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Analysis of stationary fluid queue driven by state-dependent birth-death process subject to catastrophes

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Abstract. This paper investigates an infinite buffer fluid queueing model driven by a state-dependent birth-death process prone to catastrophes. We use the Laplace-Stieltjes transform and continued fraction approaches to establish precise expression for the joint probability of the content of the buffer and the number of customers in an M/M/1 queueing model. The importance of the proposed system is that, in numerous practical situation, the service facility has defence mechanisms in place to deal with long waits. Under the strain of a significant backlog of work, the servers may improve their service rate. Therefore, considering the state-dependent character of queueing systems is of relevance. For example, congestion control technologies prevent long queues forming in computer and communication systems by adjusting packet transmission speeds based on the length of the queue (of packets) at the source or destination. Theoretical results are supported by numerical illustrations.

1. Introduction

Analysis of fluid queueing systems has received significant attention of researchers over the past few years due to their wide spread applications in the field of production and manufacturing systems, traffic management on high-speed telecommunication networks, actuarial science, environmental systems, inventory systems, and population growth. A continuous fluid enters and exits a storage device, known as a buffer, at a randomly changing rate regulated by an external stochastic environment in the fluid queue. Stochastic fluid flow models are useful instruments for evaluating numerous

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performance measures including the distribution and moments of the fluid level in the area of congestion control, communication networks, risk processes, and other related systems. Some significant works on fluid flow models and their applications can be found in Anick et al. [1], Elwalid and Mitra [2], Kapoor and Dharmaraja [3], Mitra [4], Latouche and Taylor [5], Bekker and Mandjes [6], Stern and Elwalid [7], Knessl and Morrison [8], El-Baz et al. [9], and related references therein.

The behavior of fluid queueing models examined by several researchers can be classified into two types; one is the stationary and another being the non-stationary. Virtamo and Norros [10] derived as a simple integral formula for the steady-state buffer content distribution using the spectral decomposition technique for the M/M/1 fluid queue. Later, Adan and Resing [11] have shown the results of Virtamo and Norros [10] in the form of the modified Bessel function of the first kind of order one by observing at embedded time points. Parthasarathy et al. [12] carried out explicit expressions for the steady-state distribution of buffer occupancy and buffer content of a fluid queueing model driven by an M/M/1 queue based on the continued fraction technique. Barbot and Sericola [13] derived the solution of buffer level and state of the M/M/1 fluid queueing model using the generating function approach associated with exponential matrix. Konovalov [14] studied a more general GI/GI/1 fluid queueing model, where fluid flows out of the buffer at a constant rate. Kapoor and Dharmaraja [15] obtained an explicit transient solution of the fluid queue driven through birth death process via the continued fraction approach. Kapoor et al. [16] studied the transient distribution of the buffer content of fluid queueing model driven by two distinct restricted state birth-death processes. More works on fluid queueing models with stationary and non-stationary behaviors can be found in Maki [17], Lenin and Parthasarathy [18], Kapoor and Dharmaraja [19], van Doorn and Scheinhardt [20], Parthasarathy and Vijayashree [21], Arunachalama et al. [22], and to name a few.

In recent years, many researchers investigated on fluid queueing systems subject to catastrophes or vacations. The analysis of fluid queueing system with vacations can be found in Ammar [23], Zhang and Shi [24], Xu et al. [25], Mao et al. [26], Xu et al. [27], Mao et al. [28], Vijayashree and Anjuka [29], and related references therein. Moreover, many researcher discussed the behaviour of fluid queueing systems with catastrophes. Vijayalakshmi and Thangaraj [30] derived a simple closed form transient state probability distribution for a fluid queueing model driven with an M/M/1 queue, where a chain sequence shows birth and death rates, as well as disasters. Vijayalakshmi and Thangaraj [31] used the continued fraction technique to explore the transient state probability distribution for a fluid queueing model driven through an M/M/1 queue with disaster and restoration time. Vijayashree and Anjuka [32] investigated a fluid queueing model based on an M/M/1/N queue that has been damaged and needs to be repaired. Ammar [33] developed precise expression of the stationary distribution function of the buffer content for a fluid queue driven via M/M/1 disaster queue. Further, Ammar [34] studied the stationary behavior of M/M/1 fluid queueing model in a random setting with catastrophes. He presented the buffer content distribution as a modified Bessel function of the first kind via generating function technique.

On the other hand, analysis of state-dependent fluid queueing models is not straightforward although they are more appropriate for the application. However, it appears that very little research has been done on such fluid queues that are influenced through birth and death process. Maki [17] derived the distribution function of a birth and death process in which the denominator and numerator are generic polynomials, where the rates of birth and death are rational functions of the state of the process. Parthasarathy and Vijayashree [21] carried out the distribution of buffer content of a fluid queueing model driven through birth and death process with quadratic arrival and service rates on a limited state space. Kapoor and Dharmaraja [19] established the stationary behaviour of a fluid queueing model with rational birth and death rates driven through a limited birth death process. Kumar et al. [35] studied a state-dependent queueing system with catastrophes using continued fraction technique. Guillemin and Sericola [36] investigated the behaviour of a fluid queueing model controlled with a general ergodic birth and death process using spectral theory in the area of Laplace-Stieltjes Transform (L.-S.T.).

