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A lower bound on the stretch factor of Yao graph Y_4

D. Bakhshesh[a,*] and M. Farshi[b]

a. *Department of Computer Science, University of Bojnord, Bojnord, Iran.*
b. *Combinatorial and Geometric Algorithms Lab., Department of Mathematical Sciences, Yazd University, P.O. Box 89195-747, Yazd, Iran.*

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Abstract. One of the most popular graphs in computational geometry is Yao graph, denoted by Y_k. For every point set S in the plane and an integer $k \geq 2$, Yao graph Y_k is constructed in the following. Around each point $p \in S$, the plane is divided into k regular cones with the apex at p. The set of all these cones is denoted by C_p. Then, for each cone $C \in C_p$, an edge (p, r) is added to Y_k, where r is the closest point in C to p. This study provides a lower bound of 3.8285 for the stretch factor of Y_4 which can partially solve an open problem posed by Barba et al. (L. Barba et al., "New and Improved Spanning Ratios for Yao Graphs", Journal of Computational Geometry, 6(2), pp. 19–53 2015).

1. Introduction

Assume that S is a set of points in the plane. A weighted graph $G = (S, E)$ is called *geometric* if every edge $e = (p, q)$ in G is a straight line between p and q, and the weight of e is $|pq|$, i.e., the Euclidean distance between p and q. The length of a path between two points $p, q \in S$ is defined as the sum of the weights of all edges of the path. For any two points $p, q \in S$, the *stretch factor* of the pair p and q is the ratio of the length of the shortest path between p and q over $|pq|$. The *stretch factor* (or *dilation*) of G is the maximum stretch factor between any pair of vertices of G. For a real number $t > 1$, a geometric graph G is called a *t-spanner* if the stretch factor of G is at most t. We refer the reader to the book [1] and the papers [2–10] for an overview of t-spanners and related algorithms.

One of the most popular graphs in computational geometry is *Yao graph*, denoted by Y_k, introduced by Flinchbaugh and Jones [11] and Yao [12] independently. For every point set S in the plane and an integer $k \geq 2$, the Yao graph Y_k is constructed as follows. Around each point $p \in S$, the plane is partitioned into k regular cones with the apex at p. The set of all these cones is denoted by C_p. Then, for each cone $C \in C_p$, we add an edge (p, r) to Y_k, where r is the closest point in C to p. Throughout the paper, the *nearest* point to p in C_p is called point r. Other popular graphs are *theta-graphs* (Θ_k) first introduced by Clarkson [13] and then independently by Keil [14]. The construction of Θ_k is similar to that of Yao graph Y_k, except that the definition of the nearest point is changed as follows: A point $r \in C_p$ is the nearest point to p if the orthogonal projection of r, denoted by r', onto the bisector of C_p minimizes $|pr'|$.

Clarkson [13] first proved that Y_{12} was a $(1+\sqrt{3})$-spanner. Then, Althöfer et al. [15] proved that there was an integer k such that Y_k was a t-spanner for every $t > 1$. For $k > 8$, Bose et al. [16] demonstrated that Y_k was a t-spanner with $t \leq 61/(\cos\theta - \sin\theta)$ where $\theta = \frac{2\pi}{k}$. Then, Bose et al. [17] found that for $k > 6$, Y_k

*. *Corresponding author. Tel.: +98 5832201000;*
Fax: +98 5832284605
E-mail addresses: d.bakhshesh@ub.ac.ir (D. Bakhshesh); mfarshi@yazd.ac.ir (M. Farshi)

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Table 1. Lower and upper bounds on the stretch factor of Y_k.

$\mathbf{Y_k}$	**Lower bound**	**Upper bound**
Y_6	2 [19]	5.8 [19]
Y_5	2.87 [19]	3.74 [19]
Y_4	**3.8285 (this paper)**	54.62 [20]
Y_2, Y_3	∞	Not a spanner [22]

was a $1/(1-2\sin(\theta/2))$-spanner. Damian and Raudonis [18] also proved that Y_6 had the stretch factor at most 17.64. This was later improved by Barba et al. to 5.8 [19]. The authors in [19] improved the stretch factor of Y_k for all odd values of $k \geq 5$ to $1/(1-2\sin(3\theta/8))$. Bose et al. [17] proved that Y_4 was a 663-spanner. Damian and Nelavalli [20] enhanced the upper bound of 663 on Y_4 to 54.62. In [19], Barba et al. provided the lower bounds of 2.87 and 2 on the stretch factor of Y_5 and Y_6, respectively. Some of the previous results on the stretch factor of Yao graphs are summarized in Table 1.

