

EMBEDDED SOLITONS WITH $\chi^{(2)}$ NONLINEAR SUSCEPTIBILITY

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Abstract

This paper recovers optical soliton solutions with $\chi^{(2)}$ -nonlinear susceptibility. Bright, dark, singular, bright-dark combo solitons are recovered. A variety of algorithms are implemented. These include the Riccati equation approach, exp-function expansion method, modified simple equation algorithm, sine-Gordon equation scheme, F -expansion approach, trial function method and functional variable algorithm.

Key words: embedded solitons; $\chi^{(2)}$ -nonlinearity; integrability

1 INTRODUCTION

Optical solitons that exist in the continuous regime of the scattering spectrum are referred to as embedded solitons. These are governed by quadratic nonlinearity that emerge from $\chi^{(2)}$ nonlinear susceptibility. These give way to bright, dark as well as singular solitons. These embedded solitons have been studied in the past using a variety of analytical approaches [1–11]. It is now time to revisit the same arena using a wider variety of mathematical approaches. This will yield a fresh set of soliton solutions that are of a different form namely bright-dark combo solitons and other will be revealed using a wide spectrum of analytical approaches. The results are being reported for the very first time in this paper that will encompass the previously reported results as well. These soliton solutions are enumerated with their respective existence criteria that are also presented.

1.1 GOVERNING MODEL

The governing model with the quadratic nonlinearity [1–11] reads

$$iu_t + a_1 u_{xx} + b_1 u_{xt} + c_1 u + \lambda_1 u^* v = i\alpha_1 u_x, \quad (1)$$

$$iv_t + a_2 v_{xx} + b_2 v_{xt} + c_2 v + \lambda_2 u^2 = i\alpha_2 v_x, \quad (2)$$

where x represents the spatial variable, while t denotes the temporal variable. The coefficients a_j , b_j , c_j , λ_j and α_j ($j = 1, 2$) are real valued constants. a_j stand for the coefficients of chromatic dispersion, while b_j stem from the coefficients of spatio-temporal dispersion. Next, λ_j are the coefficients of quadratic nonlinearity effect, while α_i depict the coefficients of inter-modal dispersion. The first terms are linear temporal evolution and $i = \sqrt{-1}$. The functions $u = u(x, t)$ and $v = v(x, t)$ are complex-valued functions that represents the wave profiles of the forward harmonic and second harmonic waves respectively. Lastly, $u^* = u^*(x, t)$ is the conjugate of $u = u(x, t)$.

2 MATHEMATICAL ANALYSIS

To solve (1) and (2), the wave transformations are structured as

$$u(x, t) = U_1(\vartheta) e^{i\varphi_1(x, t)}, \quad (3)$$

$$v(x, t) = U_2(\vartheta) e^{2i\varphi_2(x, t)}. \quad (4)$$

In (3) and (4), the amplitude components are $U_j(\vartheta)$ for $j = 1, 2$ and the wave variable is

$$\vartheta = \eta(x - pt), \quad (5)$$

where the real-valued constants η and p represents the soliton width and the soliton velocity respectively, while the phase components are

$$\varphi_j(x, t) = -kx + wt + \zeta, \quad (6)$$

where k , w , and ζ are real-valued constants that stands for the soliton frequency, the soliton wave number and the phase constant respectively.

Next, the real and imaginary parts are

$$\eta^2 (a_1 - pb_1) U_1'' + (c_1 - w - k^2 a_1 + kwb_1 - k\alpha_1) U_1 + \lambda_1 U_1 U_2 = 0, \quad (7)$$

$$kpb_1 - 2ka_1 + wb_1 - p - \alpha_1 = 0, \quad (8)$$

$$\eta^2 (a_2 - pb_2) U_2'' + (c_2 - 2w - 4k^2 a_2 + 4kwb_2 - 2k\alpha_2) U_2 + \lambda_2 U_1^2 = 0, \quad (9)$$

$$2kpb_2 - 4ka_2 + 2wb_2 - p - \alpha_2 = 0, \quad (10)$$

respectively as long as Eqs. (3) and (4) are put in Eqs. (1) and (2). By the use of the balancing rule, Eqs. (7)-(10) reduce to the ordinary differential equation

$$\eta^2 (a - pb) U'' + (2ak^2 - 2bkw + \alpha k + w) U + \lambda U^2 = 0, \quad (11)$$

along with the velocity

$$p = \frac{4ka - 2wb + \alpha}{2kb - 1}, \quad (12)$$

and the parameter constraints

$$\begin{aligned} U_1 = U_2 = U, \quad b_1 = 2b, \quad b_2 = b, \quad a_1 = 2a, \quad a_2 = a, \quad \alpha_1 = \alpha_2 = \alpha, \\ \lambda_1 = 2\lambda, \quad \lambda_2 = \lambda, \quad c_1 = c_2 = c, \quad c = 6ak^2 - 6bkw + 3\alpha k + 3w. \end{aligned} \quad (13)$$

In what follows, we will employ a variety of schemes that will be used to achieve the goals set for this work.

2.1 RICCATI EQUATION APPROACH

Assume that the solution structure of Eq. (11) is considered as

$$U(\vartheta) = \sum_{i=0}^N A_i V^i(\vartheta), \quad (14)$$

where A_i are constants to be established later, N is the balancing integer, $A_N \neq 0$ and also the function $V(\vartheta)$ satisfies the Riccati equation

$$V'(\vartheta) = S_2 V^2(\vartheta) + S_1 V(\vartheta) + S_0, \quad S_2 \neq 0, \quad (15)$$

where S_2 , S_1 and S_0 are constants. The solutions of Eq. (15) are listed as

$$\begin{aligned} V(\vartheta) &= -\frac{S_1}{2S_2} - \frac{\sqrt{\mu}}{2S_2} \tanh\left(\frac{\sqrt{\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu > 0, \\ V(\vartheta) &= -\frac{S_1}{2S_2} - \frac{\sqrt{\mu}}{2S_2} \coth\left(\frac{\sqrt{\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu > 0, \\ V(\vartheta) &= -\frac{S_1}{2S_2} + \frac{\sqrt{-\mu}}{2S_2} \tan\left(\frac{\sqrt{-\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu < 0, \\ V(\vartheta) &= -\frac{S_1}{2S_2} - \frac{\sqrt{-\mu}}{2S_2} \cot\left(\frac{\sqrt{-\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu < 0, \\ V(\vartheta) &= -\frac{S_1}{2S_2} - \frac{1}{S_2\vartheta + \vartheta_0}, \quad \mu = 0, \end{aligned} \quad (16)$$

where $\mu = S_1^2 - 4S_0S_2$ and ϑ_0 is an arbitrary real constant. Next, Eq. (14) can be rewritten as