To the best of our expertise, no attempt has been made to analyze the behavior of buffer content distribution in a fluid queueing model driven via state dependent birth-death process subject to catastrophes. The interest to investigate the state dependent nature of fluid queueing systems with catastrophes is due to its importance in the field of computer and communication systems, where transmission rates of packets may be controlled based on current packets at source or destination. The effect of catastrophes (for example, virus attack) in computer system which is not protected by any antivirus software leads to the system sudden damage and all the works in progress are to be lost. In this study, we discuss the fluid queueing model subject to catastrophes in which arrival and service rates may depend on the number of customers present in the system. Based on the L.-S.T. and the continued fraction techniques, we derive the steady state joint probability of the content of the buffer and the number of customers in an M/M/1 fluid queueing model. Numerical results are presented in the form of graphs to show the effect of model parameters on the stationary buffer content distribution.

The remaining sections of the paper are arranged as follows. Section 2 provides assumption and explanation of the model. The stationary solution of the model is examined in Section 3. The numerical depiction of the buffer content distribution for various model parameters is presented in Section 4. Section 5 concludes the paper.

2. Model description

Consider a fluid queue operated by a state-dependent M/M/1 queueing system with catastrophe. The customers arrive in accordance with a Poisson process with a state-dependent arrival rate σ_{2j}, $j = 0, 1, 2, ...$, when the system contains j customers before his arrivals. The state-dependent service time of a customer is exponentially distributed with service rate σ_{2j-1}, $j = 1, 2, 3, ...$, when the system contains j customers just before his service starts. Further, the catastrophe occurs according to exponential distribution with rate γ. When the system encounters a disaster, all customers, including the one now in service, are lost, and the system becomes idle. During the busy period of the server, the fluid accumulates in an infinite capacity fluid buffer which we can be interpreted as a fluid reservoir at a constant rate $r > 0$. During the server's idle phase, the credit buffer depletes the fluid at a consistent rate $r_0 < 0$ as long as the buffer is not empty. Let $N(t)$, $t \geq 0$, denote the number of customers in the background state-dependent birth-death process with catastrophe at time t. If $C(t)$, $t \geq 0$, denotes the content of the fluid buffer at time t, then $\{(N(t), C(t))\}$ is a two-dimensional Markov process with a unique stationary distribution under appropriate stability condition. The mean aggregate input rate should be negative, that is, $r_0 p_0 + r\sum_{j=1}^{\infty} p_j < 0$, to ensure that the process $\{(N(t), C(t)), t \geq 0\}$ remains stable, where p_j, $j \in \{0, 1, 2, ...\}$ signifies the stationary probability of j customers in the background state-dependent M/M/1 queueing system with catastrophe. The rate at which the content of the fluid buffer changes over time t is given by:

$$\frac{dC(t)}{dt} = \begin{cases} 0, & N(t) = 0, \quad C(t) = 0 \\ r_0, & N(t) = 0, \quad C(t) > 0 \\ r, & N(t) > 0, \quad C(t) > 0 \end{cases}$$

The steady-state combined probability distribution function of the Markov process $\{(N(t), C(t)), t \geq 0\}$ is defined as:

$$F_j(u) = \lim_{t\to\infty} P\{N(t) = j, C(t) \leq u\},$$

$$t \geq 0, \quad u \geq 0, \quad j \geq 0,$$

with boundary conditions $F_0(0) = a$, $0 < a < 1$, needs to be determined, $F_j(0) = 0$, $j \geq 1$, $F_j(\infty) = p_j$, $j \geq 0$, and $r_0 F_0(u) + r\sum_{j=1}^{\infty} F_j(u) = 1$, $u > 0$.

3. Analysis of the model

In this section, we use the L.-S.T., power series and continued fraction to determine the stationary solution of the model. The steady-state Kolmogorov differential-difference equations for the Markov process $\{(N(t), C(t)), t \geq 0\}$ are obtained by monitoring the states of the system at time epochs t and $t + dt$. Hence, we have:

$$r_0 \frac{dF_0(u)}{du} = -(\sigma_0 + \gamma)F_0(u) + \sigma_1 F_1(u) + \gamma, \qquad (1)$$

$$r \frac{dF_j(u)}{du} = -(\sigma_{2j} + \sigma_{2j-1} + \gamma)F_j(u) + \sigma_{2j-2}F_{j-1}(u) + \sigma_{2j+1}F_{j+1}(u), \quad j \geq 1. \qquad (2)$$

To determine the constant a, adding Eqs. (1) and (2), we obtain:

$$r_0 \frac{dF_0(u)}{du} + r\sum_{j=1}^{\infty} \frac{dF_j(u)}{du} = 0. \qquad (3)$$

Integrating Eq. (3) with respect to u over 0 to 1, we have:

$$r_0(p_0 - a) + r\sum_{j=1}^{\infty} p_j = 0,$$

which yields:

$$a = \frac{(r_0 - r)p_0 + r}{r_0},$$

where p_j, $j \geq 0$, can be achieved by solving the following equations:

$$0 = -(\sigma_0 + \gamma)p_0 + \sigma_1 p_1 + \gamma, \qquad (4)$$

$$0 = -(\sigma_{2j} + \sigma_{2j-1} + \gamma)p_j + \sigma_{2j-2}p_{j-1} + \sigma_{2j+1}p_{j+1}, \quad j \geq 1, \qquad (5)$$

subject to the normalization condition:

$$\sum_{j=0}^{\infty} p_j = 1, \qquad (6)$$

for the background state-dependent M/M/1 queueing system with catastrophe.

From Eq. (5), for given σ_j, $j \geq 0$ and γ, we can express p_j, $j \geq 2$, in terms of p_0 and p_1. Using these values of p_j, $j \geq 2$, in Eqs. (4) and (6), we get two independent equations in two unknowns p_0 and p_1. Solving these two independent equations, we get the values of p_0 and p_1. Finally, p_j, $j \geq 2$, can be obtained recursively from Eq. (5).

