Scientia Iranica B (2022) 29(4), 1868–1879

Sharif University of Technology
Scientia Iranica
Transactions B: Mechanical Engineering
http://scientiairanica.sharif.edu

Research Note

Generalized variance estimator using auxiliary information in the presence and absence of measurement error

M. Umair Tariq[a], M. Nouman Qureshi[1,*], and M. Hanif

Department of Statistics, National College of Business, Administration and Economics, Lahore, Pakistan.

Received 14 December 2020; received in revised form 24 November 2021; accepted 25 April 2022

KEYWORDS
Auxiliary information;
Measurement errors;
Variance estimation;
Mean square error and percentage relative efficiency.

Abstract. This study proposes a new generalized estimator based on auxiliary information for the estimation of population variance in the presence and absence of measurement error. Approximate expressions of bias and mean square error of the proposed generalized estimator are derived up to the first order. Several new and existing estimators are found to be special cases of the proposed estimator and expressed on different values of optimized and generalized constants. The proposed estimator is compared mathematically with a number of existing estimators under certain conditions. The performance of the proposed estimator is evaluated through simulation and real-data application under different sample sizes. It was observed that the proposed estimator performed better than other competing estimators in the presence and absence of measurement errors.

1. Introduction

A favorable understanding of variation is essential to obtaining efficient results. In sample surveys, efficient estimators are required to estimate different features including population mean, total, variance, etc. In the literature, various authors have proposed many efficient estimators for population parameters based on auxiliary information. Variance estimation of population is a very important issue in which variability control is difficult to achieve in its application. Researchers have found a major interest in controlling variations of objects/subjects. For instance, in health matters, body temperature, blood pressure, and pulse rate are basic diagnosis monitors that help design a treatment for control variation. In agricultural fields, sufficient knowledge about the climatic variation, rainfall, and area is required to formulate a suitable plan for cultivating a crop. In industries, manufacturers require regular information of variation in the reaction and behavior of people towards a product so that they will be able to know whether to enhance the quality of the product or increase or decrease its price.

The estimation problem of population variance is to measure the variability of the study variable which has received a significant interest from statisticians in survey sampling. In [1], the ratio and regression estimators were discussed to estimate finite population variance. In [2–4], exponential ratio-type, exponential product-type, and generalized exponential estimators of population variance were suggested. In [5], a ratio-cum-dual-type estimator was recommended for use when the population parameters of auxiliary variable were known for the estimation of population variance.

1. *Present address: School of Statistics, University of Minnesota, Minneapolis, USA.*
*. *Corresponding author.*
E-mail addresses: umairstats2@gmail.com (M. Umair Tariq); nqureshi633@gmail.com (M. Nouman Qureshi); drmainhanif@gmail.com (M. Hanif)

doi: 10.24200/sci.2022.57298.5164

Recently, the problem of variance estimation has been discussed by various authors. In [6], a population variance estimator was proposed using two auxiliary variable and generalized class of estimators under Simple Random Sampling Without Replacement (SRSWOR). In [7], chain ratio-type and chain ratio-ratio-type exponential estimators were suggested along with their generalized versions. In addition, the properties of the proposed estimators were discussed and the conditions were obtained in which the proposed estimators outperformed the existing estimators. Authors in [8] suggested a new generalized ratio-product-type estimator for population variance of study variable utilizing the information obtained from two auxiliary variables. In [9], an estimator for population variance was developed by introducing a linear combination of coefficient of kurtosis and decile mean of the auxiliary variable to achieve the efficiency of the proposed estimator. In [5], population variance of the study variable was estimated using tri-mean and third quartile of the auxiliary variable. Researchers in [10] proposed a class of ratio estimators for the estimation of population variance of the study variable using the coefficient of quartile deviation of auxiliary variable. In [11], the inter quartile range and population correlation coefficient of the auxiliary variable for populations of different characteristics were considered.