The current study provides a lower bound of 3.8285 on the stretch factor of Y_4, thus partially solving an open problem posed by Barba et al. [19].

2. Main result

The present study aimed to provide a lower bound of 3.8285 for the stretch factor of Y_4. In this regard, a point set S was presented where the stretch factor of the graph Y_4 on S was 3.8285. The construction of S is a modification of the construction of the point set $S0$, which was proposed by Barba et al. [21], for Θ_4. Let $C_0(a)$, $C_1(a)$, $C_2(a)$, and $C_3(a)$ be the four regular cones with their apex at a of angle 90° that equally partitions the plane. Assume that the first ray of $C_0(a)$ is in the direction of the positive x-axis. In addition, suppose that the cones $C_0(a)$, $C_1(a)$, $C_2(a)$, and $C_3(a)$ are in counterclockwise order around a. Let $D_i(a,b)$ be a quarter of a closed disk with the center and radius of a and $|ab|$, respectively, such that cone $C_i(a)$ contains $D_i(a,b)$ (see Figure 1).

In the following steps, the construction of the point set S is illustrated:

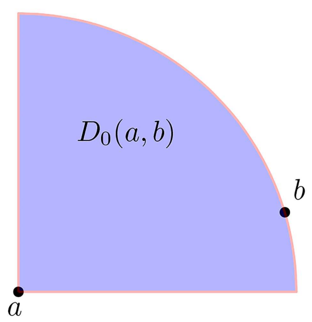

Figure 1. The region $D_0(a,b)$.

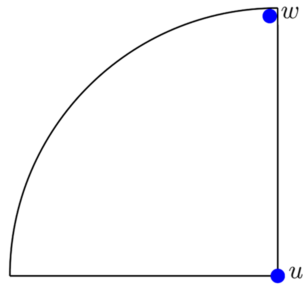

Figure 2. The point set S: Step 1.

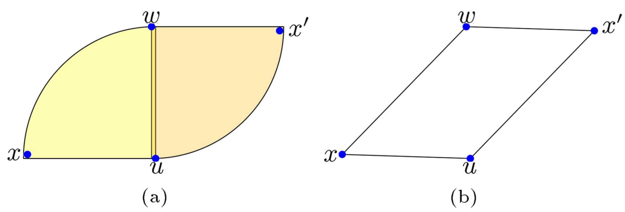

Figure 3. (a) The point set S. (b) The Yao graph Y_4 on S.

- **Step 1.** Let $u=(0,0)$ and $w=(-\epsilon_w, 1-\epsilon_w)$, where $\epsilon_w > 0$ is a small real number. The construction begins with $S=\{u,w\}$, and $C_1(u)$ contains w (see Figure 2). Clearly, the graph Y_4 on S contains the single edge (u,w). Then, the stretch factor of Y_4 on S is found to be 1. The idea to obtain the lower bound 3.8285 is that it can facilitate the extension of the shortest path between u and w by adding some extra points to S.
- **Step 2.** Let $\epsilon_x > 0$ be a small real number such that the point $x=(\epsilon_x - 1, \epsilon_x)$ is inside $D_1(u,w)$. Let M be the mid-point of the segment uw and x' the reflection of x with respect to M. Obviously, x' is inside $D_3(w,u)$. Now, add x and x' to S (see Figure 3(a)). As a result, we have $S=\{u,w,x,x'\}$. If we draw the graph Y_4 on S, then we do not have the edge (u,w), hence extension of the shortest path between u and w (see Figure 4(b)). The shortest path between u and w is $P_{uw} := uxw$ or $Q_{uw} := ux'w$. Then, the length of P_{uw} can be obtained as follows. Note that by the symmetric arguments, $|Q_{uw}| = |P_{uw}|$.

$$|P_{uw}| = |ux| + |xw| = \sqrt{(\epsilon_x - 1)^2 + \epsilon_x^2} + \sqrt{2(\epsilon_x - 1 + \epsilon_w)^2}.$$

Now, we have:

$$\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_w \to 0}} \frac{|P_{uw}|}{|uw|} = \frac{1+\sqrt{2}}{1} \approx 2.4143.$$

Compared to Step 1, upon extending the shortest paths between u and w, the stretch factor between u and w will increase up to 2.4143.

