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New decision-making methods for ranking of non-dominated points for multi-objective optimization problems

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Abstract. A Multi-objective Optimization Problem (MOP) is a simultaneous optimization of more than one real-valued conflicting objective function subject to some constraints. Most MOP algorithms try to provide a set of Pareto optimal solutions that are equally good in terms of the objective functions. The set can be infinite, and hence, the analysis and choice task of one or several solutions among the equally good solutions is hard for a Decision Maker (DM). In this paper, a new scalarization approach is proposed to select a Pareto optimal solution for convex MOPs such that the relative importance assigned to its objective functions is very close together. In addition, two decision-making methods are developed to analyze convex and non-convex MOPs based on evaluating a set of Pareto optimal solutions and the relative importance of the objective functions. These methods support the DM to rank the solutions and obtain one or several of them for real implementation without having any familiarity with MOPs.

1. Introduction

Single-objective optimization can be described as optimizing a problem by using a single objective function [1–3]. In contrast, optimizing some objective functions subject to a number of constraints is required in many practical problems (see [4–8]). These problems are called Multi-Objective Optimization Problems (MOPs) and are formulated as follows:

$$\min \quad f(x) = (f_1(x), f_2(x), ..., f_p(x))^T, \tag{1}$$

s.t.:

$$x \in X = \{x \in \mathbb{R}^n | \, g_j(x) \leq 0 \ \text{ for } \ j=1,...,m\},$$

where $X \subset \mathbb{R}^n$ is a feasible set in the decision space $\mathbb{R}^n$ and $f{:}X \to \mathbb{R}^p$ consists of the p objective functions $f_i : X \to \mathbb{R}$ for all $i = 1, ..., p$, $(p \geq 2)$. In addition, $Y = f(X) = \{f(x) \mid x \in X\}$ is a feasible objective set in the objective space $\mathbb{R}^p$. Problem (1) is stated as a convex MOP if all the objective functions and feasible set are convex. MOPs usually present a set of solutions that cannot be enhanced in each objective function without

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degrading at least one of the other objective functions. These solutions are named Pareto optimal solutions, and their images under f are named non-dominated points. The goal of solving MOPs is obtaining a set of all non-dominated points called the Pareto front. It should be noted that since the Pareto front may have an infinite number of points, particularly for continuous MOPs, it is impossible to compute it completely in practice, and hence, a discrete approximation of the Pareto front with a finite size is considered.

There are some different methods to solve MOPs in the literature. Miettinen categorized multi-objective optimization methods into four groups assuming the existence or nonexistence of a Decision Maker (DM) during the solving process: no preference, a priori, interactive, and a posteriori methods [9]. Note that the DM is an expert in the domain of these problems who can provide preference information to choose a final solution in MOPs. In no preference methods, such as the global criterion and multi-objective proximal bundle method [10], the preference information from the DM is not considered, and the MOP is solved for obtaining a Pareto optimal solution. They are reasonable for situations where there are not any special expectations of solution for DM. In a priori methods, the DM is initially requested to identify her/his preference information, and then, the information is used to formulate a parametric Single-objective Optimization Problem (SOP). Lastly, the SOP is solved in a straightforward way without any interactions with the DM to find a solution. The lexicographic ordering [11] and goal programming [12] belong to this group. In interactive methods, the solving process is iterative, and the DM determines the preference information and interacts with the method at iterations as long as an attained solution is acceptable from the viewpoint of the DM. In addition, the DM can gain some familiarity with the problem at each iteration and can correct one's preferences. The reference point [13] and (interactive weighted) Tchebycheff procedure [14] are examples of these methods.

A posteriori method first generates an approximation of the Pareto front without paying attention to the DM's preference information. Then, the DM examines all points and selects the best one among them by considering their mind priorities. Some algorithms of this class for continuous MOPs are the normal boundary intersection [15], normal constraint [16], directed search domain [17], non-dominated sorting genetic algorithm-II [18], and S-metric selection evolutionary multi-objective algorithm [19]. Also, there are algorithms to obtain non-dominated points for discrete MOPs (see [20–24]). Note that in many real-life MOPs and particularly in combinatorial MOPs, these approximations can obtain many points such that none of them can be said to be better than the others in the absence of mind priorities of the DM. In addition, the size of an approximation of the Pareto front grows proportionally to the number of objective functions. On the other hand, from a practical viewpoint, one or some particular solution corresponding to this approximation has to be selected for industrial implementation. Therefore, choosing among the equally good solutions may be a challenging problem for the DM. Hence, a decision-making support tool becomes very effective to assist the DM to choose a preferred solution among all these solutions. There are different decision-making techniques to select one or several solutions among a set of solutions obtained by a posteriori methods. A review of some decision-making algorithms is described as follows.

The Analytic Hierarchy Process (AHP) introduced by Saaty [25] needs a decision tree with the goal at the top level, criteria and sub-criteria at the middle levels, and the solutions at the bottom. In AHP, the DM makes pairwise comparisons of criteria subject to the goal and determines the relative weight of objective functions by different methods such as row sum, column sum, arithmetic mean, geometric mean, and square sum methods. Finally, pairwise comparisons of solutions with criteria are done, and the DM selects the best solution according to the highest rank between solutions. The elimination and choice translating reality (ELECTRE) method presented by Benayoun et al. [26] uses dominant relationships between solutions. It is based on outranking relationships and uses thresholds of indifference and preference for pairwise comparisons among the solutions. Srinivasan and Shocker [27] developed a linear programming technique for multidimensional analysis of preference (LINMAP) in which a solution with a minimum distance from an ideal point is selected as the best solution. Hwang and Yoon [28] proposed a Technique to Order Preferences by Similarity to an Ideal Solution (TOPSIS). According to this technique, a non-dominated point is chosen that has the smallest Euclidean distance from an ideal point and also the largest Euclidean distance from a nadir point. Afshari et al. [29] suggested a Simple Additive Weight (SAW) method, which considers a weighted sum of normalized values of objective functions for each Pareto optimal solution and selects a solution with the greater value. Note that the DM decides the weight corresponding to each objective function. Guisado et al. [30] presented Shannon's entropy method to calculate normalized weights or relative importance of each objective function by considering all solutions. According to this method, whatever dispersion in the index is greater, the index is more important. It calculates a weighted sum of each normalized solution and selects a solution with a maximum value as the ultimate solution. Opricovic and Tzeng [31] introduced the VIKOR (vlsekriterijumska optimizacija i kompromisno resenje, which in Serbian means multi-criteria optimization and compromise solution) method, which is based on a particular measure of closeness to the ideal solution. A weight vector of criteria is considered as input that can be determined by the DM or methods like AHP. It uses the concept of compromise programming and determines a compromise solution accepted by the DM. Fernando et al. [32] proposed a simple method named FUCA, which is a French acronym for "Faire Un Choix Adéquat" (make an adequate choice). This method is based on individual rankings of objective functions. For each objective function, the rank "one" is assigned to its best value and the rank "n" to the worst one. Note that n is the number of points of approximation of the Pareto front. Besides, this algorithm computes a weighted sum of ranks for each solution in which weights represent preferences and selects a final solution with the smallest values of this sum. Yoon and Hwang [33] proposed the multiplicative exponent weighting (MEW) method, in which a product of the weighted exponent of a normalized value of objective functions for each solution is calculated, and a recommended solution is considered with the largest value.

Malakooti and Raman [34] applied the unsupervised learning clustering artificial neural network with variable weights to a group of solutions. This method uses a feed-forward artificial neural network to select the best solution for each cluster. Furthermore, Malakooti and Yang [35] proposed clustering solutions into different groups such that different methods can be applied for analyzing and selecting each group. They provided theories and procedures for clustering based on similar features among solutions, ideal solutions or most representative solutions, and other preferential information given by the DM. Taboada et al. [36] reduced the size of the Pareto optimal solutions by two methods. In the first method, the objective functions are ranked non-numerically, scaled, and combined into a single objective function using randomly generated weight sets. Then, the DM can select a solution that reflects his objective functions priorities. In the second method, the k-means algorithm is applied to cluster the data based on clustering techniques used in data mining. It finds k groups of similar solutions, and the DM chooses k solutions without any objective function preference information. Cheikh et al. [37] and Zio and Bazzo [38] partitioned Pareto optimal solutions into k clusters in which each cluster contains solutions with similar properties. In these methods, the nearest solution to the ideal point is chosen in each cluster. In addition, a fuzzy scoring procedure is applied for ranking solutions in the Zio and Bazzo method. Deb and Goel [39] use a clustering method in which each solution belongs to a separate cluster. Next, the distance between all pairs of clusters is calculated by finding a centroid of each cluster and calculating the Euclidean distance between the centroids. Besides, two clusters having a minimum distance are merged together into a bigger cluster. This step is continued until the desired number of clusters is obtained. For the remaining clusters, a solution closest to the centroid of the cluster is retained, and all other solutions from each cluster are deleted.