A very common assumption about statistical data is that the values obtained from the survey are correctly measured to check their corresponding true values and that there is no error in collected observations. However, this assumption does not usually hold in practice. Generally, different types of errors contaminate the data due to many inevitable causes such as non-response from respondents, faulty questionnaire preparation, flawed collection of sampling units, inaccurate interview techniques, or/and combination of some or all these. Measurement Errors (MEs) are defined as the difference between the observations obtained from the survey and true observations of the study variable [12]. MEs include observational error, instrument error, respondent error, etc. It has severe effects on the estimation of population parameters in terms of increase in bias and variation. Thus, it is essential that the role of measurement error be studied in developing better estimation techniques and obtaining more reliable and efficient estimates of the parameters in the presence of measurement errors.

Many authors have used auxiliary information in the presence of MEs to estimate population mean. For instance, these referenced authors [13-19] studied the effects of MEs on estimation of population parameters. Many researchers including [20–25] contributed to the variance estimation of the variable of interest in the presence of MEs.

The main objective of this study is to propose a generalized variance estimator using single auxiliary variable for the estimation of population variance in the presence and absence of MEs. The proposed estimator may produce some families and sub-families of the estimators as special cases using different suitable choices of the scalar constants. Following a brief introduction and literature review, the rest of the paper is as follows. Section 2 describes the sampling methodology with some existing estimators and basic notations for variance estimation in the presence and absence of MEs. Section 3 shows approximate mathematical expressions of bias and Mean Square Error (MSE) of the proposed estimator derived in the presence and absence of MEs. In this section, some particular cases of the proposed generalized estimator are also expressed on various values of optimized and generalized constants. Section 4 presents mathematical conditions where the proposed estimator has the least MSE compared to other existing estimators. Section 5 evaluates the performance of the suggested estimators through a numerical study in the presence and absence of MEs on the data based on three different artificial populations generated by normal distribution under different sample sizes. The application to real population is presented in Section 6. Final discussion in this paper is given in Section 7.

2. Sampling methodology, notations, and basic estimators

This section describes the sampling strategy of simple random sampling along with essential notations and some associated estimators based on population variance. The measurement errors are defined for both the study and auxiliary variables.

2.1. Sampling procedure

Suppose that Y and X are the study variable and the auxiliary variable, respectively, which are defined on N identifiable but distinct units of a finite population, $U = \{U_1, U_2, U_3 ..., U_N\}$. Let n pair of observations be obtained using simple random sampling without replacement (SRSWOR) on two variables Y and X. Suppose a situation where both variables Y and X are observed with some considerable error. For the ith sampling unit, let y_i and x_i be observed instead of true observations Y_i and X_i where $(i = 1, 2, ...n)$. The MEs may be defined as follows:

$$u_i = (y_i - Y_i), \tag{1}$$

and

$$v_i = (x_i - X_i), \tag{2}$$

where u_i and v_i are stochastic in nature and are the associated MEs with constant or zero mean and known variances S_U^2 and S_V^2, respectively. After the study in [26], it is assumed that the errors u_i and v_i are independent of each other and independent of Y_i and X_i, implying that:

$$COV\ (X,Y) \neq 0,$$

$$COV\ (X,U) = COV(X,V) = COV\ (U,Y) = 0,$$

$$COV\ (V,Y) = COV\ (U,V) = 0.$$

It is also assumed that the finite population correction is neglected. Let $\left(\overline{Y},\overline{X}\right)$ and $\left(S_y^2, S_x^2\right)$ be the finite population means and variances of the variable of interest and auxiliary variable $(Y,\ X)$, respectively, and ρ_{yx} be the correlation coefficient between the subscripts.

