Considering Customer Lifetime Network Value in Oligopoly Markets With The Use of Game Theory Approach

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Abstract

This research proposes a model that calculates the customer lifetime value and simultaneously considers the network effects. An oligopoly market is also considered in which companies compete with each other. Each company has a number of buyers and sellers and offers its services to the buyers free while receiving the sellers' membership fees in return. Interestingly, the customers interact with each other and change the companies' clients. This interaction between them is known as word-of-mouth marketing, and it also exists among the sellers. It is noteworthy that the existence of both buyers and sellers is only meaningful. Indeed, the increase of buyers leads to a rise in the number of sellers and also makes the company more profitable. In fact, the network effects are categorized into four forms that are as follows: (1) buyers' effects on each other, (2) sellers' effects on each other, (3) buyers' effects on sellers, and (4) sellers' effects on their buyers. These effects are the main factors that the companies tend to take into consideration when determining the optimal marketing and pricing policies. Applying differential game theory makes it possible to receive the companies' market share, advertising, and pricing strategy. Besides, two numerical examples and sensitivity analyses are provided to demonstrate the proposed model. Finally, it was found that making the network effects stronger leads to less need for advertising and attracting more customers; thus, the total costs are minimized.

Keywords: Customer Lifetime Value, Game theory, Market share, Marketing, Pricing

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1. Introduction

Nowadays, companies are increasingly seeking to create and maintain long-term relationships with their customers. In fact, the aim of marketing is to maximize the customer lifetime value in the company. Creating an optimal combination of different levels of customers with maximum profit is the goal of many companies, and undoubtedly this goal is achieved only by devoting attention to CLV. Calculating the customer lifetime value (CLV) is one of the striking methods to evaluate customer profitability. Preliminary studies such as [1] conducted regarding CLV considered it to be a present value of the future profit stream expected over a given time horizon of transacting with the customer. The primary purpose of calculating CLV is to weigh the customers to allocate resources to them. Besides, the CLV seeks the best methods to compare the customers. When an industry is capable of offering various products, it is likely to provide better services to customers whose CLV is higher. It can also be employed as a criterion to draw new customers' attention. As a result, the companies' costs to attract them do not exceed the customer value [2, 3]. Also, CLV allows companies to strike a balance for their resources between attracting and retaining customers [4].

Overall, through the use of CLV, the companies are able to address the following questions [5]:

- On average, how much is each customer worth to the company?
- How is much investment for the different customers considered optimal?
- What is the best medium to have a relationship with the customers?

CLV calculation consists of a variety of models, including RFM, probabilistic, econometric, persistence, diffusion/growth, etc. In fact, these models are presented in answer to the following marketing questions:

- How do the organizations decide which customer to treat especially? Even in customized and unique cases.
- Which group of customers must the organizations communicate with through cheap communication channels (such as telephone or internet) or which customers must they get rid of?
- How do organizations can realize the best time to express their offers (product or purchase)?
- How do the organizations realize which customers will be more profitable in the future, and how do they decide to attract them?
- According to customers' information, how do the organizations allocate their sales resources and services to them for future business relationships with the customers?
- How do the organizations examine their customers' activity to regulate their marketing activities?

With respect to most of the studies conducted regarding CLV, the customers' values have often been implicitly assumed to be independent of the other customers, and it is not affected by them. In other words, their value is not the product of the interaction among them. In fact, most of the research in this field has considered the customers individually and ignored impacts on the communication networks [6]. On the other hand, each company's marketing and pricing strategy determination is a dynamic decision over time [7]. Since the companies need to choose their strategies to maximize their profits, a striking problem that arises for many companies is determining optimal marketing and pricing strategies. Consequently, their demand to maximize the profit and CLV of their customers will be met. Interestingly, some companies, such as online stores with the Marketplace business model (such as e-Bay), offer their services free in the hope of receiving membership fees from companies and vendor brands. According to traditional CLV models, the CLV of buyers in these stores is negative (due to the costs incurred by these buyers to the company, such as advertising and the lack of direct income from them). However, in the absence of the buyers, the existence of sellers is meaningless. In other words, the number of sellers is in direct proportion to the number of buyers. Therefore, companies aim to raise the number of their buyers by conducting marketing activities. Marketing strategy determination is a dynamic decision that is related to the CLV level of each customer. On the other hand, a company's decisions affect its competitive decisions in oligopoly markets. In accordance with the limitations and defects of the previous studies associated with CLV, the three main aims of this paper are as follows:

1. Presenting a model to calculate CLV by considering the relationship between the individuals and the companies in oligopoly markets.
2. Determining the optimal marketing and pricing strategies to maximize the profits based on the relationships between individuals and companies in oligopoly markets.
3. Calculating the value of the customer life cycle in a space such as a marketplace space (in the presence of buyers and the companies' members) through considering the network relationships and connections between them.

The aims of this research are regarded advantageous based on the remarkable results through which many questions in the field of companies' profitability are answered. The key questions that arise according to the network effects between the customers are as follows:

- How much investment is reasonable to obtain new customers with respect to network effects? This problem is crucial for the active companies in the markets with strong direct and indirect network effects.
- In fact, the network effects' value is probably time-dependent, and the customer value changes over time consequently. It is important to specify how this value changes over time.
- The exchange between buyers and sellers brings the customer value; it is essential to specify how much value is obtained through each set of customers.
- In the presence of network effects, should the companies change their policies over time?

In fact, the relationship considered in this study is between buyers and buyers (verbal advertisement), buyers and suppliers (the number of buyers affects the number of suppliers and vice versa), and suppliers and suppliers (verbal advertisement). Hence, through the use of optimal control and game theory, the optimal and dynamic marketing strategy (for buyers) and pricing (for suppliers) for companies in an oligopoly market is determined.

This paper develops a monopoly model presented by Gupta et al. in 2006 to the oligopoly form [8]. Also, the explicit closed-loop solutions are delivered here. In order to extract the feedback Nash equilibrium and obtain enough knowledge regarding the impact of the network on advertising expenditure, market share, and profitability, an infinite horizon differential game is considered. In addition, with the aim of obtaining the carryover dynamics of advertising, pricing, and competitive interactions, a differential game theory is applied. It is also noteworthy that this theory is beneficial to specify which course is the best time to take action for each firm concerning their competitors' response. Finally, two numerical examples and sensitivity analyses are presented to clarify the proposed model.

The rest of this research is organized as follow: the second section is provided as the literature review; in the third section, the model and assumptions are presented, section four discusses the obtained results, and also gives a numerical example and a sensitivity analysis, finally, the conclusions and the suggestions for the future work are presented in the last section.

2. Literature Review

Customer Relationship Management (CRM) refers to an approach adopted by a company to interact with existing and future customers. The aim of this management is to drive sales growth. It is significant to improve business relationships with customers; however, retaining customers is of considerable importance. This type of management considers both of them and entails analyzing data regarding customers' history with a company. In customer relationship management, the customer life cycle represents customer progress in purchasing and using a product or service and specifies their loyalty. In 2000, analysts such as Jim Sterne and Matt Cutler presented a matrix that divides a customer's life cycle into five categories: reach, acquisition, conversion, retention, and loyalty [9]. In 2003, Matthias Braehler et al. defined the customer lifetime value based on the supplier-oriented perspective, and it is considered an economic value for the company [10]. This definition is completely different from the demand-oriented concept for the customer value in a particular company. Hence, the customer lifetime value was regarded as a profound supplier-oriented understanding of customer value in this study. Through CLV, the profit streams of a customer across the whole customer life cycle are calculated. In 2009, Nicolas et al. considered the customer lifetime value (CLV) as the discounted value of future marginal earnings, depending on the customer’s activity [11]. According to this definition, when one’s marginal profit (CLV) declines, he or she is considered a churner. Calculating the customer lifetime value (CLV) is one of the striking methods to evaluate customer profitability. Some definitions associated with CLV are also provided in Table 1.

According to the aforementioned explanations and the papers indicated in Table 1, a variety of definitions for CLV have been presented so far. Nevertheless, the purpose of CLV for the companies is the same. In general, the CLV is of great importance for companies in terms of profitability.

In fact, a large number of existing studies in the broader literature have examined CLV. In 2020, Oğuzhan Kivrak and Cüneyt Akar investigated the effect of social media on customer lifetime value (CLV) as a recent communication channel [12]. Hence, they tried to develop both artificial neural network models and sector-specific applicable models. It was found that the social media variable was influential in minimizing the error value when calculating CLV. In 2019, Lithiya Paul and T. Radha Ramanan employed an RFM and CLV analysis to retain customers and present a solution to better their relationship management in a logistics firm [13]. Through the analysis conducted in this study, the managers can make a reasonable decision regarding marketing strategies according to the RFM scores and customer's CLV. Also, Kijpokin Kasemsap investigated the benefits of CLV in maximizing the profitability of the companies

[14]. According to this study, CLV has been proved beneficial to enhance marketing performance and achieve strategic aims in global marketing. In 2019, Siti Monalisa et al. attempted to classify CLV into various clusters based on their profitability for companies using the RFM model [15]. Hence, through clustering customers with the Fuzzy C-means algorithm, the customer transactions data is examined to specify which customers are less valuable for the company. Finally, the customers were classified as superstar customers, typical customers, and dormant customers. In 2018, Pavel Jasek et al. investigated several CLV models, including the Extended Pareto/NBD model (EP/NBD), Markov chain model, and Status Quo model to specify how much they are able to be predictive and beneficial to be employed in online shopping within e-commerce business settings [16]. For this reason, experimental statistical analysis has been conducted in this study. According to the results obtained in this study, it was concluded that the EP/NBD model is the best model in terms of stability and suitability for non-contractual relations in online shopping. Moreover, to understand the CLV's practicalities deeply and the defects in the related studies, a large number of studies in the field of CLV are illustrated in Table 2.

