Integrability and Dynamics Analysis of the Chaos Laser System

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Abstract. In this article the complex dynamics of a laser model, which externally injected class B which is described by a system of three nonlinear ordinary differential equations with two parameters for field intensity phase and population inversion, are studied. In particular, we investigate the integrability and nonintegrabilty of laser system in three dimension. We prove that system is completely integrable only when the parameters are zero. Particularly, we study polynomial, rational, Darboux and analytic first integrals of the mentioned system. Moreover, we compute all the invariant algebraic surfaces and exponential factors of this system. We find sufficient conditions for the existence of periodic orbits emanating from an equilibrium point origin of a laser differential system with a first integral.

1. Introduction and the main result

The literature on the existence and nonexistence of first integrals or generally the integrability problem of a dynamical system is one of the main open problems in the qualitative theory of differential systems, see [1, 2, 3, 4]. Many non-linear dynamical systems arise in physical and electrical engineering, etc. Finding an innovative way to illustrate and analyze these systems has been an interesting subject in the field of differential equations and dynamical systems. Chaotic behavior has been appeared in some nonlinear laser systems (see [5]). The two-level laser model and its time-rescaled variant, along with numerical results demonstrating evidence of generalized bistability, were introduced by the authors in [6]. The physics of deterministic chaos and integrability problems has been covered in many recent review contributions, both in their theoretical and experimental aspects [7]. The theory of integrability and invariant algebraic curves plays an important role in the study of the dynamics of polynomial differential systems.

We will refer to an externally injected class B which is described by a system of three nonlinear ordinary differential equations for field intensity phase and population inversion. When the

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damping rate associated with the field is much greater than that of the population, as in CO_2 laser [8-10], the equation describing a class B laser with injected signal is

$$\dot{\phi} = -\theta - \frac{E_0}{E} sin\phi,$$
$$\dot{E} = \omega E + E_0 cos\phi,$$
$$\dot{\omega} = d - E^2 - \varepsilon\left(1+E^2\right)\omega.$$

The cavity field amplitude E is normalized to its saturation value ϕ and the population inversion ω is referred to the equilibrium value. The parameter θ is the cavity mistuning normalized to the chosen time scale, $\varepsilon = \sqrt{\frac{\gamma}{k}}$, γ is the damping constant and $\frac{1}{k}$ is the relaxation time. The parameter E_0 is the Rabi frequency of the external field normalized and d is the pump parameter referred to the threshold value. For ε =0, a suitable rescaling of above system and the equations of the model describing a class B laser with injected in Cartesian coordinates are

$$\begin{aligned} \dot{x} &= z\,x + y + a, \\ \dot{y} &= y\,z - x, \\ \dot{z} &= b - x^2 - y^2, \end{aligned} \qquad (1)$$

where the time has been rescaled by θ, the field amplitude and population inversion by $\frac{1}{\theta}$, and the new parameters are $a = \frac{E_0}{\theta^2}, b = \frac{d}{\theta^2}$, see for more details [9, 10, 11, 12]. System (1) is time reversible by the invariance of the flow under the time reversal with respect t $:(x, y, z, t) \to (-x, y, -z, -t)$. Since the divergence of the vector field is $2z$, then the phase space volume is not conserved. In [8] the authors proved the existence of periodic attractors and repellers of system (1) by a symmetry breaking bifurcation. In [9] the authors gave some physically motivation of system (1). The stability of the equilibrium points in system (1) is investigated in [11], and numerical studies demonstrate that each of them is impacted by the parameters. Additionally, the system's energy is examined with the system's dynamic characteristics are found. Arecchi discussed the physics of laser chaos and related problems in [7]. For a scenario in which the polarization is adiabatically erased, the dynamical characteristics of a class-B laser system with dissipative strength are examined [12]. The numerical simulation reveals that the system contains more than one attractor and has a fold-Hopf bifurcation [12]. The dynamics of this system has been intensively studied; see for instance [9, 10, 12, 13, 14, 15, 16, 17]. In this paper we investigate the topological structure of the dynamics of system (1) by studying its integrability of the system appears in physics of laser.

The existence of a first integral in a differential system (1) reduces the dynamics analysis of this system in one dimension when the first integral's value is fixed. This makes things easier strongly the analysis of the dynamics of such systems. Moreover, if a system (1) has two independent first integrals then fixing these two first integrals we obtain the solutions curves.

In what follows, we summarize the main results related to the generalized rational and global analytic first integrals of system (1) when $a=0$ and $b=0$ or $b\neq 0$.

Theorem 1. Consider a polynomial laser system (1). Then the following statement holds.

1. If $a=b=0$, then system (1) is completely integrable with the following first integrals $H_1=x^2+y^2+z^2$ and

$$H_2=(x^2+y^2+2z^2-2z\sqrt{x^2+y^2+z^2})\,\frac{e^{2\sqrt{x^2+y^2+z^2}\,arctan\left(\frac{y}{x}\right)}}{x^2+y^2}.$$

2. If $a=0$ and $b\neq 0,$ then system (1) has generalized rational first integral $\dfrac{x^2+y^2}{e^{\frac{x^2+y^2+z^2}{b}}}$ and is not completely integrable with two functionally independent rational first integrals.

