On the quest of optimal class of estimators using ranked set sampling

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Abstract

The present paper acquaints some optimal class of estimators of population mean utilizing information on an auxiliary variable in ranked set sampling methodology. The effectiveness of acquainted estimators is studied regarding the estimators suggested by Samawi and Muttlak [1], Yu and Lam [2], Kadilar et al. [3], Singh and Solanki [4, 5], Singh et al. [6], Solanki and Singh [7], Mehta and Mandowara [8], Saini and Kumar [9], Mehta et al. [10] and Bhushan and Kumar [11] which cover most of the familiar estimators. The optimality conditions have been established and followed up by a simulation study as well as a real data application and the results are constituted a rather satisfactory showing advancement over the all prominent estimators discussed in this article.

Keywords: Bias, Mean square error, Optimal class of estimators, Ranked set sampling, Relative efficiency.

1 Introduction

It has been proven that sampling requires less cost and provides a rapid results in comparison to complete enumeration. It becomes more beneficial if the number of sampled units can be reduced from a larger set of available units to a smaller set provided the non-sampled units add some amount of information. In this aspect, McIntyre [12] deliberated the notion of ranked set sampling (*RSS*) as an effective choice over simple random sampling (*SRS*) but

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did not provide any mathematical support. Takahasi and Wakimoto [13] visited to McIntyre [12] and facilitated the required mathematical background to the theory of RSS. Both McIntyre [12] and Takahasi and Wakimoto [13] chose the instance of perfect ranking of units, whereas Dell and Clutter [14] demonstrated that the ranked set sample mean is an unbiased estimator of the population mean whether the ranking of units are perfect or imperfect, *i.e.* errors arise in ranking the unit concerning to the chosen variable. Muttlak [15] applied RSS in simple linear regression to compute the parameters. Samawi and Muttlak [1] in his seminal work proved that the ranking of denominator variables in the ratio estimators provided a more efficient results. Singh et al. [6] provided a general estimation procedure for population mean under RSS. Some modified ratio-cum-product estimators are provided by Mehta and Mandowara [8] in RSS. Khan and Shabbir [16] discussed difference-cum-exponential ratio estimator under RSS. Saini and Kumar [9] utilized quartiles as the auxiliary information and suggested a ratio estimator using RSS. Zamanzade and Mahdizadeh [17] investigated pair RSS for computing the population proportion with application to air quality monitoring. Zamanzade and Mahdizadeh [18] suggested the procedure of computing population proportion using RSS with extreme ranks. Mehta et al. [10] suggested a generalized class of estimator under RSS. Bhushan and Kumar [19] suggested some log type class of estimators under RSS. Shahzad et al. [20] introduced quantile regression-ratio-type estimators for mean estimation using complete and partial auxiliary information. Shahzad et al. [21] developed an estimation procedure of the population mean by successive using auxiliary information under median RSS. Bhushan and Kumar [11] proposed some optimal classes of estimators under RSS, whereas Bhushan and Kumar [22] suggested novel log type estimators of population mean and studied the effect of asymmetry over the efficiency of the estimators. For a more comprehensive study on RSS, the authors recommend the work of Mahdizadeh and Zamanzade [23], Bhushan et al. [24] and Bhushan and Kumar [25, 26]. In this paper, the optimality of acquainted estimators has been examined against the existing estimators based on RSS.

1.1 Procedures and notations

McIntyre [12] addressed the proposition of ranked set sampling which is the aggregation of simple random samples, usually with one unit from each of the selected simple random samples which are being quantified after imposing a type of ranking. Likely to *SRS*, the quantified units using RSS method lend independently to make an efficient illation on the population. The method of RSS is based on extracting m simple random samples, of length m units from the population and m units are being ranked inside every set as per variable of choice either by eye or by any cost-independent measure. The unit assigned rank 1 is selected for the computation of the element from the first sample and the rest units are removed. Again, the unit assigned rank 2 is selected for the computation of elements from the second sample and the rest units are removed. The process is retained in the same mode until the unit assigned rank m is selected for the quantification of elements from the m^{th} sample. The prior process builds cycle. The r repetition of this cycle will generate $n = mr$ ranked set samples.

Suppose $(X_1,Y_1),(X_2,Y_2),...,(X_n,Y_n)$ are bivariate simple random samples of length n extracted from a population of length N having a joint probability density function $f(x,y)$ and cumulative distribution function ($c.d.f.$) $F(x,y)$. Let μ_x, μ_y, σ_x^2, σ_y^2, C_x and C_y be, respectively, the population means, population variances and population coefficient of variations of auxiliary variable X and study variable Y and ρ denotes the population coefficient of correlation between variables X and Y. We suppose that the ranking is imposed on the auxiliary variable X to compute the population parameter of study variable Y. Suppose (X_{11},Y_{11}), (X_{12},Y_{12}),..., (X_{1n},Y_{1n}), (X_{21},Y_{21}), (X_{22},Y_{22}),..., (X_{2n},Y_{2n}),..., (X_{n1},Y_{n1}), (X_{n2},Y_{n2}),...,(X_{nn},Y_{nn}) are bivariate random samples quantified from the population utilizing simple random sampling without replacement possessing the same $c.d.f.$ $F(x,y)$ and let $(X_{i(1)},Y_{i[1]})$, $(X_{i(2)},Y_{i[2]})$,..., $(X_{i(n)},Y_{i[n]})$ be the order statistic of X_{i1}, X_{i2},..., X_{in} and the judgement order of Y_{i1}, Y_{i2},..., Y_{in}; $i=1,2,...,n$. To simplify the notations, we have denoted $(X_{i(i)}$,$Y_{i[i]})$ by $(X_{(i)}$,$Y_{[i]})$. Here, the parentheses () and [] used in the subscripts of X and Y show, respectively, the perfect and imperfect ranking of the units.

To derive the characteristic of the proposed estimators, let us assume that
$\varsigma_0 = (\bar{Y}_{[n]} - \mu_y)/\mu_y$, $\varsigma_1 = (\bar{X}_{(n)} - \mu_x)/\mu_x$ provided $E(\varsigma_0) = E(\varsigma_1) = 0$
and

$$\Delta_{r,s} = \frac{E\{(\bar{X}_{(n)} - \mu_x)^r (\bar{Y}_{[n]} - \mu_y)^s\}}{\mu_x^r \mu_y^s} \tag{1.1}$$

where $\bar{X}_{(n)} = \sum_{i=1}^{m} x_{(i)}/mr$ and $\bar{Y}_{[n]} = \sum_{i=1}^{m} y_{[i]}/mr$ is the ranked set sample means of variables X and Y, respectively.

Following Equation (1.1), we can write

$$E(\varsigma_0^2) = (\gamma C_y^2 - W_{y_{[i]}}^2) = \Delta_{0,2},$$
$$E(\varsigma_1^2) = (\gamma C_x^2 - W_{x_{(i)}}^2) = \Delta_{2,0},$$
$$E(\varsigma_0, \varsigma_1) = (\gamma \rho C_x C_y - W_{xy_{[i]}}) = \Delta_{1,1},$$

where $\gamma = 1/mr$, $C_x = S_x/\mu_x$, $C_y = S_y/\mu_y$, $W_{x_{(i)}}^2 = \sum_{i=1}^{m} (\mu_{x_{(i)}} - \mu_x)^2/m^2 r \mu_x^2$, $W_{y_{[i]}}^2 = \sum_{i=1}^{m} (\mu_{y_{[i]}} - \mu_y)^2/m^2 r \mu_y^2$, $W_{xy_{[i]}} = \sum_{i=1}^{m} (\mu_{x_{(i)}} - \mu_x)(\mu_{y_{[i]}} - \mu_y)/m^2 r \mu_x \mu_y$, $\mu_{x_{(i)}} = E(X_{(i)})$ and $\mu_{y_{[i]}} = E(Y_{[i]})$. It is worthwhile that the quantities $\mu_{x_{(i)}}$ and $\mu_{y_{[i]}}$ rely upon the order statistics from particular distributions which can easily be determined from Arnold et al. [27]. However, in the absence of judgement error, the quantities $\mu_{x_{(i)}}$ and $\mu_{y_{[i]}}$ are considered equal provided both the study and auxiliary variables possess the same distribution, refer, Dell and Clutter [14]. Further, the paper is molded in different sections. In Section 2, we review some prominent estimators along with their properties. The proposed class of estimators are discussed in Section 3 with their properties. In Section 4, the optimality conditions are derived which are successively assessed by a simulation experiment in Section 5 using symmetric and asymmetric populations. A real data application is provided in Section 6. Lastly, the conclusion is given in Section 7.