According to Tables 1 and 2, we argue that previous literature regarding CLV models suffers from certain limitations. Thus, presenting a comprehensive model to determine optimal marketing and pricing strategies for companies and maximizing the profits and customers' CLV with respect to the network effects is of the essence. In fact, many studies have been conducted regarding CLV so far; nevertheless, the characteristics of network effects into general CLV computational models have not been dealt with widely. In 2006, Gupta et al. reported one of the remarkable researches in this field [8]. They actually considered a company and sought to maximize the company's profits by analyzing the network effects between customers and sellers. However, as already mentioned, a company's decisions affect its competitive decisions in oligopoly markets. This paper aims to develop the model presented by Gupta et al. to an oligopoly market.

Nowadays, advertising plays a prominent role in industries and is beneficial to compete for market shares over time. Some markets such as cola drinks, beers, and cigarettes are considered examples in this regard [17]. The firms that are advertising generally seek to raise market share. On the other hand, the competitor's advertising aims to reduce their market share. As stated in the introduction, employing the differential game theory, the carryover dynamics of advertising and competitive interactions can be obtained, which is examined in the previous studies (See Table 3). In another similar study, Miriam Däs et al. indicated the advantages of technology-enabled media such as online social networks in making connections between the customers [6]. This study was conducted based on a design-oriented approach, and a model is developed for the customer lifetime network value (CLNV) according to an integrated network perspective. Finally, the CLNV was proved to be beneficial in specifying the individual customers' value.

Some studies concerning a differential game in the marketing and pricing area are highlighted in

Table 3.

According to the study conducted by Prasad et al. (2012), in oligopoly markets, each firm's market share is not only related to its own and competitors' advertising decisions [18]. This idea is also employed in this research.

In the light of recent events in CLV, there is now considerable concern about the network relations and the communication that takes place among them. In order to rectify this problem, this work seeks to calculate CLV in a space such as a marketplace (with the presence of buyers, sellers, and competing companies). It is worth mentioning that no study to date has calculated the CLV by considering this condition.

3. Notation and problem formulation

This section introduces all states and control variables, input parameters, and assumptions used in the proposed model.

3.1. Indices

The only index considered in this paper is as follows:

$i(i=1,2,\ldots,n)$	Set of Companies

3.2. States variables

The states variables considered in this research are as follows:

$N_t^{B_i}$	The number of buyers of company *i* at time t
$N_t^{S_i}$	The number of sellers of company *i* at time t

3.3. Control variables

The control variables used to analyze the problem are as follows:

A_t^i	Company $i's$ marketing cost at time t
p_t^i	Company $i's$ price at time t

3.4. Input parameters

The main parameters considered in this paper are as follows:

c_t^i	Company i's service cost at time t
M^B	The potential market size of buyers
M^S	The potential market size of Sellers
b_1	The strength of direct network effects between Sellers
b_2	The strength of direct network effects between Buyers
c_1	The strength of indirect network effects (sellers on buyers)
c_2	The strength of indirect network effects (buyers on sellers)
r_i	Company i's discount rate

$x, h, \varnothing, w$ Constant coefficients

q_j^i

$Q_j^i, rr_j^i,$

$tt_j^i, NN_j^i,$

$RR_j^i, kk^i,$

$p_j^i, qq_j^i,$

QQ_j^i and

Y_0^i.

3.5. Assumptions

The proposed model is based on the following assumptions:

1. Buyers do not directly bring revenue to the company.
2. The company's income is provided through sellers.
3. As the number of sellers increases, the number of buyers increases and vice versa.
4. The following factors influence the growth of buyers and sellers:
    - Marketing rate to attract buyers
    - The marketing rate of the company's competitors
    - Pricing amount (the money received from sellers)
    - Pricing amount of the company's competitors
    - Direct effects between buyers (word of mouth)
    - Direct effects between sellers (word of mouth)
    - Indirect effects (buyers' effects on sellers)
    - Indirect effects (sellers' effects on buyers)
    - Direct effects between buyers of the company's competitors (word of mouth)
    - Direct effects between sellers of the company's competitors (word of mouth)
    - Indirect effects of the company’s competitors (effect of buyers on sellers)
    - Indirect effects of the company’s competitors (effect of sellers on buyers)
5. An infinite time horizon is considered.
6. The potential number of buyers and sellers is stable.
7. The planning horizon is infinite.
8. Parameters are deterministic.
9. The term $a(A_t)$ indicates that a firm can accelerate its buyers' growth through advertising. Here, $\alpha(A_t) = x + h \ln A_t$ is assumed [8].
10. The term $a(P_t)$ indicates that the sellers increase based on the money the firm ask them to pay. Here, $\alpha(P_t) = \varnothing - w \ln P_t$ is assumed [8].
11. Total sellers are relatively constant. When a seller leaves one company, then he tends to join another one.

12. Total sales of the market are relatively constant since when a buyer leaves a company, he tends to join another.

3.6. Model formulation

This section aims to evaluate customer value considering the interaction between two parallel populations (such as buyers and sellers), which leads to affecting the network directly and indirectly. According to these conditions, a group of customers (such as sellers) typically provide direct financial returns to the company. For instance, the sellers offer commissions to real estate agencies. However, companies need to acquire and retain another set of customers (e.g., buyers). These customers are "free" because they bring no direct revenue. The target here is to develop a model to evaluate the value of both types of customers. For this purpose, the following model is presented as shown in Figure 1.

In accordance with Figure 1, n companies compete within an oligopoly, and each one has various buyers and sellers. Since they get a membership fee from their sellers, they offer free services to the buyers.

It is worth mentioning that the presented model is a differential game according to which several companies are competing with each other, and the game structure is dynamic. In fact, the differential games in this theory consist of a set of problems associated with the modelling and analysis of conflict in the context of a dynamic system. Based on the differential equation, the variables depend on time.

Optimal control theory is a branch of optimization issue, and its main idea is based on performing a process that affects the behaviour of the dynamic system. The main target of this theory is to minimize costs (return maximization). It is noteworthy that the basis of optimal control theory is to find a control that leads us to achieve the desired goal, i.e., cost minimization (return maximization). This control is called optimal control. In this paper, an optimal control model is presented, which is based on the followings:

1. Communication between:
    - ✓ Buyers and buyers (each company)
    - ✓ Buyers and buyers (competitors of each company)
    - ✓ Buyers and sellers (each company)
    - ✓ Buyers and sellers (competitors of each company)
    - ✓ Sellers and sellers (each company)
    - ✓ Sellers and sellers (competitors of each company)
2. pricing strategy (each company)
3. marketing strategy (each company)
4. pricing strategy (competitors of each company)
5. marketing strategy (competitors of each company)

Optimal pricing and advertising are calculated using this optimal control model and based on the buyer's and seller's growth constraints.

According to Equation 1, company *i* aims to maximize its long-run discounted profit. This amount is calculated by Equation 1.

$$\max v_i = \max CLV = \int_{t=0}^{\infty} ((p_t^i - c_t^i) N_t^{S_i} - A_t^i) e^{r_i t} dt \quad (1)$$

Subject to market share dynamics given by:

$$\frac{dN_t^{S_i}}{dt} = (\alpha(p_t^i) - \sum_{j \neq i} \alpha(p_t^j) + b_1\left(\frac{N_t^{S_i}}{M^S}\right) + c_1\left(\frac{N_t^{B_i}}{M^B}\right) - \sum_{j \neq i}\left(b_1\left(\frac{N_t^{Sj}}{M^S}\right) + c_1\left(\frac{N_t^{Bj}}{M^B}\right)\right))(M^S - N_t^{S_i}) \quad (2)$$

$$\frac{dN_t^{B_i}}{dt} = (\alpha(A_t^i) - \sum_{j \neq i} \alpha(A_t^j) + b_2\left(\frac{N_t^{B_i}}{M^S}\right) + c_2\left(\frac{N_t^{S_i}}{M^B}\right) - \sum_{j \neq i}\left(b_2\left(\frac{N_t^{Bj}}{M^S}\right) + c_2\left(\frac{N_t^{Sj}}{M^B}\right)\right))(M^B - N_t^{B_i}) \quad (3)$$