In the second main result, related to the non-existence of polynomial and Darboux first integral for system (1) and characterize all the invariant algebraic surfaces and exponential factors of this system.

Theorem 2. The following statements holds.

i. If $a\neq 0$ or $b\neq 0$, then system (1) has no polynomial first integrals.

ii. The only irreducible invariant algebraic surfaces of system (1) with non-zero cofactor are $x+i\,y=0$ and $x-i\,y=0$ with cofactors $-i+z$ and $i+z$ respectively if and only if $a=0$.

iii. If $ab\neq 0$, then $e^{x^2+y^2+z^2}$ is the only exponential factor of system (1) with cofactor $2(ax+bz)$.

iv. System (1) with $a\neq 0$ has no Darboux first integrals.

We compute the equilibrium points of system (1) which are of the following

$E_0 = (0, -a, 0)$ if $b = a^2$,

$$E_1 = \left(\frac{-\sqrt{b\left(a^2 - b\right)}}{a}, \frac{-b}{a}, \frac{\sqrt{b\left(a^2 - b\right)}}{b} \right)$$ and

$$E_2 = \left(\frac{\sqrt{b\left(a^2 - b\right)}}{a}, \frac{-b}{a}, \frac{-\sqrt{b\left(a^2 - b\right)}}{b} \right),$$ with being real when $b\left(a^2 - b\right) \geq 0$. Now, we shall study the non-existence of analytic first integrals of system (1) when $b = a^2$, $a \neq 0$ and $a^2 > b > 0$. Using Routh-Hurwitz criterion and the existence of attractor or repeller equilibrium points in order to study the non-integrability of system (1), we obtain the following results.

Theorem 3. System (1) with $a^2 \geq b > 0$ has no any global analytic first integrals.

We now will show that in the case when for system (1) has first integral can be applied for proving the existence of periodic orbits.

Theorem 4. If $a = b = 0$, then for any sufficiently small positive that ε any integral surface $x^2 + y^2 + z^2 = \varepsilon^2$, contains at least one periodic solution of system (1) whose period is close to π.

2. Definitions and preliminary results

Before we discuss our results we need to introduce some basic facts and preliminaries. Some well-known results on the Darboux theory of integrability and analytic first integrals may be found in [1, 2, 3, 18]. We characterize here integrability and non-integrability of system (1). Thus to prove the main results, we use Darboux Theorem of integrability in order to find invariant algebraic surfaces and exponential factors and characterize its local analytic first integrals of system (1) .

By χ we denote the corresponding vector field of system (1)

$$\chi = (zx + y + a)\frac{\partial}{\partial x} + (yz - x))\frac{\partial}{\partial y} - \left(x^2 + y^2 - b\right)\frac{\partial}{\partial z}.$$

A continuously differentiable function $H(x, y, z)$ in a neighborhood $U \in \mathbb{R}^3$ is said to be a first integral of the vector field (1) if $H(x, y, z)$ is a constant on the trajectories of system (1), that is

$$\chi(H) = (z\,x + y + a)\frac{\partial H}{\partial x} + (y\,z - x)\frac{\partial H}{\partial y} + \left(b - x^2 - y^2\right)\frac{\partial H}{\partial z} = 0.$$

We call H a polynomial (respectively analytic) first integral if H is polynomial (respectively analytic). The existence of Darboux first integral depends on the exponential factors and on the invariant algebraic surfaces. Hence we recall definitions of Darboux polynomial and exponential factor. Let $f \in C[x,y,z]$ be a non-constant polynomial. If f satisfy the partial differential equation

$$(zx+y+a)\frac{\partial f}{\partial x}+(yz-x)\frac{\partial f}{\partial y}+(b-x^2-y^2)\frac{\partial f}{\partial z}=kf,$$

for some polynomial $k \in \mathbb{C}[x,y,z]$. We call $f=0$ an invariant algebraic surface (and f a Darboux polynomial) of system (1), and k is the cofactor of f of degree one.

Let $f,g \in C[x,y,z]$ be relatively coprime. A non-constant function $e^{\frac{f}{g}}$ is called an exponential factor of system (1) if it satisfies the partial differential equation

$$(zx+y+a)\frac{\partial e^{\frac{f}{g}}}{\partial x}+(yz-x)\frac{\partial e^{\frac{f}{g}}}{\partial y}+(b-x^2-y^2)\frac{\partial e^{\frac{f}{g}}}{\partial z}=Le^{\frac{f}{g}},$$

for some polynomial $L \in C[x,y,z]$ of degree 1. We call L the cofactor of $e^{\frac{f}{g}}$.