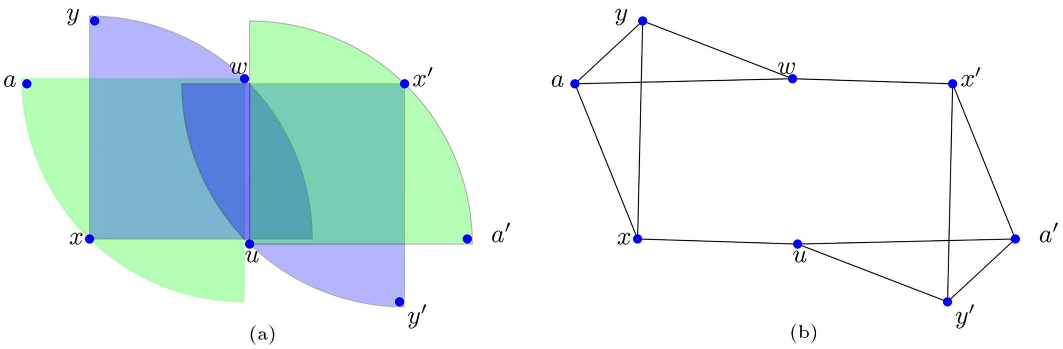

Figure 4. (a) The point set S. (b) The Yao graph Y_4 on S.

- **Step 3.** Now, add some extra points to S to extend the shortest path between u and w. Let $\epsilon_y > 0$ and $\epsilon_a > 0$ be two small real numbers such that $y = (\epsilon_y - 1, |xw| - \epsilon_y)$ and $a = (\epsilon_a - |xw|, 1 - \epsilon_a)$ are the points inside $D_0(x,w)$ and $D_2(w,x)$, respectively (see Figure 4(a)). Let y' and a' be the reflection of the points of y and a with respect to M, respectively. It is clear that y' and a' are inside $D_2(x',u)$ and $D_0(u,x')$, respectively. Now, add y, a, y', and a' to the point set S. As a result, $S = \{u, w, x, x', y, a, y', a'\}$.

  There are four short paths between u and w, including: $P_1 := uxaw$, $P_2 := uxyw$, $Q_1 := ua'x'w$, and $Q_2 := uy'x'w$. Now, compute the length of the paths P_1 and P_2 based on the following equations:

$$|P_1| = |ux| + |xa| + |aw| = \sqrt{(\epsilon_x - 1)^2 + \epsilon_x^2}$$

$$+\sqrt{(\epsilon_x - 1 - \epsilon_a + |xw|)^2 + (\epsilon_x + \epsilon_a - 1)^2}$$

$$+\sqrt{(\epsilon_a + \epsilon_w - |xw|)^2 + (\epsilon_w - \epsilon_a)^2}.$$

$$|P_2| = |ux| + |xy| + |yw| = \sqrt{(\epsilon_x - 1)^2 + \epsilon_x^2}$$

$$+\sqrt{(\epsilon_x - \epsilon_y)^2 + (\epsilon_x + \epsilon_y - |xw|)^2}$$

$$+\sqrt{(\epsilon_y + \epsilon_w - 1)^2 + (|xw| - \epsilon_y + \epsilon_w - 1)^2}.$$

  Now, we have:

$$\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_a \to 0 \\ \epsilon_w \to 0}} \frac{|P_1|}{|uw|} = \lim_{\substack{\epsilon_x \to 0 \\ \epsilon_y \to 0 \\ \epsilon_w \to 0}} \frac{|P_2|}{|uw|}$$

$$= \frac{1 + \lim_{\substack{\epsilon_x \to 0 \\ \epsilon_w \to 0}} \left(\sqrt{1 + (|xw| - 1)^2} + |xw|\right)}{1}$$

$$= 1 + \sqrt{1 + (\sqrt{2} - 1)^2} + \sqrt{2}$$

$$= 1 + \sqrt{4 - 2\sqrt{2}} + \sqrt{2} \approx 3.49661.$$

  Based on the symmetric arguments, we have $|Q_1| = |P_1|$ and $|Q_2| = |P_2|$. In comparison to Step 2, upon extending the shortest paths between u and w, the stretch factor between u and w will increase up to 3.4966.

