A Megastable Oscillator with Two Types of Attractors

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In this paper, a megastable system is designed with particular formation of attractors. It has two formations of attractors: the inner ones with a smaller amplitude, and the outer ones with the Eye of God nebula shape and larger amplitude. To the best of our knowledge, such a megastable oscillator with this special formation of attractors has not been studied before. Afterward, the oscillator is forced, and its attractors are discussed. Different dynamics of this new oscillator are investigated using tools such as bifurcation and Lyapunov exponent diagram, and basins for each attractor.

Keywords: Megastable oscillator; forced system; chaos; bifurcation analysis; multistability.

1. Introduction

Presenting novel chaotic oscillators with unique features has been a controversial subject [1-3] in the field of nonlinear dynamics and chaos. Such oscillators have many applications, such as secure communications [4], and image encryption [5-7]. Regarding the special features, oscillators with different types of equilibria has been considerably investigated [8], such as oscillators without any equilibria [9] and oscillators with a single stable equilibrium [10]. Many of such oscillators are systems with hidden attractors [11]. An existing categorization of dynamical systems is

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dividing them to self-excited and hidden attractors [12-14]. Another critical characteristic of chaotic oscillators is the topology of their strange attractor [15, 16]. As an example one can mention new chaotic oscillators with different types of symmetry [17], and with multi-scroll attractors [18].

A multistable oscillator is a system that has coexisting attractors regarding the initial conditions (ICs) [19, 20]. The state of a multistable oscillator can be different for various ICs according to the attractors' basin of attraction. The multistable phenomenon is widespread in the nature and real-world systems [21] such as the catalysis of linear arrays of three cells in chemistry [22], lasers [23], and optics [24]. Also, multistability has been seen in neuronal models which mimic the behavior of real neurons [25], as well as different disciplines of physics, like semiconductors [26]. Various types of multistability have been investigated, such as extreme multistable oscillators [27] and megastable oscillators [28, 29]. Various megastable oscillators have been studied recently. A simple megastable oscillator with a triangular term was discussed in [30]. In [31], a megastable oscillator with oyster-like dynamics was proposed. In [32], a carpet-like magastable oscillator with only sinusoidal terms was investigated. It can be seen that the formation of attractors in megastable oscillators is a core in such researches. Therefore, this paper focuses on proposing a megastable oscillator with a unique attractors' formation. A comparison of the proposed oscillator with the previous ones is shown in Table 1.

Here, a megastable two-dimensional oscillator is designed. By forcing the oscillator, its chaotic dynamics and their bifurcations are discussed. The infinite attractors include cyclic attractors and strange attractors in various parameters. The oscillator has two types of attractors, the inner attractors with smaller amplitudes and the outer ones with larger amplitudes, like the Eye of God nebula. The 2D oscillator is proposed in the next section, and its attractors are discussed. Then, applying a forcing term, the new attractors are studied. In Section 3, the forced oscillator is analyzed through its bifurcations and Lyapunov exponents regarding different coexisting attractors. Eventually, the conclusion is stated in Section 4.

2. Proposed Oscillator

Here, a two-dimensional megastable oscillator is proposed as follows:

$$\begin{aligned}\dot{x} &= \tanh(y) \\ \dot{y} &= -0.09x + y\cos(x) + xe^{-\frac{(x-30)^2}{5}}\tanh(x)\end{aligned} \tag{1}$$

The system is designed based on a simple megastable system [33]. The new system is constructed by varying and adding some terms to the oscillator and using a vast search to find the megastable dynamics. The proposed oscillator has an equilibrium point at $(0,0)$. By using the stability analysis method, the Jacobian matrix of Equation (1) at the fixed point is:

$$Jac = \begin{pmatrix} 0 & 1 \\ -0.09 & 1 \end{pmatrix} \tag{2}$$

The characteristic equations are:

$$\begin{aligned}&|\lambda I - J| = 0 \rightarrow \begin{vmatrix} \lambda & -1 \\ 0.09 & \lambda - 1 \end{vmatrix} = 0 \\ &\rightarrow \lambda^2 - \lambda + 0.09 = 0.\end{aligned} \tag{3}$$