A Darboux first integral, is a first integral of the form

$$f_1^{\lambda_1}\dots f_p^{\lambda_p}E_1^{\mu_1}\dots E_q^{\mu_q},$$

where $f_1,\dots,f_p$ are Darboux polynomials and $E_1,\dots,E_q$ are exponential factors with λ_j for $j = 1,\dots, p$ and μ_k for $k = 1,\dots, q$ are constants.

The following result restricted to the invariant algebraic surfaces goes back to Darboux which concerning the existence of Darboux first integrals, see for the addition of the exponential factors for instance [2, 3, 18].

Theorem 5. Suppose that a polynomial system (1) of degree m admits p invariant algebraic surfaces $f_i=0$ with cofactors k_i for $i = 1,\dots, p$, and q exponential factors $\exp\left(\frac{g_i}{h_i}\right)$ with cofactors L_j for $j = 1,\dots, q$. Then there exist λ_i and $\mu_i \in C$ not all zero such that $\sum_{i=1}^{p}\lambda_i k_i+\sum_{j=1}^{q}\mu_i L_i=0$,

if and only if the function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1} \right) \right)^{\mu_1} \ldots (\exp\left(\frac{g_q}{h_q} \right))^{\mu_q},$$

is a Darboux first integral of system (1).

Theorem 6. [2] The following statements hold.

(a) If $\exp\left(\frac{g}{h}\right)$ is an exponential factor for the polynomial differential system (1) and h is not a constant polynomial, then $h = 0$ is an invariant algebraic surface.

(b) Eventually $\exp(g)$ can be an exponential factor, coming from the multiplicity of the invariant plane at infinity.

Furthermore, we also recall some results that we will use later on. We first consider an analytic differential system

$$\dot{X} = f(X), \qquad (2)$$

where $f : U \to \mathbb{R}^n$ is C^2, U is an open subset of $\mathbb{R}^n$ and the dot denotes the derivative with respect to time t.

An equilibrium point of system (2) is an attractor if either it is asymptotically stable of system (2), or if it is an asymptotically stable equilibrium point of system $\dot{X} = -f(X)$.

Theorem 7. [19, 20]. If system (2) has an isolated equilibrium point q which is either attractor or repeller, then it has no C^1 first integrals defined in a neighborhood of q.

We recall that a first integral which is a rational function is called a rational first integral. A generalized rational first integral is a function which is the quotient of two analytic functions.

We also need the following result for the existence of more than one functionally independent rational first integrals.

Theorem 8.[20] Assume that the differential system (1) has p as an equilibrium point and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the linear part of system (1) at p .Then the number of functionally independent generalized rational first integrals of system (1) is at most the dimension of the minimal vector subspace of $\mathbb{R}^3$ containing the set

$\{(k_1,k_2,k_3) \in \mathbb{Z}^3 : k_1\lambda_1 + k_2\lambda_2 + \lambda_3 k_3 = 0, (k_1,k_2,k_3) \neq (0,0,0)\}$.

We also need the following result concerning complete integrability of nonlinear three dimensional differential systems [21].

Theorem 9. Assume that P,Q,R,S,H and G are meromorphic functions and there exists a point $c=(x_1,y_1)$. such that $P(c)=Q(c)=R(c)=S(c)=0$ and $\frac{\partial Q(c)}{\partial y}=0$ and at least $H(c)\neq 0$ or $G(c)\neq 0$. Then system

$$\dot{x}=P(x,y)+Q(x,y)z,$$

$$\dot{y}=R(x,y)+S(x,y)z,$$
$$\dot{z}=H(x,y)+G(x,y)z,$$

is not completely integrable with two functionally independent rational first integrals in variables x,y,z.

The following result provides the existence of periodic solutions in the case when the linearized system is degenerate, see [15, 22].

Theorem 10. Let $\dot{U}=f(U)$ be a dynamical system, U_0 an equilibrium point and $C:=(C_1,C_2,\ldots,C_k):M\to\mathbb{R}^k$ vector valued constant of motion for the above dynamical system with $C(U_0)$ a regular value of C. If

1. The eigenspace corresponding to the eigenvalue zero of the linearized system around U_0 has dimension *k*.

2.Jacobian matrix at U_0 has a pair of imaginary eigenvalues $\pm i\beta$ with $\beta\neq 0$.

3. There exist a first integral $H:M\to\mathbb{R}$ for dynamical system with $dH(U_0)=0$ and such that $d^2H(U_0)\big|_{WxW}>0$, where $\mathrm{W}=\cap_{i=1}^{k} kerC_{\mathrm{i}}(U_0)$, then for each sufficiently real small ε, any integral surface $H(U)=H(U_0)+\varepsilon^2$ contains at least one periodic solution of U whose period is close to the period of the corresponding linear system around U_0.

3. Proof of the main results

In the study of the first integrals of Darboux type of system (1), one have find polynomial first integrals, all Darboux polynomials and exponential factors of system (1) and this is due to the fact that the Darboux first integrals can be constructed using these kind of functions.