In this work, a new scalarization approach is introduced to find a Pareto optimal solution for convex MOPs such that the relative importance of each objective function is likeness. It uses the fact that whatever the relative importance assigned to the objective functions are closer together, the product of their component is greater. Note that some decision-making approaches require interactions with the DM, such as SAW, TOPSIS, VIKOR, AHP, and MEW. Therefore, these methods depend on DM to provide inputs, preference information, and final outputs, and hence, different DMs may obtain other solutions and cause errors. Some approaches, such as AHP and clustering methods, cannot get an acceptable solution without having information about problems and a set of solutions and need to analyze the solutions. Some approaches differ in expressing the preferences. For example, AHP considers weighed factors independently of solutions, while Shannon's entropy method calculates weighs by considering all solutions. In this paper, two decision-making methods are proposed to help the DM to find the most preferred solutions from a set of non-dominated points without having any familiarity with the problem. Hence, these methods are user-friendly because they do not require any input from the user. The first method ranks a set of Pareto optimal solutions or non-dominated points obtained for a convex MOP, and another method ranks a set of non-dominated points obtained for non-convex MOPs. Both methods use concepts of the weighted sum scalarization method to assign a weighted vector to each non-dominated point. These methods calculate weights by considering all solutions or some solutions and solving linear programming. Finally, the results of these methods on several examples are presented.

The remainder of this paper is organized as follows: Some basic definitions of MOPs used throughout this paper are briefly described in Section 2. After that, a new scalarization approach is introduced in Section 3. The first decision-making method is proposed to rank a set of Pareto optimal solutions of convex MOP in Section 4. Then, another decision making-method is suggested for ranking a set of Pareto optimal solutions of non-convex MOP in Section 5. Finally, Section 6 presents conclusions.

2. Preliminaries

This section presents notations, definitions, and preliminaries used during the paper in which $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p |\ y_i \geq 0 \text{ for } i \in \{1, ..., p\}\}$.

Definition 1 [40]. Given points y^1 and $y^2 \in \mathbb{R}^p$. y^1 is said to dominate y^2, and it is shown as $y^1 \preceq y^2$ if $y^1_i \leq y^2_i$ for all $i = 1, 2, ..., p$, and $y^1 \neq y^2$. In addition, $y^1 \prec y^2$ if and only if $y^1_i < y^2_i$ for all $i = 1, ..., p$.

Definition 2 [40]. A solution $\hat{x} \in X$ is called a Pareto optimal solution of Problem (1) if there is no feasible solution $x \in X$ such that $f(x)$ dominates $f(\hat{x})$. If $\hat{x} \in X$ is a Pareto optimal solution, then $f(\hat{x})$ is called a *non-dominated point*. The set of all Pareto optimal solutions of the MOP are denoted by X_E, and their images are called *Pareto front* and denoted by Y_N, respectively.

According to Definition 2, the Pareto optimal solutions do not allow improvement in one objective function without deteriorating at least one other objective function. These trade-offs among objective functions can be measured by computing the increase in objective function per unit decrease in objective function f_i. In some situations, these trade-offs can be unbounded. In order to eliminate Pareto optimal solutions that exhibit an unbounded tradeoff in their objective values to other solutions, properly Pareto optimal solutions were introduced as follows.

Definition 3 [40]. A Pareto optimal solution $\hat{x} \in X$ is called a *properly Pareto optimal solution in Geoffrion's sense* for Problem (1) if there exists a positive number M such that for each $x \in X$ and $i \in \{1, ..., p\}$ with $f_i(x) < f_i(\hat{x})$, there exists at least an index $j \in \{1, ..., p\}$ with $f_j(\hat{x}) < f_j(x)$ such that:

$$\frac{f_i(x) - f_i(\hat{x})}{f_j(\hat{x}) - f_j(x)} < M.$$

If $\hat{x} \in X$ is a properly Pareto optimal solution, $f(\hat{x})$ is called a *properly non-dominated point*.

Definition 4 [41]. Let $\hat{x} \in X$ be a Pareto optimal solution of Problem (1). The *global trade-off* $T^G_{ij}(\hat{x})$ for i and $j \in \{1, ..., p\}$, with $i \neq j$, is defined as follows:

$$T^G_{ij}(\hat{x}) = \sup_{f(x) \in Z_j^{<(f(\hat{x}))}} \frac{f_i(x) - f_i(\hat{x})}{f_j(\hat{x}) - f_j(x)},$$

where $Z_j^{<(f(\hat{x}))} = \{x \in X | f_j(x) < f_j(\hat{x}),\ f_k\ (\hat{x}) \leq f_k(x),\ k = 1, ..., p,\ k \neq j\}$.

Definition 5. Let the individual minimum of $f_i(x)$ over $x \in X$ be attained at x^i for each $i = 1, ..., p$. The *ideal point* of Problem (1) is defined as $y^I = \left(f_1(x^1), f_2(x^2), ..., f_p(x^p)\right)^T$.

Definition 6. The *nadir point* of Problem (1) is $y^N = \left(f^N_1, f^N_2, ..., f^N_p\right)^T$ where the component f^N_i is determined by $\max_{x \in X_E} f_i(x)$ for $i \in \{1, ..., p\}$ and gives an upper bound on the Pareto front.

In the following theorem, Karush-Kuhn-Tucker (KKT) optimality conditions for Problem (1) are discussed.

Theorem 1 (KKT sufficient optimality condition) [9]. Suppose that the objective functions $f_i : X \to \mathbb{R}$, $i = 1, .., p$ and the constraint functions $g_j : X \to \mathbb{R}$, $j = 1, ..., m$ are continuously differentiable and convex at a point $\hat{x} \in X$. Besides, there are $\lambda \in \mathbb{R}^p_{\succ}$ and $\mu \in \mathbb{R}^m_{\geqq}$ such that:

$$\sum\nolimits_{i=1}^{p} \lambda_i \nabla f_i(\hat{x}) + \sum\nolimits_{j=1}^{m} \mu_j \nabla g_j(\hat{x}) = 0,$$

$$\mu_j g_j(\hat{x}) = 0, \qquad j = 1, 2, ..., m,$$

$$\lambda \succ 0, \ \ \mu \geqq 0, \tag{2}$$

then, $\hat{x}$ is a Pareto optimal solution to Problem (1).

2.1. Weighted sum scalarization method

A scalarization approach is a common approach for determining solutions of the MOP in which the MOP is reformulated as a parametric SOP. Therefore, this SOP can be solved by using standard single-objective optimization techniques. The best-known and simplest method of the scalarization approaches is the weighted sum method suggested by Gass and Saaty [42]. This method associates each objective function with a weighting coefficient determined by the DM and optimizes the real-valued function of the weighted sum of the objective functions. The scalar weighted sum problem for the given weight vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in \mathbb{R}^p_{\succ}$ is written as follows:

$$\min \ \ \sum\nolimits_{i=1}^{p} \lambda_i f_i(x), \tag{3}$$

$$\text{s.t.:} \quad x \in X.$$

The weight λ_i, $i = 1, ..., p$ can be interpreted as the relative importance or worth of the objective function f_i when it is compared to the other objective functions. It is usually supposed that weights are normalized, that is, $\sum_{i=1}^{p} \lambda_i = 1$. In this method, each Pareto optimal solution of a convex MOP can be found by varying λ; see [40]. The geometrical illustration of this method is given in Figure 1 for a specified weight vector λ and a bi-objective minimization problem. Since $Y_N + \mathbb{R}^p_{\geqq}$ is a convex set in this figure, the optimal solution of the weighted sum problem is the intersection

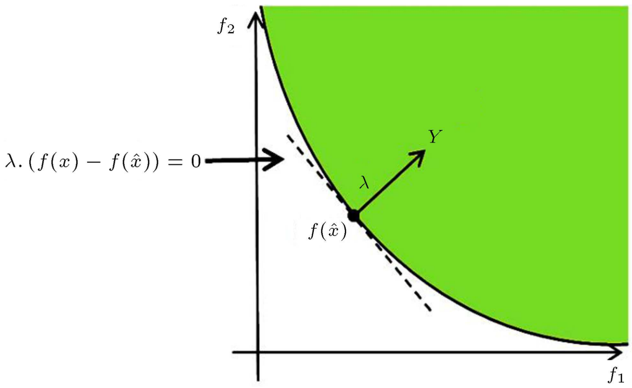

Figure 1. Feasible region of the bi-objective minimization problem in objective space: A weighted objective function appears as a linear curve.

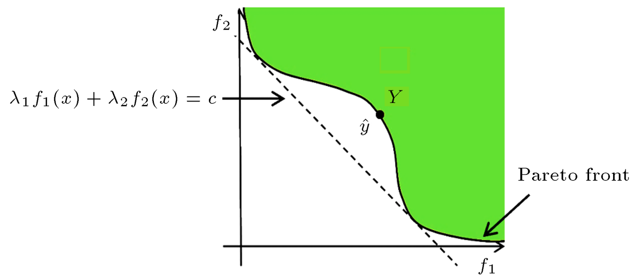

Figure 2. Inability of the weighted sum method to find a point $\hat{y}$ of a non-convex bi-objective problem.

of the supporting hyperplanes of the Pareto front at a point $f(\hat{x})$ with the normal vector λ, that is, $\lambda \cdot (f(x) - f(\hat{x})) = 0$ in which "." is the dot product. In addition, the non-dominated point $f(\hat{x})$ is obtained by pushing the contour as far to the southwest as possible until it just touches the boundary of Y.