The eigenvalues are $\lambda_1 = 0.9, \lambda_2 = 0.1$. Since both eigenvalues are positive, the equilibrium point is unstable. So, it repels all the nearby trajectories. Figure 1 shows the coexisting limit cycles for various ICs. The oscillator has two groups of coexisting limit cycles. The first group is the four inner attractors, plotted in part (a) of Fig. 1. They are plotted by nine ICs as $[x_0, 0]$ where $x_0 = 0.8889\pi, 1.7778\pi, 2.6667\pi, 3.5556\pi, 4.4444\pi, 5.3333\pi, 6.2222\pi, 7.1111\pi, 8\pi$ .The second group is the outer and larger attractors. Three of these attractors are plotted in part (b) of Fig. 1. The ICs of these attractors are $[x_0, 0]$ where $x_0 = 8.5\pi, 28.875\pi, 49.25\pi, 69.6250\pi, 90\pi$. The second group limit cycles have oscillations with larger amplitude, and they are like the Eye of God's nebula. They need more run time, and more transient time should be removed. The first group limit cycles are plotted with rum time 4000, and the transient half of it is neglected. The second group limit cycles are calculated by run time 60000, and 0.8 of the time series are removed as transient time.

Then, by adding periodically forced term $A\sin(\sqrt{3}t)$ to the system (1), we have:

$$\begin{aligned} \dot{x} &= \tanh(y) \\ \dot{y} &= -0.09x + y\cos(x) + xe^{-\frac{(x-30)^2}{5}}\tanh(x) + A\sin\sqrt{3}t \end{aligned} \tag{4}$$

Equation (4) can show more complex dynamics by adding the periodically forcing term. Figure 2 shows the dynamics of Equation (4) with ICs the same as Fig. 1 and $A = 2.7$. Figure 2a shows the attractors with ICs in the interval $x_0 \in [0, 8\pi], y_0 = 0$. The figure shows that some of the four inner attractors became chaotic with $A = 2.7$. Figure 2b presents the outer attractors with ICs in the interval $x_0 \in [8.5\pi, 90\pi], y_0 = 0$. It seems that the three attractors are not chaotic in $A = 2.7$.

3. The forced system's dynamics

To study the dynamics of the Equation (4), the evolution of the attractors is investigated using the bifurcation diagrams. A is considered a bifurcation parameter. The bifurcation diagram and Lyapunov spectrums are presented in Figure 3. Figure 3a and 3b show the bifurcation and Lyapunov of the largest attractor in the inner group, with the constant ICs $[-7.5\pi, 0]$. Figure 3a shows the bifurcation plot for the variable y of the Equation (4). By changing the bifurcation parameter A, the system shows different dynamics like quasiperiodic, chaotic, and periodic attractors regarding the bifurcation diagram in Fig. 3a. The largest Lyapunov exponent is plotted in green in Fig. 3b. The positive largest Lyapunov means that the system presents chaotic dynamics. Using Wolf's method, the Lyapunov exponents are approximated [34] with the run-time 20000.

Figure 4 demonstrates the bifurcation plot and Lyapunov exponents of the Equation (4) by changing the IC of x in the interval $x_0 \in [0, 40]$. The parameter A is set constant to 2.6. Figure 4a presents the dynamics of the logarithm of the y's maxima by varying

x_0. The set of ICs is considered as $[x_0, 0]$. Figure 4b shows the largest Lyapunov exponent. The figure illustrates the basin of attractions of different attractors in the x_0 domain. The oscillator shows different strange attractors for the interval $x_0 \in [3.87, 25.88]$. To study the Equation (4) dynamics in different ranges, seven attractors with the selected ICs are plotted in Fig. 5.

To investigate the basin of different attractors in 2D, Fig. 6 is plotted. The basin of the four inner attractors is plotted in four colors (cyan, yellow, purple, and black). The white color shows the uninvestigated region of the basin of attraction.

4. Conclusion

A new megastable oscillator was introduced. Various dynamics of the oscillator were discussed. In addition, the oscillator was studied by adding a periodically forcing term. Various periodic and chaotic attractors of the oscillator were analyzed through its attractors plot, bifurcation plot and Lyapunov exponent diagrams. The evolution of one of the attractors of the oscillator was investigated as a function of parameter A. Another bifurcation diagram was studied as a function of the initial values for the variable x. In this case, the basin of attractions of some of the attractors was discussed. The 2D basin of attraction diagram was also studied to investigate the variation of basins for two variables' ICs. The results revealed the exciting dynamics of the oscillator.