Define the L.-S.T. of $F_j(u)$ as:

$$\widetilde{F}_j(s) = \int_0^\infty e^{-su} F_j(u) du, \quad Re(s) \geq 0. \tag{7}$$

Now, multiply Eqs. (1) and (2) by e^{-su} and integrating them over 0 to ∞, using Eq. (7), we have:

$$r_0 s\widetilde{F}_0(s) = -(\sigma_0 + \gamma)\widetilde{F}_0(s) + ar_0 + \sigma_1 \widetilde{F}_1(s) + \frac{\gamma}{s}, \tag{8}$$

$$rs\widetilde{F}_j(s) = -(\sigma_{2j} + \sigma_{2j-1} + \gamma)\widetilde{F}_j(s) + \sigma_{2j-2}\widetilde{F}_{j-1}(s) + \sigma_{2j+1}\widetilde{F}_{j+1}(s), \quad j \geq 1. \tag{9}$$

Multiplying $r_0 F_0(u) + r\sum_{n=1}^{\infty} F_n(u) = 1$ by e^{-su} and integrating over 0 to ∞, using Eq. (7), we obtain:

$$r_0\widetilde{F}_0(s) + r\sum_{j=1}^{\infty}\widetilde{F}_j(s) = \frac{1}{s}. \tag{10}$$

Using Eq. (10) in Eq. (8) and simplifying, we obtain:

$$\widetilde{F}_0(s) = \frac{a + \frac{\gamma}{r_0 s}}{\left(s + \frac{\gamma}{r}\right) + \frac{\sigma_0}{r_0} - \frac{\sigma_1}{r_0}\frac{\widetilde{F}_1(s)}{\widetilde{F}_0(s)}}. \tag{11}$$

Further, from Eq. (9) for $j \geq 1$ after repeated substitution, we obtained Eq. (12) as shown in Box I.

Using Eq. (12) for $j = 1$ in Eq. (11), we obtained Eq. (13) as shown in Box II.

One may note here that the J-fractions (or Jacobi fractions) Eqs. (12) and (13) can be represented as a S-fraction in the form of:

$$\phi(z) = \cfrac{\frac{1}{z}}{1 + \cfrac{\frac{c_0}{z}}{1 + \cfrac{\frac{c_1}{z}}{1 + \cfrac{\frac{c_2}{z}}{1 + \cdots}}}}. \tag{14}$$

Theorem 1. According to the property of S-fraction, see [37–39], if the solution of Jacobi-type continued fraction Eq. (12) can be expressed in the form of formal power series of $\frac{1}{s+\frac{\gamma}{r}}$, i.e.:

$$\frac{\widetilde{F}_j(s)}{\widetilde{F}_{j-1}(s)} = \frac{\sigma_{2j-2}}{r}\sum_{m=0}^{\infty}\frac{(-1)^m H(m,j)}{(s + \frac{\gamma}{r})^{m+1}}, \quad j = 1, 2, 3, \ldots, \tag{15}$$

then the values of $H(m, j)$ are given by:

$$H(0, j) = 1, \qquad j = 1, 2, 3, \ldots, \tag{16}$$

$$H(1, j) = \frac{1}{r}\sum_{i_1=2j-1}^{2j}\sigma_{i_1}, \qquad j = 1, 2, 3, \ldots, \tag{17}$$

$$H(m, j) = \frac{1}{r^m}\sum_{i_1=2j-1}^{2j}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}\sum_{i_3=2j-1}^{i_2+1}\sigma_{i_3}$$

$$\frac{\widetilde{F}_j(s)}{\widetilde{F}_{j-1}(s)} = \frac{\frac{\sigma_{2j-2}}{r}}{\left(s + \frac{\gamma}{r}\right) + \left(\frac{\sigma_{2j-1}+\sigma_{2j}}{r}\right) - \frac{\sigma_{2j+1}}{r}\frac{\widetilde{F}_{j+1}(s)}{\widetilde{F}_j(s)}}$$

$$= \cfrac{\frac{\sigma_{2j-2}}{r}}{\left(s + \frac{\gamma}{r}\right) + \left(\frac{\sigma_{2j-1}+\sigma_{2j}}{r}\right) - \cfrac{\frac{\sigma_{2j}\sigma_{2j+1}}{r^2}}{\left(s+\frac{\gamma}{r}\right) + \left(\frac{\sigma_{2j+1}+\sigma_{2j+2}}{r}\right) - \cfrac{\frac{\sigma_{2j+2}\sigma_{2j+3}}{r^2}}{\left(s+\frac{\gamma}{r}\right) + \left(\frac{\sigma_{2j+3}+\sigma_{2j+4}}{r}\right) - \cdots}}}. \tag{12}$$

Box I

$$\widetilde{F}_0(s) = \cfrac{a + \frac{\gamma}{r_0 s}}{\left(s + \frac{\gamma}{r}\right) + \frac{\sigma_0}{r_0} - \cfrac{\frac{\sigma_0\sigma_1}{r_0 r}}{\left(s+\frac{\gamma}{r}\right) + \left(\frac{\sigma_1+\sigma_2}{r}\right) - \cfrac{\frac{\sigma_2\sigma_3}{r^2}}{\left(s+\frac{\gamma}{r}\right) + \left(\frac{\sigma_3+\sigma_4}{r}\right) - \cfrac{\frac{\sigma_4\sigma_5}{r^2}}{\left(s+\frac{\gamma}{r}\right) + \left(\frac{\sigma_5+\sigma_6}{r}\right) - \cdots}}}}. \tag{13}$$