Proof of Theorem 1. Let $H_1=x^2+y^2+z^2$ and $H_2=\frac{(x^2+y^2+2z^2-2z\sqrt{x^2+y^2+z^2})e^{2\sqrt{x^2+y^2+z^2}\arctan\left(\frac{y}{x}\right)}}{x^2+y^2}$.

It is clear that

$$\left(xz+y\right)\frac{\partial H_i}{\partial x}+\left(yz-x\right)\frac{\partial H_i}{\partial y}-\left(x^2+y^2\right)\frac{\partial H_i}{\partial z}=0.$$

Therefore, the function H_i is a constant over the solutions of the system (1) for $i=1,2$, and so system (1) is completely integrable when $a=b=0$. Hence this conclude the proof of the first statement in the theorem.

To prove the second statement, if $a=0$ and $b\neq 0$, then $x^2+y^2=0$ is an invariant algebraic surface of system (1) with cofactor $2z$, and $e^{\frac{x^2+y^2+z^2}{b}}$ is an exponential factor of system (1) with cofactor $2z$. From Theorem 5 it follows that $\frac{x^2+y^2}{e^{\frac{x^2+y^2+z^2}{b}}}$ is a Darboux first integral of system (1). Also this first integral is a global analytic first integral. Since analytic first integrals is particular case of generalized rational first integrals. Then its generalized first integral.

Comparing laser system (1) with system in Theorem 9, we obtain forms of functions P,Q,R,S,H and G, which are as follows:

$$P=y+a, Q=x, R=-x, S=y,$$

$$H=-b+x^2+y^2 \text{ and } G=0.$$

Since if $a=0$, $b\neq 0$ and $c=\left(x_1,y_1\right)=\left(0,0\right)$ then all conditions of Theorem 9 are satisfied, we directly conclude that the system (1) with $a=0$, $b\neq 0$ is not completely integrable with two functionally independent rational first integrals. □

Remark 1. i. We note that when $a=b=0$, then system (1) admit the polynomial first integral $H_1=x^2+y^2+z^2$. Then the phase spaces of these equations is foliated by the dimensional invariant algebraic surfaces $H\left(x,y,z\right)=r$, with $r\geq 0$, hence system (1) is not chaotic. This system is the chaotic for some special values of the parameters, see [7, 18].

ii. Also when $a=b=0$, system (1) has infinitely many equilibria $\left(0,0,c\right)$ for all $c\in\mathbb{R}$, which has three eigenvalues are $\lambda_1=0$ and $\lambda_{2,3}=c\pm i$. Suppose that $c\neq 0$, and there exist three integers k_1,k_2,k_3 such that $k_1\lambda_1+k_2\lambda_2+k_3\lambda_3=0$, then the set

$$S=\left\{k_1,k_2,k_3\in\mathbb{N}^*: k_1\lambda_1+k_2\lambda_2+k_3\lambda_3=0\right\},$$

has dimension one generated by $(k_1,0,0)$, hence via Theorem 8, system (1) has at most one generalized rational first integrals, which must be a function of H_1.

iii. According to Theorem 1, for $a=0$ *and* $b=0$, so the system (1) possess two first integrals H_1 and H_2 we are able to reduce the dimension of the phase space to one. Hence solves completely the problem of determining its phase portraits. Also if $a=0$ and $b\neq 0$, so the system (1) possess one first integral we are able to reduce the dimension of the phase space to two. Hence in order to get quickly insight into the dynamics of the considered system.

Proof of Theorem 2 i. Let $H(x,y,z)=\sum_{i=1}^{n} h_i(x,y,z)$ be a polynomial first integral of degree n of system (1), where each $h_i(x,y,z)$ is a homogeneous polynomial of degree i and $h_n\neq 0$, $n\geq 1$. Then H satisfies

$$(xz+y+a)\frac{\partial H}{\partial x}+(yz-x)\frac{\partial H}{\partial y}+(b-x^2-y^2)\frac{\partial H}{\partial z}=0. \quad (3)$$

The terms of degree $n+1$ in (3) satisfy

$$xz\frac{\partial h_n}{\partial x}+yz\frac{\partial h_n}{\partial y}-(x^2+y^2)\frac{\partial h_n}{\partial z}=0.$$

The solution of this linear partial differential equation we get

$$h_n=h_n\left(\frac{y}{x},x^2+y^2+z^2\right),$$

where h_n is a function in the variables x,y and z.