$$U = A_0 + A_1V + A_2V^2, \quad (17)$$

by virtue of the balance principle applied in Eq. (11). Then, the equations are recovered as

$$-6b\eta^2pA_2S_2^2 + 6a\eta^2A_2S_2^2 + \lambda A_2^2 = 0, \quad (18)$$

$$-2b\eta^2pA_1S_2^2 - 10b\eta^2pA_2S_1S_2 + 2a\eta^2A_1S_2^2 + 10a\eta^2A_2S_1S_2 + 2\lambda A_1A_2 = 0, \quad (19)$$

$$\begin{aligned} & -\eta^2A_1S_1S_0bp - 2\eta^2A_2S_0^2bp + \eta^2A_1S_1S_0a + 2\eta^2A_2S_0^2a \\ & + 2A_0ak^2 - 2A_0bkw + A_0\alpha k + \lambda A_0^2 + A_0w = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} & -2b\eta^2pA_1S_0S_2 - b\eta^2pA_1S_1^2 - 6b\eta^2pA_2S_0S_1 + 2a\eta^2A_1S_0S_2 + a\eta^2A_1S_1^2 \\ & + 6a\eta^2A_2S_0S_1 + 2ak^2A_1 - 2bkwA_1 + \alpha kA_1 + 2\lambda A_0A_1 + wA_1 = 0, \end{aligned} \quad (21)$$

$$\begin{aligned} & -3b\eta^2pA_1S_1S_2 - 8b\eta^2pA_2S_0S_2 - 4b\eta^2pA_2S_1^2 + 3a\eta^2A_1S_1S_2 + 8a\eta^2A_2S_0S_2 \\ & + 4a\eta^2A_2S_1^2 + 2ak^2A_2 - 2bkwA_2 + \alpha kA_2 + 2\lambda A_0A_2 + \lambda A_1^2 + wA_2 = 0, \end{aligned} \quad (22)$$

as long as Eq. (17) along with Eq. (15) is inserted in Eq. (11). So, from Eqs. (18)-(22) are

$$\begin{aligned} S_0 &= \pm \sqrt{-\frac{3(4A_0A_2 - A_1^2)(a - bp)}{2ak^2 - 2bkw + \alpha k + w} \frac{\left(\begin{array}{c} 12ak^2A_0A_2 - 2ak^2A_1^2 - 12bkwA_0A_2 - \alpha kA_1^2 \\ + 2bkwA_1^2 + 6\alpha kA_0A_2 + 6wA_0A_2 - wA_1^2 \end{array} \right)}{6(4A_0A_2 - A_1^2)\eta A_2(a - bp)}}, \\ S_1 &= \pm \frac{A_1}{3\eta} \sqrt{-\frac{6ak^2 - 6bkw + 3\alpha k + 3w}{(4A_0A_2 - A_1^2)(a - bp)}}, \\ S_2 &= \pm \frac{A_2}{3\eta} \sqrt{-\frac{6ak^2 - 6bkw + 3\alpha k + 3w}{(4A_0A_2 - A_1^2)(a - bp)}}, \\ \lambda &= \frac{2(2ak^2 - 2bkw + \alpha k + w)A_2}{4A_0A_2 - A_1^2}, \end{aligned}$$

and

$$\mu = S_1^2 - 4S_0S_2 = \frac{2ak^2 - 2bkw + \alpha k + w}{\eta^2(a - bp)}. \quad (23)$$

If one employs the solution set (23) with Eq. (16) in Eq. (17), dark solitons are

$$\begin{aligned} u(x, t) &= \left\{ \begin{array}{l} \frac{4A_0A_2 - A_1^2}{4A_2} - \frac{3(4A_0A_2 - A_1^2)}{4A_2} \\ \times \tanh^2 \left(\sqrt{\frac{2ak^2 - 2bkw + k\alpha + w}{4(a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) + \vartheta_0 \right) \end{array} \right\} \\ &\times e^{i(-kx + wt + \zeta)}, \end{aligned} \quad (24)$$

$$\begin{aligned} v(x, t) &= \left\{ \begin{array}{l} \frac{4A_0A_2 - A_1^2}{4A_2} - \frac{3(4A_0A_2 - A_1^2)}{4A_2} \\ \times \tanh^2 \left(\sqrt{\frac{2ak^2 - 2bkw + k\alpha + w}{4(a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) + \vartheta_0 \right) \end{array} \right\} \\ &\times e^{2i(-kx + wt + \zeta)}, \end{aligned} \quad (25)$$

with

$$(a - bp)(2ak^2 - 2bkw + k\alpha + w) > 0.$$

Singular solitons are

$$u(x, t) = \left\{ \begin{array}{l} \frac{4A_0A_2 - A_1^2}{4A_2} - \frac{3(4A_0A_2 - A_1^2)}{4A_2} \\ \times \coth^2 \left(\sqrt{\frac{2ak^2 - 2bkw + k\alpha + w}{4(a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) + \vartheta_0 \right) \end{array} \right\} \times e^{i(-kx + wt + \zeta)}, \quad (26)$$

$$v(x, t) = \left\{ \begin{array}{l} \frac{4A_0A_2 - A_1^2}{4A_2} - \frac{3(4A_0A_2 - A_1^2)}{4A_2} \\ \times \coth^2 \left(\sqrt{\frac{2ak^2 - 2bkw + k\alpha + w}{4(a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) + \vartheta_0 \right) \end{array} \right\} \times e^{2i(-kx + wt + \zeta)}, \quad (27)$$

with

$$(a - bp)(2ak^2 - 2bkw + k\alpha + w) > 0.$$

2.2 SINE-GORDON EQUATION METHOD

The solution of Eq. (11) is structured as follows:

$$U(\vartheta) = \sum_{i=1}^N \cos^{i-1}(V(\vartheta)) [B_i \sin(V(\vartheta)) + A_i \cos(V(\vartheta))] + A_0, \quad (28)$$

where A_i and B_i are real constants to be detected later, $A_i \neq 0$, $B_i \neq 0$, N is the balancing integer and the function $V(\vartheta)$ holds

$$V'(\vartheta) = \sin(V(\vartheta)). \quad (29)$$

Eq. (29) has the solutions given by

$$\begin{aligned} \sin(V(\vartheta)) &= \operatorname{sech}(\vartheta) \quad \text{or} \quad \sin(V(\vartheta)) = i \operatorname{csh}(\vartheta), \\ \cos(V(\vartheta)) &= \tanh(\vartheta) \quad \text{or} \quad \cos(V(\vartheta)) = \operatorname{coth}(\vartheta). \end{aligned} \quad (30)$$