Box II

$$... \sum_{i_m=2j-1}^{i_{m-1}+1} \sigma_{i_m}, \qquad m = 2,3,4,...,$$

$$j = 1,2,3,.... \tag{18}$$

Proof. We prove the above results by mathematical induction on m for every j. Now, using Eq. (15) in Eq. (12), we have the equation as shown in Box III, which can be written as:

$$\left[\sum_{m=0}^{\infty} \frac{(-1)^m H(m,j)}{(s+\frac{\gamma}{r})^{m+1}}\right]\left[1+\left(\frac{\sigma_{2j-1}+\sigma_{2j}}{r}\right)\frac{1}{(s+\frac{\gamma}{r})} - \frac{\sigma_{2j}\sigma_{2j+1}}{r^2}\sum_{m=0}^{\infty}\frac{(-1)^m H(m,j+1)}{(s+\frac{\gamma}{r})^{m+2}}\right] = \frac{1}{(s+\frac{\gamma}{r})}. \tag{19}$$

Now, collecting the coefficient of $\frac{1}{(s+\frac{\gamma}{r})^m}$, $m = 1,2,3,...$, from both the sides of Eq. (19), we obtain:

$$H(0,j) = 1, \qquad j = 1,2,3,\ldots, \tag{20}$$

$$H(1,j) = \frac{1}{r}\sum_{i_1=2j-1}^{2j} \sigma_{i_1}, \qquad j = 1,2,3,\ldots, \tag{21}$$

$$H(m,j) = \left(\frac{\sigma_{2j-1}+\sigma_{2j}}{r}\right) H(m-1,j) + \frac{\sigma_{2j}\sigma_{2j+1}}{r^2}\sum_{k=0}^{m-2} H(k,j+1)H(m-2-k,j),$$
$$m = 2,3,4,\ldots, \quad j = 1,2,3,\ldots. \tag{22}$$

Using Eqs. (20) and (21) in Eq. (22) for $m = 2,3$ and simplifying, we obtain:

$$H(2,j) = \frac{1}{r^2}\sum_{i_1=2j-1}^{2j}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}, \qquad j = 1,2,3,\ldots.$$

$$H(3,j) = \frac{1}{r^3}\sum_{i_1=2j-1}^{2j}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}\sum_{i_3=2j-1}^{i_2+1}\sigma_{i_3},$$
$$j = 1,2,3,\ldots.$$

Hence, Eq. (18) holds for $m = 2$ and $m = 3$.

We now assume that Eq. (18) holds true for all non-negative integers up to $m-1$, i.e.:

$$H(m-1,j) = \frac{1}{r^{m-1}}\sum_{i_1=2j-1}^{2j}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}\sum_{i_3=2j-1}^{i_2+1}\sigma_{i_3}\cdots\sum_{i_{m-1}=2j-1}^{i_{m-2}+1}\sigma_{i_{m-1}}. \tag{23}$$

Further, it can be shown by mathematical induction that:

$$\sum_{k=0}^{m-2} H(k,j+1)H(m-2-k,j) = \frac{1}{r^{m-2}}\sum_{i_1=2j-1}^{2j+2}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}\cdots\sum_{i_{m-2}=2j-1}^{i_{m-3}+1}\sigma_{i_{m-2}}, \quad m \geq 4, \tag{24}$$

provided Eq. (18) is true for any non-negative integer up to $m-1$. Using Eqs. (23) and (24) in Eq. (22), we obtain:

$$H(m,j) = \left(\frac{\sigma_{2j-1}+\sigma_{2j}}{r^m}\right)\sum_{i_1=2j-1}^{2j}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}\sum_{i_3=2j-1}^{i_2+1}\sigma_{i_3}\cdots\sum_{i_{m-1}=2j-1}^{i_{m-2}+1}\sigma_{i_{m-1}} + \frac{\sigma_{2j}\sigma_{2j+1}}{r^m}\sum_{i_1=2j-1}^{2j+2}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2},\sum_{i_3=2j-1}^{i_2+1}\sigma_{i_3}\cdots\sum_{i_{m-2}=2j-1}^{i_{m-3}+1}\sigma_{i_{m-2}},$$

$$= \frac{\sigma_{2j-1}}{r^m}\sum_{i_1=2j-1}^{2j}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}\sum_{i_3=2j-1}^{i_2+1}\sigma_{i_3}\cdots\sum_{i_{m-1}=2j-1}^{i_{m-2}+1}\sigma_{i_{m-1}} + \frac{\sigma_{2j}}{r^m}\sum_{i_1=2j-1}^{2j+1}\sigma_{i_1}\sum_{i_2=2j-1}^{i_1+1}\sigma_{i_2}\sum_{i_3=2j-1}^{i_2+1}\sigma_{i_3}\cdots\sum_{i_{m-1}=2j-1}^{i_{m-2}+1}\sigma_{i_{m-1}}, \tag{25}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m H(m,j)}{(s+\frac{\gamma}{r})^{m+1}} = \frac{1}{(s+\frac{\gamma}{r}) + \left(\frac{\sigma_{2j-1}+\sigma_{2j}}{r}\right) - \frac{\sigma_{2j}\sigma_{2j+1}}{r^2}\sum_{m=0}^{\infty}\frac{(-1)^m H(m,j+1)}{(s+\frac{\gamma}{r})^{m+1}}}.$$

Box III

$$= \frac{1}{r^m} \sum_{i_1=2j-1}^{2j} \sigma_{i_1} \sum_{i_2=2j-1}^{i_1+1} \sigma_{i_2} \sum_{i_3=2j-1}^{i_2+1} \sigma_{i_3} \cdots \sum_{i_m=2j-1}^{i_{m-1}+1} \sigma_{i_m}, \qquad j = 1, 2, 3, ... \qquad (26)$$

Thus, the result is true for all $m \geq 1$.