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Data availability statement

Data generated during the current study will be made available at reasonable request.

Declarations Conflict of interest

The authors declare that they have no conflict of interest.

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Fig. 1. Seven limit cycles for the 14 ICs to show the coexisting limit cycles of Equation (1); a) four inner limit cycles with ICs in $x_0 \in [0, 8\pi], y_0 = 0$ and run-time 4000; b) three limit cycles with ICs in $x_0 \in [8.5\pi, 90\pi], y_0 = 0$ and run-time 60000; Specifically, part (a) of the figure shows the limit cycles of System (1) by nine ICs as $[x_0, 0]$ where $x_0 = 0.8889\pi, 1.7778\pi, 2.6667\pi, 3.5556\pi, 4.4444\pi, 5.3333\pi, 6.2222\pi, 7.1111\pi, 8\pi$; for part (b) of the figure three of outer and larger attractors are plotted; the ICs of these attractors are $[x_0, 0]$ where $x_0 = 8.5\pi, 28.875\pi, 49.25\pi, 69.6250\pi, 90\pi$; Some of the ICs result in the same attractors;

Fig. 2. Seven attractors for the 14 ICs to present the coexistence of more complex dynamics like chaotic and limit cycle attractors of Equation (4) for $A = 2.7$; a) four inner dynamics with ICs in $x_0 \in [0, 8\pi], y_0 = 0$; b) three outer dynamics with ICs in $x_0 \in [8.5\pi, 90\pi], y_0 = 0$; Specifically, part (a) of the figure shows the dynamics by nine ICs as $[x_0, 0]$ where $x_0 = 0.8889\pi, 1.7778\pi, 2.6667\pi, 3.5556\pi, 4.4444\pi, 5.3333\pi, 6.2222\pi, 7.1111\pi, 8\pi$; for the part (b) of the figure three of outer and larger attractors are plotted; the ICs of these attractors are $[x_0, 0]$ where $x_0 = 8.5\pi, 28.875\pi, 49.25\pi, 69.6250\pi, 90\pi$;

Fig. 3. a) The bifurcation plot of Equation (4) by varying bifurcation parameter A; b) the three Lyapunov exponents (LEs) in which the largest is plotted in green; where the largest LE is positive, the oscillator has the chaotic dynamics; the diagrams are plotted with the constant ICs $[-7.5\pi, 0]$;

Fig. 4. a) The bifurcation plot of y of Equation (4) as a function of the value of IC of variable x to investigate the range of chaotic attractors for the values of ICs; b) the largest Lyapunov exponent ($LE_{\max}$); The parameter $A = 2.6$ is considered constant during the simulation.

Fig. 5. Attractors of Equation (4) with different ICs corresponding to the different parts of bifurcation of Fig. 4a; a) with ICs [1, 0]; b) with ICs [7, 0]; c) with ICs [15, 0]; d) with ICs [23, 0]; e) with ICs [28, 0]; f) with ICs [35, 0]; Various dynamics of the oscillator by changing ICs of the x variable are presented;

Fig. 6. Basin of attraction of Equation (4) shows the different attractors for the different values of ICs in $A = 2.6$; The basin of attraction of four different attractors is plotted in different colors; the white color shows the uninvestigated region;