Next, Eq. (28) can be rewritten as

$$U(\vartheta) = B_1 \sin(V(\vartheta)) + A_1 \cos(V(\vartheta)) + \cos(V(\vartheta)) [B_2 \sin(V(\vartheta)) + A_2 \cos(V(\vartheta))] + A_0, \quad (31)$$

by virtue of the balance principle applied in Eq. (11). Then, the derived equations are

$$-6b\eta^2 p B_2 + 6a\eta^2 B_2 + 2\lambda A_2 B_2 = 0, \quad (32)$$

$$-6b\eta^2 p A_2 + 6a\eta^2 A_2 + \lambda A_2^2 - \lambda B_2^2 = 0, \quad (33)$$

$$-2b\eta^2 p B_1 + 2a\eta^2 B_1 + 2\lambda A_1 B_2 + 2\lambda A_2 B_1 = 0, \quad (34)$$

$$-2b\eta^2 p A_1 + 2a\eta^2 A_1 + 2\lambda A_1 A_2 - 2\lambda B_1 B_2 = 0, \quad (35)$$

$$b\eta^2 p B_1 - a\eta^2 B_1 + 2ak^2 B_1 - 2bkw B_1 + \alpha k B_1 + 2\lambda A_0 B_1 + w B_1 = 0, \quad (36)$$

$$-2b\eta^2 p A_2 + 2a\eta^2 A_2 + 2ak^2 A_0 - 2bkw A_0 + \alpha k A_0 + \lambda A_0^2 + \lambda B_1^2 + w A_0 = 0, \quad (37)$$

$$5b\eta^2 p B_2 - 5a\eta^2 B_2 + 2ak^2 B_2 - 2bkw B_2 + \alpha k B_2 + 2\lambda A_0 B_2 + 2\lambda A_1 B_1 + w B_2 = 0, \quad (38)$$

$$2b\eta^2 p A_1 - 2a\eta^2 A_1 + 2ak^2 A_1 - 2bkw A_1 + \alpha k A_1 + 2\lambda A_0 A_1 + 2\lambda B_1 B_2 + w A_1 = 0, \quad (39)$$

$$\begin{aligned} & 8b\eta^2 p A_2 - 8a\eta^2 A_2 + 2ak^2 A_2 - 2bkw A_2 + \alpha k A_2 \\ & + 2\lambda A_0 A_2 + \lambda A_1^2 - \lambda B_1^2 + \lambda B_2^2 + w A_2 = 0, \end{aligned} \quad (40)$$

as long as Eq. (31) with Eq. (29) is substituted in Eq. (11). So, from Eqs. (32)-(40) are

Result-1:

$$\begin{aligned} \eta &= \pm \sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}}, \quad A_0 = \frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda}, \\ A_2 &= -\frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda}, \quad A_1 = 0, \quad B_1 = 0, \quad B_2 = 0. \end{aligned} \quad (41)$$

If one substitutes Eq. (41) along with Eq. (30) in Eq. (31), dark solitons are

$$\begin{aligned} u(x, t) &= \left\{ \begin{aligned} & \frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} - \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \\ & \times \tanh^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \end{aligned} \right\} \\ & \times e^{i(-kx+wt+\zeta)}, \end{aligned} \quad (42)$$

$$\begin{aligned} v(x, t) &= \left\{ \begin{aligned} & \frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} - \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \\ & \times \tanh^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \end{aligned} \right\} \\ & \times e^{2i(-kx+wt+\zeta)}, \end{aligned} \quad (43)$$

with

$$(a-bp)(2ak^2 - 2bkw + k\alpha + w) > 0.$$

Singular solitons are

$$\begin{aligned} u(x, t) &= \left\{ \begin{aligned} & \frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} - \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \\ & \times \coth^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \end{aligned} \right\} \\ & \times e^{i(-kx+wt+\zeta)}, \end{aligned} \quad (44)$$

$$v(x, t) = \left\{ \begin{array}{l} \frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} - \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \\ \times \coth^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \end{array} \right\} \times e^{2i(-kx+wt+\zeta)}, \quad (45)$$

with

$$(a - bp)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

Result-2:

$$\begin{aligned} \eta &= \pm \sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{a - bp}}, \quad A_0 = \frac{2(2ak^2 - 2bkw + \alpha k + w)}{\lambda}, \quad B_1 = 0, \\ A_2 &= -\frac{3(2ak^2 - 2bkw + \alpha k + w)}{\lambda}, \quad B_2 = \pm \frac{3i(2ak^2 - 2bkw + \alpha k + w)}{\lambda}, \quad A_1 = 0. \end{aligned} \quad (46)$$

Combo singular solitons are

$$u(x, t) = \left\{ \begin{array}{l} \frac{2(2ak^2 - 2bkw + \alpha k + w)}{\lambda} \\ + \coth \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{a - bp}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \\ \left(\begin{array}{l} \pm \frac{3(2ak^2 - 2bkw + \alpha k + w)}{\lambda} \\ \times \operatorname{csch} \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{a - bp}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \\ - \frac{3(2ak^2 - 2bkw + \alpha k + w)}{\lambda} \\ \times \coth \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{a - bp}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \end{array} \right) \end{array} \right\} \times e^{i(-kx+wt+\zeta)}, \quad (47)$$

$$\begin{aligned}
v(x, t) = & \left\{ \begin{aligned} & \frac{2(2ak^2 - 2bkw + \alpha k + w)}{\lambda} \\ & + \coth \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{a - bp}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \\ & \times \left(\begin{aligned} & \pm \frac{3(2ak^2 - 2bkw + \alpha k + w)}{\lambda} \\ & \times \operatorname{csch} \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{a - bp}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \\ & - \frac{3(2ak^2 - 2bkw + \alpha k + w)}{\lambda} \\ & \times \coth \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{a - bp}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \end{aligned} \right) \end{aligned} \right\} \\
& \times e^{2i(-kx+wt+\zeta)}, \tag{48}
\end{aligned}$$

with

$$(a - bp)(2ak^2 - 2bkw + k\alpha + w) > 0.$$

2.3 FUNCTIONAL VARIABLE METHODOLOGY

This subsection will apply the functional variable methodology for overcoming Eq. (11). Eq. (11) can be written as

$$\frac{\eta^2(a - pb)\{V^2(U)\}'}{2} + (2ak^2 - 2bkw + \alpha k + w)U + \lambda U^2 = 0, \tag{49}$$

by employing of a functional variable form

$$U' = V(U). \tag{50}$$

Thus, the important result emerged from Eq. (49) is

$$V(U_j) = \pm \sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{\eta^2(pb - a)}} U \sqrt{1 + \frac{2\lambda}{3(2ak^2 - 2bkw + \alpha k + w)}} U. \tag{51}$$