2 Review of prominent estimators

The usual mean estimator under RSS is given as

$$t_m = \bar{Y}_{[n]} \tag{2.1}$$

Samawi and Muttlak [1] envisaged the classical ratio estimator under RSS as

$$t_r = \bar{Y}_{[n]} \left(\frac{\mu_x}{\bar{X}_{(n)}} \right) \tag{2.2}$$

Yu and Lam [2] provided the classical regression estimator under RSS as

$$t_{lr} = \bar{Y}_{[n]} + \beta(\mu_x - \bar{X}_{(n)}) \tag{2.3}$$

where β is the regression coefficient of Y on X.
Kadilar et al. [3], motivated by Prasad [28] and Samawi and Muttlak [1], investigated a ratio estimator under RSS as

$$t_{kc} = k\bar{Y}_{[n]} \left(\frac{\mu_x}{\bar{X}_{(n)}} \right) \tag{2.4}$$

where k is a suitably chosen scalar.
Following Singh and Solanki [4], one may consider a family of estimators under RSS as

$$t_{ss_1} = \theta_{s_1}\bar{Y}_{[n]} \left\{ \frac{\alpha(a\bar{X}_{(n)}+b)+(1-\alpha)(a\mu_x+b)}{(a\mu_x+b)} \right\}^{\delta} + \Theta_{s_1}\bar{Y}_{[n]} \left\{ \frac{(a\mu_x+b)}{\alpha(a\bar{X}_{(n)}+b)+(1-\alpha)(a\mu_x+b)} \right\}^{g} \tag{2.5}$$

where δ, g and α are real constants, whereas θ_{s_1} and Θ_{s_1} are suitably chosen scalars. Moreover, $a(\neq 0)$ and b possess either real values or the functions of prescribed parameters based on the auxiliary variable X, namely, standard deviation S_x, coefficient of variation C_x, coefficient of correlation ρ, coefficient of skewness $\beta_1(x)$ and coefficient of kurtosis $\beta_2(x)$.
On the lines of Singh and Solanki [5], one may consider the following class of estimators under RSS as

$$t_{ss_2} = \theta_{s_2}\bar{Y}_{[n]} \left\{ \frac{\mu_x^*}{\alpha\bar{X}^*_{(n)} + (1-\alpha)\mu_x^*} \right\}^{g} + \Theta_{s_2}\bar{Y}_{[n]} \exp\left\{ \frac{\delta(\mu_x^* - \bar{X}^*_{(n)})}{(\mu_x^* + \bar{X}^*_{(n)})} \right\} \tag{2.6}$$

where g and δ are real constants, θ_{s_2}, Θ_{s_2} are suitably chosen scalars to be obtained later and g takes values 1 and -1 to produce ratio and product type estimators. Moreover, $\mu_x^* = a\mu_x + b$, $\bar{X}^*_{(n)} = a\bar{X}_{(n)} + b$.
Motivated by Upadhyaya et al. [29], Khoshnevisan et al. [30] and Koyuncu and Kadilar [31], Singh et al. [6] examined a family of estimators under RSS as

$$t_s = \lambda_1\bar{Y}_{[n]} + \lambda_2\bar{Y}_{[n]} \left\{ \frac{\mu_x^*}{\theta\bar{X}^*_{(n)} + (1-\theta)\mu_x^*} \right\}^{g} \tag{2.7}$$

where λ_1 and λ_2 are duly chosen scalars. Also, authors have noted that more than 184 estimators can be tabulated from estimator t_s.
Motivated by Solanki and Singh [7], one may develop the following estimators in RSS as

$$t_{ss_3} = \theta_{s_3}\bar{Y}_{[n]}\left(\frac{\mu_x^*}{\bar{X}_{(n)}^*}\right)^{\alpha} \exp\left\{\frac{\beta(\mu_x^* - \bar{X}_{(n)}^*)}{(\mu_x^* + \bar{X}_{(n)}^*)}\right\} + \Theta_{s_3}\bar{Y}_{[n]}\left(\frac{\bar{X}_{(n)}^*}{\mu_x^*}\right)^{\delta} \exp\left\{\frac{\lambda(\bar{X}_{(n)}^* - \mu_x^*)}{(\mu_x^* + \bar{X}_{(n)}^*)}\right\} \tag{2.8}$$

where α, β, δ, λ are constants considering quantities $(-1,0,1)$ to construct several estimators and θ_{s_3}, Θ_{s_3} are suitably chosen scalars.
Mehta and Mandowara [8] introduced following estimators using RSS as

$$t_{mm_1} = \bar{Y}_{[n]}\left(\frac{\mu_x + C_x}{\bar{X}_{(n)} + C_x}\right) \tag{2.9}$$

$$t_{mm_2} = \bar{Y}_{[n]}\left(\frac{\mu_x + \beta_2(x)}{\bar{X}_{(n)} + \beta_2(x)}\right) \tag{2.10}$$

$$t_{mm_3} = \bar{Y}_{[n]}\left(\frac{\mu_x C_x + \beta_2(x)}{\bar{X}_{(n)} C_x + \beta_2(x)}\right) \tag{2.11}$$

$$t_{mm_4} = \bar{Y}_{[n]}\left(\frac{\bar{X}_{(n)} C_x + \beta_2(x)}{\mu_x C_x + \beta_2(x)}\right) \tag{2.12}$$

$$t_{mm_5} = \bar{Y}_{[n]}\left\{\phi\left(\frac{\mu_x C_x + \beta_2(x)}{\bar{X}_{(n)} C_x + \beta_2(x)}\right) + (1-\phi)\left(\frac{\bar{X}_{(n)} C_x + \beta_2(x)}{\mu_x C_x + \beta_2(x)}\right)\right\} \tag{2.13}$$

where ϕ is an optimizing scalar.
Saini and Kumar [9] investigated the ratio type estimator under RSS as

$$t_{sk_i} = \bar{Y}_{[n]}\left(\frac{\mu_x - \bar{X}_{(n)} + q_i}{\mu_x + \bar{X}_{(n)} + q_i}\right), \quad i = 1,3 \tag{2.14}$$

Mehta et al. [10] investigated Singh and Agnihotri [32] estimator in RSS as

$$t_v = \delta\left(\frac{a\mu_x + b}{a\bar{X}_{(n)} + b}\right)^p + (1-\delta)\left(\frac{a\bar{X}_{(n)} + b}{a\mu_x + b}\right) \tag{2.15}$$

where δ is a suitably chosen optimizing scalar and p is a real constant to design different estimators.
Recently, Bhushan and Kumar [11] proposed some optimal classes of estimators under RSS

as

$$t_1 = \alpha_1 \bar{Y}_{[n]} + \beta_1 (\bar{X}_{(n)} - \mu_x) \tag{2.16}$$

$$t_2 = \alpha_2 \bar{Y}_{[n]} \left(\frac{\mu_x}{\bar{X}_{(n)}} \right)^{\beta_2} \tag{2.17}$$

$$t_3 = \alpha_3 \bar{Y}_{[n]} \left\{ \frac{\mu_x}{\mu_x + \beta_3 (\bar{X}_{(n)} - \mu_x)} \right\} \tag{2.18}$$

$$t_4 = \alpha_4 \bar{Y}_{[n]} + \beta_4 (\bar{X}^*_{(n)} - \mu^*_x) \tag{2.19}$$

$$t_5 = \alpha_5 \bar{Y}_{[n]} \left(\frac{\mu^*_x}{\bar{X}^*_{(n)}} \right)^{\beta_5} \tag{2.20}$$

$$t_6 = \alpha_6 \bar{Y}_{[n]} \left\{ \frac{\mu^*_x}{\mu^*_x + \beta_6 (\bar{X}^*_{(n)} - \mu^*_x)} \right\} \tag{2.21}$$

where α_i and β_i; $i = 1,2,...,6$ are suitably chosen scalars.
The readers are referred to Appendix A for a quick review of the mean square error (MSE) of all discussed estimators.

3 Proposed estimators

Many estimators are being proposed on each passing day, which actually raises an important question that "whether the proposed estimators are still optimal in some sense?" Most of the papers do not address this important aspect and that they are mere adaptions of some important estimators. An important aspect pertaining to optimal estimators in the scale of minimum MSE is that they are merely achieved the MSE of classical regression estimators. These includes various adaptions of important estimators like difference, Srivastava [33] and Walsh [34] type estimators. If any estimator merely attains the minimum MSE of the regression estimator, then they do not add to the knowledge even with some information. This raises another important question that "whether the best linear unbiased (BLU) estimator can or can not be improved upon?" This paper has been specially written to answer this question. Keeping this aspect in mind, we propose a class of estimators under RSS to estimate the population mean μ_y given by

$$t_{b_1} = \eta_1 \bar{Y}_{[n]} \left(\frac{\mu_x}{\bar{X}_{(n)}} \right)^{\theta_1} + \psi_1 \bar{Y}_{[n]} \left\{ \frac{\mu_x}{\mu_x + \zeta_1 (\bar{X}_{(n)} - \mu_x)} \right\} \tag{3.1}$$

$$t_{b_2} = \eta_2 \bar{Y}_{[n]} \left(\frac{\mu^*_x}{\bar{X}^*_{(n)}} \right)^{\theta_2} + \psi_2 \bar{Y}_{[n]} \left\{ \frac{\mu^*_x}{\mu^*_x + \zeta_2 (\bar{X}^*_{(n)} - \mu^*_x)} \right\} \tag{3.2}$$

where η_i, ψ_i, θ_i and ζ_i, $i = 1,2$ are duly chosen characterizing scalars. The values of θ_i and ζ_i can be obtained from the respective classical estimators. The suggested estimators

$t_{b_i},\ i=1,2$ deforms into:

(i). Usual mean estimator t_m, for (η_1,θ_1,ψ_1)=(1,0,0)

(ii). Samawi and Muttlak [1] estimator t_r, for (η_2,θ_2,ψ_2)=(1,1,0)

(iii). Kadilar et al. [3] estimator t_{kc}, for (η_1,θ_1,ψ_1)=(k,1,0)

(iv). Mehta and Mandowara [8] estimator t_{mm_1}, for $(\eta_2,\theta_2,\psi_2,a,b)$=(1,1,0,1,$C_x$)

(v). Mehta and Mandowara [8] estimator t_{mm_2}, for $(\eta_2,\theta_2,\psi_2,a,b)$=(1,1,0,1,$\beta_2(x)$)

(vi). Mehta and Mandowara [8] estimator t_{mm_3}, for $(\eta_2,\theta_2,\psi_2,a,b)$=(1,1,0,$C_x$,$\beta_2(x)$)

(vii). Mehta and Mandowara [8] estimator t_{mm_2}, for $(\eta_2,\theta_2,\psi_2,a,b)$=(1,-1,0,$C_x$,$\beta_2(x)$)

(viii). Bhushan and Kumar [11] estimator t_2, for (η_1,θ_1,ψ_1)=$(\alpha_2,\beta_2,0)$

(ix). Bhushan and Kumar [11] estimator t_3, for (η_1,θ_1,ψ_1)=$(0,\alpha_3,\beta_3)$

(x). Bhushan and Kumar [11] estimator t_5, for (η_2,θ_2,ψ_2)=$(\alpha_5,\beta_5,0)$

(xi). Bhushan and Kumar [11] estimator t_6, for (η_2,θ_2,ψ_2)=$(0,\alpha_6,\beta_6)$

Theorem 3.1. *The MSE and minimum MSE of the class of proposed estimators $t_{b_i},\ i=1,2$ are expressed by*

$$MSE(t_{b_i}) = \mu_y^2(1+\eta_i^2 A_i + \psi_i^2 B_i + 2\eta_i\psi_i C_i - 2\eta_i D_i - 2\psi_i E_i) \tag{3.3}$$
$$minMSE(t_{b_i}) = \mu_y^2(1-\Lambda) \tag{3.4}$$

where $\Lambda = (A_iE_i^2 + B_iD_i^2 - 2C_iD_iE_i)/(A_iB_i - C_i^2)$.