- ✓ Equation 1 is a definition of CLV derived from the Gupta model [8].
- ✓ The term $\alpha(p_t^i)$ in Equation 2 shows that sellers' growth depends on the price the company asks them to pay [8].
- ✓ Similarly, the term $\alpha(A_t^i)$ in Equation 3 shows that a company can expedite the buyers' growth by advertising [8].
- ✓ The second term in Equation 2 is the market share loss due to competitive pricing.
- ✓ Similarly, the second term in Equation 3 denotes the market share loss with respect to competitive advertising.
- ✓ The direct network or word-of-mouth effect between sellers is represented by the third term in Equation 2 [8].
- ✓ The third term in Equation 3 denotes the direct network or word-of-mouth effect between the buyers [8].
- ✓ As a company acquires many buyers, more sellers are motivated to join that and vice versa. The fourth term in Equation 2 and 3 represent this indirect network effect [8].
- ✓ Direct and indirect effects between competitors' buyers and sellers are captured by fifth and sixth terms in Equation 2 and Equation 3.
- ✓ With respect to Equation 2, $(M^S - N_t^{S_i})$ represents a fraction of sellers who are not members of the company. The aim is to join these buyers to the company; thus, this coefficient is multiplied by Equation 2. In fact, this coefficient is the sum of pricing effects, direct and indirect network effects associated with the company and its competitors.
- ✓ According to Equation 3, $(M^B - N_t^{B_i})$ denotes a fraction of buyers who are not members of the company. The aim is to join these customers to the company; thus, this coefficient is multiplied by Equation 3. In fact, this coefficient is the sum of marketing effects, direct and indirect network effects associated with the company and its competitors.
- ✓ According to the proposed model, p_t^i is the membership fee. It is noteworthy that determining the market strategy and advertising is considered a dynamic decision for

each company. Besides, the companies adapt their strategy based on obtaining maximum profit.

- ✓ Since the membership fee depends on time, it is reasonable to consider a continuous function for a membership fee.

By substituting the values $\alpha(A_t)$ and $\alpha(P_t)$ in equations (2) and (3), the equations become (4) and (5), respectively.

$$\frac{dN_t^{S_i}}{dt} = (\text{Ø} - w\ln p_t^i - \sum_{j\neq i}(\text{Ø} + w\ln p_t^j) + b_1\left(\frac{N_t^{S_i}}{M^S}\right) + c_1\left(\frac{N_t^{B_i}}{M^B}\right) - \sum_{j\neq i}\left(b_1\left(\frac{N_t^{Sj}}{M^S}\right) + c_1\left(\frac{N_t^{Bj}}{M^B}\right)\right))(M^S - N_t^S) \quad (4)$$

$$\frac{dN_t^{B_i}}{dt} = (x + h\ln A_t^i - \sum_{j\neq i}(x + h\ln A_t^j) + b_2\left(\frac{N_t^{B_i}}{M^S}\right) + c_2\left(\frac{N_t^{S_i}}{M^B}\right) - \sum_{j\neq i}\left(b_2\left(\frac{N_t^{Bj}}{M^S}\right) + c_2\left(\frac{N_t^{Sj}}{M^B}\right)\right))(M^B - N_t^{B_i}) \quad (5)$$

To solve the model, the Hamilton-Jacobi-Bellman (HJB) equation is used to obtain the optimal values of A_t^i ,p_t^i , $N_t^{B_i} and\ N_t^{S_i}$ for the Company.

3.7 The presented theorems

Two theorems are considered in this paper, which are described here.

Theorem 1 provides optimal marketing, optimal pricing, profit, and market share for each company at time t.

Theorem 1

Let $q_j^i, Q_j^i, rr_j^i, tt_j^i, NN_j^i, R_{\alpha j}^i, kk^i, p_j^i, qq_j^i, QQ_j^i$ and Y_0^i be determined from the relations: (Refer to Appendix 1)

$$r_i Y_0^i = M^S w q_i^i - Q_i^i M^B h \quad (6)$$

$$r_i \sum_{j=1}^{n} q_j^i = 2\sum_j q_j^i b_1 - \sum_j Q_j^i c_2 \quad (7)$$

$j \neq i$

$$r_i q_i^i = -wq_i^i - c_t^i = 2b_1 + Q_i^i c_2 \quad (8)$$

$$j = i$$

$$r_i \sum_{j=1}^{n} Q_j^i = \sum_j Q_j^i M^B b_2 \left(\frac{1}{M^S} \right)$$

(9)

$$j \neq i$$

$$r_i Q_i^i = Q_i^i h + 3M^S \left(\frac{c_1}{M^B} \right) q_i^i + Q_i^i \left(\frac{M^B b_2}{M^S} \right) \quad (10)$$

$$j = i$$

$$r_i \sum_{j=1}^{n} p_j^i = -\sum_j \left(\frac{2q_j^i c_1}{M^B} \right) - \sum_j \left(\frac{2Q_j^i c_2}{M^B} \right) \quad (11)$$

$$r_i (qq_i^i) = -q_i^i \left(\frac{2b_1}{M^S} \right) \quad (12)$$

$$r_i \sum_{j=1}^{n} QQ_j^i = -\sum_j Q_j^i b_2 \left(\frac{1}{M^S} \right)$$

(13)

$$r_i \sum_j tt_j^i = -\sum_j q_j^i W M^S$$

(14)

$$t_j = -q_j^j$$

$$r_j^i = q_j^j M^S$$

$$r_i \sum_j rr_j^i = \sum_j q_j^i W$$

(15)

$$r_i \sum_j RR_j^i = -\sum_j Q_j^i M^B h$$

(16)

$$r_i TT^i = 2M^B h Q_i^i \quad (17)$$

$$T_i = Q_i^i M^B h \quad (18)$$

$$R_i^i = -Q_i^i h$$

$$r_i kk^i = -2hQ_i^i$$

$$r_i \sum_j NN_j^i = -\sum_j Q_j^i h$$

(19)

According to the differential game presented by (1), (4), and (5), the optimal feedback marketing rate, pricing, and function value for company *i* are: (Refer to Appendix 1)

$$p^{i*} = \frac{(M^S - N^{s_i})w}{N^{s_i}} q_i^i \tag{20}$$

And

$$A^{i*} = Q_i^i (M^B - N^{B_i})h \tag{21}$$

$$\begin{aligned} V^{i*} &= Y_0^i + \sum_{j=1}^{n} q_j^i N_t^{s_j} + \sum_{j=1}^{n} Q_j^i N_t^{B_j} + \sum_{j=1}^{n} rr_j^i N_t^{s_i} ln\left(\frac{r_j^i}{N_t^{s_j}} + t_j\right) \\ &+ \sum_{j=1}^{n} tt_j^i ln\left(\frac{r_j^i}{N_t^{s_j}} + t_j\right) + \sum_{j=1}^{n} NN_j^i ln(N_t^{B_i} R_j^i + T_j) N_t^{B_i} \\ &+ \sum_{j=1}^{n} RR_j^i ln(N_t^{B_i} R_j^i + T_j) \\ &+ kk^i N_t^{B_i} ln(N_t^{B_i} R_j^i + T_i) + (N_t^{B_i} R_i^i + T_i) \\ &+ \sum_{j=1}^{n} N_t^{sj} N_t^{Bj} p_j^i + \sum_{j=1}^{n} N_t^{sj^2} qq_j^i + \sum_{j=1}^{n} N_t^{Bj^2} QQ_j^i \end{aligned} \tag{22}$$

Theorem 2

The steady-state market shares of company i are given by (20) and (21) (refer to Appendix 2). The main parameters in relation to Theorem 2 are as follows:

$$\begin{aligned} & N_t^{\bar{B}_i} T + h \ln N^{\bar{B}i} - \frac{N_t^{\bar{B}_i}}{M^s} b_2 N_t^{\bar{B}_i} \\ & = h^2 lnM^B - h^2 lnN_t^{\bar{B}_i} + c_2 \left(\frac{N_t^{\bar{S}_i}}{M^B}\right) - \frac{c_2}{M^B} N_t^{\bar{B}_i} N_t^{\bar{S}_i} + c_2 N_t^{\bar{S}_i} + U \end{aligned} \tag{23}$$

$$-WlnN^{\bar{S_i}}wq_i^i + wln(N_t^{\bar{S_i}}) + N_t^{\bar{S_i}}G + N_t^{\bar{S_i}}\frac{b_1}{M^s}N_t^{\bar{S_i}}$$

$$= \frac{c_1}{M^B}N_t^{\bar{B_i}} + \frac{c_1}{M^B}M^s N_t^{\bar{B_i}} - \frac{c_1}{M^B}N_t^{\bar{S_i}}N_t^{\bar{B_i}} + F \quad (24)$$

Where:

$$\left(-b_2\left(\frac{1}{M^S}\right) + \frac{1}{M^S}b_2M^B - c_2 - b_2\frac{M^B}{M^S}\right) + h(lnQ_i^i) + hlnh = T \quad (25)$$

$$hlnQ_i^iM^Bh - \sum_{j\neq i}hlnQ_j^jM^Bh - b_2\frac{M^B}{M^S}(M^B) - c_2M^B + h^2\sum_{j\neq i}hlnQ_j^j) = U$$
(26)

$$G = -\frac{b_1}{M^S} - 2b_1 - c_1 \quad (27)$$

$$-wlnM^Swq_i^i + wlnM^Sw - wlnM^s + wlnM^s - b_1M^s - \frac{c_1}{M^B}M^sM^B$$
$$-\sum_{j\neq i}hln(wq_j^j) = F \quad (28)$$

4. Results and discussion

Two numerical examples and the sensitivity analysis of parameters b_1, b_2, c_1, and c_2 are presented to illustrate the model's significant features.