- **Step 4** (**final step**). This step shows how to extend the shortest path between u and w by adding some other extra points to S. Let $\epsilon_b, \epsilon_c, \epsilon_d, \epsilon_e > 0$ be the four small real numbers such that the points:

$$b = (\epsilon_b - |ux| - |xa|, \epsilon_b),$$

$$c = (\epsilon_c + \epsilon_a - |xw|, -\epsilon_c),$$

$$d = (-\epsilon_d, 1 + |wy| - \epsilon_d),$$

$$e = (|wy| + \epsilon_y - \epsilon_e - 1, |xw| - \epsilon_y - \epsilon_e)$$

  are inside $D_1(x,a)$, $D_3(a,x)$, $D_2(w,y)$, and $D_3(y,w)$, respectively (see Figure 5(a)). Let b', c', d', and e' be the reflections of the points b, c, d, and e with respect to M, respectively. Now, add the points $\{b, c, d, e, b', c', d', e'\}$ to S. Consequently, we have:

$$S = \{u, w, x, x', a, y, a', y', b, c, d, e, b', c', d', e'\}.$$

  Figure 5(b) shows the Yao graph Y_4 on the point set S. Now, compute the length of the short paths between u and w. Let $P_1 := uxyew$, $P_2 := uxcaw$, $Q_1 := ue'y'x'w$, and $Q_2 := ua'c'x'w$. According to Figure 5(b), at least one of the paths P_1, P_2, Q_1, and Q_2 is the shortest path between u and w. The length of the paths P_1 and P_2 can be calculated through the equations shown in Box I. Hence, the equation, shown in Box II, is obtained. Since $\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_w \to 0}} |xw| = \sqrt{2}$ and $\lim_{\substack{\epsilon_y \to 0 \\ \epsilon_w \to 0}} |wy| = \sqrt{4 - 2\sqrt{2}}$, we have:

$$\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_y \to 0 \\ \epsilon_e \to 0 \\ \epsilon_w \to 0}} \frac{|P_1|}{|uw|} = 1 + \sqrt{2} + \sqrt{4 - 2\sqrt{2}}$$

$$+\sqrt{(\sqrt{4 - 2\sqrt{2}} - 1)^2 + (\sqrt{2} - 1)^2}$$

$$\approx 3.919.$$

  For the path P_2, we have:

$$|P_1| = |ux| + |xy| + |ye| + |ew| = \sqrt{(\epsilon_x - 1)^2 + \epsilon_x^2} + \sqrt{(\epsilon_x - \epsilon_y)^2 + (\epsilon_x + \epsilon_y - |xw|)^2} + \sqrt{(\epsilon_e - |wy|)^2 + (\epsilon_e)^2}$$
$$+ \sqrt{(|wy| + \epsilon_y - \epsilon_e - 1 + \epsilon_w)^2 + (|xw| - \epsilon_y - \epsilon_e - 1 + \epsilon_w)^2}$$
$$|P_2| = |ux| + |xc| + |ca| + |aw| = \sqrt{(\epsilon_x - 1)^2 + \epsilon_x^2} + \sqrt{(\epsilon_x - 1 - \epsilon_c - \epsilon_a + |xw|)^2 + (\epsilon_x + \epsilon_c)^2}$$
$$+ \sqrt{(\epsilon_c)^2 + (-\epsilon_c - 1 + \epsilon_a)^2} + \sqrt{(\epsilon_a + \epsilon_w - |xw|)^2 + (\epsilon_w - \epsilon_a)^2}.$$