Proof. The precis of the derivations and the definition of parametric functions A_i, B_i, C_i, D_i and $E_i,\ i=1,2$ are given in Appendix B. □

Corollary 3.1. *The acquainted estimators $t_{b_i},\ i=1,2$ have always smaller MSE than the classical regression estimator t_{lr}. i.e.*

$$MSE(t_{lr}) > MSE(t_{b_i}),\ \forall\, i=1,2 \tag{3.5}$$

4 Optimality conditions

To establish the optimality conditions, we compare the MSE of the acquainted class of estimators with the MSE of the existing estimators.

(i). From Equation (3.4) and Equation (A.1), we get

$$MSE(t_{b_i}) < MSE(t_m) \implies \Lambda > 1-\Delta_{0,2} \tag{4.1}$$

(ii). From Equation (3.4) and Equation (A.2), we get

$$MSE(t_{b_i}) < MSE(t_r) \implies \Lambda > 1 - \Delta_{0,2} - \Delta_{2,0} + 2\Delta_{1,1} \tag{4.2}$$

(iii). From Equation (3.4) and Equation (A.3), Equation (A.5), Equation (A.12), we get

$$MSE(t_{b_i}) < MSE(t) \implies \Lambda > 1 - \Delta_{0,2} + \frac{\Delta_{1,1}^2}{\Delta_{2,0}} \tag{4.3}$$

where $t = t_{lr},\ t_s,\ t_v$

(iv). From Equation (3.4) and Equation (A.4), we get

$$MSE(t_{b_i}) < MSE(t_{kc}) \implies \Lambda > \begin{bmatrix} 1 - (k^* - 1)^2 - \Delta_{2,0} \\ -k^{*2}\Delta_{0,2} + 2k^*\Delta_{1,1} \end{bmatrix} \tag{4.4}$$

(v). From Equation (3.4) and Equation (A.6), we get

$$MSE(t_{b_i}) < MSE(t_s) \implies \Lambda > \frac{\left(B_s - 2C_sD_s + A_sD_s^2 \right)}{\left(A_sB_s - C_s^2\right)} \tag{4.5}$$

(vi). From Equation (3.4) and Equation (A.7), we get

$$MSE(t_{b_i}) < MSE(t_{ss_i}) \implies \Lambda > \frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \tag{4.6}$$

(vii). From Equation (3.4) and Equation (A.8), we get

$$MSE(t_{b_i}) < MSE(t_{mm_j}) \implies \Lambda > 1 - \Delta_{0,2} - \delta_j^2\Delta_{2,0} + 2\delta_j\Delta_{1,1} \tag{4.7}$$

(viii). From Equation (3.4) and Equation (A.9), we get

$$MSE(t_{b_i}) < MSE(t_{mm_4}) \implies \Lambda > 1 - \Delta_{0,2} - \delta_4^2\Delta_{2,0} - 2\delta_4\Delta_{1,1} \tag{4.8}$$

(ix). From Equation (3.4) and Equation (A.10), we get

$$MSE(t_{b_i}) < MSE(t_{mm_5}) \implies \Lambda > \begin{Bmatrix} 1 - \Delta_{0,2} - (1 - 2\phi_0)^2 t_3^2 \Delta_{2,0} \\ -2(1 - 2\phi_0)t_3\Delta_{1,1} \end{Bmatrix} \tag{4.9}$$

(x). From Equation (3.4) and Equation (A.11), we get

$$MSE(t_{b_i}) < MSE(t_{sk_j}) \implies \Lambda > 1 - \Delta_{0,2} - I_j^2\Delta_{2,0} + 2I_j\Delta_{1,1} \tag{4.10}$$

(xi). From Equation (A.13), Equation (A.14) and Equation (3.4), we get

$$MSE(t_{b_i}) < MSE(t_j) \implies \Lambda > \frac{Q_j^2}{P_j} \tag{4.11}$$

It is worth mentioning that only under the above conditions, we can ensure the optimality of the proposed class of estimators $t_{b_i},\ i=1,2$. Further, to increase the credibility of aforesaid conditions, a simulation study and a real data application are conducted over different populations.

5 Simulation study

To have a clear cut perception about the properties of the proposed methodology, following Singh and Horn [35], a simulation study is performed over hypothetically generated one family of symmetric population such as `Normal` and one family of asymmetric population such as `Log-Normal` with variables X and Y which are obtained using the following models.

$$Y = 7.8+\sqrt{(1-\rho^2)}Y^*+\rho\left(\frac{S_y}{S_x}\right)X^* \tag{5.1}$$

$$X = 7.2+X^* \tag{5.2}$$

where X^* and Y^* denote the independent variates of the respective distributions used to construct different populations. The description of the population is given below.

1. A Normal population of size $N=1000$ is drawn using Equation (5.1) and Equation (5.2) such that $X^*\sim N(10,40)$ and $Y^*\sim N(20,50)$.

2. A Normal population of size $N=170$ is drawn using Equation (5.1) and Equation (5.2) such that $X^*\sim N(5,20)$ and $Y^*\sim N(7,25)$.

3. A Log-normal population of size $N=1000$ is drawn using Equation (5.1) and Equation (5.2) such that $X^*\sim LN(1.5,2)$ and $Y^*\sim LN(2.5,3)$.

4. A Log-normal population of size $N=170$ is drawn using Equation (5.1) and Equation (5.2) such that $X^*\sim LN(0.5,5)$ and $Y^*\sim LN(0.7,7)$.

We have taken various amounts of correlation coefficients ρ=0.65,0.75,0.85,0.95 in each population to observe the behaviour of the proposed estimators. We have drawn ranked set samples using RSS procedure discussed in Section 1. We have taken set sizes m=3,4 and 5 and the numbers of cycles r=4, 6 such that $n=mr$=12,16,20,18,24 and 30 units from populations of large sizes whereas set sizes m=3,4 and 5 and numbers of cycles r=3,4 such that $n=mr$=9,12,15,12,16 and 20 units from populations of small sizes. We have considered 15000 iterations and obtained the relative efficiency (RE) of the proposed class of estimators regarding the usual mean estimator by utilizing the following expressions.

$$MSE(T)=\frac{1}{15000}\sum_{i=1}^{15000}(T-\mu_y)^2 \tag{5.3}$$

$$RE=\frac{MSE(t_m)}{MSE(T)} \tag{5.4}$$

where T=t_m, t_r, t_{lr}, t_{kc}, t_s, t_{ss_i} $i=1,2,3$, t_{mm_i} $i=1,2...,5$, t_{sk_i} $i=1,3$, t_v, t_i, $i=1,2,...,6$ and t_{b_i} $i=1,2$. The outcomes of the simulation study for each population are disclosed from Table 1 to Table 4 by RE for various amounts of correlation coefficient ρ.

[Tables 1-4 Here]

6 Real data application

To obtain a comprehensive appreciation about the properties of the acquainted estimators, a real data application is provided using three real data sets. The description of the data sets is given below.

Population 1: Origin: (Sarndal et al. [36], pp. 652-659), Y=total number of seats (S82) in municipal council in 1982, X=number of conservative seats (CS82) in municipal council in 1982, N=284, μ_y=46.0704, μ_x=9.095, S_y=12.5918, S_x=4.9364 and ρ=0.6878.

Population 2: The data is considered from Kadilar and Cingi [37]. (Origin: Institute of Statistics, Republic of Turkey), Y=amount of apple yield in East Anatolia region, X=quantity of apple trees in East Anatolia region, N=104, μ_y=625.36, μ_x=14274.95, S_y=1167.00, S_x=23016.15 and ρ=0.8852.

Population 3: Origin: (Singh [38], pp. 1115), Y=season average price per pound during 1996, X=season average price per pound during 1994, N=36, $\bar{Y}$=0.2032, $\bar{X}$=0.1708, S_y=0.0803, S_x=0.0633 and ρ=0.8577.

From each data set discussed above, we have taken 12 ranked set samples, each having set size $m=3$ and number of cycles $r=4$, provided $n=mr=12$. For the chosen ranked set samples, we have tabulated the RE of different estimators regarding the mean per unit estimator by utilizing Equation (5.4). The outcomes are displayed in Table 5 which reveal the superiority of the acquainted estimators in comparison to the existing estimators discussed in the earlier section.

[Table 5 Here]

7 Conclusion

This paper started with a quest for optimal estimators of population mean under RSS which covers the usual mean estimator, Samawi and Muttlak [1] estimator, Yu and Lam [2] estimator, Kadilar et al. [3] estimator, Mehta and Mandowara [8] estimator and Bhushan and Kumar [11] estimator for suitably chosen values of scalars and provide an improvement over the classical regression (BLU) estimator under RSS. From Corollary 3.1, it is clear that the acquainted estimators challenge the traditional optimality and repress it with a new class of estimators which utilize the same amount of information and provide more accurate results. In order to have a clear appreciation about the properties of the proposed methodologies, a simulation study is conducted over artificially generated symmetric and asymmetric populations for different values of set size m, number of cycles r and correlation coefficient ρ. A real data application using three populations is also added for illustration. The results of the simulation study and real data application are summarized from Table 1 to Table 5. On the basis of the results of theoretical study, simulation study and real data applications, the important findings are reported below in a pointwise fashion.