Since h_n is a homogenous polynomial of degree $n\geq 1$, we conclude that

$$h_n=c_n\left(x^2+y^2+z^2\right)^{\frac{n}{2}},$$

where c_n is a constant. Similarly computing the homogenous terms of degree n in (3) we get $h_{n-1}=c_{n-1}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}$,

where c_{n-1} is a constant. Now computing the terms of degree $n-1$ in (3) yields

$$xz\frac{\partial h_{n-2}}{\partial x}+yz\frac{\partial h_{n-2}}{\partial y}-(x^2+y^2)\frac{\partial h_{n-2}}{\partial z}+$$
$$y\frac{\partial h_{n-1}}{\partial x}-x\frac{\partial h_{n-1}}{\partial y}+a\frac{\partial h_n}{\partial x}+b\frac{\partial h_n}{\partial z}=0.$$

Solving it we obtain

$$h_{n-2} = -\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1} nbc_n \ln x + F_1\left(\frac{y}{x}, x^2+y^2+z^2\right) - \frac{\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1} nac_n \arctan\left(\frac{\sqrt{1+\frac{y^2}{x^2}}\,x}{z}\right)}{\sqrt{1+\frac{y^2}{x^2}}},$$

where F_1 is an arbitrary function. From the hypothesis $a \neq 0$ or $b \neq 0$ and taking into account that h_{n-2} must be a polynomial, we conclude that $nc_n = 0$. This is a contradiction and hence the result follows. □

Proof of Theorem 2 ii. It is easy to show that when $a = 0$ after straightforward computations $x + iy = 0$ and $x - iy = 0$ are two irreducible invariant algebraic surfaces of system (1) with respective cofactors $-i + z$ and $i + z$. First we show that the cofactor of invariant algebraic surface of system (1) is of the form $\alpha + \beta z$, for some $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{N}$.

Since system (1) is quadratic then the cofactor must be of the form $k = \alpha + \mu x + \delta y + \beta z$ where $\alpha, \mu, \delta, \beta \in \mathbb{C}$. Let $f(x,y,z) = \sum_{i=1}^{n} f_i(x,y,z)$ be an invariant algebraic surface of system (1) of degree $n \geq 1$, where each $f_i(x,y,z)$ is a homogeneous polynomial of degree i. Then it satisfies the partial differential equation

$$\left(xz + y + a\right)\frac{\partial f}{\partial x} + \left(yz - x\right)\frac{\partial f}{\partial y} - \left(x^2 + y^2 - b\right)\frac{\partial f}{\partial z} = \left(\alpha + \mu x + \delta y + \beta z\right) f.$$

Computing the terms of degree $n+1$ in the above equation and from the Euler's Theorem of homogenous function for $f_n(x,y,z)$ we obtain

$$nz f_n(x,y,z) - \left(x^2+y^2+z^2\right)\frac{\partial f_n(x,y,z)}{\partial z} = \left(\mu x + \delta y + \beta z\right) f_n(x,y,z).$$

Solving this linear differential equation, we obtain

$$f_n(x,y,z) = \left(z^2 - x^2 - y^2\right)^{\frac{\beta - n}{2}} e^{\frac{-\operatorname{arctanh}\left(\frac{z}{\sqrt{x^2+y^2}}\right)(\delta y + \mu x)}{\sqrt{x^2+y^2}}} F_1(x,y),$$

where F_1 is a polynomial function. Since f_n is a polynomial then we must $\mu = \delta = 0$ and

$\alpha \in \mathbb{C}, \beta \in \mathbb{N}$.

We now show that system (1) has no irreducible invariant algebraic surfaces of degree two or more. By the change of variables $u = x + i\,y$ and $v = x - i\,y$, system (1) becomes

$$\begin{aligned} \dot{u} &= -i\,(i - z)u + a, \\ \dot{v} &= (i + z)v, \qquad (4) \\ \dot{z} &= b - uv \ . \end{aligned}$$

Let $f\,(u,v,z)$ be an invariant algebraic surface of system (1) with cofactor $k = \alpha + \beta z$ where $\alpha \in \mathbb{C}, \beta \in \mathbb{N}$, then it must be satisfies

$$\left((1 + i\,z)u + a\right)\frac{\partial f}{\partial u} + \left((i + z)v\right)\frac{\partial f}{\partial v} +$$
$$(b - uv)\frac{\partial f}{\partial z} = (\alpha + \beta z)f \ . \ (5)$$

First if $a = 0$, then $u = 0$ and $v = 0$ are invarint algebraic surfaces of system (4) with cofactors $1 + i\,z$ and $(i + z)$ respectively, then we can write system (4) restricted to $u = 0, v = 0$:

$$\begin{aligned} \dot{u} &= 0, \\ \dot{v} &= 0, \\ \dot{z} &= b \ . \end{aligned}$$

Let $g\,(0,0,z)$ be an invariant algebraic surface of the above system with cofactor $k = \alpha + \beta z$, where $\alpha \in \mathbb{C}, \beta \in \mathbb{N}$. Then

$$b\,\frac{dg\,(z)}{dz} = (\alpha + \beta z)\,g\,(z),$$

its solution is give by $g\,(z) = c_1 e^{\frac{z(2\alpha + \beta z)}{2b}}$, where c_1 is a constant. Since g i s a polynomial then must be $\alpha = \beta = 0$. Hence has no irreducible invariant algebraic surfaces of degree two or more with non-zero cofactor.