2.2. Notations

Let $\bar{y} = \frac{1}{n}\sum_i^n y_i$ and $\bar{x} = \frac{1}{n}\sum_i^n x_i$ be the sample mean estimators that are unbiased to the population means $\left(\overline{Y},\overline{X}\right)$, respectively. However, under MEs, $s_y^2 = \frac{1}{n-1}\sum_i^n \left(y_i - \bar{y}\right)^2$ and $s_x^2 = \frac{1}{n-1}\sum_i^n \left(x_i - \bar{x}\right)^2$ are not unbiased estimators of $\left(S_y^2, S_x^2\right)$, respectively, where the expected values of s_y^2 and s_x^2 in the presence of MEs are as follows:

$$E\left(s_y^2\right) = S_y^2 + S_u^2, \qquad E\left(s_x^2\right) = S_x^2 + S_v^2.$$

Error variances and S_v^2 associated with respective study variable and the auxiliary variable are known. In such situations, the unbiased estimators of S_y^2 and S_x^2 are respectively given by:

$$\hat{s}_y^2 = s_y^2 - S_u^2 > 0, \qquad \hat{s}_x^2 = s_x^2 - S_v^2 > 0.$$

Now, let us define:

$$\hat{s}_y^2 = S_y^2\left(1 + e_o\right), \qquad \hat{s}_x^2 = S_x^2\left(1 + e_1\right).$$

Therefore:

$$E\left(e_o\right) = E\left(e_1\right) = 0, \quad E\left(e_o^2\right) = \frac{A_y}{n}, \quad E\left(e_1^2\right) = \frac{A_x}{n},$$

$$E\left(e_o e_1\right) = \frac{\delta - 1}{n}$$

where:

$$A_y = \gamma_{2y} + \gamma_{2u}\frac{S_u^4}{S_y^4} + 2\left(1 + \frac{S_u^2}{S_y^2}\right)^2,$$

$$A_x = \gamma_{2x} + \gamma_{2v}\frac{S_v^4}{S_x^4} + 2\left(1 + \frac{S_v^2}{S_x^2}\right)^2,$$

$$\gamma_{2z} = \beta_{2z} - 3, \qquad \beta_{2z} = \mu_{4z}\Big/\mu_{2z}^2,$$

$$\mu_{rz} = E(z_i - \mu_z)^r, \quad \theta_x = S_x^2\Big/S_x^2 - S_v^2,$$

$$\delta = \frac{\mu_{22}\left(X,Y\right)}{S_x^2 S_y^2},$$

where $z = Y, X, U$, and V.

If there is no measurement error, then $E\left(e_o\right) = E\left(e_1\right) = 0$, and $E\left(e_o^2\right) = \frac{\beta_{2y}-1}{n} = V_{40}$, whereas $E\left(e_1^2\right) = \frac{\beta_{2x}-1}{n} = V_{04}$, and $E\left(e_o e_1\right) = \frac{\mu_{22}-1}{n} = V_{22}$.

2.3. Basic estimators

In this sub-section, some classical estimators are given for population variance in the presence and absence of MEs.

The unbiased estimator $\hat{s}_y^2$ for population variance in the presence of MEs is as follows:

$$t_o = \hat{s}_y^2.$$

The variances of t_0 with and without MEs are respectively given as follows:

$$Var\left(t_o\right) = \frac{S_y^4}{n}A_y.$$

and:

$$var\left(t_o\right) = S_y^4 V_{40}.$$

Classical ratio estimator under measurement errors is defined as follows [1]:

$$t_1 = \hat{s}_y^2\left(\frac{S_x^2}{\hat{s}_x^2}\right).$$

The expressions of approximate bias and MSE of t_1 in the presence and absence of MEs are respectively given as:

$$Bias\left(t_1\right) \cong \frac{S_y^2}{n}\left(1 + A_x - \delta\right),$$

$$Bias\left(t_1\right) \cong S_y^2\left(V_{04} - V_{22}\right),$$

and:

$$MSE\left(t_1\right) \cong \frac{S_y^4}{n}\left(2 + A_y + A_x - 2\delta\right),$$

$$MSE\left(t_1\right) \cong S_y^4\left(V_{40} + V_{04} - 2V_{22}\right).$$

As proposed in [2], the modified ratio-type estimator in the presence of MEs is as follows:

$$t_2 = \hat{s}_y^2\left(\frac{S_x^2 + C_x}{\hat{s}_x^2 + C_x}\right).$$

The expressions of approximate bias and MSE of t_2 in the presence and absence of MEs are respectively given as:

$$Bias\left(t_2\right) \cong \frac{S_y^2}{n}B\left(1 + BA_x - \delta\right),$$

$$Bias\left(t_2\right) \cong S_y^2 B\left(BV_{04} - V_{22}\right),$$

and:

$$MSE\left(t_2\right) \cong \frac{S_y^4}{n}\left(A_y + B^2 A_x - 2B\left(\delta - 1\right)\right),$$

$$MSE\left(t_2\right) \cong S_y^4\left(V_{40} + B^2 V_{04} - 2BV_{22}\right),$$

where:

$B = S_x^2 \Big/ (S_x^2 + C_x)$.

The exponential ratio estimator suggested by [4] in the presence of MEs is:

$$t_3 = \hat{s}_y^2 \exp\left(\frac{S_x^2 - \hat{s}_x^2}{S_x^2 + \hat{s}_x^2}\right).$$

The equations of bias and MSE of t_3 with and without MEs are respectively, given as:

$$Bias\,(t_3) \cong \frac{S_y^2}{n}\left(\frac{3A_x}{4} - (\delta - 1)\right),$$

$$Bias\,(t_3) \cong \frac{S_y^2}{2}\left(\frac{3V_{04}}{4} - V_{22}\right),$$

and:

$$MSE\,(t_3) \cong \frac{S_y^4}{n}\left(A_y + \frac{A_x}{4} - (\delta - 1)\right),$$

$$MSE\,(t_3) \cong S_y^4\left(V_{40} + \frac{V_{04}}{4} - V_{22}\right).$$

3. Proposed generalized estimator

In this section, a generalized estimator is proposed for population variance in the presence of MEs using simple random sampling technique. The proposed estimator produces many special cases (summarized in Appendix A) as a family of the proposed estimators among which the estimator suggested in [7] is a particular case for different values of properly chosen constants. The proposed estimator in [7] is a less generalized estimator for population variance than the one given here and it acts regardless of the MEs. The proposed estimator is defined as follows:

$$t_{pi} = \hat{s}_y^2\left[M_1 + M_2\left(S_x^2 - \hat{s}_x^2\right)\right]\left(\frac{aS_x^2 + b}{a\hat{s}_x^2 + b}\right)^{\alpha}$$
$$\exp\left[\beta\left(\frac{a\left(S_x^2 - \hat{s}_x^2\right)}{a\left(S_x^2 - \hat{s}_x^2\right) + 2b}\right)\right], \quad (3)$$

where α and β are suitable constants that can assume $\{(0,0),\ (1,1),\ (-1,-1),\ (1,0),\ (0,1),\ (0,-1),\ (-1,0),\ (1,-1),\ (-1,1)\}$, whereas ($a$ 0; b) are real numbers or some known parameters of the auxiliary variable and M_1 and M_2 are the optimized constants which minimize the MSE of the proposed estimators t_{pi}.

To obtain the mathematical expressions for bias and MSE of t_{pi}, Eq. (3) can be rewritten in e terms as follows:

$$t_{pi} = S_y^2\left(1 + e_o\right)\left[M_1 + M_2\left(S_x^2 - S_x^2\left(1 + e_1\right)\right)\right]$$
$$\left(1 + ke_1\right)^{-\alpha}\exp\left(\frac{-\beta ke_1}{2}\left(1 + \frac{ke_1}{2}\right)^{-1}\right), \quad (4)$$

where $k = aS_x^2 \Big/ (aS_x^2 + b)$.

Expanding Eq. (4) through Taylor series, we have:

$$t_{pi} = S_y^2\left(1 + e_o\right)\left[M_1 + M_2 S_x^2 e_1\right]$$
$$\left(1 - \alpha ke_1 + \frac{\alpha\left(\alpha + 1\right)k^2 e_1^2}{2}\right)$$
$$\exp\left(\frac{-\beta ke_1}{2}\left(1 + \frac{ke_1}{2}\right)^{-1}\right). \quad (5)$$

Upon the simplification of Eq. (5), we get Eq. (6) as shown in Box I.