4.1. Numerical example

In order to clarify the proposed model, two numerical examples are illustrated here. It is noteworthy that devoting attention to these examples is of great significance to understand the model’s practicality.

4.1.1. Example

To illustrate the application of Theorem 1 and Theorem 2, a triopoly is examined. Input parameters are as Table 4.

Based on Theorem 1, company 1 function value is as follow: (Equation 29)

$$v_1 = Y_0^1 + q_1^1N_t^{s_1} + q_2^1N_t^{s_2} + q_3^1N_t^{s_3} + Q_1^1N_t^{s_1} + Q_2^1N_t^{s_2} + Q_3^1N_t^{s_3}$$

$$
\begin{aligned}
&+rr_1^1 N_t^{s_1} ln\left(\frac{r_1^1}{N_t^{s_1}}+t_1\right)+rr_2^1 N_t^{s_1} ln\left(\frac{r_2^1}{N_t^{s_2}}+t_2\right)+rr_3^1 N_t^{s_1} ln\left(\frac{r_3^1}{N_t^{s_3}}+t_3\right)\\
&+tt_1^1 N_t^{s_1} ln\left(\frac{r_1^1}{N_t^{s_1}}+t_1\right)+tt_2^1 N_t^{s_1} ln\left(\frac{r_2^1}{N_t^{s_2}}+t_2\right)+tt_3^1 N_t^{s_1} ln\left(\frac{r_3^1}{N_t^{s_3}}+t_3\right)\\
&+NN_1^1 ln(N_t^{B_1} R_1^1+T_1) N_t^{B_1}+NN_2^1 ln(N_t^{B_2} R_2^1+T_2) N_t^{B_1}\\
&+NN_3^1 ln(N_t^{B_3} R_3^1+T_3) N_t^{B_1}+RR_1^1 ln(N_t^{B_1} R_1^1+T_1)\\
&+RR_2^1 ln(N_t^{B_2} R_2^1+T_2)+RR_3^1 ln(N_t^{B_3} R_3^1+T_3)\\
&+kk^1 N_t^{B_1} ln(N_t^{B_1} R_1^1+T_1)+TT^1 ln(N_t^{B_1} R_1^1+T_1)+N_t^{s_1} N_t^{B_1} p_1^1\\
&+N_t^{s_2} N_t^{B_2} p_2^1+N_t^{s_3} N_t^{B_3} p_3^1+N_t^{s_1 2} qq_1^1+N_t^{s_2 2} qq_2^1+N_t^{s_3 2} qq_3^1\\
&+N_t^{B_1 2} QQ_1^1+N_t^{B_2 2} QQ_2^1+N_t^{B_3 2} QQ_3^1
\end{aligned} \qquad (29)
$$

Equation 29 is also true for companies 2 and 3.

By placing Table 3 values in Equations 6 to 19 and solving these equations simultaneously, the results in Table 5 are obtained.

Given the same conditions for the three companies, the number of sellers, the number of buyers, the optimal marketing and pricing strategies, and the companies' value are equal.

4.1.2. Example

Input parameters are as Table 6.

By placing Table 6 values in Equations 6 to 19 and solving the equations simultaneously, the results in Table 7 are obtained.

The network effects considered in Example 2 are more tangible than those in Example 1 (positive effect). The positive effects of the network have resulted in lower advertising and pricing costs compared to Example 1.

4.2. Sensitivity analysis

In this section, sensitivity analysis is performed on the value of b_1, b_2, c_1, and c_2. Table 8 and Figure 2 highlight the results of the sensitivity analysis on b_1.
With respect to Table 8 and Figure 2, it can be concluded that a rise in the amount of b_1 has a considerable effect on reducing the cost of advertising, increasing the amount of money received from sellers, and consequently increasing the companies' profit. Interestingly, as b_1 increases up to 0.09, companies can keep their advertising to a minimum amount (and even zero).

Figures 3-5 demonstrate the sensitivity analysis results on b_2, c_1, and c_2, respectively.

As shown in the diagrams above, the power of direct effects between buyers (b_2) in association with change in advertising, profit, and pricing is more tangible compared to other network effects (related to variables b_1, c_1, and c_2). On the other hand, in general, the impact of direct effects is greater than indirect ones. In accordance with Figures 3-5, as the network parameter increases, the amount of pricing and advertising decreases, but the profit increases. This effect is greater for the indirect network effects compared to indirect ones. In other words, a rise in the network effects constitutes less need for advertising which even affects the pricing and leads to reduced prices.

5. Conclusion

As mentioned in the introduction, the main aims of this research are calculating CLV and determining optimal marketing and pricing strategies, while the relationships between buyers and companies in oligopoly markets are considered. As regards the use of an infinite horizon differential game, it was found that how much the network affects advertising expenditure, market share, and profitability. Besides, differential game theory is taken to illustrate the carryover dynamics of advertising, pricing, and competitive interactions. An optimal control model is presented based on the communication between buyers and buyers, buyers and sellers, sellers and sellers, pricing, and marketing strategy for each company and its competitors. Moreover, the interaction between companies is modeled as a static game to obtain the Nash equilibrium. In order to illustrate the application of the model, a triopoly market is examined. The main conclusions that can be drawn in this paper are as follows:

- When the network effects become stronger, marketing plays less of a role in attracting buyers and sellers. Consequently, the need to advertise is reduced. Thus, managers are advised to increase the networking effects between their buyers and sellers rather than marketing to attract new customers.
- In fact, it is better to have an initial cost to create network effects first. These network effects increase over time and minimize the company's need for advertising.
- Overall, according to the proposed model, a rise in b_1, b_2, c_1, and c_2 leads to reducing advertising and pricing and increasing profit. Therefore, the managers whose aim is to maximize profits are advised to pay attention to these parameters (See Table 8 and Figure 2).
- Increasing the network effects is of great importance for a manager seeking to become a winner in the competitive market because this is the best way to minimize prices as well as attracting more customers (See Figures 3-5).

It is worth mentioning that one of the main innovative aspects of this paper is examining the network effects that existed among customers. The remarkable benefits of the interaction between the customers are clarified through the findings of this study, while the other previous studies failed to illuminate the importance of network effects. The results are regarded to be practical and advantageous according to the following reasons:

- It is specified that how much a company should spend to obtain new customers according to the network effects.

- It is illustrated that how much a company should spend to obtain a "free" customer who does not bring any direct financial returns to the company.
- It is specified that how the company’s marketing efforts change over time with respect to the network effects.

In this study, the total number of buyers and sellers in a fixed industry is considered. For future research, it is recommended that buyers and sellers can be churn from the industry. Also, in this research, companies, and buyers are considered quite similar. Future studies on the current topic are therefore required, and the subsequent researches can focus on different sellers and buyers. By relaxing constraints on growth by one customer, his/ her lifetime value can impute to the firm. In addition, this model can also be extended to multi-sided markets. For example, YouTube is a three-sided market with viewers, content providers, and advertisers. Concerning this paper, we also suggest obtaining the open-loop control and compare it with the feedback control and present a more practical and precise model in future studies.

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Table 1. Some definitions of CLV

Authors	Definition
[19]	✓ The whole net benefit of a customer within his lifetime
[20]	✓ The net benefit and loss of a company during the time of transacting a costumer
[2]	✓ The present value of all the future benefits gained from one costumer
[21]	✓ The collection of revenue generated by customers after deducting the costs of acquisition, sales, and service during the lifetime of the interaction with the company, taking into account the time value of money

Table 2. literature review on CLV

Authors	Considered elements / defined model	Network Effects
[20]	✓ Two steps should be taken to calculate CLV: 1. Predicting the net cash flows that the customers are likely to bring for the companies. 2. Computing the existing value of that stream of cash flows $CLV = Revenue - (cost\ of\ sales + promption\ expense)$	-
[22]	✓ A method to specify CLV consisting of customer-specific brand switching metrics. ✓ The Markov model is employed in this study to model the customer's probability associated with switching from one brand to another by a transition matrix.	-
[23]	✓ Three main components constitute CLV which are as follows: return on the acquisition, return on retention, and return on cross-selling	-
[24]	✓ They showed that after losing a customer, a company will lose the cash flow it can earn from that customer in the future and lose the cash flow that can be gained from other customers due to less customer attraction as a result of reduced social impact.	✓
[21]	✓ Three main factors are examined, which were mainly neglected according to the previous studies conducted on CLV. These factors are as follows: past profit contribution, potential benefit, and defection probabilities of the customer. ✓ Besides, a framework is presented to analyze the customers' values and categorize them in accordance with their values.	-
[25]	✓ Customers are grouped based on three factors of RFM.	-
[8]	✓ Development of a joint model of buyer and seller growth to calculate the customers' value. ✓ Three sources are considered for this growth which is as follows: marketing actions (price and advertising), direct network effects (such as buyer to buyer effects), and indirect network effects (such as buyer to seller effects). ✓ The problem of specifying optimal pricing and advertising for the firm with respect to the constraints on customer growth is simultaneously solved by employing this growth model.	✓