Second if $a \neq 0$, we restrict equation (5), to $v = 0$ and we obtain

$$\left(-i\,(i - z)u + a\right)\frac{\partial f\,(u,z)}{\partial u} + + (b)\frac{\partial f\,(u,z)}{\partial z} =$$
$$(\alpha + \beta z)f \ .$$

Solving the above equation we obtain

$$f\,(u,z) = F_1\left(\frac{1}{2\sqrt{b\pi}}\left(e^{\frac{-1}{2b}}\left(\begin{array}{l} 2u\,e^{\frac{1 - z^2 + 2Iz}{2b}}\sqrt{b\pi} + \\ a\mathrm{erf}\left(\dfrac{\sqrt{2}(-z + I)}{2\sqrt{b}}\right)\sqrt{2\pi} \end{array}\right)\right)\right)e^{\frac{z(2\alpha + \beta z)}{2b}},$$

where F_1 is an arbitrary function and erf is an error function. Since $f\left(u,z\right)$ is not polynomial, then has no invariant algebraic surfaces if $a\neq 0$.This completes the proof of the theorem. □

Proof of Theorem 2 iii. From Theorems 2 ii and 6 we can write the exponential factors of system (1) of the form $E=e^{g(x,y,z)}$, where $g\left(x,y,z\right)$ is a polynomial in its variables. Since system (1) is quadratic, then the cofactor must be of the form $L=\alpha+\mu x+\delta y+\beta z$,where $\alpha,\mu,\delta,\beta\in\mathbb{C}$. First we suppose that g be a polynomial of degree $n\geq 3$. Let $g\left(x,y,z\right)=\sum_{i=0}^{n}g_i\left(x,y,z\right)$, where each $g_i\left(x,y,z\right)$ is a homogeneous polynomial of degree i and $g_n\neq 0$. Then E satisfies the partial differential equation

$$\left(xz+y+a\right)\frac{\partial e^g}{\partial x}+\left(yz-x\right)\frac{\partial e^g}{\partial y}-\left(x^2+y^2-b\right)\frac{\partial e^g}{\partial z}=Le^g.$$

Hence

$$\left(xz+y+a\right)\frac{\partial g}{\partial x}+\left(yz-x\right)\frac{\partial g}{\partial y}-\left(x^2+y^2-b\right)\frac{\partial g}{\partial z}=L.(6)$$

Computing the terms of degree $n+1$ in equation (6) we obtain

$$xz\frac{\partial g_{n(x,y,z)}}{\partial x}+yz\frac{\partial g_{n(x,y,z)}}{\partial y}-\left(x^2+y^2\right)\frac{\partial g_{n(x,y,z)}}{\partial z}=0.$$

It is general soution is

$$g_n=g_n\left(\frac{y}{x},x^2+y^2+z^2\right).$$

Since g_n is a homogenous polynomial of degree n ,we must take $g_n=k_1(x^2+y^2+z^2)^{\frac{n}{2}}$, where k_1 is a constant. Concerning the terms of degree n in equation (6) we obtain

$$xz\frac{\partial g_{n-1(x,y,z)}}{\partial x}+yz\frac{\partial g_{n-1(x,y,z)}}{\partial y}+y\frac{\partial g_{n(x,y,z)}}{\partial x}-$$
$$x\frac{\partial g_{n(x,y,z)}}{\partial y}-\left(x^2+y^2\right)\frac{\partial g_{n-1(x,y,z)}}{\partial z}=0,$$

and therefore $g_{n-1}=g_{n-1}\left(\frac{y}{x},x^2+y^2+z^2\right)$.

In the same way as the previous we must take $g_{n-1}=k_2(x^2+y^2+z^2)^{\frac{n-1}{2}}$,
where k_2 is a constant.
From equation (6), the equation of degree $n-1$ is

$$xz\frac{\partial g_{n-2(x,y,z)}}{\partial x}+yz\frac{\partial g_{n-2(x,y,z)}}{\partial y}+y\frac{\partial g_{n-1(x,y,z)}}{\partial x}$$
$$-x\frac{\partial g_{n-1(x,y,z)}}{\partial y}+a\frac{\partial g_{n(x,y,z)}}{\partial x}+b\frac{\partial g_{n(x,y,z)}}{\partial z}$$
$$-\left(x^2+y^2\right)\frac{\partial g_{n-2(x,y,z)}}{\partial z}=0.$$

Solving it we obtain

$$g_{n-2(x,y,z)}=\frac{1}{\sqrt{1+\frac{y^2}{x^2}}}(-\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}k_1nb\ln(x)\sqrt{1+\frac{y^2}{x^2}}$$
$$-ak_1n\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}arctan\left(\frac{\sqrt{1+\frac{y^2}{x^2}}\,x}{z}\right)$$
$$+F_1\left(\frac{y}{x},x^2+y^2+z^2\right)\sqrt{1+\frac{y^2}{x^2}},$$

where F_1 is an arbitrary function. Since g_{n-2} is a polynomial and $ab\neq 0, n\geq 3$, then must be $k_1=0$, therefore $g_n=0$, which is a contradiction . Hence $g(x,y,z)$ is a polynomial of degree at most two satisfying equation (6). Easy computation we get that $g=x^2+y^2+z^2$. Then $e^{x^2+y^2+z^2}$ be the only exponential factor of the system (1) with cofactor $2(ax+bz)$.