Simplifying and taking expectations on both sides of Eq. (6), we get the bias of the estimator t_{pi} in the presence of the MSE. Here, $c = \alpha + (\beta/2)$.

$$Bias\,(t_{pi}) \cong S_y^2$$
$$\left[(M_1 - 1) + \frac{A_x}{n}\left(M_2 S_x^2 kc + \frac{M_1 c\left(c + 1\right)k^2}{2}\right)\right.$$
$$\left. - \frac{(\delta - 1)}{n}\left(M_2 S_x^2 + M_1 kc\right)\right]. \quad (7)$$

If ME is zero, then the expression of bias of the estimator t'_{pi} will be:

$$Bias\,(t'_{pi}) \cong S_y^2$$
$$\left[(M_1 - 1) + V_{04}\left(M_2 S_x^2 kc + \frac{M_1 c\left(c + 1\right)k^2}{2}\right)\right.$$
$$\left. - V_{22}\left(M_2 S_x^2 + M_1 kc\right)\right]. \quad (8)$$

$$\left(t_{pi} - S_y^2\right)$$
$$= S_y^2\left[\begin{array}{l}(M_1 - 1) + M_1 e_o - e_1\left(M_2 S_x^2 + M_1\alpha k + \frac{M_1\beta k}{2}\right) - e_o e_1\left(M_2 S_x^2 + M_1\alpha k + \frac{M_1\beta k}{2}\right) + e_1^2 \\ \left(M_2 S_x^2\alpha k + \frac{M_1\alpha(\alpha+1)k^2}{2} + \frac{M_2 S_x^2\beta k}{2} + \frac{M_1\alpha\beta k^2}{2} + \frac{M_1\beta k^2}{4} + \frac{M_1\beta^2 k^2}{8}\right)\end{array}\right]. \quad (6)$$

Box I

$$\left(t_{pi}-S_y^2\right)^2=S_y^4$$
$$\begin{bmatrix} (M_1-1)+M_1e_o-e_1\left(M_2S_x^2+M_1\alpha k+\frac{M_1\beta k}{2}\right)-e_oe_1\left(M_2S_x^2+M_1\alpha k+\frac{M_1\beta k}{2}\right)+e_1^2 \\ \left(M_2S_x^2\alpha k+\frac{M_1\alpha(\alpha+1)k^2}{2}+\frac{M_2S_x^2\beta k}{2}+\frac{M_1\alpha\beta k^2}{2}+\frac{M_1\beta k^2}{4}+\frac{M_1\beta^2k^2}{8}\right)\end{bmatrix}^2. \quad (9)$$

Box II

$$MSE\left(t_{pi}\right)\cong S_y^4$$
$$\begin{bmatrix} 1+M_1^2\left\{1+\frac{A_y}{n}+\frac{A_xk^2c(2c+1)}{n}-4\left(\frac{\delta-1}{n}\right)kc\right\}+M_2^2S_x^4\frac{A_x}{n}+(4M_1M_2-2M_2) \\ S_x^2\left\{\frac{A_x}{n}kc-\left(\frac{\delta-1}{n}\right)\right\}-2M_1\left\{1+\frac{A_xk^2c(c+1)}{2n}-\left(\frac{\delta-1}{n}\right)kc\right\}\end{bmatrix}. \quad (10)$$

Box III

To get the MSE of the proposed estimator, squaring and applying Taylor series on Eq. (5), we have Eq. (9) shown in Box II. Upon the simplification of Eq. (6), we have the final expression of MSE in the presence of ME of the proposed estimator (Eq. (10) shown in Box III).