[26]	✓ A framework is considered to analyze the customer's value and segment them based on their value. ✓ After the segmentation, strategies associated with each segment are proposed. ✓ Customer defection and cross-selling opportunities are considered in this article.	-
[27]	✓ A model is proposed to determine CLV. ✓ This determination is conducted with respect to the Markov chain model and classification and regression tree.	-
[28]	✓ Developing RFM model by including the time of first purchase and probability (RFMTC)	-
[29]	✓ A model is developed for customer valuation, referred to as the customer lifetime network value (CLNV), consisting of an integrated network perspective. ✓ According to the net network contribution of customers, the CLNV reallocates values between the customers based on social impact without the overall network value (a firm's customer equity)	✓
[2]	✓ Customer lifetime network value is divided into two parts: (1) The existing value of individual cash flows and (2) The existing value of network contribution	✓
[30]	✓ They indicated how much the word-of-mouth (WOM) effects are beneficial to the CLV anticipation in start-up businesses.	✓
[12]	✓ This study's primary goal is to investigate whether social media has an impact on CLV as a recent communication channel.	✓

Table 3. Differential game in the marketing and pricing area

Authors	Considered elements / defined model
[31], [32]	✓ Development of the oligopoly models in which the sales increase over time by innovation diffusion dynamics. After a while, the market reaches saturation and will be affected by competitive advertising. Thus the sales are influenced indirectly. ✓ They stated that the advertising must reduce over time due to the saturation effect.
[33]	✓ He indicated that firms must reduce advertising unless for the market share leader, as the number of firms increases.
[34]	✓ Employing the method of dynamic conjectural variations to examine oligopoly markets. Three symmetric competitors and zero discount rates are considered.
[35]	✓ A model of new-product adoption consisting of price and advertising effects is proposed. ✓ An optimal control problem is explicitly solved that employs the model as its dynamics. The aim is to obtain the optimal price and advertising effort over time.
[7]	✓ This paper analyzes dynamic advertising and pricing policies in a durable goods duopoly. ✓ The differential game theory is employed to analyze two different demand specifications – linear demand and isoelastic demand – for symmetric and asymmetric competitors.
[36]	✓ Marketing and operations decisions are modeled as a no-cooperative differential game.

	✓ A feedback Nash equilibrium is derived for the game
[18]	✓ They incorporate the effects of churn in a dynamic model of advertising for oligopoly markets. ✓ Each firm's market share depends not only on its own and competitors' advertising decisions but also on market churn. ✓ Applying differential game theory, they derive a feedback Nash equilibrium under symmetric and asymmetric competition.

Figure 1. The model of this study

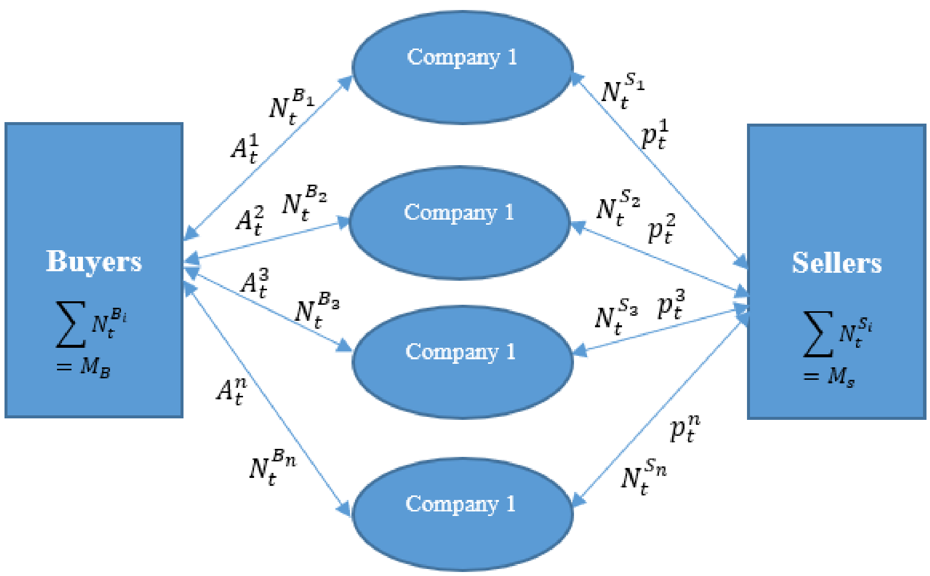


Table 4. The value of input parameters

$M^B = 100 M^s = 12$	$b_1 = -0.06$ $b_2 = -0.01$	$c_1 = 0.019$ $c_2 = 0.095$
$r_1 = r_2 = r_3 = 0.05$ $c_1 = c_2 = c_3 = 0.01$	$x = -0.004$ $h = -0.0003$	$\O = -0.037$ $w = 0.015$

Table 5. The obtained results of the numerical example

$N^{B_1 *} = N^{B_2 *} = N^{B_3 *}$ $= 20$	$N^{S_1 *} = N^{S_2 *}$ $= N^{S_3 *} = 2$	$A^{1^*} = A^{2*} =$ $A^{3*} = 0.15$	$P^{1*} = P^{2*} = P3^*$ $= 0.08$	$v^{1*} = v^{2*} = v^{3*}$ $= 0.12$

Table 6. The value of input parameters

$M^{B} = 1200 M^{s} = 60$	$b_1 = +0.03$ $b_2 = +0.05$	$c_1 = 0.02$ $c_2 = 0.03$
$r_1 = r_2 = r_3 = 0.05$ $c_1 = c_2 = c_3 = 0.01$	$x = -0.004$ $h = -0.0003$	$\emptyset = -0.037$ $w = 0.015$

Table 7. The obtained results of numerical example

$N^{B_1*} = N^{B_2*} = N^{B_3*} = 400$	$N^{s_1*} = N^{s_2*} = N^{s_3*} = 20$	$A^{1*} = A^{2*} = A^{3*} = 0.01$	$P^{1*} = P^{2*} = P^{3*} = 0.02$	$v^{1*} = v^{2*} = v^{3*} = 0.25$

Table 8. Sensitive analysis on b_1

$b_1 = 0.09$	$v^{1*} = v^{2*} = v^{3*} = 0.151$	$P^{1*} = P^{2*} = P^{3*} = 0.095$	$A^{1*} = A^{2*} = A^{3*} = 0.011$	$N^{s_1*} = N^{s_2*} = N^{s_3*} = 2$	$N^{B_1*} = N^{B_2*} = N^{B_3*} = 20$
$b_1 = 0.04$	$v^{1*} = v^{2*} = v^{3*} = 0.142$	$P^{1*} = P^{2*} = P^{3*} = 0.093$	$A^{1*} = A^{2*} = A^{3*} = 0.054$	$N^{s_1*} = N^{s_2*} = N^{s_3*} = 2$	$N^{B_1*} = N^{B_2*} = N^{B_3*} = 20$
$b_1 = -0.01$	$v^{1*} = v^{2*} = v^{3*} = 0.138$	$P^{1*} = P^{2*} = P^{3*} = 0.091$	$A^{1*} = A^{2*} = A^{3*} = 0.101$	$N^{s_1*} = N^{s_2*} = N^{s_3*} = 2$	$N^{B_1*} = N^{B_2*} = N^{B_3*} = 20$
$b_1 = -0.06$	$v^{1*} = v^{2*} = v^{3*} = 0.122$	$P^{1*} = P^{2*} = P^{3*} = 0.080$	$A^{1*} = A^{2*} = A^{3*} = 0.152$	$N^{s_1*} = N^{s_2*} = N^{s_3*} = 2$	$N^{B_1*} = N^{B_2*} = N^{B_3*} = 20$
$b_1 = -0.11$	$v^{1*} = v^{2*} = v^{3*} = 0.111$	$P^{1*} = P^{2*} = P^{3*} = 0.062$	$A^{1*} = A^{2*} = A^{3*} = 0.172$	$N^{s_1*} = N^{s_2*} = N^{s_3*} = 2$	$N^{B_1*} = N^{B_2*} = N^{B_3*} = 20$
$b_1 = -0.16$	$v^{1*} = v^{2*} = v^{3*} = 0.104$	$P^{1*} = P^{2*} = P^{3*} = 0.051$	$A^{1*} = A^{2*} = A^{3*} = 0.185$	$N^{s_1*} = N^{s_2*} = N^{s_3*} = 2$	$N^{B_1*} = N^{B_2*} = N^{B_3*} = 20$