It should be noted that an optimal solution of the weighted sum problem with $\lambda \in \mathbb{R}^p_{\succ}$ is a properly Pareto optimal solution of the MOP, and any Pareto optimal solution of a convex MOP can be found by the weighted sum method and varying λ. In addition, this method misses non-dominated points on non-convex parts of the Pareto front for non-convex MOPs; see Figure 2. This is due to the fact that this method is often implemented as a convex combination of the objective functions, where the sum of all weights is constant and negative weights are not allowed. A new scalarization approach for convex MOPs.

As mentioned in the introduction, a discrete approximation of optimal solutions is obtained by solving an MOP in a posteriori method. Since the size of this approximation is usually large, choosing a solution among these solutions is difficult. In addition, it is important that a solution can be obtained as a recommended solution that considers a relation between the objective functions without exploring all Pareto optimal solutions. For this purpose, a new scalarization approach is presented for convex MOPs in which the KKT sufficient optimality condition is used. The optimal solution of the proposed SOP is a Pareto optimal solution with the characteristic that the relative importance of the objective functions is very close together and, in particular, is equal. This approach uses the fact that whatever n positive numbers are closer together, their product is greater (see Theorem 2 and Lemma 1).

Theorem 2 **[43].** The product of n positive real numbers with a constant sum is maximal when all the numbers are equal.

Lemma 1. Let S is an arbitrary set of positive real numbers with a constant sum σ, i.e., $S = \{s = (s_1, s_2, ..., s_n) \in \mathbb{R}^n \mid \sum_{i=1}^n s_i = \sigma\}$. Then, whatever $\Delta^s = \sum_{i=1}^n |s_i - \frac{\sigma}{n}|$ is smaller, the product of its component, i.e., $\gamma^s = \prod_{i=1}^n s_i$ is bigger for $s \in S$.

Proof. It is obvious by considering Theorem 2.

Lemma 1 and the KKT sufficient optimality condition have an important role in formulating this scalarization approach. The Constraints 2 are used in order to obtain a solution belonging to the Pareto optimal set of the MOP. Hence, suppose that the objective functions and constraint functions of the MOP are convex and continuously differentiable on X and consider the following SOP:

$$\begin{aligned}
&\max \quad \prod_{i=1}^{p} \lambda_i \\
&\text{s.t.:} \quad \sum_{i=1}^{p} \lambda_i \nabla f_i(x) + \sum_{j=1}^{m} \mu_j \nabla g_j(x) = 0, \\
&\mu_j g_j(x) = 0, \qquad j = 1, 2, ..., m, \\
&\mu_j \geq 0, \qquad j = 1, 2, ..., m, \\
&\sum_{i=1}^{p} \lambda_i = 1,
\end{aligned} \tag{4}$$

where $\lambda \geq 0$ and $x \in X$. It is obvious that an optimal solution $(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^p_{\succ} \times \mathbb{R}^m_{\succeq}$ to Problem (4) is a solution in that the weights λ_i, $i = 1, ..., p$, are very close together, and x is a Pareto optimal solution of the MOP. Since the maximum of the product of λ_i, $i = 1, ..., p$ is calculated, λ_i is always positive. Problem (4) is a Multiplicative Programming Problem (MPP) in which an objective function is a product of several linear functions. Moreover, the MPP is known as both a global optimization problem and an NP-hard problem, even in special cases where its objective function is the product of two linear functions subject to linear constraints. Hence, Problem (4) can be solved by algorithms presented for convex MPP, such as an objective space cut and bound algorithm

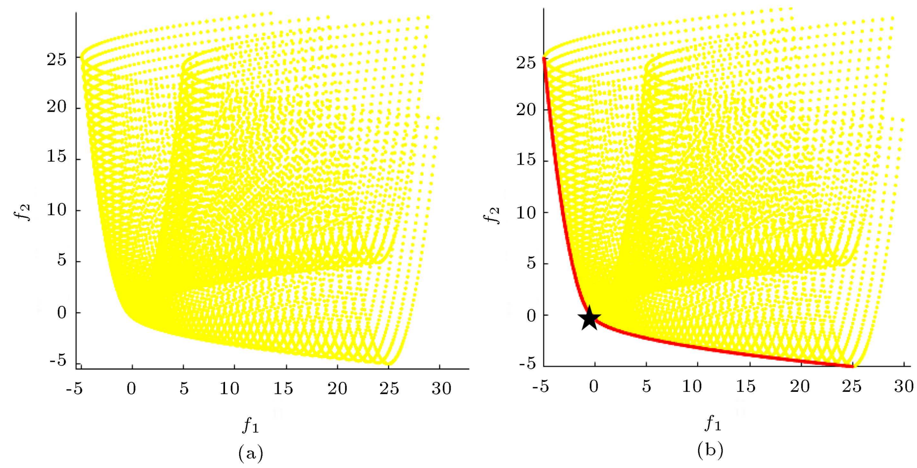

Figure 3. (a) The objective space of the bi-objective problem corresponding to Example 1 and (b) Pareto front of Example 2 and obtained solutions of the Problem (6).

proposed by Shao and Ehrgott [44] that obtains an ε-optimal solution for Problem (4) with a specified approximation error $\varepsilon > 0$. A technique is proposed to solve this problem exactly. Since the function $\ln(x)$ is a strictly increasing function, the objective function of Problem (4) is replaced by $\sum_{i=1}^{p} \ln(\lambda_i)$. In addition, changing variables $z_i = \ln(\lambda_i)$, and as a result $\lambda_i = e^{z_i}$ is applied in this problem, and the following SOP is obtained. It should be noted that since $0 < \lambda_i < 1$, $i = 1, ..., p$, $z_i < 0$. Then, by this logarithmic transformation, Problem (5) can be solved for a globally optimum solution quicker and easier.

$$\begin{aligned}
&\max \quad \sum\nolimits_{i=1}^{p} z_i,\\
&\text{s.t.:} \quad \sum\nolimits_{i=1}^{p} e^{z_i} \nabla f_i(x) + \sum\nolimits_{j=1}^{m} \mu_j \nabla g_j(x) = 0,\\
&\mu_j g_j(x) = 0, \qquad j = 1, 2, ..., m,\\
&\mu_j \geq 0, \qquad j = 1, 2, ..., m,\\
&\sum\nolimits_{i=1}^{p} e^{z_i} = 1,\\
&z_i \leq 0, \qquad i = 1, 2, ..., p,\\
&x \in X. \qquad (5)
\end{aligned}$$

Example 1. Consider the following bi-objective optimization problem:

$$\begin{aligned}
&\min \quad f_1(x) = x_1^2 + x_2,\\
&\min \quad f_2(x) = x_2^2 + x_1,\\
&\text{s.t.:} \quad -5 \leq x_1, x_2 \leq 5.
\end{aligned}$$

According to Problem (5), Example 1 can be formulated as follows:

$$\begin{aligned}
&\max \quad z_1 + z_2,\\
&\text{s.t.:} \quad e^{z_1} \nabla f_1(x) + e^{z_2} \nabla f_2(x) = 0,\\
&\sum_{i=1}^{p} e^{z_i} = 1,\\
&z_1, z_2 \leq 0, \quad -5 \leq x_1, \quad x_2 \leq 5. \qquad (6)
\end{aligned}$$

All implementations in this paper are done by MATLAB 2016 on a laptop with Pentium 4 at 2.3 GHz and 4 GB RAM run. $(x_1, x_2, \lambda_1, \lambda_2) = (-0.5, -0.5, 0.50, 0.50)$ is an obtained solution to this problem. A feasible objective set of the objective space is shown by dots points calculated on a regular grid in Figure 3(a).

In addition, the Pareto front of Example 1 is illustrated by a continuous curve shown by red dots points, and a non-dominated point obtained by solving Problem (5) is $(-0.25, -0.25)$ shown by a star in Figure 3(b). As seen in this figure, the obtained solution corresponds to a point on the Pareto front, and the relative importance of each objective function is equal.