$$MSE\left(t_{pi}\right)\cong S_y^4\Big[1+M_1^2C_1+M_2^2C_2+\left(4M_1M_2-2M_2\right)C_3-2M_1C_4\Big], \quad (11)$$

where:

$$C_1=\left\{1+\frac{A_y}{n}+\frac{A_xk^2c\left(2c+1\right)}{n}-4\left(\frac{\delta-1}{n}\right)kc\right\},$$

$$C_2=S_x^4\frac{A_x}{n},$$

$$C_3=S_x^2\left\{\frac{A_x}{n}kc-\left(\frac{\delta-1}{n}\right)\right\},$$

$$C_4=\left\{1+\frac{A_xk^2c\left(c+1\right)}{2n}-\left(\frac{\delta-1}{n}\right)kc\right\}.$$

If ME is assumed negligible, then Eq. (11) will be substitute in Eq. (12), as shown in Box IV.

$$MSE\left(t'_{pi}\right)\cong S_y^4\Big[1+M_1^2C'_1+M_2^2C'_2+\left(4M_1M_2-2M_2\right)C'_3-2M_1C'_4\Big], \quad (13)$$

where:

$$C'_1=\left\{1+V_{40}+V_{04}k^2c\left(2c+1\right)-4V_{22}kc\right\},$$

$$C'_2=S_x^4V_{04},$$

$$C'_3=S_x^2\left\{V_{04}kc-V_{22}\right\}$$

$$C'_4=\left\{1+\frac{V_{04}k^2c\left(c+1\right)}{2}-V_{22}kc\right\}.$$

For the optimum values of M_1 and M_2, we partially differentiate Eq. (13) with respect to M_1 and M_2 and equate them to zero:

$$\frac{\partial MSE\left(t_{pi}\right)}{\partial M_1}=S_y^4\left[2M_1C_1-2C_4+4M_2C_3\right]=0,$$

$$MSE\left(t'_{pi}\right)\cong S_y^4$$
$$\begin{bmatrix} 1+M_1^2\left\{1+V_{40}+V_{04}k^2c\left(2c+1\right)-4V_{22}kc\right\}+M_2^2S_x^4V_{04}+\left(4M_1M_2-2M_2\right)S_x^2\left\{V_{04}kc-V_{22}\right\} \\ -2M_1\left\{1+\frac{V_{04}k^2c(c+1)}{2}-V_{22}kc\right\}\end{bmatrix}. \quad (12)$$

Box IV

$$M_1C_1 + 2M_2C_3 = C_4. \quad (14)$$

Similarly, we have:

$$\frac{\partial MSE(t_{pi})}{\partial M_2} = S_y^4 \left[2M_2C_2 - 2C_3 + 4M_1C_3\right] = 0.$$

$$2M_1C_3 + M_2C_2 = C_3. \quad (15)$$

Solving both Eq. (14) and Eq. (15) simultaneously, we have:

$$\begin{bmatrix} C_1 & 2C_3 \\ 2C_3 & C_2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \begin{bmatrix} C_4 \\ C_3 \end{bmatrix}.$$

The optimum values of M_1 and M_2 are as follows:

$$M_1 = \frac{C_4C_2 - 2{C_3}^2}{C_1C_2 - 4{C_3}^2} \qquad M_2 = \frac{C_1C_3 - 2C_4C_3}{C_1C_2 - 4{C_3}^2}.$$

We substitute the values of M_1 and M_2 in Eq. (11) and get the minimum MSE of the proposed estimator as:

$$MSE(t_{pi})_{\min} \cong S_y^4 \left[1 - A\right],$$

where:

$$A = \frac{C_1{C_3}^2 + C_2{C_4}^2 - 4C_4{C_3}^2}{C_1C_2 - 4{C_3}^2} \quad \text{(with ME)}. \quad (16)$$

Similarly, if ME is negligible, then the MSE of the estimator t'_{pi} is:

$$MSE(t'_{pi})_{\min} \cong S_y^4 \left[1 - A'\right], \quad (17)$$

where:

$$A' = \frac{C'_1{C'_3}^2 + C'_2{C'_4}^2 - 4C'_4{C'_3}^2}{C'_1C'_2 - 4{C'_3}^2} \quad \text{(without ME)}.$$