□

We now prove our main result.

Proof of Theorem 2 iv. Suppose that H is a Darboux first integral of system (1). From Theorems 2 iii and 5 then $H=e^{\mu\left(x^2+y^2+z^2\right)}$ where $\mu\in\mathbb{C}$. So H satisfies

$$\left(xz+y+a\right)\frac{\partial H}{\partial x}+\left(yz-x\right)\frac{\partial H}{\partial y}-\left(x^2+y^2-b\right)\frac{\partial H}{\partial z}=0.$$

Then $2\mu\left(ax+bz\right)H=0$, since $a\neq 0$, then must be $\mu=0.$ Therefore H is a constant, this is contradiction. □

System (1) has a rational first integral if it is has two different invariant algebraic surfaces with the same cofactor. From above analysis then system (1) has no rational first integrals via Theorems 1 and 2 when $b\neq 0.$ □

Proof of Theorem 3. First suppose that $b=a^2$ with $a\neq 0$. Then $(0,-a,0)$ be the only equilibrium point of system (1). We transform the equilibrium point $(0,-a,0)$ to $(0,0,0)$ by the change of variables $X=x, Y=y+a, Z=z$, hence system (1) becomes

$$\begin{aligned}\dot{x}&=xz+y,\\ \dot{y}&=-x+z(y-a), \qquad (7)\\ \dot{z}&=2ay-x^2-y^2,\end{aligned}$$

where we have written again (x,y,z) instead (X,Y,Z).Suppose that $H=\sum_{i\geq 0}H_i(x,y,z)$ is a local analytic first integral of system (7), where H_i is a homogenous polynomial of degree i for $i=0,1,2,\dots$. Thus H must be satisfy

$$\begin{aligned}&(xz+y)\frac{\partial H}{\partial x}+(-x+z(y-a))\frac{\partial H}{\partial y}\\ &+(2ay-x^2-y^2)\frac{\partial H}{\partial z}=0.\end{aligned} \qquad (8)$$

We now use mathmatical induction to show that $H_i=0$ for $i=1,2,3,\dots$.
The homogenous parts of degree one in equation (8) is a partial differential equation

$$y\frac{\partial H_1}{\partial x}-(x+az))\frac{\partial H_1}{\partial y}+2ay\frac{\partial H_1}{\partial z}=0.$$

The general solution of this equation is

$$H_1=F_1\left(z-2ax, 2a^2x^2+2a(-2ax+z)x+x^2+y^2\right),$$

where F_1 is an arbitrary function. Since H_1 is a homogenous polynomial of degree one, then $H_1=\alpha_1(z-2ax)$, where α_1 is a constant.
Similarly computing the homogenous parts of degree two in equation (8) satisfy

$$\begin{aligned}&y\frac{\partial H_2}{\partial x}-(x+az))\frac{\partial H_2}{\partial y}+2ay\frac{\partial H_2}{\partial z}\\ &-\alpha_1\left(2axz+x^2+y^2\right)=0.\end{aligned}$$

By the hypothesis $a\neq 0$, then the above equation has a polynomial solution of degree 2 only if $\alpha_1=0$ and we get $H_1=0,$ in this case

$$H_2=\alpha_2(z-2ax)^2+\beta_2\left(2a^2x^2+2a(-2ax+z)x+x^2+y^2\right),$$

where α_2 and β_2 are constants. Now computing the homogenous parts of degree three in equation (8) satisfy

$$y\frac{\partial H_3}{\partial x}-\left(x+az\right)\frac{\partial H_3}{\partial y}+2ay\frac{\partial H_3}{\partial z}+xz\,(-4\alpha_2 a\left(z-2ax\right)$$
$$+\alpha_3\left(4a^2x\ \ +2a\left(-2ax+z\right)x+x^2\right)=0.$$

In the same way the above equation has a polynomial solution of degree 3 only if $\alpha_2=\beta_2=0$ and we get $H_2=0$, in this case $H_3=\alpha_3(z-2ax)^3$, where α_3 is a constant. Suppose that $H_1=H_2=\ldots=H_{n-1}=0$. We consider two cases.

Case 1. If n is even number, then

$$H_n=\alpha_n\left(z-2ax\right)^n+\beta_n\left(2a^2x^2+2a\left(-2ax+z\right)x+x^2+y^2\right)^{\frac{n}{2}},$$

where α_n and β_n are constants. So the terms of degree $n+1$ in equation (8) satisfy

$$y\frac{\partial H_{n+1}}{\partial x}-\left(x+az\right)\frac{\partial H_{n+1}}{\partial y}+2ay\frac{\partial H_{n+1}}{\partial z}+$$
$$xz\frac{\partial H_n}{\partial x}+z\,y\frac{\partial H_n}{\partial y}-\left(x^2+y^2\right)\frac{\partial H_n}{\partial z}=0.$$

The above equation has a polynomial solution only if $\alpha_n=\beta_n=0$, and we get $H_n=0$ and $H_{n+1}=\alpha_{n+1}\left(z-2ax\right)^{n+1}$, where α_{n+1} is a constant.