(i). From the results of the simulation study reported in Tables 1-4, the proposed class of estimators $t_{b_i},\ i=1,2$ perform better than the traditional mean estimator t_m, classical ratio and regression estimators t_r and t_{lr}, Kadilar et al. [3] estimator t_{kc}, Singh and Solanki (2013a, b) type estimators t_{ss_1}, t_{ss_2}, Singh et al. [6] estimator t_s, Solanki and Singh [7] type estimator t_{ss_3}, Mehta and Mandowara [8] estimators $t_{mm_i},\ i=1,2,...,5$, Saini and Kumar [9] estimators $t_{sk_i},\ i=1,3$, Mehta et al. [10] estimator t_v and Bhushan and Kumar [11] estimators $t_i,\ i=1,2,...,6$. The RE of the proposed estimators gradually increases as the value of correlation coefficient increases. Moreover, RE also increases with the increase of the set size m.

(ii). From the results of the three real data sets summarized in Table 5, a similar conclusions can be made.

(iii). Since, the proposed class of estimators $t_{b_i},\ i=1,2$ become superior than Singh et al. [6] estimator t_s in both artificial and real populations, so these will also become superior than those 184 estimators which are the members of Singh et al. [6] estimator t_s.

Further, by following Samawi and Muttlak [1], the whole study can be iterated when the ranking is performed over the study variable Y.

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References

[1] Samawi, H.M. and Muttlak, H.A. "Estimation of ratio using ranked set sampling", *Biom. J.*, **38**, pp. 753-764 (1996).

[2] Yu, L.H. and Lam, K. "Regression estimator in ranked set sampling", *Biometrics*, **53**, pp. 1070-1080 (1997).

[3] Kadilar, C. Unyazici, Y. and Cingi, H. "Ratio estimator for the population mean using ranked set sampling", *Stat Pap*, **50**, pp. 301-309 (2009).

[4] Singh, H.P. and Solanki, R.S. “Efficient ratio and product estimators in stratified random sampling”, *Commun. Stat. Theory Methods*, **42(6)**, pp. 1008-1023 (2013).

[5] Singh, H.P. and Solanki, R.S. “An efficient class of estimators for the population mean using auxiliary information”, *Commun. Stat. Theory Methods*, **42(1)**, pp. 145-163 (2013).

[6] Singh, H.P., Tailor, R. and Singh, S. “General procedure for estimating the population mean using ranked set sampling”, *J. Stat. Comput. Simul.*, **84(5)**, pp. 931-945 (2014).

[7] Solanki, R.S. and Singh, H.P. “An improved estimation in stratified random sampling”, *Commun. Stat. Theory Methods*, **45(7)**, pp. 2056-2070 (2016).

[8] Mehta, N. and Mandowara, V.L. “A modified ratio-cum-product estimator of finite population mean using ranked set sampling”, *Commun. Statis. Theory Methods*, **45(2)**, pp. 267-276 (2016).

[9] Saini, M. and Kumar, A. “Ratio estimators for the finite population mean under simple random sampling and ranked set sampling”, *Int. J. Syst. Assur. Eng. Manag.*, **8(2)**, pp. 488-492 (2016).

[10] Mehta, V., Singh, H.P. and Pal, S.K. “A general procedure for estimating finite population mean using ranked set sampling”, *Rev. Inv. Ope.*, **41(1)**, pp. 80-92 (2020).

[11] Bhushan, S. and Kumar, A. “On optimal classes of estimators under ranked set sampling”, *Commun. Statis. Theory Methods*, **51(8)**, pp. 2610-2639 (2022).

[12] McIntyre, G.A. “A method of unbiased selective sampling using ranked set”, *Aust. J. Agric. Res.*, **3**, pp. 385-390 (1952).

[13] Takahasi, K. and Wakimoto, K. “On unbiased estimates of the population mean based on the sample stratified by means of ordering”, *Ann. Inst. Stat. Math.*, **20**, pp. 1-31 (1968).

[14] Dell, T.R. and Clutter, J.L. “Ranked set sampling theory with order statistics background”, *Biometrics*, **28**, pp. 545-555 (1972).

[15] Muttlak, H.A. “Parameter estimation in simple linear regression using ranked set sampling”, *Biom. J.*, **37(7)**, pp. 799-810 (1995).

[16] Khan, L. and Shabbir, J. “An efficient class of estimators of finite population mean under ranked set sampling”, *Open J. Stat.*, **6**, pp. 426-435 (2016).

[17] Zamanzade, E. and Mahdizadeh, M. “Estimating the population proportion in pair ranked set sampling with application to air quality monitoring”, *J. Appl. Stat.*, **45**, pp. 426-437 (2018).

[18] Zamanzade, E. and Mahdizadeh, M. “Using ranked set sampling with extreme ranks in estimating the population proportion”, *Stat. Methods Med. Res.*, **29**, pp. 165-177 (2020).

[19] Bhushan, S. and Kumar, A. "Log type estimators of population mean under ranked set sampling", *Predic. Analy. Stat. and Big Data: Conc. Model.*, **28**, pp. 47-74 (2020). DOI:10.2174/9789811490491120010007

[20] Shahzad, U., Hanif, M., Sajjad, I. and Anas, M.M. "Quantile regression-ratio-type estimators for mean estimation under complete and partial auxiliary information", *Sci. Iran.*, **In Press**, (2020).

[21] Shahzad, U., Ahmad, I., Oral, E., Hanif, M. and Almanjahie, I.M. "Estimation of the population mean by successive use of an auxiliary variable in median ranked set sampling", *Math. Popul. Stud.*, **28(3)**, pp. 176-199 (2021).

[22] Bhushan, S. and Kumar, A. "Novel log type class of estimators under ranked set sampling", *Sankhya, B*, **84**, pp. 421-447 (2022).

[23] Mahdizadeh, M. and Zamanzade, E. "Smooth estimation of a reliability function in ranked set sampling", *Stats*, **52(4)**, pp. 750-768 (2018).

[24] Bhushan, S., Kumar, A. and Lone, S.A. "On some novel classes of estimators under ranked set sampling", *Alex. Eng. J.*, **61**, pp. 5465-5474 (2021).

[25] Bhushan, S. and Kumar, A. "Predictive estimation approach using difference and ratio type estimators in ranked set sampling", *J. Comput. Appl. Math.*, **410**, 1-18 (2022).

[26] Bhushan, S. and Kumar, A. "An efficient class of estimators based on ranked set sampling", *Life Cycle Relia. Saf. Eng.*, **11**, pp. 39-48 (2022).

[27] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. "A first course in order statistics", Wiley, New York (1993).

[28] Prasad, B. "Some improved ratio type estimators of population mean and ratio in finite population sample surveys", *Commun. Stat. Theory Methods*, **18**, pp. 379-392 (1989).

[29] Upadhyaya, L.N. Singh, H.P. and Vos, J.W.E. "On the estimations of population means and ratios using supplementary information", *Stat. Neerl.*, **39(3)**, pp. 309-318 (1985).

[30] Khoshnevisan, M. Singh, R. Chauhan, P. Sawan, N. and Smarandache, F. "A general family of estimators for estimating population mean using known value of some population parameter(s)", *Far East J. Theoret. Stat.*, **22**, pp. 181-191 (2007).

[31] Koyuncu, N. and Kadilar, C. "Ratio and product estimator in stratified random sampling", *J. Statist. Plann. Inference*, **139**, pp. 2552-2558 (2009).

[32] Singh, H.P. and Agnihotri, N. "A general procedure of estimating population mean using auxiliary information in sample surveys", *Stat. Transit.-new series*, **9**, pp. 71-87 (2008).

[33] Srivastava, S.K. "An estimator using auxiliary information", *Calcutta Statist. Assoc. Bull.*, **16**, pp. 121-132 (1967).

[34] Walsh, J.E. “Generalization of ratio estimator for population total. *Sankhya, A*, **32**, pp. 99-106 (1970).

[35] Singh, H.P. and Horn, S. “An alternative estimator for multi-character surveys”, *Metrika*, **48**, pp. 99-107 (1998).

[36] Sarndal, C.E., Swensson, B. and Wretman, J. “Model Assisted Survey sampling”, Springer-Verlag, Newyork (2003).

[37] Kadilar, C. and Cingi, H. “Ratio estimators in stratified random sampling”, *Biom. J.*, **45**, pp. 218-225 (2003).

[38] Singh, S. “Advanced sampling theory with applications: How Michael selected Amy”, In The *Netherlands: Kluwer Academic Publishers*, **1&2**, (2003).

Appendix A

The MSE expressions and the optimum values of the scalars of existing estimators are given below.

$$MSE(t_m) = \mu_y^2 \Delta_{0,2} \tag{A.1}$$

$$MSE(t_r) = \mu_y^2 (\Delta_{0,2} + \Delta_{2,0} - 2\Delta_{1,1}) \tag{A.2}$$

$$minMSE(t_{lr}) = \mu_y^2 \left(\Delta_{0,2} - \frac{\Delta_{1,1}^2}{\Delta_{2,0}} \right) \tag{A.3}$$

$$minMSE(t_{kc}) = \mu_y^2 \left\{ (k^* - 1)^2 + \Delta_{2,0} + k^{*2}\Delta_{0,2} - 2k^*\Delta_{1,1} \right\} \tag{A.4}$$

$$minMSE(t_s)_I = \mu_y^2 \left(\Delta_{0,2} - \frac{\Delta_{1,1}^2}{\Delta_{2,0}} \right), \text{ when } \lambda_1 + \lambda_2 = 1 \tag{A.5}$$

$$minMSE(t_s)_{II} = \mu_y^2 \left\{ 1 - \frac{\left(B_s - 2C_sD_s + A_sD_s^2 \right)}{\left(A_sB_s - C_s^2 \right)} \right\} \text{ when } \lambda_1 + \lambda_2 \neq 1 \tag{A.6}$$

$$minMSE(t_{ss_i}) = \mu_y^2 \left\{ 1 - \frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \right\}, \;\; i = 1,2,3 \tag{A.7}$$

$$MSE(t_{mm_i}) = \mu_y^2 \left(\Delta_{0,2} + \delta_i^2\Delta_{2,0} - 2\delta_i\Delta_{1,1} \right), \; i = 1,2,3 \tag{A.8}$$

$$MSE(t_{mm_4}) = \mu_y^2 \left(\Delta_{0,2} + \delta_4^2\Delta_{2,0} + 2\delta_4\Delta_{1,1} \right) \tag{A.9}$$

$$minMSE(t_{mm_5}) = \mu_y^2 \left\{ \Delta_{0,2} + (1 - 2\phi_0)^2 t_3^2 \Delta_{2,0} + 2(1 - 2\phi_0) t_3 \Delta_{1,1} \right\} \tag{A.10}$$

$$MSE(t_{sk_i}) = \mu_y^2 \left(\Delta_{0,2} + I_i^2\Delta_{2,0} - 2I_i\Delta_{1,1} \right), \; i = 1,3 \tag{A.11}$$

$$minMSE(t_v) = \mu_y^2 \left(\Delta_{0,2} - \frac{\Delta_{1,1}^2}{\Delta_{2,0}} \right) \tag{A.12}$$

$$minMSE(t_i) = \mu_y^2 \left(1 - \alpha_{1i(opt)} \right) = \mu_y^2 \left(1 - \frac{Q_i^2}{P_i} \right), \; i = 1,4 \tag{A.13}$$

$$minMSE(t_i) = \mu_y^2 \left(1 - \frac{Q_i^2}{P_i} \right), \; i = 2,3,5,6 \tag{A.14}$$