Many special cases of the proposed estimators are obtained and presented in Appendix Table A.1 Some special cases of estimator t_{pi} are:

Case 1: For $M_2 = 0$, the estimator is reduced to [7]:

$$t_{pi1} = M_{11}\hat{s}_y^2 \left(\frac{aS_x^2 + b}{a\hat{s}_x^2 + b}\right)^{\alpha},$$

$$\exp\left[\beta\left(\frac{a\left(S_x^2 - \hat{s}_x^2\right)}{a\left(S_x^2 - \hat{s}_x^2\right) + 2b}\right)\right]. \quad (18)$$

The approximate bias and MSE of the estimator t_{pi1} defined in Eq. (13) under ME are:

$$Bias\left(t_{pi1}\right) \cong S_y^2\Bigg[\left(M_{11} - 1\right) + \frac{M_{11}kc}{n}\left(\frac{A_x\left(c{+}1\right)k}{2} - (\delta - 1)\right)\Bigg]. \quad (19)$$

and:

$$MSE\left(t_{pi1}\right) \cong S_y^4\Bigg[1 + M_{11}^2\Bigg\{1 + \frac{A_y}{n} + \frac{A_xk^2c\left(2c+1\right)}{n} - 4\left(\frac{\delta - 1}{n}\right)kc\Bigg\} - 2M_{11}\Bigg\{1 + \frac{A_xk^2c\left(c+1\right)}{2n} - \left(\frac{\delta - 1}{n}\right)kc\Bigg\}\Bigg]. \quad (20)$$

The MSE of the estimator t'_{pi1} without ME is:

$$MSE\left({t'}_{pi1}\right) \cong S_y^4\Bigg[1 + M_{11}^2 \left\{1 + V_{40} + V_{04}k^2c\left(2c+1\right) - 4V_{22}kc\right\} - 2M_{11}\left\{1 + \frac{V_{04}k^2c\left(c+1\right)}{2} - V_{22}kc\right\}\Bigg]. \quad (21)$$

Case 2: For $M_1 = 0$, the estimator is:

$$t_{pi2} = M_{12}\hat{s}_y^2\left(S_x^2 - \hat{s}_x^2\right)\left(\frac{aS_x^2 + b}{a\hat{s}_x^2 + b}\right)^{\alpha} \exp\left[\beta\left(\frac{a\left(S_x^2 - \hat{s}_x^2\right)}{a\left(S_x^2 - \hat{s}_x^2\right) + 2b}\right)\right]. \quad (22)$$

The approximate bias and MSE of the estimator t_{pi2} defined in Eq. (14) under ME are:

$$Bias\left(t_{pi2}\right) \cong S_y^2\left[\frac{M_{12}S_x^2}{n}\left\{A_xkc - (\delta - 1)\right\} - 1\right]. \quad (23)$$

and

$$MSE\left(t_{pi2}\right) \cong S_y^4 \left[1 + M_{12}^2S_x^4\frac{A_x}{n} - \frac{2M_{12}S_x^2}{n}\left\{A_xkc - (\delta - 1)\right\}\right]. \quad (24)$$

The MSE of estimator t'_{pi2} without ME is:

$$MSE\left({t'}_{pi2}\right) \cong S_y^4 \left[1 + M_{12}^2S_x^4V_{04} - 2M_{12}S_x^2\left\{V_{04}kc - V_{22}\right\}\right]. \quad (25)$$