If we integrate Eq. (51), bright solitons are

$$\begin{aligned}
u(x, t) = & -\frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{sech}^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(pb - a)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \\
& \times e^{i(-kx+wt+\zeta)}, \tag{52}
\end{aligned}$$

$$v(x, t) = -\frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{sech}^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(pb - a)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \times e^{2i(-kx + wt + \zeta)}, \quad (53)$$

with

$$(pb - a)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

Singular solitons are

$$u(x, t) = \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{csch}^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(pb - a)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \times e^{i(-kx + wt + \zeta)}, \quad (54)$$

$$v(x, t) = \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{csch}^2 \left[\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(pb - a)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right] \times e^{2i(-kx + wt + \zeta)}, \quad (55)$$

with

$$(pb - a)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

2.4 F -EXPANSION PRINCIPLE

The formal solution of Eq. (11) is given as

$$U(\vartheta) = \sum_{i=0}^N \mu_i F^i(\vartheta), \quad (56)$$

where μ_i are constants that need to be detected, $\mu_N \neq 0$, N is the balancing integer and also $F(\vartheta)$ ensures

$$F'(\vartheta) = \sqrt{PF^4(\vartheta) + QF^2(\vartheta) + R}, \quad (57)$$

where P , Q and R are constants. The solutions of Eq. (57) are presented as below:

$$\begin{aligned} F(\vartheta) &= \operatorname{sn}(\vartheta) = \tanh(\vartheta), \quad P = m^2, \quad Q = -(1 + m^2), \quad R = 1, \quad m \rightarrow 1, \\ F(\vartheta) &= \operatorname{ns}(\vartheta) = \coth(\vartheta), \quad P = 1, \quad Q = -(1 + m^2), \quad R = m^2, \quad m \rightarrow 1, \\ F(\vartheta) &= \operatorname{sc}(\vartheta) = \tan(\vartheta), \quad P = 1 - m^2, \quad Q = 2 - m^2, \quad R = 1, \quad m \rightarrow 0, \\ F(\vartheta) &= \operatorname{cs}(\vartheta) = \cot(\vartheta), \quad P = 1, \quad Q = 2 - m^2, \quad R = 1 - m^2, \quad m \rightarrow 0, \\ F(\vartheta) &= \operatorname{cn}(\vartheta) = \operatorname{sech}(\vartheta), \quad P = -m^2, \quad Q = 2m^2 - 1, \quad R = 1 - m^2, \quad m \rightarrow 1, \\ F(\vartheta) &= \operatorname{ds}(\vartheta) = \operatorname{csch}(\vartheta), \quad P = 1, \quad Q = 2m^2 - 1, \quad R = -m^2(1 - m^2), \quad m \rightarrow 1, \\ F(\vartheta) &= \operatorname{nc}(\vartheta) = \sec(\vartheta), \quad P = 1 - m^2, \quad Q = 2m^2 - 1, \quad R = -m^2, \quad m \rightarrow 0, \\ F(\vartheta) &= \operatorname{ns}(\vartheta) = \operatorname{csc}(\vartheta), \quad P = 1, \quad Q = -(1 + m^2), \quad R = m^2, \quad m \rightarrow 0, \\ F(\vartheta) &= \operatorname{ns}(\vartheta) \pm \operatorname{ds}(\vartheta) = \coth(\vartheta) \pm \operatorname{csch}(\vartheta), \quad P = \frac{1}{4}, \quad Q = \frac{m^2 - 2}{2}, \quad R = \frac{m^2}{4}, \quad m \rightarrow 1, \end{aligned}$$

$$F(\vartheta) = \operatorname{sn}(\vartheta) \pm \operatorname{icn}(\vartheta) = \tanh(\vartheta) \pm \operatorname{isech}(\vartheta), \quad P = \frac{m^2}{4}, \quad Q = \frac{m^2 - 2}{2}, \quad R = \frac{m^2}{4}, \quad m \rightarrow 1,$$

$$F(\vartheta) = \operatorname{ns}(\vartheta) \pm \operatorname{cs}(\vartheta) = \operatorname{csc}(\vartheta) \pm \operatorname{cot}(\vartheta), \quad P = \frac{1}{4}, \quad Q = \frac{1 - 2m^2}{2}, \quad R = \frac{1}{4}, \quad m \rightarrow 0,$$

$$F(\vartheta) = \operatorname{nc}(\vartheta) \pm \operatorname{sc}(\vartheta) = \operatorname{sec}(\vartheta) \pm \operatorname{tan}(\vartheta), \quad P = \frac{1 - m^2}{4}, \quad Q = \frac{1 + m^2}{2}, \quad R = \frac{1 - m^2}{4}, \quad m \rightarrow 0. \quad (58)$$

Next, Eq. (56) can be rewritten as

$$U = \mu_0 + \mu_1 F + \mu_2 F^2, \quad (59)$$

by virtue of the balance principle applied in Eq. (11). Then, the strategic equations are found as the following:

$$-6\eta^2 P b p \mu_2 + 6\eta^2 P a \mu_2 + \lambda \mu_2^2 = 0, \quad (60)$$

$$-2\eta^2 P b p \mu_1 + 2\eta^2 P a \mu_1 + 2\lambda \mu_1 \mu_2 = 0, \quad (61)$$

$$-\eta^2 Q b p \mu_1 + \eta^2 Q a \mu_1 + 2ak^2 \mu_1 - 2bkw \mu_1 + \alpha k \mu_1 + 2\lambda \mu_0 \mu_1 + w \mu_1 = 0, \quad (62)$$

$$-2\eta^2 R b p \mu_2 + 2\eta^2 R a \mu_2 + 2ak^2 \mu_0 - 2bkw \mu_0 + \alpha k \mu_0 + \lambda \mu_0^2 + w \mu_0 = 0, \quad (63)$$

$$-4\eta^2 Q b p \mu_2 + 4\eta^2 Q a \mu_2 + 2ak^2 \mu_2 - 2bkw \mu_2 + \alpha k \mu_2 + 2\lambda \mu_0 \mu_2 + \lambda \mu_1^2 + w \mu_2 = 0, \quad (64)$$

as long as Eq. (59) along with Eq. (57) is inserted in Eq. (11). So, from Eqs. (60)-(64) are

$$\begin{aligned} \mu_1 &= 0, \quad \eta = \pm \sqrt[4]{-\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{16(a - bp)^2(3PR - Q^2)}}, \\ \mu_0 &= -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{-\frac{Q^2(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2(3PR - Q^2)}}, \\ \mu_2 &= \pm \sqrt{-\frac{9P^2(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2(3PR - Q^2)}}. \end{aligned} \quad (65)$$