Figure 2. Sensitive analysis on b_1

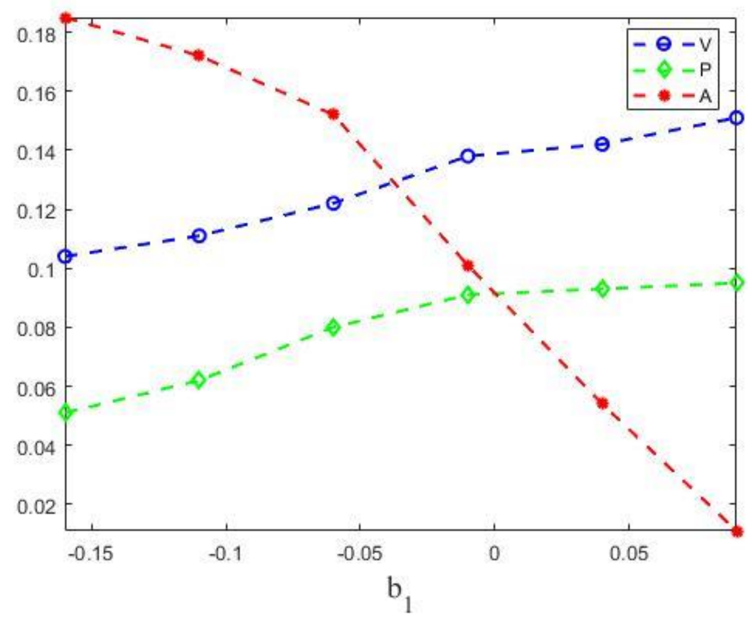


Figure 3. Sensitive analysis on b_2

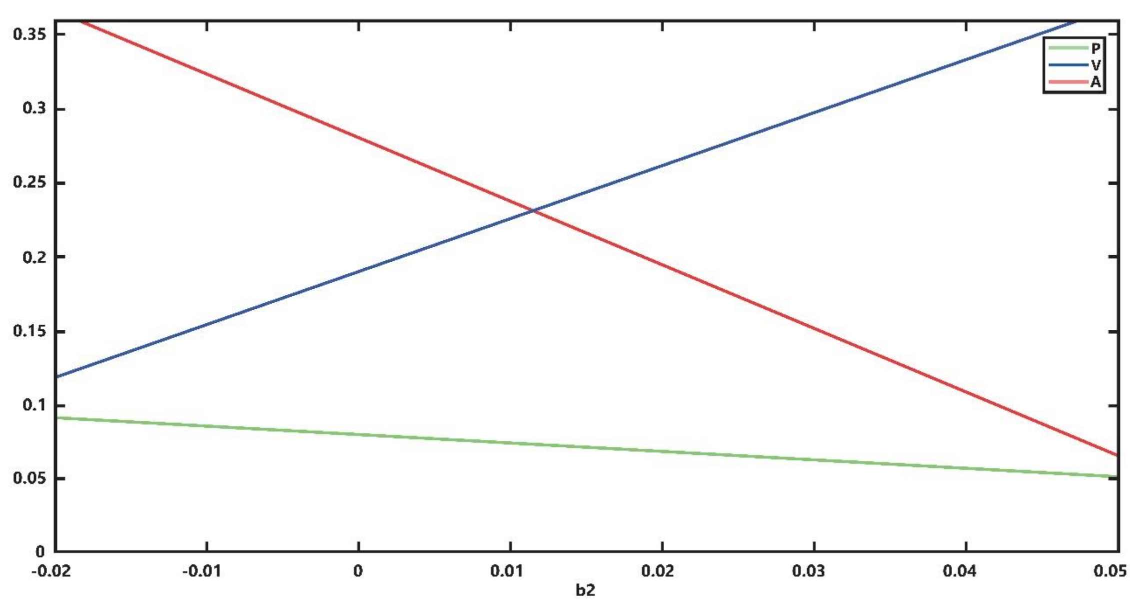


Figure 4. Sensitive analysis on c_1

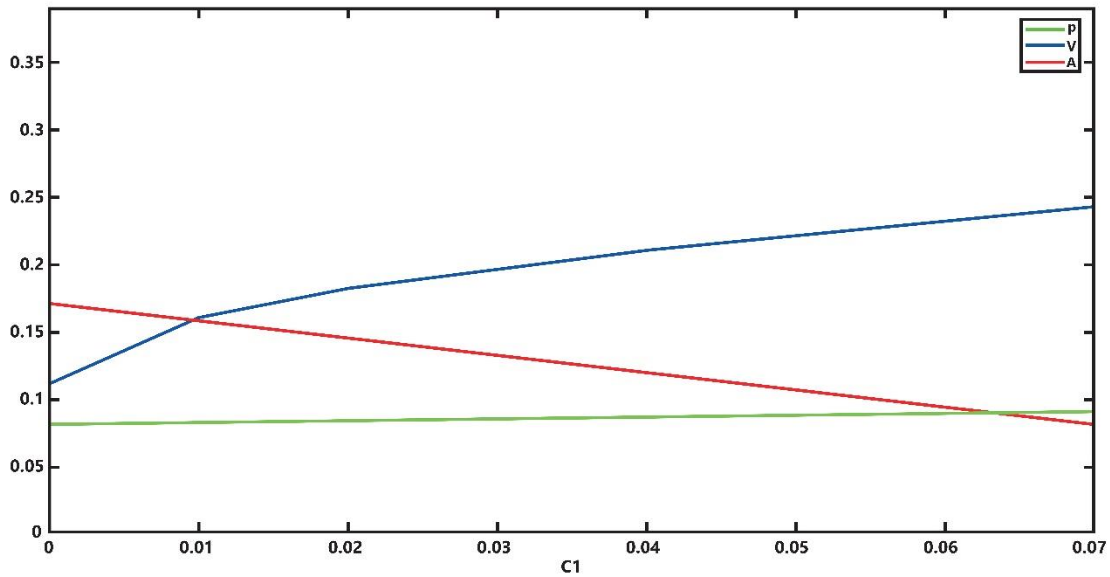
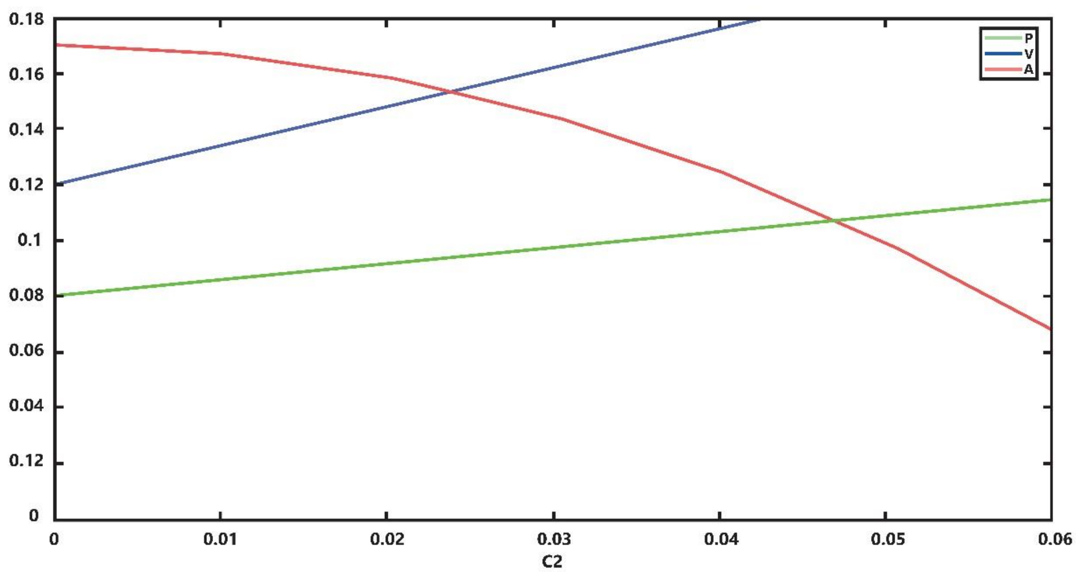


Figure 5. Sensitive analysis on c_2



Appendix 1

The Hamilton-Jacobi-Bellman (HJB) equation for company i is given by:

$$r_i v_i = max p_t^i N_t^{s_i} - c_t^i N_t^{s_i} - A_t^i + \sum_j \frac{dv}{dN_t^{s_j}} (M^s - N_t^{(s_i)})(\text{Ø} - wlnp_{t\cdot}^i - \sum_{j \neq i} (\text{Ø} + wlnp_t^j) + b_1 \left(\frac{N_t^{s_i}}{M^s} \right) + c_1 \left(\frac{N_t^{B_i}}{M^B} \right)$$

$$- \sum_{j \neq i} \left(b_1 \left(\frac{N_t^{S_j}}{M^s} \right) + c_1 \left(\frac{N_t^{B_j}}{M^B} \right) \right))$$

$$+ \sum_j \frac{dv_i}{dN_t^{B_j}} (M^B - N_t^{B_i})(x + hlnA_t^i - \sum_{j \neq i} (x + hlnA_t^j) + b_2 \left(\frac{N_t^{B_i}}{M^s} \right) + c_2 \left(\frac{N_t^{s_i}}{M^B} \right) - \sum_{j \neq i} \left(b_2 \left(\frac{N_t^{B_j}}{M^s} \right) + c_2 \left(\frac{N_t^{s_j}}{M^B} \right) \right))$$