Theorem 3. Assume (x^*, λ^*, μ^*) be an optimal solution of Problem (4), then x^* is the optimal solution of a following weighted sum problem:

$$\begin{aligned}
&\min \quad \sum\nolimits_{i=1}^{p} \lambda_i^* f_i(x),\\
&\text{s.t.:} \quad x \in X. \qquad (7)
\end{aligned}$$

Proof. Since (x^*, λ^*, μ^*) is the optimal solution to Problem (7), we have the following:

$$\begin{aligned}
&\sum\nolimits_{i=1}^{p} \lambda_i^* \nabla f_i(x^*) + \sum\nolimits_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0,\\
&\mu_j^* g_j(x^*) = 0, \qquad j = 1, 2, ..., m.
\end{aligned}$$

Therefore, x^* is satisfied with the KKT optimality conditions of the weight sum Problem (7), and since this problem is convex, x^* is the optimal solution to this problem. □

Theorem 4. Assume (x^*, λ^*, μ^*) be an optimal solution of Problem (4), then the global trade-off $T_{ij}^G(x^*)$ for all i and $j \in \{1, ..., p\}$ that $i \neq j$ is calculated as follows:

$$T_{ij}^G(x^*) \leq \frac{\max_j \lambda_j^*}{\min_i \lambda_i^*}.$$

Proof. Let (x^*, λ^*, μ^*) be the optimal solution of Problem (4), then x^* is the optimal solution of Problem (7). Then,

$$\sum_{i=1}^{p} \lambda_i^* f_i(x^*) \leq \sum_{i=1}^{p} \lambda_i^* f_i(x)$$

$$\Rightarrow \sum_{i\neq j}^{p} \lambda_i^* \left(f_i(x^*) - f_i(x)\right) + \lambda_j^* \Big(f_j(x^*)$$

$$-f_j(x) \Big) \leq 0, \Rightarrow \sum_{i\neq j}^{p} \lambda_i^* \left(f_i(x^*) - f_i(x)\right)$$

$$\leq \lambda_j^* \left(f_j(x) - f_j(x^*)\right).$$

Let $\bar{\lambda} = \min_{i\neq j}\{\lambda_i^*\}$, then $\bar{\lambda} \sum_{i\neq j}^{p} \left(f_i(x^*) - f_i(x)\right) \leq \lambda_j^* \left(f_j(x) - f_j(x^*)\right)$. Moreover, we have:

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq \sum_{i\neq j}^{p} \frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq \frac{\lambda_j^*}{\bar{\lambda}}$$

$$\leq \frac{\max_{j\neq i}\lambda_j^*}{\bar{\lambda}} = \frac{\max_{j\neq i}\lambda_j^*}{\min_{i\neq j}\lambda_i^*} \leq \frac{\max_j \lambda_j^*}{\min_i \lambda_i^*}.$$

As a result $T_{ij}^G(x^*) \leq \frac{\max_j \lambda_j^*}{\min_i \lambda_i^*} = M$. The vector λ^* which corresponds to the optimal solution x^* to Problem (4) has a maximum value $\prod_{i=1}^{p} \lambda_i^*$, or the components of λ^* are very close together. It means that $\max_i \{\lambda_i^*\} - \min_i \{\lambda_i^*\}$ has a minimum value. In addition, the smallest lower bound for M occurs when $\max_i \{\lambda_i^*\} - \min_i \{\lambda_i^*\}$ has a minimum value. Then, it is possible to obtain the smallest lower bound for $T_{ij}^G(x^*)$ for all i and $j \in \{1, ..., p\}$ that $i \neq j$ by having the vector λ^*.

3. A method for ranking non-dominated points of convex MOPs

The DM's preferences are important in MOPs, especially when the goal is to help the DM to find the most preferred solutions and make a suitable decision. Many algorithms are proposed for generating an approximation of the Pareto front, while the number of methods presented for ranking obtained solutions is relatively small. In addition, some information on an MOP is needed in several decision-making methods, such as obtaining the ideal or nadir points. We presented a new decision-making method for ranking a set of non-dominated points of a convex MOP without any information about the problem. The proposed method uses concepts of the weighted sum method and is based on this method; whatever the relative importance of the objective functions in a point is closer together, the point is more important, or the product of weights corresponding to the point is greater (see Theorem 2 and Lemma 1).

Suppose $PF = \{y^1, y^2, ..., y^N\}$ is an obtained approximation of k non-dominated points of the convex MOP that covers all regions in the actual Pareto front. As mentioned before, there is a unique positive weight λ_i, $i = 1, 2..., p$, in the weighted sum problem for each Pareto optimal solution. It should be noted that there is not necessarily a unique positive weight for each Pareto optimal solution for a discrete set of piecewise linear points, and there may exist a set of weights for each Pareto optimal solution. The goal is to find a weight vector λ corresponding to a non-dominated point of the set PF whose corresponding components λ are very close together. In other words, this vector has a maximum product of its components. By considering the mechanism of the sum weighted method, λ is determined for each y^k such that $\lambda . y^k$ has a minimum value in the set PF. Hence, the following problem is solved:

$$\begin{aligned} \min \quad & \lambda \cdot y^k, \\ \text{s.t.:} \quad & \lambda \cdot y^k \leq \lambda \cdot y^j, \qquad j = 1, 2, ..., N, \ j \neq k, \\ & \sum_{i=1}^{p} \lambda_i = 1, \ \lambda_i \geq 0, \quad i = 1, 2, ..., p. \end{aligned} \tag{8}$$

After solving the linear Problem (8), vector λ^k, which is an optimal solution corresponding to y^k is obtained, and $\omega_k = \prod_{i=1}^{p} \lambda_i^k$ is calculated. Now, a set $\{\omega_1, \omega_2, ..., \omega_N\}$ is sorted in ascending order, and the set PF is ranked. Hence, a non-dominated point, which its corresponding ω has greater value, is better.

Example 2. Consider the following convex bi-objective problem:

$$\begin{aligned} \min \quad & f(x) = (x_1, x_2), \\ \text{s.t.:} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Figure 4(a) shows an obtained approximation of the Pareto front for Example 2. Then, the proposed method for ranking the obtained approximations of the Pareto front is applied, and a following ranking

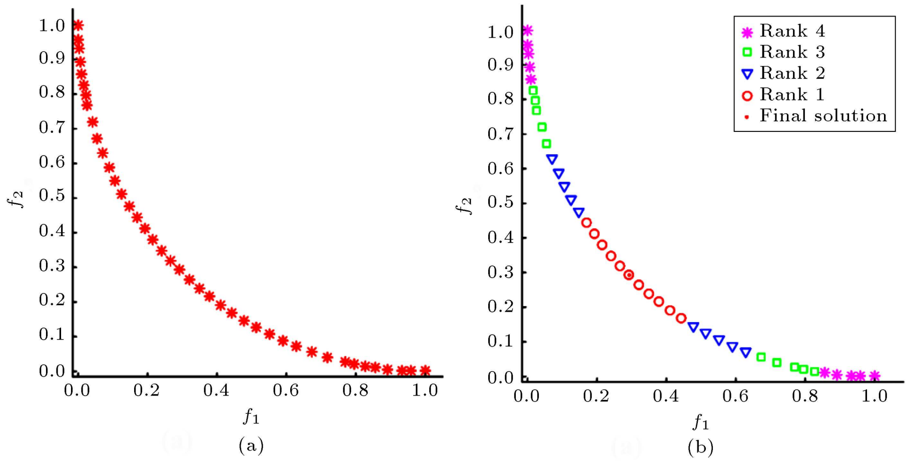

Figure 4. (a) The approximation of the Pareto front for Example 2 and (b) Ranking of the approximation of the Pareto front by the proposed method for Example 2.

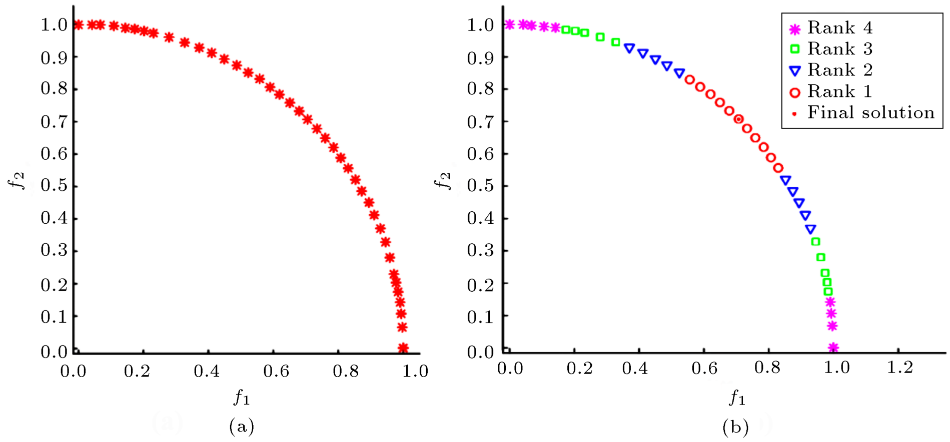

Figure 5. (a) The approximation of the Pareto front for Example 3 and (b) Ranking of the approximation of the Pareto front by the proposed method for Example 3.

of the points displayed in this figure is obtained. We divided the obtained non-dominated points into four groups. The points of this approximation are ranked into four groups, which are indicated by symbols of circle, triangle, square, and star, respectively. It should be noted that the lower ranks include points with more preference for selection. In addition, a final solution (0.2929,0.2929) corresponding to a weight vector (0.4904,0.5096) is obtained.