If one utilizes the solution set given by (65) along with Eq. (58) in Eq. (59), dark solitons are

$$\begin{aligned} u(x, t) &= \left\{ \begin{aligned} &-\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{\lambda^2}} \\ &\pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ &\times \tanh^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{aligned} \right\} \\ &\times e^{i(-kx + wt + \zeta)}, \end{aligned} \quad (66)$$

$$v(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \tanh^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\} \\ \times e^{2i(-kx+wt+\zeta)}, \quad (67)$$

with

$$(a - bp)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

Singular solitons are

$$u(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \coth^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\} \\ \times e^{i(-kx+wt+\zeta)}, \quad (68)$$

$$v(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \coth^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\} \\ \times e^{2i(-kx+wt+\zeta)}, \quad (69)$$

with

$$(a - bp)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

Bright solitons are

$$u(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \operatorname{sech}^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\} \\ \times e^{i(-kx+wt+\zeta)}, \quad (70)$$

$$v(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \operatorname{sech}^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\} \\ \times e^{2i(-kx+wt+\zeta)}, \quad (71)$$

with

$$(a - bp)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

Singular solitons are

$$u(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \operatorname{csch}^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\} \\ \times e^{i(-kx+wt+\zeta)}, \quad (72)$$

$$v(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \operatorname{csch}^2 \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\} \times e^{2i(-kx+wt+\zeta)}, \quad (73)$$

with

$$(a - bp)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

Combo singular solitons are

$$u(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \left\{ \begin{array}{l} \coth \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{-bp + a}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \\ \pm \operatorname{csch} \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{-bp + a}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\}^2 \end{array} \right\} \times e^{i(-kx+wt+\zeta)}, \quad (74)$$

$$v(x, t) = \left\{ \begin{array}{l} -\frac{2ak^2 - 2bkw + \alpha k + w}{2\lambda} \pm \sqrt{\frac{(2ak^2 - 2bkw + \alpha k + w)^2}{\lambda^2}} \\ \pm \sqrt{\frac{9(2ak^2 - 2bkw + \alpha k + w)^2}{4\lambda^2}} \\ \times \left\{ \begin{array}{l} \coth \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{-bp + a}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \\ \pm \operatorname{csch} \left(\sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{-bp + a}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \end{array} \right\}^2 \end{array} \right\} \times e^{2i(-kx+wt+\zeta)}, \quad (75)$$

with

$$(a - bp)(2ak^2 - 2bkw + \alpha k + w) > 0.$$

2.5 EXP-EXPANSION FUNCTION

The formal solution of Eq. (11) is taken to be

$$U(\vartheta) = \sum_{i=0}^N A_i \{\exp(-V(\vartheta))\}^i, \quad (76)$$

where the coefficients A_i are constants to be designated later, such that $A_N \neq 0$, N is the balancing integer and also $V(\vartheta)$ satisfies

$$V'(\vartheta) = \exp(-V(\vartheta)) + S \exp(V(\vartheta)) + R, \quad (77)$$

where S and R are constants. Eq. (77) has the following strategic solutions:

$$\begin{aligned} V(\vartheta) &= \ln \left[-\frac{R}{2S} - \frac{\sqrt{\mu}}{2S} \tanh \left(\frac{\sqrt{\mu}}{2} (\vartheta + \vartheta_0) \right) \right], \quad S \neq 0, \quad \mu > 0, \\ V(\vartheta) &= \ln \left[-\frac{R}{2S} - \frac{\sqrt{\mu}}{2S} \coth \left(\frac{\sqrt{\mu}}{2} (\vartheta + \vartheta_0) \right) \right], \quad S \neq 0, \quad \mu > 0, \\ V(\vartheta) &= \ln \left[-\frac{R}{2S} + \frac{\sqrt{-\mu}}{2S} \tan \left(\frac{\sqrt{-\mu}}{2} (\vartheta + \vartheta_0) \right) \right], \quad S \neq 0, \quad \mu < 0, \\ V(\vartheta) &= \ln \left[-\frac{R}{2S} - \frac{\sqrt{-\mu}}{2S} \cot \left(\frac{\sqrt{-\mu}}{2} (\vartheta + \vartheta_0) \right) \right], \quad S \neq 0, \quad \mu < 0, \end{aligned} \quad (78)$$

where $\mu = R^2 - 4S$ and ϑ_0 is an arbitrary real constant. Next, Eq. (76) can be rewritten as

$$U_j = A_0 + A_1 \exp(-V(\vartheta)) + A_2 \exp(-2V(\vartheta)), \quad (79)$$

by virtue of the balance principle applied in Eq. (11). Then, the revealed equations are as follows:

$$-6b\eta^2 p A_2 + 6a\eta^2 A_2 + \lambda A_2^2 = 0, \quad (80)$$

$$-10Rb\eta^2 p A_2 + 10Ra\eta^2 A_2 - 2b\eta^2 p A_1 + 2a\eta^2 A_1 + 2\lambda A_1 A_2 = 0, \quad (81)$$

$$\begin{aligned} -RSb\eta^2 p A_1 - 2S^2 b\eta^2 p A_2 + RSa\eta^2 A_1 + 2S^2 a\eta^2 A_2 \\ + 2ak^2 A_0 - 2bkw A_0 + \alpha k A_0 + \lambda A_0^2 + w A_0 = 0, \end{aligned} \quad (82)$$

$$\begin{aligned} -R^2 b\eta^2 p A_1 - 6RSb\eta^2 p A_2 + R^2 a\eta^2 A_1 + 6RSa\eta^2 A_2 - 2Sb\eta^2 p A_1 \\ + 2Sa\eta^2 A_1 + 2ak^2 A_1 - 2bkw A_1 + \alpha k A_1 + 2\lambda A_0 A_1 + w A_1 = 0, \end{aligned} \quad (83)$$

$$\begin{aligned} -4R^2 b\eta^2 p A_2 + 4R^2 a\eta^2 A_2 - 3Rb\eta^2 p A_1 - 8Sb\eta^2 p A_2 + 3Ra\eta^2 A_1 \\ + 8Sa\eta^2 A_2 + 2ak^2 A_2 - 2bkw A_2 + \alpha k A_2 + 2\lambda A_0 A_2 + \lambda A_1^2 + w A_2 = 0, \end{aligned} \quad (84)$$

as long as Eq. (79) along with Eq. (77) is put in Eq. (11). So, from Eqs. (80)-(84) are