The optimum values of scalars are given below for ready reference.

$$\beta_{(opt)} = \frac{R\Delta_{1,1}}{\Delta_{0,2}} \tag{A.15}$$

$$k_{(opt)} = \frac{\left(1 + \Delta_{1,1} \right)}{\left(1 + \Delta_{2,0} \right)} = k^* (\text{say}) \tag{A.16}$$

$$\lambda_{1_{(opt)}} = 1 - \frac{\Delta_{1,1}}{g\theta\alpha\Delta_{0,2}}, \text{ when } \lambda_1 + \lambda_2 = 1 \tag{A.17}$$

$$\lambda_{1_{(opt)}} = \frac{(B_s - C_sD_s)}{(A_sB_s - C_s^2)}, \text{ when } \lambda_1 + \lambda_2 \neq 1 \tag{A.18}$$

$$\lambda_{2_{(opt)}} = \frac{(A_sD_s - C_s)}{(A_sB_s - C_s^2)} \tag{A.19}$$

$$\theta_{s_i(opt)} = \frac{(G_iI_i - H_iJ_i)}{(F_iG_i - H_i^2)},\ i = 1,2,3 \tag{A.20}$$

$$\Theta_{s_i(opt)} = \frac{(F_iJ_i - H_iI_i)}{(F_iG_i - H_i^2)},\ i = 1,2,3 \tag{A.21}$$

$$\phi_{(opt)} = \frac{(t_3 + k)}{2t_3} = \phi_0(\text{say}) \tag{A.22}$$

$$\delta_{(opt)} = -\frac{\Delta_{1,1}}{\Delta_{0,2}} \tag{A.23}$$

$$\alpha_{i(opt)} = \frac{Q_i}{P_i},\ i = 1,2,...,6 \tag{A.24}$$

$$\beta_{1(opt)} = -\frac{\mu_y}{\mu_x}\frac{\Delta_{1,1}}{\Delta_{2,0}}\alpha_{1(opt)} \tag{A.25}$$

$$\beta_{i(opt)} = \frac{\Delta_{1,1}}{\Delta_{2,0}},\ i = 2,3 \tag{A.26}$$

$$\beta_{4(opt)} = -\frac{\mu_y}{\mu_x}\frac{\Delta_{1,1}}{\nu\Delta_{2,0}}\alpha_{1(opt)} \tag{A.27}$$

$$\beta_{i(opt)} = \frac{\Delta_{1,1}}{\nu\Delta_{2,0}},\ i = 5,6 \tag{A.28}$$

where

$$A_s = 1 + \Delta_{0,2}$$
$$B_s = 1 + \Delta_{0,2} + g(2g+1)\theta^2\alpha^2\Delta_{2,0} - 4g\theta\alpha\Delta_{1,1}$$
$$C_s = 1 + \Delta_{0,2} - 2g\theta\alpha\Delta_{1,1} + \frac{g(g+1)}{2}\theta^2\alpha^2\Delta_{2,0}$$
$$D_s = 1 + \frac{g(g+1)}{2}\theta^2\alpha^2\Delta_{2,0} - g\theta\alpha\Delta_{1,1}$$
$$F_1 = 1 + \Delta_{0,2} + 4\alpha\delta\nu\Delta_{1,1} + \delta(2\delta-1)\alpha^2\nu^2\Delta_{2,0}$$
$$G_1 = 1 + \Delta_{0,2} - 4\alpha g\nu\Delta_{1,1} + g(2g+1)\alpha^2\nu^2\Delta_{2,0}$$
$$H_1 = 1 + \Delta_{0,2} + 2\alpha(\delta-g)\nu\Delta_{1,1} + \left(\frac{\alpha^2\nu^2}{2}\right)(\delta-g)(\delta-g-1)\Delta_{2,0}$$
$$I_1 = 1 + \alpha\delta\nu\Delta_{1,1} + \frac{\delta(\delta-1)}{2}\alpha^2\nu^2\Delta_{2,0}$$
$$J_1 = 1 - \alpha g\nu\Delta_{1,1} + \frac{g(g+1)}{2}\alpha^2\nu^2\Delta_{2,0}$$
$$F_2 = 1 + \Delta_{0,2} + \alpha^2\nu^2(2g^2+g)\Delta_{2,0} - 4\alpha\nu g\Delta_{1,1}$$
$$G_2 = 1 + \Delta_{0,2} + \frac{\nu^2(\delta^2+\delta)}{2}\Delta_{2,0} - 2\alpha\nu\Delta_{1,1}$$
$$H_2 = 1 + \Delta_{0,2} + \left(\frac{F_2\nu^2}{8}\right) - \nu(2\alpha g+\delta)\Delta_{1,1}$$
$$I_2 = 1 + \frac{\alpha^2\nu^2(g^2+g)}{2}\Delta_{2,0} - \alpha\nu g\Delta_{1,1}$$

$$J_2 = 1 + \left\{ \frac{(\delta^2 + 2\delta)}{8} \nu^2 \right\} \Delta_{2,0} - \left(\frac{\delta \nu}{2} \right) \Delta_{1,1}$$

$$F_3 = 1 + \Delta_{0,2} - 2\Theta_1 a \Delta_{1,1} + \frac{\Theta_1(\Theta_1 + 1)}{2} a^2 \Delta_{2,0}$$

$$G_3 = 1 + \Delta_{0,2} + 2\Theta_2 a \Delta_{1,1} + \frac{\Theta_2(\Theta_2 - 1)}{2} a^2 \Delta_{2,0}$$

$$H_3 = 1 + \Delta_{0,2} + (\Theta_2 - \Theta_1) a \Delta_{1,1} + \frac{(\Theta_2 - \Theta_1)(\Theta_2 - \Theta_1 - 2)}{8} a^2 \Delta_{2,0}$$

$$I_3 = 1 - \frac{\Theta_1}{2} \left\{ a \Delta_{1,1} - \frac{(\Theta_1 + 2)}{4} a^2 \Delta_{2,0} \right\}$$

$$J_3 = 1 - \frac{\Theta_2}{2} \left\{ a \Delta_{1,1} + \frac{(\Theta_2 - 2)}{4} a^2 \Delta_{2,0} \right\}$$

$$\Theta_1 = 2\alpha + \beta$$

$$\Theta_2 = 2\delta + \lambda$$

$$I_i = \frac{(\mu_x + q_i)}{(2\mu_x + q_i)^2}, \; i = 1,3$$

$$\delta_1 = \frac{\mu_x}{(\mu_x + C_x)}$$

$$\delta_2 = \frac{\mu_x}{(\mu_x + \beta_2(x))}$$

$$\delta_3 = \delta_4 = t_3 = \frac{\mu_x C_x}{(\mu_x C_x + \beta_2(x))}$$

$$P_i = 1 + \Delta_{0,2} - \frac{\Delta_{1,1}^2}{\Delta_{2,0}}, \; i = 1,3,4,6$$

$$Q_i = 1, \; i = 1,3,4,6$$

$$P_2 = 1 + \Delta_{0,2} + \Delta_{1,1} - \frac{2\Delta_{1,1}^2}{\Delta_{2,0}}$$

$$Q_2 = 1 + \frac{\Delta_{1,1}}{2} - \frac{\Delta_{1,1}^2}{2\Delta_{2,0}}$$

$$P_5 = 1 + \Delta_{0,2} + \nu \Delta_{1,1} - \frac{2\Delta_{1,1}^2}{\Delta_{2,0}}$$

$$Q_5 = 1 + \frac{\nu \Delta_{1,1}}{2} - \frac{\Delta_{1,1}^2}{2\Delta_{2,0}}$$

Appendix B

This section considers the precis of Theorem 3.1.
Using the notations defined in earlier section, we obtain the MSE of the estimators $t_{b_i}, \; i = 1,2$

as

$$MSE(t_{b_i}) = \mu_y^2(1+\eta_i^2 A_i + \psi_i^2 B_i + 2\eta_i\psi_i C_i - 2\eta_i D_i - 2\psi_i E_i) \tag{B.29}$$

By minimizing Equation (B.29) w.r.t. η_i and ψ_i, we get

$$\eta_{i(opt)} = \frac{(B_i D_i - C_i E_i)}{(A_i B_i - C_i^2)}$$
$$\psi_{i(opt)} = \frac{(A_i E_i - C_i D_i)}{(A_i B_i - C_i^2)}$$