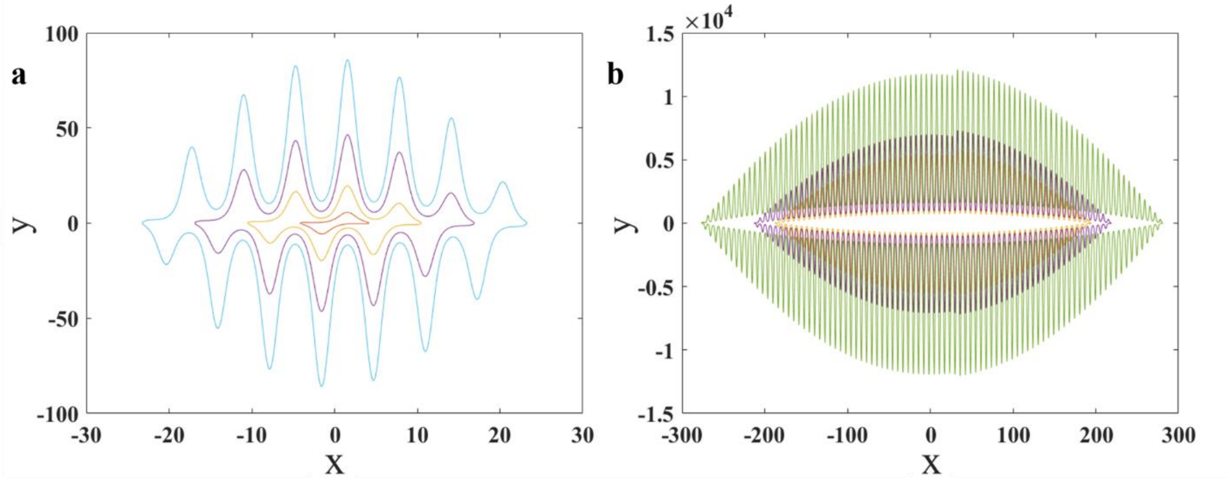
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Fig. 1.

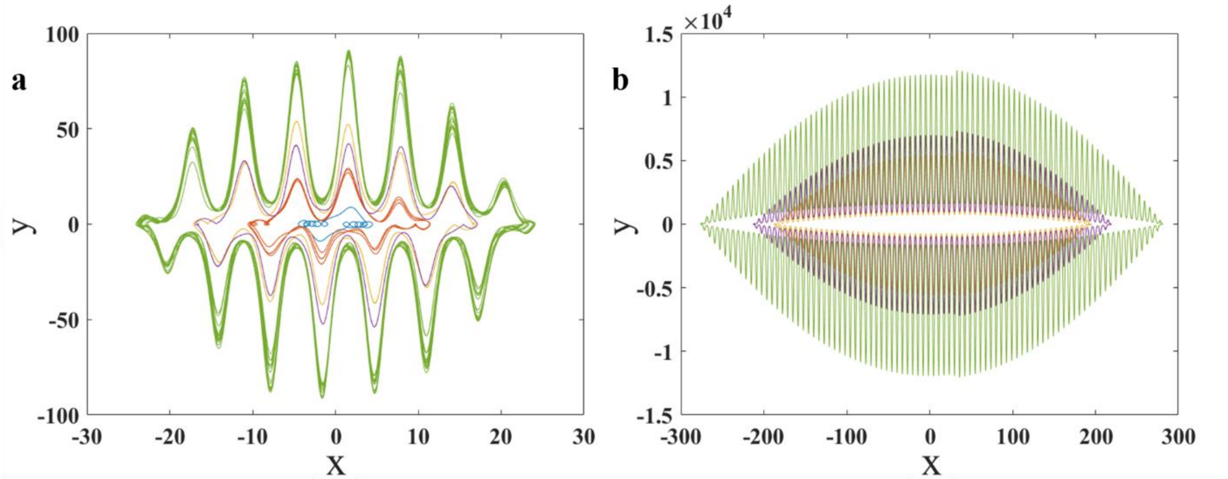
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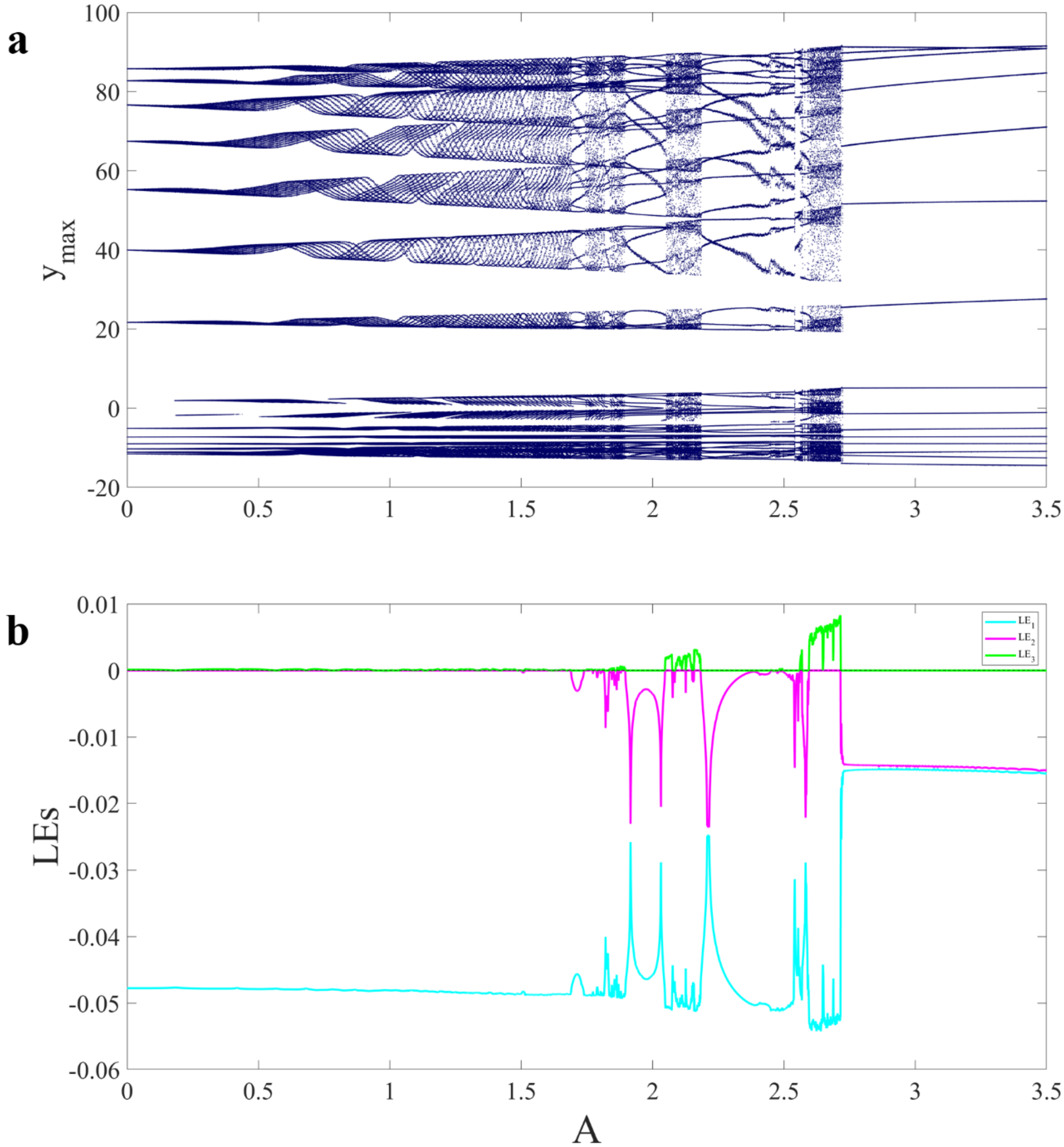
Fig. 2.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b
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LEs
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Fig. 3.