Example 3. The 10-variable DTLZ2 problem has a non-convex and continuous Pareto front and is formulated as follows:

$$\min \quad f_1(x) = \cos\left(\frac{\pi}{2}x_1\right)(1+g(x)),$$

$$\min \quad f_2(x) = \sin\left(\frac{\pi}{2}x_1\right)(1+g(x)),$$

$$g(x) = \sum_{i=2}^{n} (x_i - 0.5)^2,$$

$$\text{s.t.:} \quad x_i \in [0,\, 1], \qquad \forall i = 1, 2, ..., n.$$

The weighted sum method is not able to find non-dominated points of this problem. To assign a weight vector to each point of the approximation, the following problem can be solved in which a weight vector λ is determined such that $\lambda.y^k$ has a maximum value among all points of approximation.

$$\begin{aligned} &\max \quad \lambda \cdot y^k \\ &\text{s.t.:} \quad \lambda \cdot y^k \geq \lambda \cdot y^j, \qquad j = 1, 2, ..., k,\ j \neq k, \\ &\qquad \textstyle\sum_{i=1}^{p} \lambda_i = 1,\ \lambda_i \geq 0, \qquad i = 1, 2, ..., p. \end{aligned} \tag{9}$$

Then, the proposed method can be used for MOPs in which the convexity of the Pareto front curve on the set X is downward with this difference that Problem (9) is solved. Hence, the method can be used in Example 3. An approximation of the Pareto front is illustrated in Figure 5(a), and the proposed method for ranking these

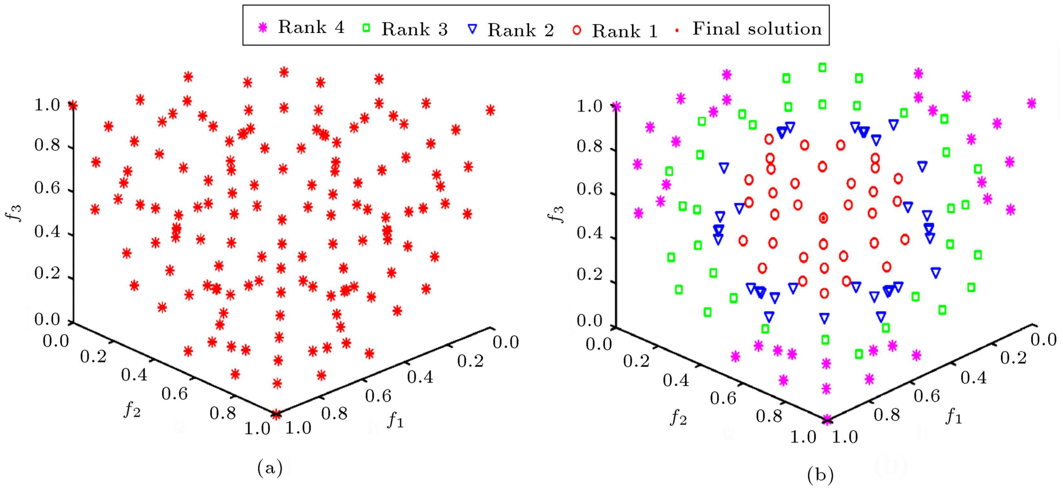

Figure 6. (a) The approximation of the Pareto front for Example 4 and (b) Ranking of the approximation of the Pareto front by the proposed method for Example 4.

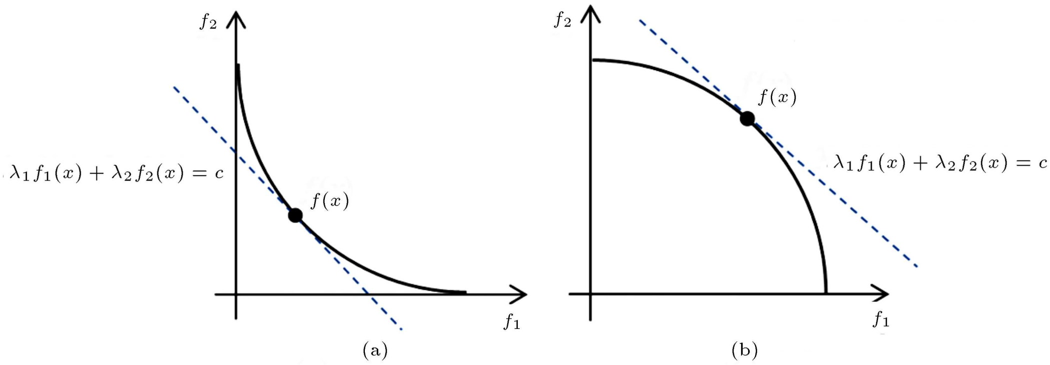

Figure 7. (a) The downward concavity and (b) upward concavity of the Pareto front curve for a bi-objective optimization problem.

approximations is applied and a ranking of the points is obtained by partitioning the point into four groups as Example 2. In addition, a final solution (0.7071,0.7071) corresponding to a weight vector (0.5096,0.4904) is obtained.

Example 4. Consider the following convex three-objective problem:

$$\min \quad f(x) = (x_1, x_2, x_3),$$

$$\text{s.t.:} \quad (x_1-1)^2 + (x_2-1)^2 + (x_3-1)^2 \leq 1,$$

$$x_1, x_2, x_3 \geq 0.$$

The feasible set and feasible objective space set are as a sphere with center (1, 1, 1) and radius 1. The Pareto front is convex, represented by a lower boundary of a unit sphere, i.e.,

$$Y_N = X_E = \{x \in \mathbb{R}^3 \mid 0 \leq x_i \leq 1, \forall i = 1,2,3 \quad \text{and}$$

$$(x_1-1)^2 + (x_2-1)^2 + (x_3-1)^2 = 1\}.$$

An approximation of the Pareto front for Example 4 is demonstrated in Figure 6(a). Then, the proposed method for ranking the obtained approximations of the Pareto front is applied, and a following ranking of the points is obtained. We divided the obtained non-dominated points into four groups; see Figure 6(b). In addition, a final solution (0.4226,0.4226,0.4226) corresponding to a weight vector (0.3570,0.2860,0.3570) is obtained.

4. A method for ranking non-dominated points of non-convex MOPs

As mentioned before, the weighted-sum method cannot find non-dominated points that lie in non-convex regions of the Pareto front, and it may duplicate points with different weight vectors. Consider three following cases:

1. The concavity of the Pareto front curve is downward, and hence, supporting (hyper) planes with the normal vector λ are located under the curve at each point $f(x)$ (see Figure 7(a));
2. The concavity of the Pareto front curve is upward,

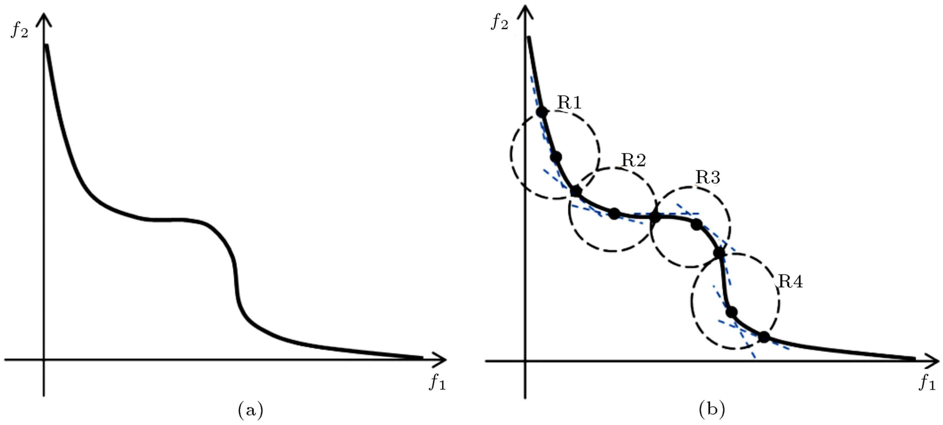

Figure 8. The Pareto front curve for a bi-objective problem with regions with upward and downward concavity.

and hence, supporting (hyper) planes with the normal vector λ are located above the curve at each point $f(x)$ (see Figure 7(b));

3. The Pareto front curve is composed of regions with upward and downward concavity; see Figure 8. As seen in this figure, the concavity of regions R1, R2, R3, and R4 is upward or downward.

Suppose $PF = \{y^1, y^2, ..., y^N\}$ is an obtained approximation of N non-dominated points of a nonlinear MOP. The Pareto front curve may be composed of regions with upward and downward concavity. There is a positive weight in the weighted sum problem in a neighborhood of each non-dominated point of the nonlinear MOP (see Figure 8(b)).

For obtaining these neighborhoods of each non-dominated point, indifferent regions are considered defined as follows.