Result-1:

$$R = \frac{A_1}{A_2}, \quad S = \frac{A_0}{A_2}, \quad \eta = \pm \sqrt{-\frac{\lambda A_2}{6(a - bp)}},$$

$$w = \frac{12 ak^2 A_2 + 6 \alpha k A_2 + 4 \lambda A_0 A_2 - \lambda A_1^2}{6 A_2 (2 bk - 1)},$$

and

$$\mu = R^2 - 4S = -\frac{4 A_0 A_2 - A_1^2}{A_2^2}. \quad (85)$$

If one employs (85) along with (78) in (79), singular solitons are

$$u(x, t) = \left\{ \begin{array}{l} A_0 - \frac{2 A_1 A_0}{A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right)} \\ + \frac{4 A_2 A_0^2}{\left(A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right) \right)^2} \end{array} \right\} \times e^{i \left(-kx + \frac{12 ak^2 A_2 + 6 \alpha k A_2 + 4 \lambda A_0 A_2 - \lambda A_1^2}{6 A_2 (2 bk - 1)} t + \zeta \right)}, \quad (86)$$

$$v(x, t) = \left\{ \begin{array}{l} A_0 - \frac{2 A_1 A_0}{A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right)} \\ + \frac{4 A_2 A_0^2}{\left(A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right) \right)^2} \end{array} \right\} \times e^{2i \left(-kx + \frac{12 ak^2 A_2 + 6 \alpha k A_2 + 4 \lambda A_0 A_2 - \lambda A_1^2}{6 A_2 (2 bk - 1)} t + \zeta \right)}, \quad (87)$$

with

$$A_2 (a - bp) (4 A_0 A_2 - A_1^2) \lambda > 0.$$

Dark solitons are

$$u(x, t) = \left\{ \begin{array}{l} A_0 - \frac{2 A_1 A_0}{A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \coth \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right)} \\ + \frac{4 A_2 A_0^2}{\left(A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \coth \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right) \right)^2} \end{array} \right\} \times e^{i \left(-kx + \frac{12 ak^2 A_2 + 6 \alpha k A_2 + 4 \lambda A_0 A_2 - \lambda A_1^2}{6 A_2 (2 bk - 1)} t + \zeta \right)}, \quad (88)$$

$$v(x, t) = \left\{ \begin{array}{l} A_0 - \frac{2 A_1 A_0}{A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \coth \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right)} \\ + \frac{4 A_2 A_0^2}{\left(A_1 + \sqrt{A_1^2 - 4 A_0 A_2} \coth \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{24 A_2 (a - bp)}} \left(x - \frac{4 ka - 2 wb + \alpha t}{2 kb - 1} \right) \right) \right)^2} \end{array} \right\} \times e^{2i \left(-kx + \frac{12 ak^2 A_2 + 6 \alpha k A_2 + 4 \lambda A_0 A_2 - \lambda A_1^2}{6 A_2 (2 bk - 1)} t + \zeta \right)}, \quad (89)$$

with

$$A_2 (a - bp) (4 A_0 A_2 - A_1^2) \lambda > 0.$$

Result-2:

$$R = \frac{A_1}{A_2}, \quad S = \frac{6 A_0 A_2 - A_1^2}{2 A_2^2}, \quad \eta = \pm \sqrt{-\frac{\lambda A_2}{6 (a - bp)}},$$

$$w = \frac{4 a k^2 A_2 + 2 \alpha k A_2 - 4 \lambda A_0 A_2 + \lambda A_1^2}{2 A_2 (2 b k - 1)},$$

and

$$\mu = R^2 - 4S = -\frac{3 (4 A_0 A_2 - A_1^2)}{A_2^2}. \quad (90)$$

If one uses (90) along with (78) in (79), singular solitons are

$$u(x, t) = \left\{ \begin{array}{l} A_0 - \frac{A_1 (6 A_0 A_2 - A_1^2)}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4 A_0 A_2)} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{8 A_2 (a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)} \\ + \frac{(6 A_0 A_2 - A_1^2)^2}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4 A_0 A_2)} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{8 A_2 (a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)^2} \end{array} \right\} \times e^{i \left(-kx + \frac{4 a k^2 A_2 + 2 \alpha k A_2 - 4 \lambda A_0 A_2 + \lambda A_1^2}{2 A_2 (2 b k - 1)} t + \zeta \right)}, \quad (91)$$

$$v(x, t) = \left\{ \begin{array}{l} A_0 - \frac{A_1 (6 A_0 A_2 - A_1^2)}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4 A_0 A_2)} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{8 A_2 (a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)} \\ + \frac{(6 A_0 A_2 - A_1^2)^2}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4 A_0 A_2)} \tanh \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{8 A_2 (a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)^2} \end{array} \right\} \times e^{2i \left(-kx + \frac{4 a k^2 A_2 + 2 \alpha k A_2 - 4 \lambda A_0 A_2 + \lambda A_1^2}{2 A_2 (2 b k - 1)} t + \zeta \right)}, \quad (92)$$

with

$$A_2 (a - bp) (4 A_0 A_2 - A_1^2) \lambda > 0.$$

Dark solitons are

$$u(x, t) = \left\{ \begin{array}{l} A_0 - \frac{A_1 (6 A_0 A_2 - A_1^2)}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4 A_0 A_2)} \coth \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{8 A_2 (a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)} \\ + \frac{(6 A_0 A_2 - A_1^2)^2}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4 A_0 A_2)} \coth \left(\sqrt{\frac{(4 A_0 A_2 - A_1^2) \lambda}{8 A_2 (a - bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)^2} \end{array} \right\} \times e^{i \left(-kx + \frac{4 a k^2 A_2 + 2 \alpha k A_2 - 4 \lambda A_0 A_2 + \lambda A_1^2}{2 A_2 (2 b k - 1)} t + \zeta \right)}, \quad (93)$$

$$v(x, t) = \left\{ \begin{array}{l} A_0 - \frac{A_1(6A_0A_2 - A_1^2)}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4A_0A_2)} \coth \left(\sqrt{\frac{(4A_0A_2 - A_1^2)\lambda}{8A_2(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)} \\ + \frac{(6A_0A_2 - A_1^2)^2}{A_2 \left(A_1 + \sqrt{3(A_1^2 - 4A_0A_2)} \coth \left(\sqrt{\frac{(4A_0A_2 - A_1^2)\lambda}{8A_2(a-bp)}} \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \right) \right)^2} \end{array} \right\} \times e^{2i \left(-kx + \frac{4ak^2A_2 + 2\alpha kA_2 - 4\lambda A_0A_2 + \lambda A_1^2}{2A_2(2bk - 1)} t + \zeta \right)}, \quad (94)$$

with

$$A_2(a - bp)(4A_0A_2 - A_1^2)\lambda > 0.$$

2.6 TRIAL EQUATION

The solution of Eq. (11) is introduced as below:

$$(U')^2 = J(U) = \sum_{i=0}^N \mu_i U^i, \quad (95)$$

where μ_i are constants to be detected, $\mu_N \neq 0$ and N is the balancing integer. Now, one rewrites (95) by the integral form

$$\pm (\vartheta - \vartheta_0) = \int \frac{dU}{\sqrt{\sum_{i=0}^N \mu_i U^i}}. \quad (96)$$