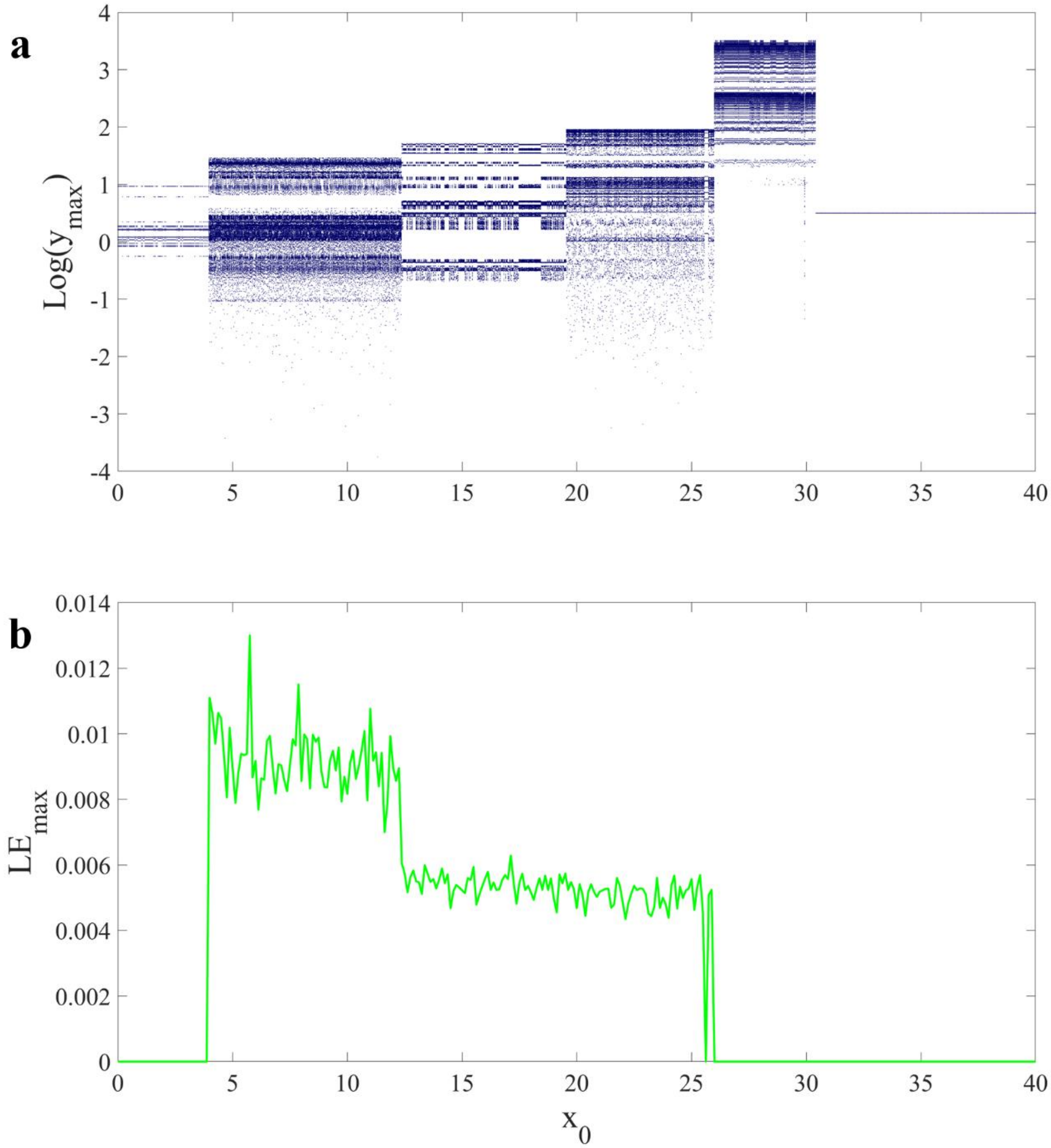

Fig. 4.