Case 2. If n is odd number, then $H_n=\delta_n\left(z-2ax\right)^n$, where δ_n is a constant. So the terms of degree $n+1$ in equation (8) satisfy

$$y\frac{\partial H_{n+1}}{\partial x}-\left(x+az\right)\frac{\partial H_{n+1}}{\partial y}+2ay\frac{\partial H_{n+1}}{\partial z}-$$
$$2\left(n+1\right)\delta_n a\,xz\left(z-2ax\right)^n- \qquad .$$
$$\left(n+1\right)\delta_n\left(x^2+y^2\right)\left(z-2ax\right)^n=0.$$

Also the above equation has a polynomial solution of degree $n+1$ only if $\delta_n=0$ and we get $H_n=0$. We have by induction under the degree of homogeneity that $H_k=0$ for all $k\geq 1$, then we obtain H is constant, hence system (1) has no local analytic first integrals at the equilibrium point $(0,-a,0)$, so the result follows.

Second if $a^2 > b > 0$, the characteristic equation of the Jacobian matrix at the equilibrium point E_2 of system (1) is given by

$$\lambda^3 + \frac{2\sqrt{b\left(a^2-b\right)}}{b}\lambda^2 + \frac{\left(2b^2+a^2\right)}{b}\lambda + 2\sqrt{b\left(a^2-b\right)} = 0.$$

Reversing time in system (1), we have the following system

$$\dot{x} = -z\,x - y - a,\quad \dot{y} = -y\,z + x,\quad \dot{z} = x^2 + y^2 - b, \qquad (9)$$

we obtain that the characteristic equation of the Jacobian matrix at the equilibrium point E_1 of system (9) is given by

$$\lambda^3 + \frac{2\sqrt{b\left(a^2-b\right)}}{b}\lambda^2 + \frac{\left(2b^2+a^2\right)}{b}\lambda + 2\sqrt{b\left(a^2-b\right)} = 0.$$

By the hypothesis and Routh-Hurwitz criterion, the zeros of the above two characteristic equations have negative real parts, hence the equilibrium point E_1 is attractor and E_2 is repeller, so system (1) has no local analytic first integrals at the equilibrium points E_1 and E_2, by Theorem 7, and consequently the system (1) with $a^2 \geq b > 0$ has no any global analytic first integrals. □

Proof of Theorem 4. We now suppose that $a = b = 0$, then system (1) has straight line of equilibrium point $(0,0,m)$ and its characteristic equation of system (1) at $(0,0,m)$ is

$$\lambda^3 - 2m\,\lambda^2 + \left(m^2+1\right)\lambda = 0,$$

so the eigenvalues of Jacobian matrix at $(0,0,m)$ are zero and $m \pm i$. Hence eigenvalues are pure imaginary if and only if $m = 0$. The eigenspace corresponding to eigenvalue zero is $V_{\lambda=0} = Span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. Via Theorem 1, $H_1 = x^2 + y^2 + z^2$ is a first integral l of system (1) and satisfies $dH_1(0,0,0)=0$ and $d^2H_1(0,0,0)\big|_{WXW} = 6>0$, where $W = Span\left\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$.Then

satisfies all conditions of Theorem 10, so for any sufficiently small ε the energy surface $x^2+y^2+z^2=\varepsilon^2$ contains at least one periodic solution of system (1) whose period is close to π. □

4. Conclusion

This study analyzed the existence of Darboux and analytic first integrals of three dimensional chaos laser systems. Firstly, we proved that system (1) is completely integrable when $a=b=0$ and is not completely integrable with two functionally independent rational first integrals for $a=0$ and $b\neq 0$. We also shown that the laser differential system near to origin, the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1- periodic solution whose period is close to π when the parameters $a=0$ and $b=0$ (see Theorem 4).

Moreover we proved that system (1) has only two irreducible Darboux polynomials, when the parameter a is zero. We also showed that the system has neither a polynomial first integral nor a rational first integral where $a\neq 0$ or $b\neq 0$. Subsequently, we proved that the system contains only one exponential factor when ab is not zero. Additionally, we proved that the system is not Darboux integrable when the parameter a is not zero. We also proved the existence of periodic orbit emanating from the origin when $a=b=0$. Finally we verified that the system has no local analytic first integrals in a neighborhood at the equilibrium points of system (1) where $b=a^2, a\neq 0$ or $a^2>b>0$. Hence system (1) with $a^2\geq b>0$ has no any global analytic first integrals. We also proved the existence of periodic orbits via first integral.

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Biography

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