Definition 7 [45]. Let $\hat{f}$ and f^* be two different non-dominated points. By Definition 2, $\hat{f}$ belongs to a region $f^* + \left(\mathbb{R}^p - (-\mathbb{R}^p_{\geqq} \cup \mathbb{R}^p_{\geqq})\right)$, in which $\mathbb{R}^p - (-\mathbb{R}^p_{\geqq} \cup \mathbb{R}^p_{\geqq})$ is defined as follows:

$$(\bigcup_{i_1=1}^{p} R_{i_1}) \cup (\bigcup_{i_1,i_2=1}^{p} R_{i_1,i_2}) \cup \cdots \cup (\bigcup_{i_1,i_2,...,i_{p-1}=1}^{p} R_{i_1,i_2,...,i_{p-1}}),$$

where $i_j \neq i_k$ for $j \neq k$, and:

$$R_{i_1} = \{w \in \mathbb{R}^p |\ w_{i_1} > 0,\ w^{T_{i_1}} \preceq 0\} \text{ with } T_{i_1} = \{1, ..., p\} \backslash \{i_1\},$$

$$R_{i_1,i_2} = \{w \in \mathbb{R}^p |\ w_{i_1} > 0,\ w_{i_2} > 0,\ w^{T_{i_1,i_2}} \preceq 0\} \text{ with } T_{i_1,i_2} = \{1, ..., p\} \backslash \{i_1, i_2\}$$

$$\vdots$$

$$R_{i_1,i_2,...,i_{p-1}} = \{w \in \mathbb{R}^p |\ w_{i_1} > 0,\ w_{i_2} > 0,\ \ldots,\ w_{i_{p-1}} > 0,\ w^{T_{i_1,i_2,...,i_{p-1}}} \preceq 0\} \text{ with } T_{i_1,i_2,...,i_{p-1}} = \{1, ..., p\} \backslash \{i_1, i_2, ..., i_{p-1}\}.$$

Note that $w^{T_{i_1,i_2,...,i_{p-1}}}$ shows elements of the vector w, whose their index belongs to $T_{i_1,i_2,...,i_{p-1}}$. In addition, R_{i_1}, $R_{i_1,i_2}, ..., R_{i_1,i_2,...,i_{p-1}}$, are $2^p - 2$ convex cones called indifferent regions. Besides, following indifferent regions with respect to the non-dominated points f^* are defined as follows:

$$f^* + R_{i_1},\ f^* + R_{i_1,i_2},\ ...,\ f^* + R_{i_1,i_2,...,i_{p-1}}.$$

By considering the definition of the indifferent regions, an approximation PF of the MOP with p objective functions is partitioned into $2^p - 2$ clusters by considering a non-dominated point such as f^* from PF. Figure 9 shows the partitioning of approximations of the Pareto front into 2 and 6 clusters by considering a non-dominated point f^* for Example 2 and Example 4, respectively.

Afterward, the following steps are done for each non-dominated point $f^* \in PF$:

Step 1: The nearest point to the point f^* is selected from each cluster and is considered as set A (see Figure 10).

If there is an indifferent region corresponding to the point f^* that does not contain any points, then consider a point close enough to f^* in that area and add it to the set A. Figure 11 displays some points of PF as circles for Examples 2 and 4, where at least one of the corresponding indifferent regions is empty.

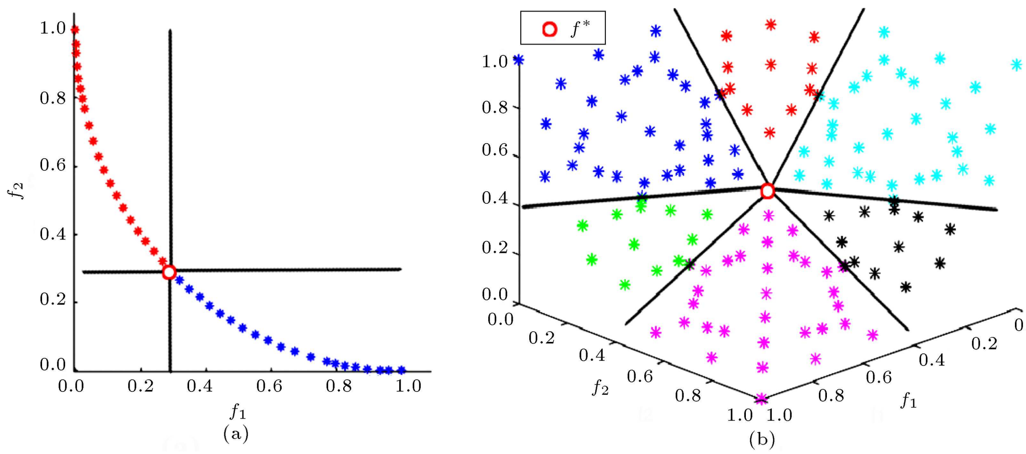

Figure 9. Partitioning of the approximation of the Pareto for (a) Example 2 and (b) Example 4.

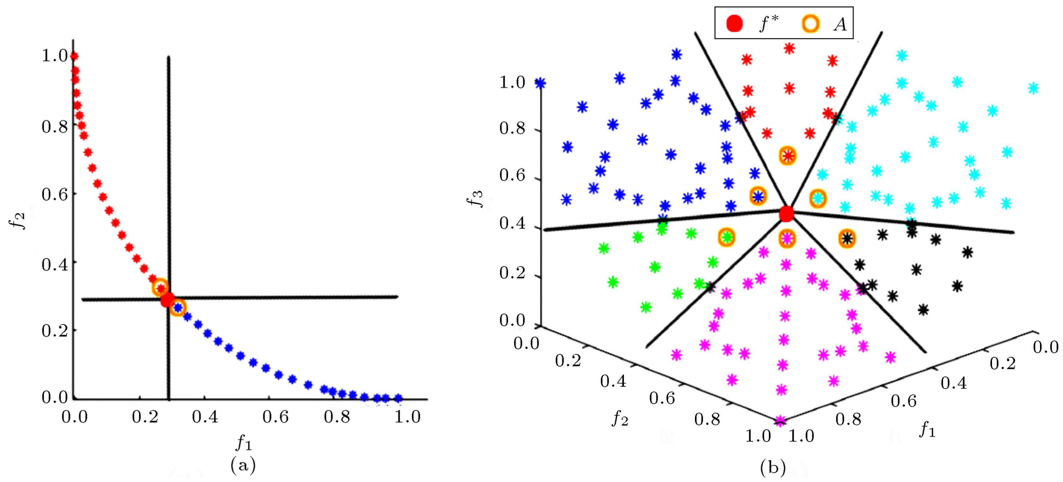

Figure 10. Illustration of the set A for (a) Example 2 and (b) Example 4.

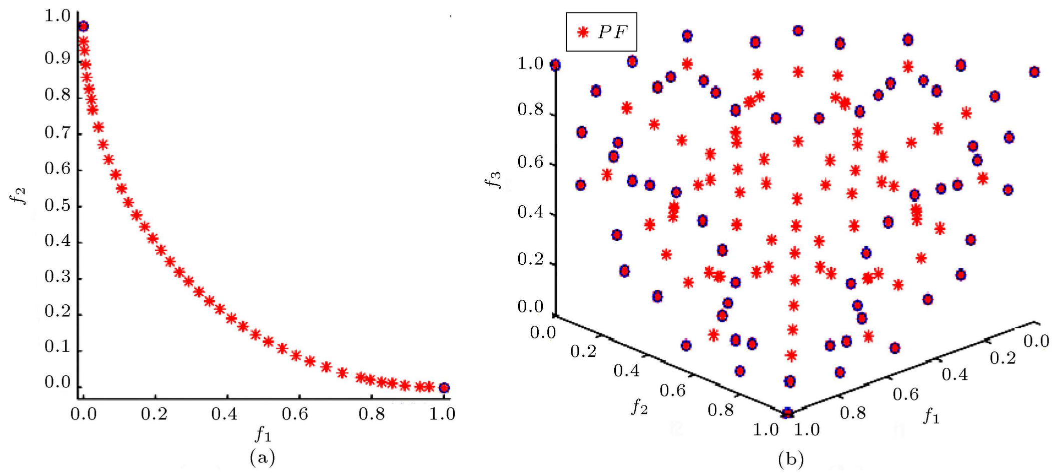

Figure 11. Illustration of some points of PF with at least one empty indifferent region for (a) Example 2 and (b) Example 4.

Step 2: Two points, c and d, are selected from set A. Consider vector b, perpendicular to the line passing through c and d. Hyper plane $H = \{y \in R^p | \, b^T y = \beta\}$ with $\beta = b^T c$ is considered passing through c and parallel to the normal vector b. Now, point $y^* \in \mathbb{R}^p$ is obtained so that y_i^* has a minimum value of ith component among the set PF. In addition, y^* satisfies a constraint $b^T y \leq \beta$. If $b^T f^* \leq \beta$, there is a neighborhood of f^* where the concavity of the curve is downward. Otherwise, there is a neighborhood of

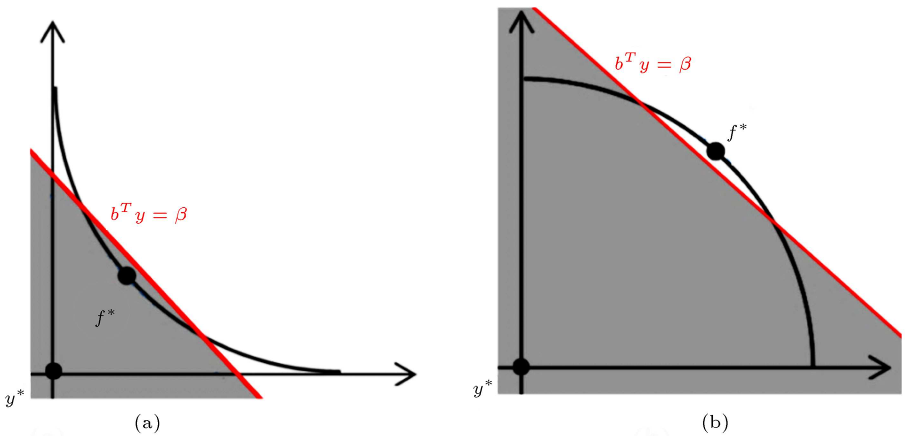

Figure 12. Determine (a) the downwards concavity and (b) concavity upwards in a neighborhood of f^*.

f^* where the concavity of the curve is upward (see Figure 12).