Next, Eq. (95) can be rewritten as

$$(U')^2 = \mu_0 + \mu_1 U + \mu_2 U^2 + \mu_3 U^3, \quad (97)$$

by virtue of the balance principle applied in Eq. (11). Then, the equations are extracted as

U^2 Coeff.:

$$3\eta^2(a - pb)\mu_3 + 2\lambda = 0, \quad (98)$$

U Coeff.:

$$\eta^2(a - pb)\mu_2 + (2ak^2 - 2bkw + \alpha k + w) = 0, \quad (99)$$

U^0 Coeff.:

$$\eta^2(a - pb)\mu_1 = 0, \quad (100)$$

as long as Eq. (97) is substituted in Eq. (11). So, from Eqs. (98)-(100) are

$$\mu_1 = 0, \quad \mu_2 = -\frac{2ak^2 - 2bkw + \alpha k + w}{\eta^2(a - pb)}, \quad \mu_3 = -\frac{2\lambda}{3\eta^2(a - pb)}. \quad (101)$$

If we use the results (101) in Eq. (96), we get:

$$\pm (\vartheta - \vartheta_0) = \int \frac{dU}{\sqrt{\mu_0 - \frac{2ak^2 - 2bkw + \alpha k + w}{\eta^2(a - pb)} U^2 - \frac{2\lambda}{3\eta^2(a - pb)} U^3}}. \quad (102)$$

Lastly, optical solitons with the quadratic nonlinearity are recovered by using of $\mu_0 = 0$ in Eq. (102). Bright solitons are

$$u(x, t) = -\frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{sech}^2 \left[\begin{array}{l} \sqrt{-\frac{2ak^2 - 2bkw + \alpha k + w}{4(a - pb)}} \\ \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \end{array} \right] e^{i(-kx + wt + \zeta)}, \quad (103)$$

$$v(x, t) = -\frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{sech}^2 \left[\begin{array}{l} \sqrt{-\frac{2ak^2 - 2bkw + \alpha k + w}{4(a - pb)}} \\ \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \end{array} \right] e^{2i(-kx + wt + \zeta)}, \quad (104)$$

with

$$(a - pb)(2ak^2 - 2bkw + \alpha k + w) < 0.$$

Singular solitons are

$$u(x, t) = \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{csch}^2 \left[\begin{array}{l} \sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a - pb)}} \\ \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \end{array} \right] e^{i(-kx + wt + \zeta)}, \quad (105)$$

$$v(x, t) = \frac{3(2ak^2 - 2bkw + \alpha k + w)}{2\lambda} \operatorname{csch}^2 \left[\begin{array}{l} \sqrt{\frac{2ak^2 - 2bkw + \alpha k + w}{4(a - pb)}} \\ \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1} t \right) \end{array} \right] e^{2i(-kx + wt + \zeta)}, \quad (106)$$

with

$$(a - pb)(2ak^2 - 2bkw + \alpha k + w) < 0.$$

2.7 MODIFIED SIMPLE EQUATION

Assume that Eq. (11) has the solution in the form

$$U(\vartheta) = \sum_{i=0}^N \mu_i \left(\frac{Q'(\vartheta)}{Q(\vartheta)} \right)^i, \quad (107)$$

where μ_i are constants to be detected, such that $\mu_N \neq 0$, and $Q(\vartheta)$ is an unknown function to be established later. Next, Eq. (107) can be rewritten as

$$U(\vartheta) = \mu_0 + \mu_1 \left(\frac{Q'(\vartheta)}{Q(\vartheta)} \right) + \mu_2 \left(\frac{Q'(\vartheta)}{Q(\vartheta)} \right)^2, \quad (108)$$

by virtue of the balance principle applied in Eq. (11). Then, the strategic equations are

$$Q^{-4} \text{ coeff.:} \quad \mu_2 (Q')^4 (-6 b \eta^2 p + 6 a \eta^2 + \lambda \mu_2) = 0, \quad (109)$$

$$Q^{-3} \text{ coeff.:} \quad 2 (Q')^2 ((5 b \eta^2 p \mu_2 - 5 a \eta^2 \mu_2) Q'' + (-b \eta^2 p \mu_1 + a \eta^2 \mu_1 + \lambda \mu_1 \mu_2) Q') = 0, \quad (110)$$

$$Q^{-2} \text{ coeff.:} \quad (-2 b \eta^2 p \mu_2 + 2 a \eta^2 \mu_2) Q' Q''' + (-2 b \eta^2 p \mu_2 + 2 a \eta^2 \mu_2) (Q'')^2 + (3 b \eta^2 p \mu_1 - 3 a \eta^2 \mu_1) Q' Q'' \\ + (2 a k^2 \mu_2 - 2 b k w \mu_2 + \alpha k \mu_2 + 2 \lambda \mu_0 \mu_2 + \lambda \mu_1^2 + w \mu_2) (Q')^2 = 0, \quad (111)$$

$$Q^{-1} \text{ coeff.:} \quad \mu_1 ((-b \eta^2 p + a \eta^2) Q''' + (2 a k^2 - 2 b k w + \alpha k + 2 \lambda \mu_0 + w) Q') = 0, \quad (112)$$

$$Q^0 \text{ coeff.:} \quad 2 a k^2 \mu_0 - 2 b k w \mu_0 + \alpha k \mu_0 + \lambda \mu_0^2 + w \mu_0 = 0, \quad (113)$$

as long as (108) is inserted in (11). So, from Eqs. (109)-(113) are

$$\mu_0 = 0, \quad \mu_1 = \pm \sqrt{-\frac{36 \eta^2 (a - b p) (2 a k^2 - 2 b k w + \alpha k + w)}{\lambda^2}}, \quad \mu_2 = -\frac{6 \eta^2 (a - b p)}{\lambda}, \quad (114)$$

and one gets

$$Q'' = \pm \sqrt{-\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)}} Q', \quad (115)$$

$$Q''' = -\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)} Q'. \quad (116)$$