Putting $\eta_{i(opt)}$ and $\psi_{i(opt)}$ in Equation (B.29), we obtain the minimum MSE to the first order of approximation as

$$minMSE(t_{b_i}) = \mu_y^2 \left\{ 1 - \frac{(A_i E_i^2 + B_i D_i^2 - 2C_i D_i E_i)}{(A_i B_i - C_i^2)} \right\}$$

where

$$A_1 = 1 + \Delta_{0,2} + (2\theta_1^2 + \theta_1)\Delta_{2,0} - 4\theta_1\Delta_{1,1}$$
$$B_1 = 1 + \Delta_{0,2} + 3\zeta_1^2\Delta_{2,0} - 4\zeta_1\Delta_{1,1}$$
$$C_1 = 1 + \Delta_{0,2} + \left\{ \zeta_1^2 + \theta_1\zeta_1 + \frac{\theta_1(\theta_1+1)}{2} \right\}\Delta_{2,0} - 2(\theta_1 - \zeta_1)\Delta_{1,1}$$
$$D_1 = 1 - \theta_1\Delta_{1,1} + \frac{\theta_1(\theta_1+1)}{2}\Delta_{2,0}$$
$$E_1 = 1 - \zeta_1\Delta_{1,1} + \zeta_1^2\Delta_{2,0}$$
$$A_2 = 1 + \Delta_{0,2} + (2\theta_2^2\nu^2 + \theta_2\nu^2)\Delta_{2,0} - 4\theta_2\nu\Delta_{1,1}$$
$$B_2 = 1 + \Delta_{0,2} + 3\zeta_2^2\nu^2\Delta_{2,0} - 4\zeta_2\nu\Delta_{1,1}$$
$$C_2 = 1 + \Delta_{0,2} + \left\{ \zeta_2^2\nu^2 + \theta_2\zeta_2\nu^2 + \frac{\theta_2(\theta_2+1)}{2}\nu^2 \right\}\Delta_{2,0} - 2\nu(\theta_2 - \zeta_2)\Delta_{1,1}$$
$$D_2 = 1 - \theta_2\nu\Delta_{1,1} + \frac{\theta_2(\theta_2+1)}{2}\nu^2\Delta_{2,0}$$
$$E_2 = 1 - \zeta_2\nu\Delta_{1,1} + \zeta_2^2\nu^2\Delta_{2,0}$$
$$\nu = \frac{a\mu_x}{(a\mu_x + b)}$$

Table 1: RE of different estimators with respect to usual mean estimator for Normal population of size $N = 1000$

m	r	t_r	t_{lr}	t_{kc}	t_{ss_1}	t_s	t_{ss_2}	t_{ss_3}	t_{mm_1}	t_{mm_2}	t_{mm_3}	t_{mm_4}	t_{mm_5}	t_{sk_1}	t_{sk_3}	t_i, $i=1,3,4,6$	t_i $i=2,5$	$\mathbf{t_{b_i}}$ $\mathbf{i=1,2}$
										$\rho = 0.65$								
3	4	0.266	1.000	0.278	1.152	1.152	1.095	1.108	0.302	0.307	0.288	0.279	0.969	0.288	0.388	1.099	1.090	1.167
	6	0.383	1.028	0.414	1.221	1.221	1.266	1.224	0.439	0.425	0.401	0.734	1.019	0.633	0.699	1.264	1.243	1.297
4	4	0.356	1.019	0.361	1.154	1.154	1.204	1.146	0.403	0.411	0.381	0.700	0.990	0.439	0.608	1.227	1.177	1.231
	6	0.790	1.220	0.987	1.348	1.348	1.465	1.385	0.807	0.811	0.805	0.987	1.219	0.945	0.962	1.604	1.420	1.636
5	4	0.466	1.094	0.585	1.327	1.327	1.336	1.232	0.492	0.520	0.498	0.811	1.093	0.646	0.703	1.336	1.334	1.367
	6	0.827	1.278	1.288	1.585	1.585	1.552	1.586	0.840	0.851	0.843	1.226	1.274	0.968	0.975	1.669	1.627	1.684
										$\rho = 0.75$								
3	4	0.246	1.001	0.256	1.176	1.176	1.108	1.154	0.304	0.310	0.297	0.288	0.971	0.292	0.392	1.160	1.159	1.190
	6	0.442	1.045	0.497	1.265	1.265	1.302	1.220	0.463	0.471	0.461	0.776	1.024	0.649	0.708	1.303	1.280	1.335
4	4	0.361	1.020	0.365	1.197	1.197	1.255	1.210	0.448	0.434	0.388	0.718	1.001	0.447	0.611	1.289	1.224	1.296
	6	0.810	1.265	0.998	1.366	1.366	1.493	1.423	0.827	0.831	0.825	0.991	1.264	0.958	0.976	1.655	1.436	1.691
5	4	0.458	1.103	0.578	1.340	1.340	1.320	1.278	0.494	0.512	0.508	0.857	1.104	0.653	0.710	1.341	1.338	1.372
	6	0.851	1.323	1.463	1.827	1.827	1.813	1.827	0.851	0.866	0.871	1.268	1.323	0.977	0.988	1.848	1.795	1.869
										$\rho = 0.85$								
3	4	0.310	1.008	0.327	1.113	1.113	1.129	1.170	0.349	0.355	0.333	0.301	0.988	0.375	0.340	1.138	1.165	1.185
	6	0.452	1.078	0.575	1.304	1.304	1.316	1.239	0.477	0.507	0.483	0.786	1.028	0.656	0.721	1.350	1.319	1.380
4	4	0.382	1.028	0.385	1.170	1.170	1.221	1.227	0.475	0.460	0.410	0.729	1.020	0.484	0.619	1.290	1.257	1.309
	6	0.842	1.155	1.030	1.396	1.396	1.530	1.471	0.859	0.864	0.858	1.012	1.271	0.963	0.984	1.720	1.492	1.731
5	4	0.536	1.108	0.656	1.337	1.337	1.358	1.281	0.556	0.563	0.553	0.993	1.116	0.830	0.724	1.413	1.342	1.428
	6	0.872	1.383	1.769	2.192	2.192	2.186	2.073	0.887	0.894	0.889	1.361	1.383	0.983	0.996	2.164	2.099	2.204
										$\rho = 0.95$								
3	4	0.428	1.021	0.449	1.155	1.155	1.165	1.187	0.472	0.479	0.455	0.356	1.019	0.589	0.683	1.183	1.173	1.213
	6	0.461	1.096	0.587	1.356	1.356	1.336	1.246	0.549	0.532	0.490	0.797	1.069	0.642	0.808	1.370	1.350	1.406
4	4	0.444	1.039	0.467	1.187	1.187	1.237	1.241	0.481	0.499	0.478	0.738	1.062	0.604	0.705	1.216	1.180	1.310
	6	0.905	1.183	1.180	1.974	1.974	1.820	1.720	0.920	0.924	0.919	1.096	1.307	0.992	1.016	1.921	1.742	2.002
5	4	0.705	1.119	1.113	1.471	1.471	1.586	1.543	0.721	0.726	0.719	1.026	1.135	0.938	0.947	1.817	1.555	1.866
	6	0.915	1.469	2.599	3.013	3.013	3.050	2.936	0.921	0.927	0.923	1.462	1.469	1.020	1.037	3.049	2.952	3.154

Table 2: RE of different estimators with respect to usual mean estimator for Normal population of size $N = 170$

m	r	t_r	t_{lr}	t_{kc}	t_{ss_1}	t_s	t_{ss_2}	t_{ss_3}	t_{mm_1}	t_{mm_2}	t_{mm_3}	t_{mm_4}	t_{mm_5}	t_{sk_1}	t_{sk_3}	t_i, $i=1,3,4,6$	t_i $i=2,5$	$\mathbf{t_{b_i}}$ $\mathbf{i=1,2}$
										$\rho = 0.65$								
3	3	0.312	1.013	0.314	1.049	1.049	1.037	1.049	0.374	0.439	0.220	0.229	1.077	0.245	0.389	1.038	1.042	1.068
	4	0.355	1.125	0.355	1.272	1.272	1.263	1.110	0.815	0.472	0.430	0.459	1.112	0.522	0.715	1.303	1.304	1.323
4	3	0.352	1.021	0.353	1.178	1.178	1.169	1.082	0.420	0.445	0.398	0.319	1.097	0.507	0.518	1.179	1.171	1.183
	4	1.026	1.153	1.245	1.534	1.534	1.437	1.162	1.071	1.119	1.113	0.948	1.136	1.145	1.138	1.743	1.740	1.758
5	3	0.745	1.145	1.116	1.320	1.320	1.317	1.158	1.056	0.805	0.772	0.614	1.124	1.021	1.018	1.363	1.329	1.375
	4	1.068	1.250	1.646	3.173	3.173	3.267	1.513	1.205	1.200	1.138	1.069	1.212	1.228	1.237	3.433	3.216	3.504
										$\rho = 0.75$								
3	3	0.319	1.029	0.335	1.067	1.067	1.041	1.054	0.390	0.446	0.231	0.272	1.060	0.301	0.401	1.056	1.061	1.090
	4	0.369	1.134	0.386	1.301	1.301	1.272	1.131	0.835	0.486	0.457	0.481	1.142	0.547	0.750	1.308	1.334	1.349
4	3	0.363	1.038	0.371	1.203	1.203	1.170	1.119	0.560	0.470	0.418	0.357	1.113	0.518	0.548	1.190	1.185	1.214
	4	1.040	1.174	1.260	1.601	1.601	1.547	1.193	1.089	1.185	1.135	0.976	1.160	1.173	1.177	1.791	1.770	1.817
5	3	0.752	1.161	1.132	1.377	1.377	1.353	1.180	1.067	0.841	0.817	0.663	1.156	1.053	1.041	1.406	1.360	1.437
	4	1.104	1.280	1.661	3.504	3.504	3.400	1.533	1.242	1.226	1.410	1.091	1.233	1.247	1.256	3.490	3.247	3.605
										$\rho = 0.85$								
3	3	0.342	1.051	0.347	1.090	1.090	1.074	1.087	0.427	0.471	0.259	0.299	1.107	0.336	0.424	1.068	1.079	1.126
	4	0.379	1.151	0.389	1.320	1.320	1.299	1.192	0.852	0.507	0.469	0.499	1.170	0.576	0.778	1.326	1.347	1.357
4	3	0.371	1.047	0.387	1.226	1.226	1.193	1.166	0.575	0.497	0.434	0.372	1.129	0.549	0.560	1.267	1.249	1.296
	4	1.100	1.195	1.272	2.903	2.903	2.880	2.099	1.137	1.200	1.160	1.020	1.188	1.193	1.192	2.949	2.884	3.160
5	3	0.766	1.183	0.850	1.391	1.391	1.380	1.240	1.070	0.887	0.850	0.720	1.178	1.083	1.092	1.495	1.496	1.574
	4	1.136	1.306	1.683	3.593	3.593	3.471	3.172	1.308	1.249	1.452	1.117	1.251	1.280	1.289	3.450	3.268	3.655
										$\rho = 0.95$								
3	3	0.361	1.096	0.369	1.109	1.109	1.100	1.099	0.453	0.509	0.379	0.322	1.138	0.424	0.576	1.091	1.088	1.153
	4	0.408	1.192	0.410	1.341	1.341	1.319	1.210	0.878	0.561	0.493	0.528	1.201	1.015	1.029	1.351	1.340	1.372
4	3	0.399	1.082	0.406	1.240	1.240	1.219	1.201	0.628	0.692	0.474	0.387	1.160	0.862	1.012	1.285	1.261	1.315
	4	1.147	1.260	1.152	2.987	2.987	2.892	2.119	1.155	1.236	1.187	1.057	1.226	1.217	1.220	2.995	2.903	3.335
5	3	0.910	1.226	0.918	1.596	1.596	1.424	1.284	1.087	0.913	0.881	0.753	1.210	1.109	1.113	1.613	1.600	1.825
	4	1.184	1.357	1.197	3.497	3.497	3.499	3.307	1.352	1.276	1.524	1.148	1.289	1.310	1.317	3.507	3.487	3.720