Theorem 2. The solution of Jacobi-type continued Eq.(11) can be expressed in the following way:

$$F_0(u) = ae^{-\frac{\gamma u}{r}} \sum_{m=0}^{\infty} \frac{(-1)^m T(m,0) u^m}{m!} + \frac{\gamma}{r_0 r} \sum_{m=0}^{\infty} \frac{(-1)^m T(m,0)}{m!} I_m(u), \qquad (27)$$

where $I_m(u) = \int_0^u e^{-\frac{\gamma y}{r}} y^m dy$ is possible to evaluate as:

$$I_0(u) = \frac{r}{\gamma} \left(1 - e^{-\frac{\gamma u}{r}}\right),$$

$$I_m(u) = \frac{mr}{\gamma} I_{m-1}(u) - \frac{ru^m}{\gamma} e^{-\frac{\gamma u}{r}}, \quad m = 1, 2, 3, \ldots.$$

and the values of $T(m,0)$ are given by:

$$T(0,0) = 1, \qquad (28)$$

$$T(1,0) = b_0, \qquad (29)$$

$$T(m,0) = b_0 \sum_{i_1=0}^{1} b_{i_1} \sum_{i_2=0}^{i_1+1} b_{i_2} \cdots \sum_{i_{m-1}=0}^{i_{m-2}+1} b_{i_{m-1}},$$

$$m = 2, 3, 4, \ldots. \qquad (30)$$

where $i_0 = 0$, $b_0 = \frac{\sigma_0}{r_0}$, and $b_j = \frac{\sigma_j}{r}$, $j = 1, 2, 3, ...$

Proof. According to the property of S-fraction, see [37–39], the Jacobi-type continued fraction (13) can be represented as formal power series of $\frac{1}{s+\frac{\gamma}{r}}$ with explicit coefficients such as:

$$\widetilde{F}_0(s) = \left(a + \frac{\gamma}{r_0 r s}\right) \sum_{m=0}^{\infty} \frac{(-1)^m T(m,0)}{(s+\frac{\gamma}{r})^{m+1}}. \qquad (31)$$

Using Eq. (15), for $j = 1$, and Eq. (31) in Eq. (11), we have:

$$\left[\sum_{m=0}^{\infty} \frac{(-1)^m T(m,0)}{(s+\frac{\gamma}{r})^{m+1}}\right] \left[\left(s + \frac{\gamma}{r}\right) + \frac{\sigma_0}{r_0} - \frac{\sigma_0 \sigma_1}{r_0 r} \sum_{m=0}^{\infty} \frac{(-1)^m H(m,1)}{(s+\frac{\gamma}{r})^{m+1}}\right] = 1. \qquad (32)$$

Now, collecting the coefficient of $\frac{1}{(s+\frac{\gamma}{r})^m}$, $m = 0, 1, 2, ...,$ from both the sides of Eq. (32), we obtain:

$$T(0,0) = 1, \qquad (33)$$

$$T(1,0) = \frac{\sigma_0}{r_0}, \qquad (34)$$

$$T(m,0) = \frac{\sigma_0}{r_0} \left[T(m-1,0) + \frac{\sigma_1}{r} \sum_{j=0}^{m-2} T(j,0) H(m-2-j,1)\right], \qquad m = 2, 3, 4, \ldots. \qquad (35)$$

Substitute $m = 2, 3$ in Eq. (35) and simplifying, we obtain:

$$T(2,0) = \frac{\sigma_0}{r_0} \left(\frac{\sigma_0}{r_0} + \frac{\sigma_1}{r}\right) = b_0 \sum_{i_1=0}^{1} b_{i_1},$$

$$T(3,0) = b_0 \left[b_0 \sum_{i_1=0}^{1} b_{i_1} + b_1 \sum_{i_1=0}^{2} b_{i_1}\right] = b_0 \sum_{i_1=0}^{1} b_{i_1} \sum_{i_2=0}^{i_1+1} b_{i_2}.$$

Assume that the result Eq. (30) holds true for all non-negative integers up to $m-1$, i.e.:

$$T(m-1,0) = b_0 \sum_{i_1=0}^{1} b_{i_1} \sum_{i_2=0}^{i_1+1} b_{i_2} \cdots \sum_{i_{m-2}=0}^{i_{m-3}+1} b_{i_{m-2}}. \qquad (36)$$

Further, it can be shown by mathematical induction that:

$$\sum_{j=0}^{m-2} T(j,0) H(m-2-j,1) = \sum_{i_1=0}^{2} b_{i_1} \sum_{i_2=0}^{i_1+1} b_{i_2} \cdots \sum_{i_{m-2}=0}^{i_{m-3}+1} b_{i_{m-2}}, \quad m \geq 4, \qquad (37)$$

provided Eq. (30) is true for any non-negative integer up to $m-1$. Using Eqs. (36) and (37) in Eq. (35), we obtain:

$$T(m,0) = b_0 \left[b_0 \sum_{i_1=0}^{1} b_{i_1} \sum_{i_2=0}^{i_1+1} b_{i_2} \cdots \sum_{i_{m-2}=0}^{i_{m-3}+1} b_{i_{m-2}} + b_1 \sum_{i_1=0}^{2} b_{i_1} \sum_{i_2=0}^{i_1+1} b_{i_2} \cdots \sum_{i_{m-2}=0}^{i_{m-3}+1} b_{i_{m-2}}\right]$$

$$= b_0 \sum_{i_1=0}^{1} b_{i_1} \sum_{i_2=0}^{i_1+1} b_{i_2} \cdots \sum_{i_{m-1}=0}^{i_{m-2}+1} b_{i_{m-1}}.$$

Thus, the result is true for all $m \geq 1$.