After simplification, the following equation is obtained:

$$r_i v_i = max p_t^i N_t^{s_i} - c_t^i N_t^{s_i} - A_t^i$$

$$+ \sum_j \frac{dv_i}{dN_t^{s_j}} \left(\text{Ø} M^s - M^s wlnp_{t\cdot}^i . - \sum_{j \neq i} (\text{Ø} M^s + M^s wlnp_t^j) + M^s b_1 \left(\frac{N_t^{s_i}}{M^s} \right) + M^s c_1 \left(\frac{N_t^{B_i}}{M^B} \right) \right)$$

$$- \sum_{j \neq i} \left(M^s b_1 \left(\frac{N_t^{s_j}}{M^s} \right) + M^s c_1 \left(\frac{N_t^{B_j}}{M^B} \right) \right) \Bigg)$$

$$- \sum_j \frac{dv_i}{dN_t^{s_j}} \left(\text{Ø} N_t^{s_i} - N_t^{s_i} wlnp_{t\cdot}^i . - \sum_{j \neq i} (\text{Ø} N_t^{s_i} + N_t^{s_i} wlnp_t^j) + N_t^{s_i} b_1 \left(\frac{N_t^{s_i}}{M^s} \right) + N_t^{s_i} c_1 \left(\frac{N_t^{B_i}}{M^B} \right) \right.$$

$$\left. - \sum_{j \neq i} \left(M^s b_1 \left(\frac{N_t^{s_j}}{M^s} \right) + M^s c_1 \left(\frac{N_t^{B_j}}{M^B} \right) \right) \right)$$

$$+\sum_j \frac{dv_i}{dN_t^{B_j}}\left(xM^B - M^B h \ln A_t^i - \sum_{j\neq i}(xM^B + xM^B h \ln A_t^j) + M^B b_2\left(\frac{N_t^{B_i}}{M^s}\right) + M^B c_2\left(\frac{N_t^{s_i}}{M^B}\right)\right.$$

$$\left. - \sum_{j\neq i}\left(M^B b_2\left(\frac{N_t^{B_j}}{M^s}\right) + M^B c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right)\right)$$

$$-\sum_j \frac{dv_i}{dN_t^{B_j}}\left(xN^{B_i} - N^{B_i} h \ln A_t^i - \sum_{j\neq i}(xN^{B_i} + xN^{B_i} h \ln A_t^j) + N^{B_i} b_2\left(\frac{N_t^{B_i}}{M^s}\right) + N^{B_i} c_2\left(\frac{N_t^{s_i}}{M^B}\right)\right.$$

$$\left. - \sum_{j\neq i}\left(N_t^{B_i} b_2\left(\frac{N_t^{B_j}}{M^s}\right) + N_t^{B_i} c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right)\right))$$

The optimal feedback controls are obtained:

$$P^{i*} = \frac{(M^S - N^{(s_i)})w}{N^{(s_i)}}\frac{dv_i}{dN^{s_i}}$$

$$A^{i*} = \frac{dv_i}{dN^{B_i}}(M^B - N^{B_i})h$$

These controls are inserted into the HJB equations, for company i:

$$r_i v_i = \frac{(M^S - N_t^{s_i})w}{N_t^{s_i}}\frac{dv_i}{dN_t^{s_i}} N_t^{s_i} - cN_t^{s_i} - \frac{dv_i}{dN_t^{Bi}}(M^B - N^{Bi})h + \sum_j \frac{dv_i}{dN_i}\left((\O M^s - M^s wln(\frac{(M^S - N^{s_i})w}{N^{s_i}}\frac{dv_i}{dN^{s_i}}). - \sum_{j\neq i}(\O M^s + M^s wln(\frac{(M^S - N^{s_j})w}{N^{s_j}}\frac{dv_j}{dN^{s_j}})\right) +$$

$$M^s b_1\left(\frac{N_t^{s_i}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_i}}{M^B}\right) - \sum_{j\neq i}\left(M^s b_1\left(\frac{N_t^{s_j}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_j}}{M^B}\right)\right)\Bigg) - \sum_j \frac{dv_i}{dN_t^{s_j}}\left(\O N_t^{s_i} w \ln(\frac{(M^S - N^{s_i})w}{N^{s_i}}\frac{dv_i}{dN^{s_i}}) - \sum_{j\neq i}\left(\O N_t^{s_i} + N_t^{s_i} wln(\frac{(M^S - N^{s_j})w}{N^{s_j}}\frac{dv_j}{dN^{s_j}})\right)\right.$$

$$\left. + N_t^{s_i} b_1\left(\frac{N_t^{s_i}}{M^s}\right) + N_t^{s_i} c_1\left(\frac{N_t^{B_i}}{M^B}\right) - \sum_{j\neq i}\left(M^s b_1\left(\frac{N_t^{s_j}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_j}}{M^B}\right)\right)\right)\Bigg) + \sum_j \frac{dv_i}{dN_t^{B_j}}\left(xM^B + M^B h \ln(\frac{dv_i}{dN^{B_i}}(M^B - N^{B_i})h\right) -$$

$$\sum_{j\neq i}\left(xM^B + M^B h \ln \frac{dv_i}{dN^{B_j}}(M^B - N^{B_j})h)\right) + M^B b_2\left(\frac{N_t^{B_i}}{M^s}\right) + M^B c_2\left(\frac{N_t^{s_i}}{M^B}\right) - \sum_{j\neq i}\left(M^B b_2\left(\frac{N_t^{B_j}}{M^s}\right) + M^B c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right)\Bigg) - \sum_j \frac{dv_i}{dN_t^{B_j}}(xN_t^{B_i} +$$

$$N_t^{B_i} hln(\frac{dv_i}{dN^{B_i}}(M^B - N^{B_i})h) - \sum_{j\neq i}\left(xN_t^{B_i} + N_t^{B_i} h \ln(\frac{dv_j}{dN^{B_j}}(M^B - N^{B_j})h)\right) + N_t^{B_i} b_2\left(\frac{N_t^{B_i}}{M^s}\right) + N_t^{B_i} c_2\left(\frac{N_t^{s_i}}{M^B}\right) - \sum_{j\neq i}\left(N_t^{B_i} b_2\left(\frac{N_t^{B_j}}{M^s}\right) + N_t^{B_i} c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right))$$

Which is equivalent to the following equation:

$$r_i v_i = \frac{(M^S - N_t^{s_i})w}{N_t^{s_i}}\frac{dv_i}{dN_t^{s_i}} N_t^{s_i} - cN_t^{s_i} - \frac{dv_i}{dN_t^{Bi}}(M^B - N^{Bi})h$$

$$+\sum_j \frac{dv_i}{dN_i}\left(ØM^s - \sum_{j=i}\left(ØM^s + M^s wln(\frac{(M^S - N^{s_i})w}{N^{s_i}}\frac{dv_i}{dN^{s_i}})\right) + M^s b_1\left(\frac{N_t^{s_i}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_i}}{M^B}\right) - \sum_{j=i}\left(M^s b_1\left(\frac{N_t^{s_j}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_j}}{M^B}\right)\right)\right.$$

$$\left. +\left(2M^s b_1\left(\frac{N_t^{s_i}}{M^s}\right) + 2M^s c_1\left(\frac{N_t^{B_i}}{M^B}\right)\right)\right) - \sum_j \frac{dv_i}{dN_t^{s_j}}\left(ØN_t^{s_i} - \sum_{j=i}\left(ØN_t^{s_i} + N_t^{s_i} w \ln(\frac{(M^S - N^{s_j})w}{N^{s_j}}\frac{dv_j}{dN^{s_j}}) + 2N_t^{s_i} b_1\left(\frac{N_t^{s_i}}{M^s}\right) + 2N_t^{s_i} c_1\left(\frac{N_t^{B_i}}{M^B}\right)\right)\right.$$

$$\left. -\sum_{j=i}\left(M^s b_1\left(\frac{N_t^{s_j}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_j}}{M^B}\right)\right)\right) + \sum_j \frac{dv_i}{dN_t^{B_j}}\left(xM^B + 2M^B h \ln(\frac{dv_i}{dN^{B_i}}(M^B - N^{B_i})h\right) -$$

$$\sum_{j=i}\left(xM^B + M^B h \ln\frac{dv_i}{dN^{B_j}}(M^B - N^{B_j})h)\right) + 2M^B b_2\left(\frac{N_t^{B_i}}{M^s}\right) + 2M^B c_2\left(\frac{N_t^{s_i}}{M^B}\right)\Bigg)\Bigg) - \sum_{j=i}\left(M^B b_2\left(\frac{N_t^{B_j}}{M^s}\right) + M^B c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right)\Bigg) - \sum_j \frac{dv_i}{dN_t^{B_j}}(xN_t^{B_i} +$$

$$2N_t^{B_i} hln(\frac{dv_i}{dN^{B_i}}(M^B - N^{B_i})h) - \sum_{j=i}\left(xN_t^{B_i} + N_t^{B_i} h \ln(\frac{dv_j}{dN^{B_j}}(M^B - N^{B_j})h)\right) + 2N_t^{B_i} b_2\left(\frac{N_t^{B_i}}{M^s}\right) + 2N_t^{B_i} c_2\left(\frac{N_t^{s_i}}{M^B}\right) - \sum_{j=i}\left(N_t^{B_i} b_2\left(\frac{N_t^{B_j}}{M^s}\right) + N_t^{B_i} c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right))$$