Table 3: RE of different estimators with respect to usual mean estimator for Log-Normal population of size $N = 1000$

m	r	t_r	t_{lr}	t_{kc}	t_{ss_1}	t_s	t_{ss_2}	t_{ss_3}	t_{mm_1}	t_{mm_2}	t_{mm_3}	t_{mm_4}	t_{mm_5}	t_{sk_1}	t_{sk_3}	t_i, $i=1,3,4,6$	t_i $i=2,5$	$\mathbf{t_{b_i}}$ $\mathbf{i=1,2}$
										$\rho = 0.65$								
3	4	0.232	1.007	0.245	1.096	1.096	1.108	1.047	0.245	0.282	0.251	0.319	1.005	0.903	0.910	1.110	1.106	1.126
	6	0.594	1.017	0.674	1.182	1.182	1.192	1.120	0.613	0.668	0.636	0.745	1.011	0.946	0.951	1.198	1.169	1.208
4	4	0.425	1.014	0.447	1.176	1.176	1.159	1.107	0.444	0.504	0.473	0.687	1.007	0.908	0.918	1.164	1.157	1.199
	6	0.876	1.025	1.067	1.261	1.261	1.272	1.261	0.882	0.903	0.911	0.821	1.030	0.983	0.989	1.263	1.266	1.285
5	4	0.600	1.018	0.752	1.206	1.206	1.200	1.204	0.720	0.713	0.656	0.809	1.023	0.955	0.968	1.206	1.197	1.219
	6	0.951	1.036	1.087	1.416	1.416	1.423	1.422	0.957	0.969	0.971	1.024	1.039	1.021	1.029	1.428	1.399	1.439
										$\rho = 0.75$								
3	4	0.237	1.009	0.251	1.099	1.099	1.119	1.050	0.251	0.290	0.259	0.325	0.010	0.912	0.919	1.125	1.110	1.130
	6	0.600	1.028	0.682	1.194	1.194	1.197	1.124	0.620	0.675	0.649	0.755	1.030	0.954	0.957	1.203	1.173	1.212
4	4	0.427	1.024	0.449	1.180	1.180	1.165	1.115	0.457	0.516	0.483	0.691	1.019	0.917	0.927	1.167	1.160	1.202
	6	0.885	1.039	1.070	1.272	1.272	1.270	1.221	0.893	0.914	0.920	0.829	1.048	0.988	0.993	1.279	1.271	1.288
5	4	0.609	1.025	0.777	1.211	1.211	1.216	1.214	0.735	0.725	0.667	0.813	1.041	0.961	0.970	1.210	1.201	1.224
	6	0.963	1.046	1.098	1.452	1.452	1.457	1.436	0.962	0.972	0.983	1.113	1.052	1.030	1.031	1.442	1.425	1.469
										$\rho = 0.85$								
3	4	0.242	1.011	0.256	1.100	1.100	1.120	1.065	0.256	0.294	0.261	0.330	1.028	0.922	1.010	1.131	1.130	1.166
	6	0.610	1.035	0.690	1.206	1.206	1.198	1.220	0.631	0.682	0.651	0.765	1.039	0.975	1.026	1.227	1.194	1.232
4	4	0.429	1.028	0.451	1.188	1.188	1.171	1.120	0.468	0.520	0.492	0.675	1.030	0.937	1.015	1.174	1.180	1.260
	6	0.890	1.045	1.079	1.280	1.280	1.297	1.267	0.899	0.925	0.945	0.976	1.066	1.008	1.031	1.299	1.296	1.318
5	4	0.617	1.039	0.801	1.216	1.216	1.220	1.221	0.757	0.731	0.675	0.829	1.057	0.183	1.028	1.236	1.221	1.244
	6	0.970	1.051	1.105	1.488	1.488	1.481	1.440	0.976	0.980	0.996	1.150	1.071	1.055	1.045	1.479	1.453	1.506
										$\rho = 0.95$								
3	4	0.247	1.036	0.270	1.112	1.112	1.129	1.081	0.281	0.350	0.301	0.345	1.041	0.976	1.025	1.147	1.161	1.175
	6	0.624	1.041	0.711	1.217	1.217	1.206	1.230	0.648	0.698	0.700	0.774	1.051	0.986	1.046	1.239	1.207	1.245
4	4	0.431	1.035	0.470	1.200	1.200	1.189	1.134	0.475	0.546	0.515	0.69	1.048	0.979	1.034	1.184	1.194	1.227
	6	0.938	1.060	1.087	1.305	1.305	1.309	1.270	0.950	0.956	0.981	0.991	1.071	1.020	1.045	1.301	1.307	1.327
5	4	0.635	1.056	0.824	1.229	1.229	1.234	1.236	0.780	0.811	0.717	0.840	1.067	0.999	1.050	1.240	1.234	1.256
	6	0.995	1.069	1.124	1.486	1.486	1.478	1.451	0.991	0.999	1.026	1.164	1.087	1.071	1.059	1.487	1.480	1.502

Table 4: RE of different estimators with respect to usual mean estimator for Log-Normal population of size $N = 170$

m	r	t_r	t_{lr}	t_{kc}	t_{ss_1}	t_s	t_{ss_2}	t_{ss_3}	t_{mm_1}	t_{mm_2}	t_{mm_3}	t_{mm_4}	t_{mm_5}	t_{sk_1}	t_{sk_3}	t_i, $i=1,3,4,6$	t_i $i=2,5$	$\mathbf{t_{b_i}}$ $\mathbf{i=1,2}$
										$\rho = 0.65$								
3	3	0.312	1.013	0.314	1.049	1.049	1.037	1.049	0.374	0.440	0.220	0.229	1.009	0.245	0.389	1.038	1.042	1.068
	4	0.352	1.125	0.355	1.272	1.272	1.263	1.140	0.815	0.472	0.430	0.459	1.064	0.522	0.715	1.303	1.304	1.323
4	3	0.352	1.021	0.353	1.178	1.178	1.169	1.082	0.420	0.451	0.398	0.319	1.015	0.507	0.518	1.179	1.171	1.183
	4	1.026	1.153	1.245	1.814	1.814	1.434	1.372	1.071	1.119	1.113	0.948	1.136	1.145	1.138	1.813	1.800	1.848
5	3	0.747	1.145	1.116	1.320	1.320	1.317	1.207	1.056	0.815	0.772	0.614	1.124	1.021	1.018	1.343	1.329	1.375
	4	1.068	1.250	1.646	3.503	3.503	1.567	1.381	1.205	1.200	1.138	1.069	1.212	1.228	1.237	3.513	3.507	3.534
										$\rho = 0.75$								
3	3	0.326	1.020	0.321	1.053	1.053	1.041	1.051	0.380	0.455	0.384	0.232	1.016	0.261	0.406	1.040	1.050	1.083
	4	0.401	1.142	0.404	1.331	1.331	1.272	1.162	0.823	0.787	0.458	0.528	1.084	0.610	0.776	1.314	1.315	1.343
4	3	0.374	1.073	0.375	1.203	1.203	1.199	1.106	0.447	0.503	0.450	0.322	1.023	0.571	0.636	1.201	1.195	1.216
	4	1.040	1.180	1.694	2.030	2.030	2.008	1.387	1.084	1.212	1.127	0.952	1.157	1.150	1.142	2.080	2.066	2.104
5	3	0.759	1.157	1.144	1.377	1.377	1.353	1.213	1.067	0.820	0.789	0.626	1.136	1.43	1.020	1.426	1.427	1.501
	4	1.054	1.301	1.793	3.537	3.537	3.342	3.030	1.300	1.319	1.228	1.076	1.243	1.268	1.241	3.567	3.541	3.621
										$\rho = 0.85$								
3	3	0.331	1.046	0.346	1.081	1.081	1.056	1.087	0.399	0.479	0.407	0.249	1.029	0.299	0.463	1.089	1.076	1.119
	4	0.476	1.170	0.480	1.332	1.332	1.283	1.176	0.871	0.812	0.583	0.574	1.100	0.637	0.810	1.326	1.325	1.357
4	3	0.424	1.091	0.429	1.262	1.262	1.225	1.142	0.505	0.566	0.513	0.342	1.048	0.594	0.671	1.267	1.249	1.296
	4	1.047	1.200	1.753	2.831	2.831	2.880	1.391	1.107	1.225	1.136	0.997	1.193	1.173	1.193	2.980	2.908	3.160
5	3	0.797	1.183	1.162	1.482	1.482	1.400	1.295	1.091	0.886	0.827	0.653	1.158	1.073	1.051	1.495	1.496	1.569
	4	1.203	1.356	2.009	3.664	3.664	3.471	3.172	1.328	1.321	1.337	1.099	1.270	1.377	1.309	3.450	3.268	3.755
										$\rho = 0.95$								
3	3	0.355	1.077	0.396	1.126	1.126	1.098	1.100	0.443	0.511	0.450	0.289	1.049	0.312	0.576	1.107	1.095	1.157
	4	0.616	1.192	0.544	1.346	1.346	1.308	1.190	1.060	0.856	0.610	0.611	1.138	0.668	1.019	1.351	1.350	1.372
4	3	0.539	1.116	0.468	1.277	1.277	1.237	1.185	0.628	0.630	0.556	0.381	1.071	0.617	1.012	1.285	1.261	1.315
	4	1.172	1.233	1.851	3.853	3.853	3.000	1.416	1.191	1.269	1.160	1.057	1.226	1.200	1.199	3.995	3.573	4.035
5	3	0.827	1.206	0.876	1.596	1.596	1.484	1.308	1.155	0.932	0.853	0.704	1.180	1.097	1.092	1.623	1.609	1.825
	4	1.352	1.407	2.401	4.097	4.097	4.082	4.006	1.350	1.374	1.370	1.131	1.298	1.613	1.600	4.103	4.027	4.151