Now, taking the inverse Laplace transform of Eq. (31), we have the desired result.

Theorem 3. The steady-state joint probability distribution function $F_j(u)$, $j = 1, 2, \ldots$, $u > 0$, can be written as:

$$F_j(u) = \left(\frac{\sigma_0\sigma_2 \ldots \sigma_{2j-2}}{r^j}\right) \left[a e^{-\frac{\gamma u}{r}} \sum_{m=0}^{\infty} \frac{(-1)^m A(m,j) u^{m+j}}{(m+j)!} + \frac{\gamma}{r_0 r} \sum_{m=0}^{\infty} \frac{(-1)^m A(m,j)}{(m+j)!} I_{m+j}(u) \right], \quad j \geq 1, \tag{38}$$

where:

$$A(m,1) = \sum_{r=0}^{m} H(r,1) T(m-r,0), \quad m \geq 0, \tag{39}$$

$$A(m,j) = \sum_{r=0}^{m} H(r,j) A(m-r,j-1), \quad m \geq 0, \quad j \geq 2. \tag{40}$$

Proof. Using Eq. (31) in Eq. (15) and after repeated substitution, we obtain:

$$\widetilde{F}_j(s) = \left(\frac{\sigma_0\sigma_2 \ldots \sigma_{2j-2}}{r^j}\right) \left(a + \frac{\gamma}{r r_0 s}\right) \sum_{m=0}^{\infty} \frac{(-1)^m A(m,j)}{\left(s + \frac{\gamma}{r}\right)^{m+j+1}}, \quad j \geq 1. \tag{41}$$

Now, taking the inverse Laplace transform of Eq. (41) we have the desired result.

According to the S-fraction property, see [11,12,25,34], the stationary probability distribution of the buffer content for the fluid model under discussion is given by:

$$F(u) = P\{C \leq u\} = 1 - \sum_{j=1}^{\infty} F_j(u), \quad u \geq 0. \tag{42}$$

4. Numerical illustration

This section shows how varying values of model parameters affect the buffer content distribution of a fluid queueing model operated via an M/M/1 queue with catastrophe. For this purpose, the interarrival and service times are assumed to be exponentially distributed with parameters λ and μ, respectively. We further assume that the disaster occurs at the service facility as a Poisson process with an occurrence rate of γ. Moreover, we assume that $F_j(0) = 0$, $j = 1, 2, \ldots$ and $F_0(0) = 0.03$. The buffer content distribution's behavior in comparison to the buffer content u is depicted in Figure 1 for $\mu = 1.3$, $\gamma = 0.7$, $r = 4$, and $r_0 = -2.5$. It is noticed that $F(u)$ increases in a monotonic manner when the buffer content u increases. In addition, we can see in Figure 1 that the buffer content distribution grows as the arrival rate λ rises. Figure 2 shows how the buffer content distribution $F(u)$ behaves in relation to the buffer content u for $\lambda = 0.9$, $\gamma = 0.7$, $r = 4$, and $r_0 = -2.5$. The performance of the buffer content distribution $F(u)$ in relation to the buffer content u is depicted in Figure 3 for $\lambda = 1$, $\mu = 1.5$, $r = 4$, and $r_0 = -2.5$. It is also observed in Figures 2 and 3 that $F(u)$ increases monotonically with the increase of buffer content u. On the contrary, we noticed in Figures 2 and 3 that the buffer content distribution decrease with the increase of the service rate μ and disaster rate γ, respectively.

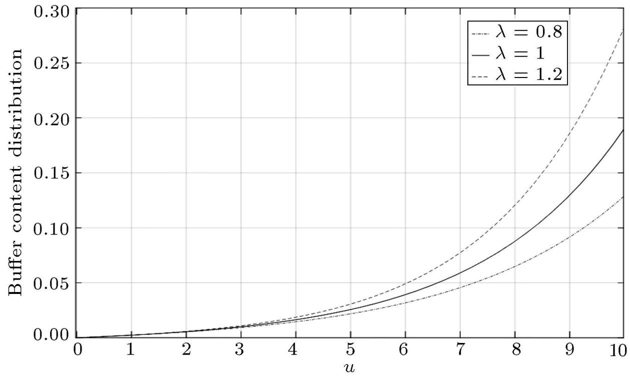

Figure 1. Demonstrates the influence of the arrival rate on $F(u)$ with $\mu = 1.3$, $\gamma = 0.7$, $r = 4$, and $r_0 = -2.5$.

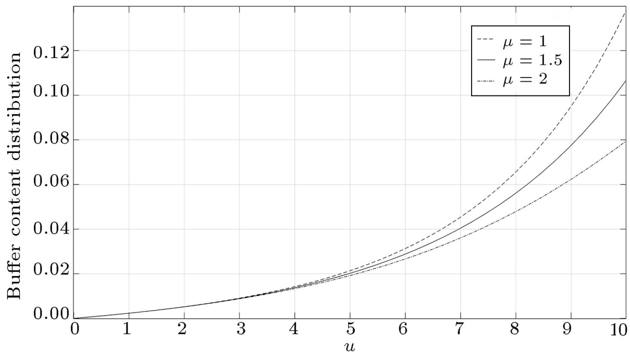

Figure 2. Illustrates the effect of service rate on $F(u)$ with $\lambda = 0.9$, $\gamma = 0.7$, $r = 4$, and $r_0 = -2.5$.