To solve these n partial differential equations simultaneously, the following functions is used:

$$v_i = Y_0^i + \sum_{j=1}^{n} q_j^i N_t^{s_j} + \sum_{j=1}^{n} Q_j^i N_t^{B_j} + \sum_{j=1}^{n} rr_j^i N_t^{s_i} \ln\left(\frac{r_j^i}{N_t^{s_j}} + t_j\right)$$

$$+\sum_{j=1}^{n} tt_j^i \ln\left(\frac{r_j^i}{N_t^{s_j}} + t_j\right) + \sum_{j=1}^{n} (NN_j^i \ln N_t^{B_j} R_j^i + T_j) N_t^{B_i} + \sum_{j=1}^{n} RR_j^i \ln\left(N_t^{B_j} R_j^i + T_j\right)$$

$$kk^i N_t^{B_i} \ln(N_t^{B_i} R_i^i + T_i) + TT^i \ln(N_t^{B_i} R_i^i + T) + \sum_{j=1}^{n} N_t^{s_j} N_t^{B_j} p_j^i + \sum_{j=1}^{n} N_t^{s_j 2} qq_j^i + \sum_{j=1}^{n} N_t^{B_j 2} QQ_j^i$$

With inserting the value function into the HJB equation, the following equation is obtained:

$$r_i v_i = r_i (Y_0^i + \sum_{j=1}^{n} q_j^i N_t^{s_j} + \sum_{j=1}^{n} Q_j^i N_t^{B_j} + N_t^{s_i} \ln\left(\frac{r_j^i}{N_t^{s_i}} + t_j\right) + N_t^{B_i} \ln\left(\frac{R_j^i}{N_t^{B_i}} + T\right) + \sum_{j=1}^{n} N_t^{s_j} N_t^{B_j} q_j^i + \sum_{j=1}^{n} N_t^{s_j^2} qq_j^i + \sum_{j=1}^{n} N_t^{B_j^2} QQ_j^i) =$$

$$(M^s - N_t^{s_i})w\frac{dv_i}{dN_t^{s_i}} - c_t^i N_t^{s_i} - \frac{dv_i}{dN_t^{B_i}}(M^B - M^{B_i}) + \sum_j \frac{dv_i}{dN_i}\left(ØM^s - \sum_j\left(ØM^s + M^s wln(\frac{(M^S - N^{s_i})w}{N^{s_i}}\frac{dv_i}{dN^{s_i}})\right) + M^s b_1\left(\frac{N_t^{s_i}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_i}}{M^B}\right) - \sum_j\left(M^s b_1\left(\frac{N_t^{s_j}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_j}}{M^B}\right)\right)\right.$$

$$\left. +\left(2M^s b_1\left(\frac{N_t^{s_i}}{M^s}\right) + 2M^s c_1\left(\frac{N_t^{B_i}}{M^B}\right)\right)\right) - \sum_j \frac{dv_i}{dN_t^{s_j}}\left(ØN_t^{s_i} - \sum_j\left(ØN_t^{s_i} + N_t^{s_i} w \ln(\frac{(M^S - N^{s_j})w}{N^{s_j}}\frac{dv_j}{dN^{s_j}}) + 2N_t^{s_i} b_1\left(\frac{N_t^{s_i}}{M^s}\right) + 2N_t^{s_i} c_1\left(\frac{N_t^{B_i}}{M^B}\right)\right)\right.$$

$$\left. -\sum_j\left(M^s b_1\left(\frac{N_t^{s_j}}{M^s}\right) + M^s c_1\left(\frac{N_t^{B_j}}{M^B}\right)\right)\right) + \sum_j \frac{dv_i}{dN_t^{B_j}}\left(xM^B + 2M^B h \ln(\frac{dv_i}{dN^{B_i}}(M^B - N^{B_i})h\right) -$$

$$\sum_j\left(xM^B + M^B h \ln\frac{dv_i}{dN^{B_j}}(M^B - N^{B_j})h)\right) + 2M^B b_2\left(\frac{N_t^{B_i}}{M^s}\right) + 2M^B c_2\left(\frac{N_t^{s_i}}{M^B}\right) - \sum_j\left(M^B b_2\left(\frac{N_t^{B_j}}{M^s}\right) + M^B c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right)\Bigg)\Bigg) - \sum_j \frac{dv_i}{dN_t^{B_j}}(xN_t^{B_i} +$$

$$2N_t^{B_i} hln(\frac{dv_i}{dN^{B_i}}(M^B - N^{B_i})h) - \sum_j\left(xN_t^{B_i} + N_t^{B_i} h \ln(\frac{dv_j}{dN^{B_j}}(M^B - N^{B_j})h)\right) + 2N_t^{B_i} b_2\left(\frac{N_t^{B_i}}{M^s}\right) + 2N_t^{B_i} c_2\left(\frac{N_t^{s_i}}{M^B}\right) - \sum_j\left(N_t^{B_i} b_2\left(\frac{N_t^{B_j}}{M^s}\right) + N_t^{B_i} c_2\left(\frac{N_t^{s_j}}{M^B}\right)\right))$$

+Equating powers of Constants coefficients, $N_t^{s_j}, N_t^{B_j}, N_t^{s_j} N_t^{B_j}, N_t^{s_j} N_t^{s_j}, N_t^{B_j} N_t^{B_j}, \sum_{j=1}^{n} N_t^{s_i} ln\left(\frac{r_j^i}{N_t^{s_j}} + t_j\right)$

$\sum_{j=1}^{n} N_t^{s_i} ln\left(\frac{r_j^i}{N_t^{s_j}} + t_j\right), \sum_{j=1}^{n} ln\left(N_t^{B_j} R_j^i + T_j\right), \ln\left(N_t^{B_i} R_i^i + T_i\right), \sum_{j=1}^{n} ln\left(N_t^{B_j} R_j^i + T_j\right) N_t^{B_i}, N_t^{B_i} \ln\left(N_t^{B_i} R_j^i + T_i\right)$

; (6), (7) , … (19) are obtained.

Then, for the differential game given by (1), (4) and (5), the optimal feedback marketing rate and pricing for company i are:

$$P^{i*} = \frac{(M^S - N^{s_i})w}{N^{s_i}} q_i^i$$

and

$$A^{i*} = Q_i^i (M^B - N^{B_i})h$$

Appendix 2

In the steady state $\frac{dN_t^{s_i}}{dt} = 0$ and $\frac{dN_t^{B_i}}{dt} = 0$. Let $N_t^{\bar{s}_i}$ and $N_t^{\bar{B}_i}$ denote the market share of sellers and buyers in the steady state for company *i*, then:

$$0 = hln(Q_i^i(M^B - N^{\overline{B_i}}))h - \sum_{j\neq i} hln(Q_j^j(M^B - N^{\overline{B_j}})h) + b_2\left(\frac{N_t^{\overline{B_i}}}{M^s}\right) + c_2\left(\frac{N_t^{\overline{S_i}}}{M^B}\right) - \sum_{j\neq i}\left(b_2\left(\frac{N_t^{\overline{B_j}}}{M^s}\right) + c_2\left(\frac{N_t^{\overline{S_j}}}{M^B}\right)\right)(M^B - N_t^{\overline{B_i}})$$

And

$$0 = -w\ln\left(\frac{(M^s - N^{\bar{s}_i})w}{N^{\bar{s}_i}} q_i^i\right) - \sum_{j\neq i}\left(w\ln\left(\frac{(M^s - N^{\bar{s}_j})w}{N^{\bar{s}_j}} q_i^i\right) + b_1\left(\frac{N_t^{\bar{s}_i}}{M^s}\right) + c_1\left(\frac{N_t^{\bar{B}_i}}{M^B}\right)\right.$$

$$\left. - \sum_{j\neq i}\left(b_1\left(\frac{N_t^{\bar{s}_j}}{M^s}\right) + c_1\left(\frac{N_t^{\bar{B}_j}}{M^B}\right)\right)\right)(M^s - N^{\bar{s}_i})$$

$\sum_{j\neq i} N^{\bar{s}_j} = M^s - N_t^{\bar{s}_i}$ And $\sum_{j\neq i} N^{\bar{B}_j} = M^B - N_t^{\bar{B}_i}$

In the above relations, the relations (20) and (21) are obtained.