Box I

$$\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_y \to 0 \\ \epsilon_e \to 0 \\ \epsilon_w \to 0}} \frac{|P_1|}{|uw|} = \frac{\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_y \to 0 \\ \epsilon_w \to 0}} \left(1 + |xw| + |wy| + \sqrt{(|wy| - 1)^2 + (|xw| - 1)^2}\right)}{1}.$$

Box II

$$\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_c \to 0 \\ \epsilon_a \to 0 \\ \epsilon_w \to 0}} \frac{|P_2|}{|uw|} = \frac{\lim_{\substack{\epsilon_x \to 0 \\ \epsilon_y \to 0 \\ \epsilon_w \to 0}} (1 + 2|xw|)}{1}$$
$$= 1 + 2\sqrt{2} \approx 3.8285.$$

Since $3.8285 < 3.919$, the stretch factor between u and w is obtained as 3.8285. Based on the symmetric argument, we have $|Q_1| = |P_1|$ and $|Q_2| = |P_2|$.

Consequently, the following theorem can be proved.

Theorem 1. *The stretch factor of Y_4 is at least 3.8285.*

3. Conclusion

The current research aimed to provide a lower bound of 3.8285 on the stretch factor of Y_4 which could partially solve an open problem. In this respect, efforts were made to improve the upper bound. For instance, it was attempted to prove that the point set presented in the current paper for the lower bound was the only point set where the stretch factor of the Yao graph Y_4 between two points u and w would reach its apex; however, we failed to prove it. Hence, improvement of the upper bound 54.62 on the stretch factor of Yao graph Y_4 remained an unsolved issue. Of note, it is an interesting open problem to know if 3.8285 is the best lower bound for the stretch factor of Y_4.

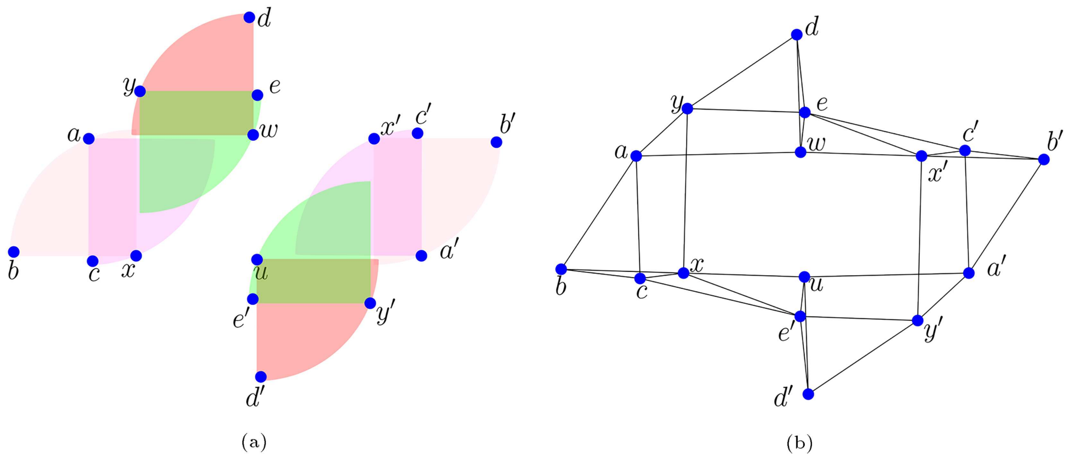

Figure 5. (a) The point set S. (b) The Yao graph Y_4 on S.

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Biographies

Davood Bakhshesh is an Assistant Professor at the Department of Computer Science at University of Bojnord, Bojnord, Iran. He has been working at University of Bojnord since 2012. He received his BSc degree in Computer Science from Vali-e-Asr University of Rafsanjan in 2007, and MSc degree in Computer Science from Sharif University of Technology in 2009, and PhD Degree in Computer Science from Yazd University in 2017.

Mohammad Farshi is an Associate Professor at the Department of Mathematical Sciences, Yazd University, Yazd, Iran. He has been working at Yazd University since 1999. He received his BSc degree in Computer Science from Yazd University in 1996, and MSc degree in Pure Mathematics from Shiraz University in 1999, and PhD degree in Computer Science from Eindhoven University of Technology (TU/e) in 2008.