5. Conclusion

In this study, we investigated a stationary fluid queue operated by a state-dependent birth-death process that is prone to catastrophes. We developed precise analytical formulas in terms of power series coefficients

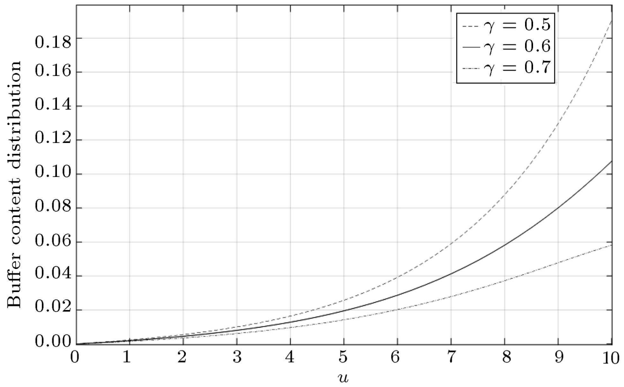

Figure 3. Shows the impact of the catastrophe rate on $F(u)$ with $\lambda = 1$, $\mu = 1.5$, $r = 4$, and $r_0 = -2.5$.

for the combined probability of the buffer content and the number of customers in an M/M/1 queueing model. Numerical illustrations are employed to support these theoretical insights.

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References

1. Anick, D., Mitra, D., and Sondhi, M.M. "Stochastic theory of a data-handling system with multiple sources", *Bell Syst. Tech. J.*, **61**, pp. 1871–1894 (1982). DOI: 10.1002/j.1538-7305.1982.tb03089.x
2. Elwalid, A.I. and Mitra, D. "Analysis and design of rate-based congestion control of high speed networks, I: Stochastic fluid models, access regulation", *Queueing Syst.*, **9**, pp. 29–63 (1991).
3. Kapoor, S. and Dharmaraja, S. "Applications of fluid queues in rechargeable batteries", In: *Applied Probability and Stochastic Processes*, Joshua, V., Varadhan, S., Vishnevsky, V. (Eds), Infosys Science Foundation Series, pp. 91–101, Springer, Singapore (2020). https://doi.org/10.1007/978-981-15-5951-8_7
4. Mitra, D. "Stochastic theory of a fluid model of producers and consumers coupled by a buffer", *Adv. Appl. Probab.*, **20**(3), pp. 646–676 (1988).
5. Latouche, G. and Taylor, P.G. "A stochastic fluid model for an ad hoc mobile network", *Queueing Syst.*, **63**, pp. 109–129 (2009). DOI: 10.1007/s11134-009-9153-6
6. Bekker, R. and Mandjes, M. "A fluid model for a relay node in an ad hoc network: The case of heavy-tailed input", *Math. Methods Oper. Res.*, **70**, pp. 357–384 (2009). DOI: 10.1007/s00186-008-0272-3
7. Stern, T.E. and Elwalid, A.I. "Analysis of separable Markov-modulated rate models for information-handling systems", *Adv. Appl. Probab.*, **23**(1), pp. 105–139 (1991). DOI: https://doi.org/10.2307/1427514
8. Knessl, C. and Morrison, J.A. "Heavy-traffic analysis of a data-handling system with many sources", *SIAM J. Appl. Math.*, **51**(1), pp. 187–213 (1991).
9. El-Baz, A.H., Tarabia, A.M.K., and Darwiesh, A.M. "Cloud storage facility as a fluid queue controlled by Markovian queue", *Probab. Eng. Inf. Sci.*, **36**(2), pp. 1–17 (2020).
10. Virtamo, J. and Norros, I. "Fluid queue driven by an $M/M/1$ queue", *Queueing Syst.*, **16**, pp. 373–386 (1994). https://doi.org/10.1007/BF01158963
11. Adan, I. and Resing, J. "Simple analysis of a fluid queue driven by an $M/M/1$ queue", *Queueing Syst.*, **22**, pp. 171–174 (1996). https://doi.org/10.1007/BF01159399
12. Parthasarathy, P.R., Vijayashree, K.V., and Lenin, R.B. "An $M/M/1$ driven fluid queue - continued fraction approach", *Queueing Syst.*, **42**, pp. 189–199 (2002). https://doi.org/10.1023/A:1020157021703
13. Barbot, N. and Sericola, B. "Stationary solution to the fluid queue fed by an $M/M/1$ queue", *J. Appl. Probab.*, **39**, pp. 359–369 (2002). DOI: 10.1239/jap/1025131431
14. Konovalov, V. "Fluid queue driven by a $GI/G/1$ queue, stable problems for stochastic models", *J. Math. Sci.*, **91**, pp. 2971–2930 (1998).
15. Kapoor, S. and Dharmaraja, S. "On the exact transient solution of fluid queue driven by a birth death process with specific rational rates and absorption", *Opsearch*, **52**(4), pp. 746–755 (2015) https://doi.org/10.1007/s12597-015-0199-4
16. Kapoor, S., Dharmaraja, S., and Arunachalam, V. "Transient solution of fluid queue modulated by two independent birth-death processes", *Int. J. Oper. Res.*, **36**(1), pp. 1–11 (2019). DOI: 10.1504/IJOR.2019.10023654
17. Maki, D.P. "On birth-death processes with rational growth rates", *SIAM J. Math. Anal.*, **7**, pp. 29–36 (1976). https://doi.org/10.1137/0507004
18. Lenin, R.B. and Parthsarathy, P.R. "A computational approach for fluid queues driven by truncated birth-death processes", *Methodol. Comput. Appl. Probab.*, **2**, pp. 373–392 (2000). https://doi.org/10.1023/A:1010010201531
19. Kapoor, S. and Dharmaraja, S. "Steady state analysis of fluid queues driven by birth death processes with rational rates", *Int. J. Oper. Res.*, **37**(4), pp. 562–578 (2020). https://doi.org/10.1504/IJOR.2020.105768