If one employs the equations (115)-(116), one reveals:

$$Q' = \pm \sqrt{-\frac{\eta^2 (a - b p)}{2 a k^2 - 2 b k w + \alpha k + w}} k_1 e^{\pm \sqrt{-\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)}} \vartheta}, \quad (117)$$

and

$$Q = -\frac{\eta^2 (a - b p)}{2 a k^2 - 2 b k w + \alpha k + w} k_1 e^{\pm \sqrt{-\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)}} \vartheta} + k_2, \quad (118)$$

with k_1 and k_2 integration constants. Lastly, if one puts Eqs. (114), (117), (118) into Eq. (108), optical solitons with the quadratic nonlinearity are recovered as

$$u(x, t) = \left\{ \begin{array}{l} \pm \sqrt{-\frac{36 \eta^2 (a - b p) (2 a k^2 - 2 b k w + \alpha k + w)}{\lambda^2}} \\ \times \left(\frac{\pm \sqrt{-\frac{\eta^2 (a - b p)}{2 a k^2 - 2 b k w + \alpha k + w}} k_1 e^{\pm \sqrt{-\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)}} \eta(x - p t)} \\ - \frac{\eta^2 (a - b p)}{2 a k^2 - 2 b k w + \alpha k + w} k_1 e^{\pm \sqrt{-\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)}} \eta(x - p t)} + k_2 \right) \\ - \frac{6 \eta^2 (a - b p)}{\lambda} \\ \times \left(\frac{\pm \sqrt{-\frac{\eta^2 (a - b p)}{2 a k^2 - 2 b k w + \alpha k + w}} k_1 e^{\pm \sqrt{-\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)}} \eta(x - p t)} \\ - \frac{\eta^2 (a - b p)}{2 a k^2 - 2 b k w + \alpha k + w} k_1 e^{\pm \sqrt{-\frac{2 a k^2 - 2 b k w + \alpha k + w}{\eta^2 (a - b p)}} \eta(x - p t)} + k_2 \right)^2 \end{array} \right\} \\ \times e^{i(-k x + w t + \zeta)}, \quad (119)$$

$$v(x, t) = \left\{ \begin{array}{l} \pm \sqrt{\frac{36\eta^2(a-bp)(2ak^2-2bkw+\alpha k+w)}{\lambda^2}} \\ \times \left(\frac{\pm \sqrt{\frac{\eta^2(a-bp)}{2ak^2-2bkw+\alpha k+w}} k_1 e^{\pm \sqrt{\frac{-2ak^2-2bkw+\alpha k+w}{\eta^2(a-bp)}} \eta(x-pt)}}{-\frac{\eta^2(a-bp)}{2ak^2-2bkw+\alpha k+w} k_1 e^{\pm \sqrt{\frac{-2ak^2-2bkw+\alpha k+w}{\eta^2(a-bp)}} \eta(x-pt)} + k_2} \right) \\ -\frac{6\eta^2(a-bp)}{\lambda} \\ \times \left(\frac{\pm \sqrt{\frac{\eta^2(a-bp)}{2ak^2-2bkw+\alpha k+w}} k_1 e^{\pm \sqrt{\frac{-2ak^2-2bkw+\alpha k+w}{\eta^2(a-bp)}} \eta(x-pt)}}{-\frac{\eta^2(a-bp)}{2ak^2-2bkw+\alpha k+w} k_1 e^{\pm \sqrt{\frac{-2ak^2-2bkw+\alpha k+w}{\eta^2(a-bp)}} \eta(x-pt)} + k_2} \right)^2 \end{array} \right\} \\ \times e^{2i(-kx+wt+\zeta)}, \quad (120)$$

and also if we set

$$k_1 = -\frac{2ak^2-2bkw+\alpha k+w}{\eta^2(a-bp)} e^{\pm \sqrt{\frac{-2ak^2-2bkw+\alpha k+w}{\eta^2(a-bp)}} \vartheta_0}, \quad k_2 = \pm 1, \quad (121)$$

in Eqs. (119) and (120), bright solitons are

$$u(x, t) = -\frac{3(2ak^2-2bkw+\alpha k+w)}{2\lambda} \operatorname{sech}^2 \left[\begin{array}{l} \sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4(a-bp)}} \\ \times \left(x - \frac{4ka-2wb+\alpha}{2kb-1} t \right) \end{array} \right] e^{i(-kx+wt+\zeta)}, \quad (122)$$

$$v(x, t) = -\frac{3(2ak^2-2bkw+\alpha k+w)}{2\lambda} \operatorname{sech}^2 \left[\begin{array}{l} \sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4(a-bp)}} \\ \times \left(x - \frac{4ka-2wb+\alpha}{2kb-1} t \right) \end{array} \right] e^{2i(-kx+wt+\zeta)}, \quad (123)$$

with

$$(a-bp)(2ak^2-2bkw+\alpha k+w) < 0.$$

Singular solitons are

$$u(x, t) = \frac{3(2ak^2-2bkw+\alpha k+w)}{2\lambda} \operatorname{csch}^2 \left[\begin{array}{l} \sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4(a-bp)}} \\ \times \left(x - \frac{4ka-2wb+\alpha}{2kb-1} t \right) \end{array} \right] e^{i(-kx+wt+\zeta)}, \quad (124)$$

$$v(x, t) = \frac{3(2ak^2-2bkw+\alpha k+w)}{2\lambda} \operatorname{csch}^2 \left[\begin{array}{l} \sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4(a-bp)}} \\ \times \left(x - \frac{4ka-2wb+\alpha}{2kb-1} t \right) \end{array} \right] e^{2i(-kx+wt+\zeta)}, \quad (125)$$

with

$$(a-bp)(2ak^2-2bkw+\alpha k+w) < 0.$$

3 CONCLUSIONS

This paper revisited embedded solitons that was studied with $\chi^{(2)}$ -nonlinear susceptibility. Several integration schemes revealed a wide range of solitons and especially the combo solitons that are visible in this work are being reported for the first time in this paper. The wide spectrum of solitons based on these diverse integration schemes are summarized in Table 1. This gives a visual perspective to the range of solitons that are available from the schemes.

These results therefore pave way for additional pathways to venture. These would include studying the model with variational principle using these combo solitons or various additional form of solitons. The conservation laws for the model are yet to be extracted. A wide variety of rich mathematical methodologies are available for implementation [12–25]. Thus, there are plentiful issues that need to be addressed with the dynamics of embedded solitons. Such a lot is floating over the horizon!

| Solitons | Bright | Dark | Singular | Straddled |
|--------------------------|--------|------|----------|-----------|
| Riccati equation | N | Y | Y | N |
| Sine–Gordon equation | N | Y | Y | Y |
| Functional Variable | Y | N | Y | N |
| F -expansion | Y | Y | Y | Y |
| Exp–function expansion | N | Y | Y | N |
| Trial equation | Y | N | Y | N |
| Modified simple equation | Y | N | Y | N |

Table 1: Soliton solutions via the integration schemes

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