Table 5: RE of different estimators with respect to usual mean estimator for real populations

Estimators Samples	t_r	t_{lr}	t_{kc}	t_{ss_1}	t_s	t_{ss_2}	t_{ss_3}	t_{mm_1}	t_{mm_2}	t_{mm_3}	t_{mm_4}	t_{mm_5}	t_{sk_1}	t_{sk_3}	$t_i,\ i=1,3,4,6$	t_i $i=2,5$	$\mathbf{t_{b_i}}$ $\mathbf{i=1,2}$
									Population 1								
1	1.847	3.835	1.849	3.066	3.066	3.876	1.222	1.007	1.671	2.547	0.714	3.263	1.081	1.101	3.889	3.850	3.959
2	0.661	1.894	0.661	1.925	1.925	1.894	1.906	0.750	1.056	1.368	0.959	1.855	1.117	1.207	1.930	1.910	1.960
3	0.470	1.881	0.470	1.924	1.924	1.881	1.923	0.537	0.783	1.069	0.220	1.881	1.623	1.760	1.887	1.901	1.987
4	1.454	2.688	1.454	2.716	2.716	2.691	2.682	1.645	2.210	2.580	0.303	2.098	1.160	1.253	2.694	2.703	2.794
5	0.453	1.898	0.453	1.914	1.914	1.898	1.366	0.518	0.759	1.044	0.215	1.898	1.060	1.073	1.904	1.919	1.944
6	1.710	2.303	1.710	2.349	2.349	2.304	2.294	0.816	1.203	1.624	0.240	2.190	1.084	1.104	2.309	2.324	2.379
7	1.989	3.209	1.990	3.274	3.274	3.218	3.303	1.159	1.799	2.493	0.241	2.663	1.099	1.123	3.275	3.236	3.315
8	1.455	1.973	1.455	2.012	2.012	1.973	2.015	0.521	0.770	1.068	0.211	1.973	1.059	1.072	1.979	1.995	2.037
9	1.552	2.004	1.552	2.047	2.047	2.004	2.042	0.632	0.922	1.251	0.230	1.986	1.071	1.087	2.010	2.024	2.091
10	1.487	1.864	1.488	1.894	1.894	1.864	1.894	0.556	0.807	1.094	0.225	1.862	1.065	1.079	1.870	1.883	1.917
11	1.870	2.078	1.870	2.107	2.107	2.078	2.082	0.984	1.364	1.710	0.279	1.936	1.110	1.136	2.084	2.093	2.128
12	1.472	3.380	1.472	3.421	3.421	3.390	3.331	0.555	0.903	1.406	0.176	3.308	1.052	1.064	3.385	3.423	3.455
									Population 2								
1	7.024	7.269	7.501	7.351	7.351	2.784	1.730	7.024	7.018	7.020	0.300	7.149	1.036	1.031	7.555	7.437	7.575
2	7.926	8.160	8.389	8.242	8.242	2.495	0.712	7.925	7.918	7.921	0.293	8.062	1.037	1.032	8.438	8.325	8.457
3	3.996	4.108	4.334	4.230	4.230	3.628	0.501	3.995	3.993	3.994	0.335	4.041	1.031	1.026	4.350	4.286	4.369
4	4.041	4.048	4.287	4.267	4.267	4.069	0.591	4.041	4.040	4.041	0.314	4.048	1.033	1.028	4.302	4.284	4.325
5	3.058	3.073	3.265	3.229	3.229	3.242	0.202	3.058	3.058	3.058	0.346	3.069	1.029	1.025	3.266	3.247	3.282
6	4.812	4.812	5.035	5.080	5.080	4.764	0.673	4.812	4.812	4.812	0.297	4.802	1.035	1.031	5.071	5.060	5.099
7	4.67	4.701	4.985	4.905	4.905	4.740	0.722	4.670	4.668	4.669	0.310	4.695	1.034	1.029	4.988	4.945	5.016
8	3.143	3.152	3.287	3.376	3.376	3.292	0.890	3.144	3.144	3.144	0.318	3.127	1.032	1.027	3.345	3.361	3.392
9	2.750	3.146	2.767	3.467	3.467	3.383	3.388	2.750	2.753	2.752	0.263	2.665	1.038	1.033	3.276	3.359	3.487
10	6.522	6.805	7.022	6.878	6.878	3.034	0.739	6.522	6.516	6.518	0.306	6.647	1.035	1.030	7.094	6.968	7.115
11	7.543	8.817	8.388	8.824	8.824	1.821	0.742	7.542	7.529	7.535	0.317	7.844	1.034	1.029	9.107	8.870	9.130
12	4.617	4.626	4.897	4.868	4.868	3.754	0.734	4.617	4.616	4.616	0.305	4.626	1.034	1.030	4.917	4.892	4.934
									Population 3								
1	4.040	4.043	4.056	4.056	4.056	4.037	4.049	1.667	0.960	0.602	1.831	3.955	2.521	2.329	4.056	4.056	4.076
2	8.111	8.260	8.134	8.265	8.265	8.282	8.235	1.810	1.090	0.579	2.037	7.336	3.143	2.806	8.273	8.269	8.303
3	6.996	7.067	7.016	7.075	7.075	7.030	7.062	1.788	1.119	0.582	2.004	6.472	3.033	2.724	7.080	7.077	7.101
4	3.663	3.722	3.670	3.740	3.740	3.761	3.358	1.702	1.890	0.589	1.875	3.720	2.582	2.390	3.735	3.737	3.785
5	4.109	4.170	4.117	4.189	4.189	4.234	4.194	1.742	1.860	0.582	1.931	4.163	2.731	2.508	4.183	4.186	4.263
6	3.604	3.681	3.611	3.700	3.700	3.718	3.711	1.708	1.870	0.586	1.882	3.681	2.591	2.399	3.694	3.697	3.754
7	2.262	2.764	2.263	2.782	2.782	2.768	2.760	1.751	1.701	0.556	1.921	2.512	2.491	2.360	2.771	2.776	2.817
8	6.094	6.638	6.125	6.639	6.639	6.628	6.172	1.677	1.104	1.610	1.854	5.409	2.658	2.421	6.651	6.643	6.699
9	7.671	10.263	7.725	10.270	10.270	10.239	10.145	1.672	2.109	1.617	1.850	6.306	2.689	2.437	10.275	10.263	10.297
10	5.622	5.637	5.632	5.653	5.653	5.617	5.558	1.803	2.860	0.574	2.020	5.532	3.019	2.726	5.650	5.651	5.680
11	4.671	5.251	4.698	5.272	5.272	5.241	5.225	1.581	1.117	0.636	1.723	4.200	2.339	2.163	5.283	5.276	5.302
12	3.796	6.060	3.798	6.110	6.110	6.103	6.107	2.261	1.560	0.491	2.657	4.796	4.557	4.014	6.066	6.083	6.136

List of Table captions

Table 1: RE of different estimators with respect to usual mean estimator for Normal population of size $N = 1000$

Table 2: RE of different estimators with respect to usual mean estimator for Normal population of size $N = 170$

Table 3: RE of different estimators with respect to usual mean estimator for Log-Normal population of size $N = 1000$

Table 4: RE of different estimators with respect to usual mean estimator for Log-Normal population of size $N = 170$

Table 5: RE of different estimators with respect to usual mean estimator for real populations

Authors' Biographies

Shashi Bhushan did his Ph. D. from University of Lucknow after qualifying CSIR NET/JRF at first attempt in appearing category. He has taught at several prestigious Indian Universities like University of Lucknow, Lucknow, Mizoram University, Aizawl, Central University of (South) Bihar, Patna and Babasaheb Bhimrao Amedkar University, Lucknow, Dr. Shakuntala Misra University before joining University of Lucknow in 2022. He has published more than 100 research papers in the refereed national/international journals, 16 chapters in the books and one book besides presenting papers in more than fifty seminars and conferences. He has also edited a research monograph published by an international publisher. His current research interests are in the areas of Applied Statistics, Non-Response, Missing Data, Measurement Errors, Robust Sampling Strategies, Data Science etc. He has received several research grants in these areas. He has also supervised three research projects funded by UGC. His papers are well cited in many journals and books. He has successfully supervised seven Ph.D. and three M.Phil. scholars apart from many masters students for their dissertations.

Anoop Kumar did his Ph.D. from Dr. Shakuntala Misra National Rehabilitation University, Lucknow, India. Currently, he is working as an Assistant Professor in the Department of Statistics, Amity University, Lucknow, U.P., India. He has qualified UGC NET twice in 2017. His main area of research interests are survey sampling, missing data, measurement errors, etc. He has publications in several prestigious national and international journals like, Sankhya B, AEJ-Alexandria Engineering Journal, Communications in Statistics - Theory and Methods, Communications in Statistics - Simulation and Computation, Journal of Computation and Applied Mathematics, AIMS Mathematics and many others.