20. Van Doorn, E.A. and Scheinhardt, W.R.W. "A fluid queue driven by an infinite-state birth-death process", *Teletraffic Science and Engineering*, **2**, Part A, pp. 465–475 (1997). https://doi.org/10.1016/S1388-3437(97)80050-9

21. Parthasarathy, P.R. and Vijayashree, K.V. "Fluid queues driven by birth and death processes with quadratic rates", *Int. J. Comput. Math.*, **80**, pp. 1385–1395 (2003). https://doi.org/10.1080/0020716031000120836

22. Arunachalam, V., Gupta, V., and Dharmaraja, S. "A fluid queue modulated by two independent birth-death processes", *Comput. Math. with Appl.*, **60**(8), pp. 2433–2444 (2010). DOI: 10.1016/j.camwa.2010.08.039

23. Ammar, S.I. "Analysis of an $M/M/1$ driven fluid queue with multiple exponential vacations", *Appl. Math. Comput.*, **227**, pp. 227–334 (2014). DOI: 10.1016/j.amc.2013.10.084

24. Zhang, H. and Shi, D. "The $M/M/1$ queue with Bernoulli-schedule-controlled vacation and vacation interruption", *Int. J. Inf. Manage.*, **20**, pp. 579–587 (2009).

25. Xu, X., Geng, J., Liu, M., et al. "Stationary analysis for fluid model driven by the M/M/C working vacation queue", *J. Math. Anal. Appl.*, **403**, pp. 423–433 (2013). https://doi.org/10.1016/j.jmaa.2013.02.021

26. Mao, B.W., Wang, F.W., and Tian, N.S. "Fluid model driven by an $M/M/1$ queue with multiple vacations and N-policy", *J. Appl. Math. Comput.*, **38**, pp. 119–131 (2012).

27. Xu, X., Guo, H., Zhao, Y., et al. "The fluid model driven by the $M/M/1$ queue with working vacations and vacation interruption", *J. Comput. Inf. Syst.*, **8**, pp. 7643–7651 (2012).

28. Mao, B., Wang, F., and Tian, N. "Fluid model driven by an $M/M/1/N$ queue with multiple exponential vacations", *J. Comput. Inf. Syst.*, **6**, pp. 1809–1816 (2010). DOI:10.1109/ITCS.2010.138

29. Vijayashree, K.V., and Anjuka, A. "Stationary analysis of a fluid queue driven by an $M/M/1$ queue with working vacation", *Qual. Technol. Quant. Manag.*, **13**, pp. 1–22 (2016).

30. Vijayalakshmi, T. and Thangaraj, V. "Transient analysis of a fluid queue driven by a chain sequenced birth and death process with catastrophes", *Int. J. Math. Oper.*, **8**(2), pp. 164–184 (2016). http://localhost:8080/jspui/handle/1/84782

31. Vijayalakshmi, T. and Thangaraj, V. "A fluid model driven by an $M/M/1$ queue with catastrophe and restoration time", *Int. J. Appl. Math.*, **26**, pp. 123–135 (2013). DOI: 10.12732/ijam.v26i2.1

32. Vijayashree, K.V. and Anjuka, A. "Stationary analysis of a fluid queue driven by an $M/M/1/N$ queue with disaster and subsequent repair", *Int. J. Oper. Res.*, **31**(4), pp. 461–477 (2018). DOI: 10.1504/IJOR.2018.10011462

33. Ammar, S.I. "Fluid queue driven by an $M/M/1$ disasters queue", *Int. J. Comput. Math.*, **91**, pp. 1497–1506 (2014). https://doi.org/10.1080/00207160.2013.844799

34. Ammar, S.I. "Fluid $M/M/1$ catastrophic queue in a random environment", *RAIRO Oper. Res.*, **55**, pp. 2677–2690 (2021). https://doi.org/10.1051/ro/2020100

35. Kumar, B.K., Vijayakumar, A., and Sophia, S. "Transient analysis for state-dependent queues with catastrophes", *Stoch. Anal. Appl.*, **26**, pp. 1201–1217 (2008).

36. Guillemin, F. and Sericola, B. "On the fluid queue driven by an ergodic birth and death process", *Telecommunications Networks Current Status and Future Trends*, pp. 379–404 (2012).

37. Henrici, P. "Applied and computational complex analysis", *Special Functions, Integral Transforms, Asymptotics, Continued Fractions*, **2**, John Wiley & Sons, Inc. (1977). https://doi.org/10.1080/07362990802405786

38. Flajolet, P. and Guillemin, F. "The formal theory of birth-and-death processes, lattice path combinatorics and continued fractions", *Adv. Appl. Probab.*, **32**(3), pp. 750–778 (2000). DOI: 10.1017/S0001867800010247

39. Guillemin, F. and Pinchon, D. "Excursions of birth and death processes, orthogonal polynomials, and continued fractions", *J. Appl. Probab.*, **36**(3), pp. 752–770 (1999). https://www.jstor.org/stable/3215439

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