If the concavity of the curve is downward, solve Problem (10); otherwise, solve Problem (11) defined as follows:

$$\begin{aligned} &\max \quad \lambda \cdot f^*, \\ &\text{s.t.:} \quad \lambda \cdot y \leq \lambda \cdot f^*, \qquad \forall y \in A, \\ &\qquad \sum_{i=1}^{p} \lambda_i = 1, \ \lambda_i \geq 0, \quad i = 1, 2, ..., p, \end{aligned} \tag{10}$$

and:

$$\begin{aligned} &\min \quad \lambda \cdot f^*, \\ &\text{s.t.:} \quad \lambda \cdot f^* \leq \lambda \cdot y\,, \qquad \forall y \in A, \\ &\qquad \sum_{i=1}^{p} \lambda_i = 1, \ \lambda_i \geq 0, \quad i = 1, 2, ..., p. \end{aligned} \tag{11}$$

Then, the weighted vector λ^* is obtained as an optimal solution to Problem (10) or Problem (11) and set $\omega = \prod_{i=1}^{p} \lambda_i^*$.

For each $y^k \in PF$, $k \in \{1, 2, ..., N\}$, an optimal solution λ^l is obtained by solving Problem (10) or Problem (11), and the value $\omega_k = \Pi_{i=1}^{p} \lambda_i^k$ is calculated. Then, a set $\{\omega_1, \omega_2, ..., \omega_N\}$ is sorted in ascending order, and the set PF is ranked. Hence, a non-dominated point, which its corresponding ω has greater value, is better. The proposed method is applied for Examples 2, 3, and 4, and the final solutions obtained by this method are similar to the method proposed for ranking non-dominated points of convex MOPs (see Figure 13).

Example 5. A following problem introduced by Tanaka et al. [46] with a non-convex and disconnected Pareto front is formulated as follows:

$$\begin{aligned} &\min \quad f(x) = (x_1, \ x_2), \\ &\text{s.t.:} \quad x_1^2 + x_2^2 \geq 1 + 0.1 \cos\left(16 \arctan(x_1/x_2)\right), \\ &\qquad (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.5, \\ &\qquad 0 \leq x_1, x_2 \leq \pi. \end{aligned}$$

An approximation of the Pareto front with 80 points is illustrated in Figure 14(a). Then, the proposed method for ranking the obtained approximations of the Pareto front of a non-convex bi-objective problem is applied, and a ranking of points is obtained as follows. We divided the obtained non-dominated points into four groups with 20 points; see Figure 14(b). In addition, (0.1222,0.9712) is a final solution corresponding to a weight vector (0.4983,0.5017) is obtained. Table 1 reports information about the ranking of 20 points of the approximation in Rank 1 for Example 5.

It is worth mentioning that other criteria can be used to rank points of approximations of the Pareto front according to weighted vectors obtained in this method. In the following, the SAW and TOPSIS are explained, and final solutions to Examples 2 and 5 are obtained by these methods. In these methods, a weighted vector is determined by the DM, while the proposed methods obtain weighted vectors and do not depend on the DM.

4.1. The SAW method

The SAW method, referred to as a scoring method, is based on the concept of the weighted sum method. In this method, the DM directly determines a weighted vector $W = (w_1, w_2, ..., w_p)$ where w_i is the relative importance assigned to the ith objective function. The main principle of the SAW method is obtaining a score for each solution by getting a weighted sum of the normalized objective values and selecting a solution with the greater value as the final solution. The steps of this method are as follows:

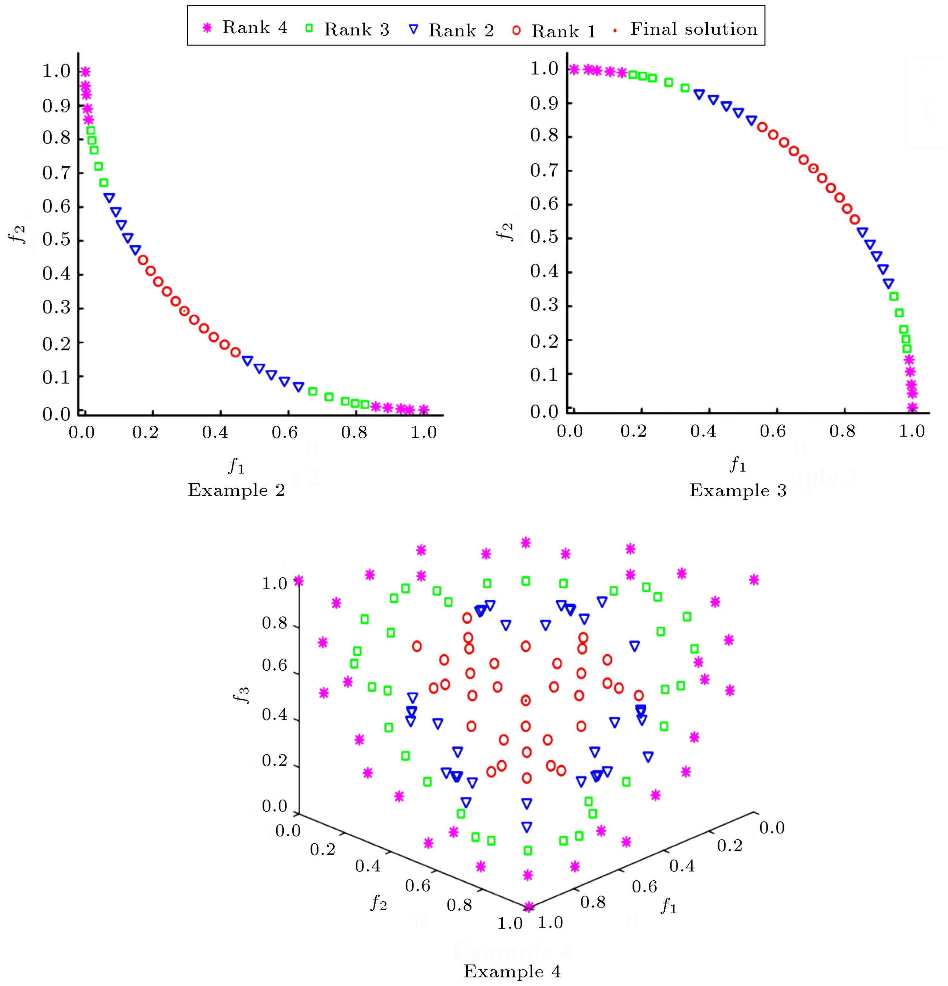

Figure 13. Ranking of the approximations of the Pareto front by the proposed method for non-convex problems for Examples 2, 3, and 4.

Table 1. Ranking of 20 points of the approximation in Rank 1 for Example 5.

k	y^k	λ^k	ω_k	k	y^k	λ^k	ω_k
1	(0.1222,0.9712)	(0.4983,0.5017)	0.2500	11	(0.1483,0.9481)	(0.4532,0.5468)	0.2478
2	(0.9712,0.1222)	(0.5017,0.4983)	0.2500	12	(0.9481,0.1483)	(0.5468,0.4532)	0.2478
3	(0.9834,0.1099)	(0.5023,0.4977)	0.2500	13	(0.7979,0.5159)	(0.4532,0.5468)	0.2478
4	(0.1099,0.9834)	(0.4977,0.5023)	0.2500	14	(0.5159,0.7979)	(0.5468,0.4532)	0.2478
5	(0.0976,0.9956)	(0.4853,0.5147)	0.2498	15	(0.7416,0.7416)	(0.4517,0.5483)	0.2477
6	(0.9956,0.0976)	(0.5147,0.4853)	0.2498	16	(0.7873,0.5294)	(0.5588,0.4412)	0.2465
7	(0.1348,0.9593)	(0.4850,0.5150)	0.2498	17	(0.5294,0.7873)	(0.4412,0.5588)	0.2465
8	(0.9593,0.1348)	(0.5150,0.4850)	0.2498	18	(1.0187,0.0719)	(0.5761,0.4239)	0.2442
9	(0.0851,1.0074)	(0.4614,0.5386)	0.2485	19	(0.0719,1.0187)	(0.4239,0.5761)	0.2442
10	(1.0074,0.0851)	(0.5386,0.4614)	0.2485	20	(0.9167,0.4545)	(0.3811,0.6189)	0.2359

Step 1: Generate a decision matrix $Y = [y_{ij}]_{N\times p}$ by using $PF = \{y^1, y^2, ..., y^N\}$ such that the ith row of Y is y^i.

Step 2: Generate a normalized decision matrix $Y' = [y'_{ij}]_{N\times p}$ in which $y'_{ij} = \frac{y_{ij}}{\min_{k=1,...,N} y_{kj}}$.

Step 3: Generate the weighted normalized matrix $Y'' = [y''_{ij}]_{N\times p}$ in which $y''_{ij} = w_j \times y'_{ij}$.

Step 4: Find the score of each point as $S_i = \sum_{j=1}^{p} {y''}_{ij}, \quad i = 1, ..., N$.

Step 5: Rank points based on values of S_i, $i = 1, ..., N$. Hence, a point with the largest value is a recommended point, and its corresponding solution in X is the final solution.