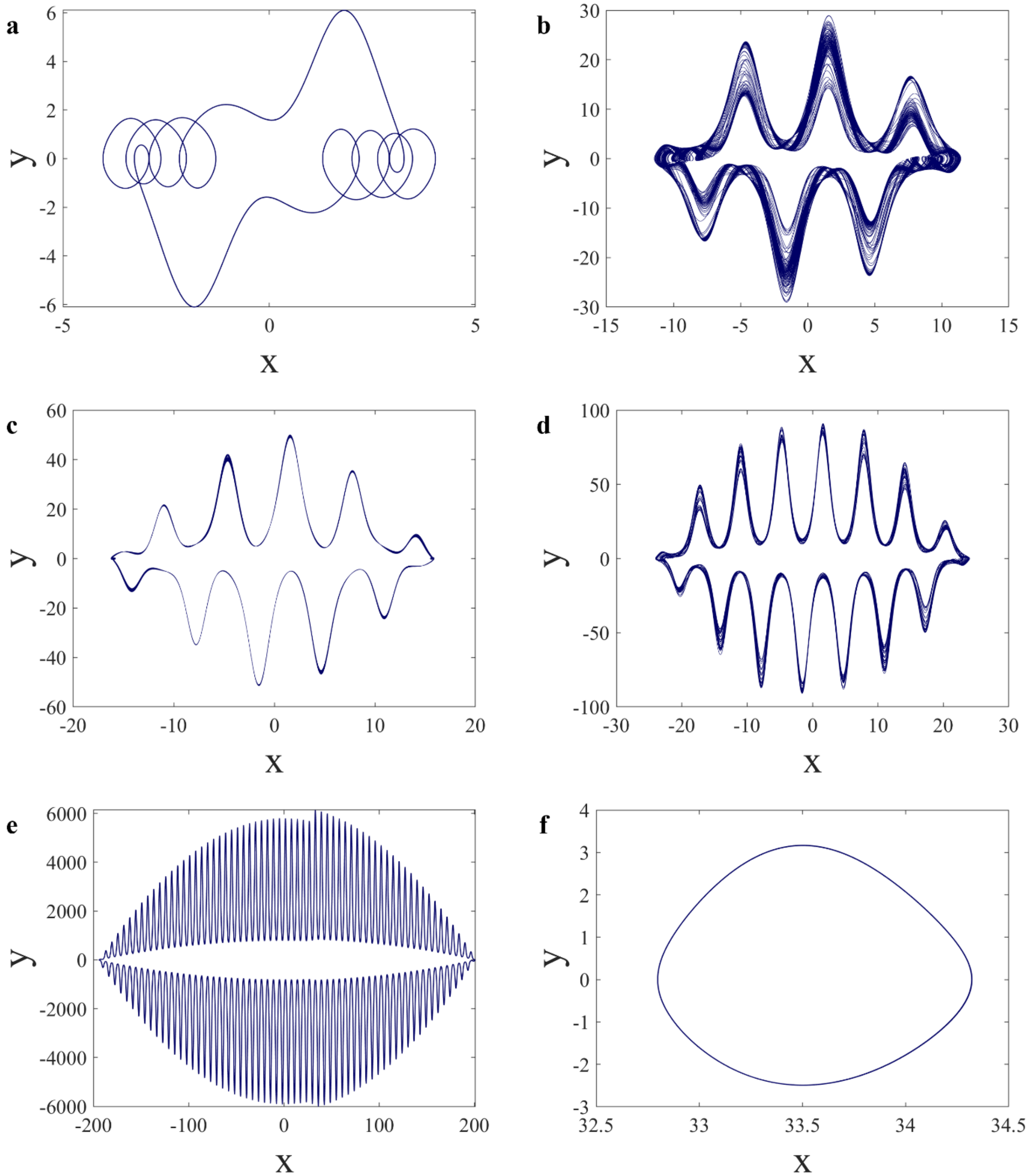

Fig. 5.

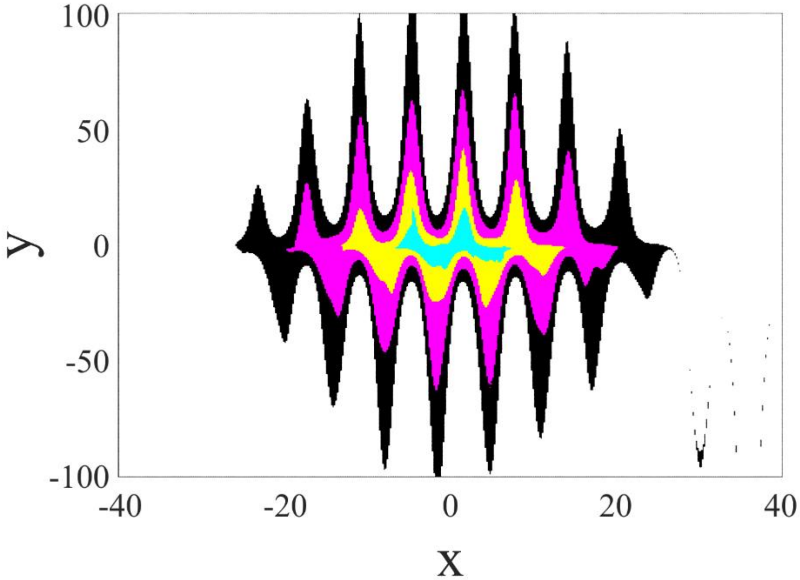

Fig. 6.

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Table 1.

#	Reference	# of trigonometric terms	Dynamics
1	[31]	2	Oyster-like
2	[32]	4	Carpet-like
3	[35]	1	Jagged-like
4	[36]	5	-
5	This work	1	Eye of God nebula

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