Structures*, **94**(3), pp. 1144–1154 (2012). https://doi.org/10.1016/j.compstruct.2011.11.012

30. Jabbari, M., Sohrabpour, S., and Eslami, M. "General solution for mechanical and thermal stresses in a functionally graded hollow cylinder due to nonaxisymmetric steady-state loads", *Journal of Applied Mechanics*, **70**(1), pp. 111–118 (2003). https://doi.org/10.1115/1.1509484

31. Moosaie, A. "A nonlinear analysis of thermal stresses in an incompressible functionally graded hollow cylinder with temperature-dependent material properties", *European Journal of Mechanics-A/Solids*, **55**, pp. 212–220 (2016). https://doi.org/10.1016/j.euromechsol.2015.09.005

Biographies

Mehdi Dehghan received his MSc degree in Mechanical Engineering from Shahid Bahonar University of Kerman in 2011, Iran and, his PhD degree in Mechanical Engineering from Yasouj University, Iran in 2018. He is interested in advanced numerical methods for the nonlinear analysis of three-dimensional shells of revolution in curvilinear coordinate systems.

Amin Moosaie received his BSc in Mechanical Engineering from Iran University of Science and Technology (IUST), Tehran, Iran, in 2006. He earned his MSc in Computational Mechanics and his PhD in Fluid Dynamics and Aerodynamics from Technische Universität Müunchen (TUM), Germany, in 2008 and 2011, respectively. He then started post-doctoral studies at TUM in the same subject, during which he took a sabbatical leave to the Polymer Physics Department at the ETH-Zurich, Switzerland, for three months in the Summer of 2011. He joined the Mechanical Engineering Department of Yasouj University, Iran, as a faculty member in 2012, where he is currently an Associate Professor. He is the Khwarizmi Award laureate of 2006 and has supervised the best MSc thesis of 2015 as elected by the Iranian Society of Mechanical Engineers (ISME). His primary research interests are computational mechanics, continuum mechanics, turbulence simulation (DNS/LES), drag reduction, and polymer dynamics in fluid flows.

Mohammad Zamani Nejad is an Associate Professor in the Department of Mechanical Engineering, College of Engineering, at Yasouj University, Iran. He received his PhD in College of Engineering from Tarbiat Modares University, Tehran, Iran. Moreover, He received his BSc and MSc degrees in Mechanical Engineering from Shiraz University and Mazandaran University, respectively. His research interest areas include thermo-elasto-plastic analysis and creep analysis of beams, plates, and shells. He has published several papers in international journals and serves as an editorial board member for some international journals.