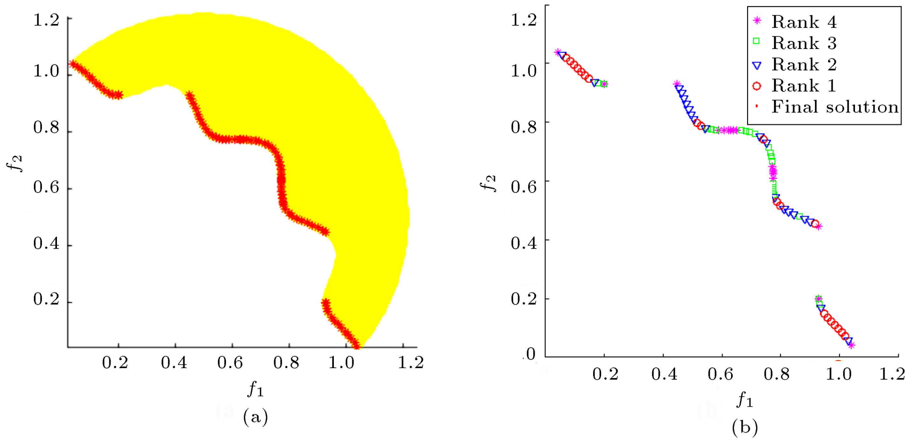

Figure 14. (a) The approximation of the Pareto front for Example 5 and (b) Ranking of the approximation of the Pareto front by the proposed method for Example 5.

Table 2. The final solutions of the proposed method, SAW and TOPSIS for Examples 2 and 5.

	The proposed method	**SAW with $W = (0.5, 0.5)$**	**TOPSIS with $W = (0.5, 0.5)$**
The final solution for Example 2	(0.2929,0.2929)	(0.0000,0.9995)	(0.2929,0.2929)
		SAW with $W = (0.25, 0.75)$	**TOPSIS with $W = (0.25, 0.75)$**
		(0.9995,0.0000)	(0.5693,0.0975)
	The proposed method	**SAW with $W = (0.5, 0.5)$**	**TOPSIS with $W = (0.5, 0.5)$**
The final solution for Example 2	(0.1222,0.9712)	(0.0417,1.0384)	(0.0417,1.0384)
		SAW with $W = (0.25, 0.75)$	**TOPSIS with $W = (0.25, 0.75)$**
		(1.0384,0.04166)	(1.0384,0.04166)

4.2. The TOPSIS method

The TOPSIS method selects a point with the smallest Euclidean distance from an ideal point and the largest Euclidean distance from a nadir point. Note that the nadir point consists of the worst values for all objectives in the set PF, while the ideal point combines the best values for all objectives. Note that the DM directly considers weights $\{w_1, w_2, ..., w_p\}$ of relative importance for each objective function. This method involves the following steps:

Step 1: Generate a decision matrix $Y = [y_{ij}]_{N \times p}$ as the same as the SAW method.

Step 2: Generate a normalized decision matrix $Y' = [y'_{ij}]_{N \times p}$ in which:

$$y'_{ij} = \frac{y_{ij}}{\sqrt{\sum_{k=1}^{N} y_{kj}^2}}, \qquad i = 1, ..., N, \quad j = 1, ..., p.$$

Step 3: Generate a weighted normalized matrix Y'' $= [y''_{ij}]_{N \times p}$ in which $y''_{ij} = w_j \times y'_{ij}$.

Step 4: Obtain an ideal point y^{Ideal} and nadir point y^{Nadir} by the weighted normalized matrix.

Step 5: Compute the distance between each point y^k from the ideal and nadir points as follows:

$$d_k^+ = \sqrt{\sum_{j=1}^{p} (y''_{kj} - y_j^{Ideal})^2} \quad \text{and}$$

$$d_k^- = \sqrt{\sum_{j=1}^{p} (y''_{kj} - y_j^{Nadir})^2} \quad \text{for} \quad k = 1, ..., N.$$

Step 6: Calculate the relative closeness of each point to the ideal point as follows:

$$C_i = \frac{d_i^-}{d_i^- + d_i^+} \quad \text{for } i = 1, ..., N.$$

Step 7: Rank the points based on the value of C; hence, the point having the largest C is the recommended point, and its corresponding solution in X is the final solution.

Figure 15 shows the final solutions obtained by the proposed method, SAW, and TOPSIS for Examples 2 and 5. Note that $W = (0.5, 0.5)$ and $W = (0.25, 0.75)$ are considered in SAW and TOPSIS. In addition, Table 2 reports the final solutions for these problems.

5. Conclusions

A posteriori method usually obtains many equally good

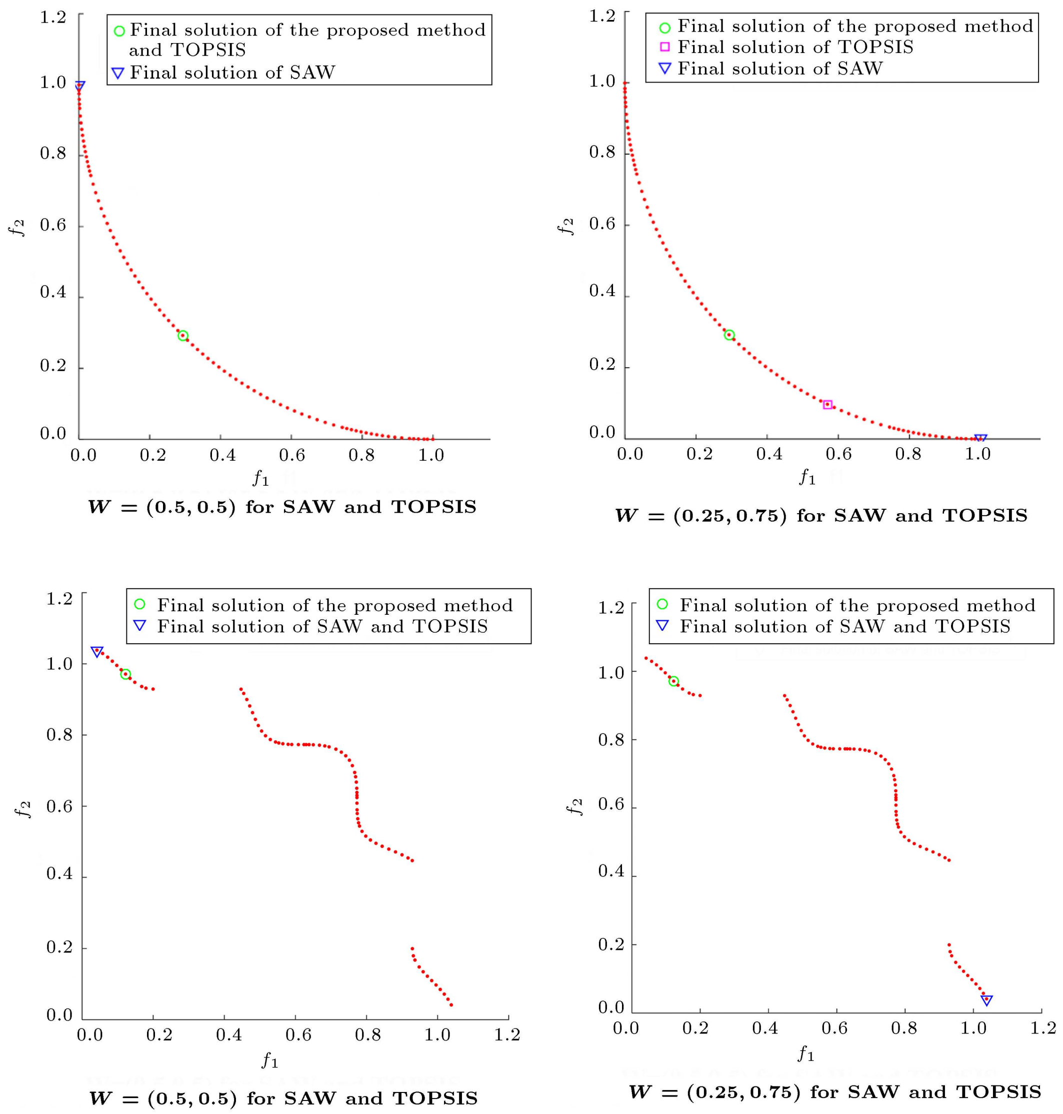

Figure 15. Determine the final solutions of the proposed method, SAW, and TOPSIS for Examples 2 and 5.

non-dominated points of an Multi-Objective Optimization Problem (MOP), and it is difficult for the Decision Maker (DM) to analyze all points and select the best one. To help the DM, a new scalarization approach was proposed in this paper to select a Pareto optimal solution for convex MOPs such that the relative importance assigned to its objective functions is the same. In addition, concepts of the weighted sum scalarization method were used to present two other decision-making approaches without any information about the MOP. The first one was proposed for analysis approximations of the Pareto front obtained for convex MOPs, and the second approach was considered for approximations of the Pareto front obtained for non-linear MOPs in the general case. These methods assigned a weight vector to each non-dominated point by considering all or some points of approximation and ranked them based on the product of the component of their weight vectors. Note that whatever the product of the component of a weight vector is greater, the component of its weight vector is very close together. Accordingly, a point with a maximum value of this product is a preferred point among the other points.

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