4. Efficiency comparisons

This section mathematically compares the MSE of the proposed estimator with those of some existing estimators (t_0, t_1, t_2, and t_3). By definition, the proposed estimator would be more efficient than the existing estimators if the MSE of t_{pi} was smaller than those of the existing estimators. The optimum conditions for t_{pi} that are given in Eq. (11) with respect to t_o, t_1, t_2, and t_3 are orderly given below:

$$MSE\left(t_o\right)-MSE(t_{pi})_{\min}>0,$$

if $S_y^4\left[A'+\frac{A_y}{n}-1\right]>0.$

$$MSE\left(t_1\right)-MSE(t_{pi})_{\min}>0,$$

if $S_y^4\left[A'+\frac{1}{n}\left\{A_y+A_x-2\left(\delta-1\right)\right\}-1\right]>0.$

$$MSE\left(t_2\right)-MSE(t_{pi})_{\min}>0,$$

if $S_y^4\left[A'+\frac{1}{n}\left\{A_y+B^2A_x-2B\left(\delta-1\right)\right\}-1\right]>0.$

$$MSE\left(t_3\right)-MSE(t_{pi})_{\min}>0,$$

if $S_y^4\left[A'+\frac{1}{n}\left\{A_y+\frac{A_x}{4}-\left(\delta-1\right)\right\}-1\right]>0.$

5. Simulation study

In this section, the efficiencies of the proposed estimator and the existing estimators mentioned in this study are comparatively examined in numerical terms. It is carried out using R-software developed by R [27]. The samples are generated using SRSWOR from the three finite populations, each of size $N=5000$. The populations are generated using multivariate normal distribution with the same ME variances as $S_U^2=S_V^2=9$. The details of population means and variances are:

- **Population I:** $\mu_{YX}=\begin{bmatrix}6 & 5\end{bmatrix}$, $S_y^2=125$, $S_x^2=100$, $\rho_{YX}=0.898$.
- **Population II:** $\mu_{YX}=\begin{bmatrix}6 & 5\end{bmatrix}$, $S_y^2=157$, $S_x^2=121$, $\rho_{YX}=0.881$.
- **Population III:** $\mu_{YX}=\begin{bmatrix}6 & 5\end{bmatrix}$, $S_y^2=149$, $S_x^2=100$, $\rho_{YX}=0.826$.

For each population, different sample sizes are considered as $n=50$, 100, 300, and 500. The Percent Relative Efficiencies (PREs) can be computed for all the estimators using the following formula:

$$PREs\left(t_i,t_o\right)=\frac{MSE\left(t_o\right)}{MSE\left(t_i\right)}\times 100,$$

where $t_i=t_1,t_2,t_3,t_{p11},t_{p12},t_{pi}$.

The results of MSE and PRE of the proposed estimator along with the existing estimators are evaluated in the presence and absence of ME and summarized in Tables 1 to 3.

Table 1. The MSE and PRE (with and without ME) of different estimators for Population I.

Sample size	Estimator	Without ME		With ME		Amount of ME
		MSE	PRE	MSE	PRE	MSE
50	t_1	271.451	233.48	343.396	184.56	71.946
	t_2	264.880	239.27	331.397	191.24	66.517
	t_3	290.793	217.95	317.135	199.84	26.342
	t_{pi1}'	220.498	287.43	321.311	197.25	100.812
	t_{pi2}'	173.262	365.8	239.157	265.01	65.895
	t_{pi}'	145.515	435.54	213.646	296.65	68.132
100	t_1	129.395	234.15	159.491	189.97	30.096
	t_2	126.567	239.38	154.725	195.82	28.158
	t_3	137.914	219.69	149.984	202.01	12.070
	t_{pi1}'	102.712	294.98	142.543	212.56	39.831
	t_{pi2}'	81.3528	372.43	107.344	282.25	25.991
	t_{pi}'	67.7830	446.99	94.626	320.19	26.843
300	t_1	41.0270	248.14	50.766	200.53	9.7391
	t_2	40.039	254.26	49.116	207.26	9.077
	t_3	44.017	231.28	47.907	212.5	3.889
	t_{pi1}'	33.805	301.15	47.062	216.32	13.257
	t_{pi2}'	27.277	373.22	35.984	282.91	8.707
	t_{pi}'	22.496	452.53	31.448	323.72	8.952
500	t_1	22.520	232.59	28.136	186.17	5.616
	t_2	22.068	237.35	27.330	191.65	5.262
	t_3	24.286	215.68	26.451	198.02	2.165
	t_{pi1}'	17.058	307.07	24.252	215.98	7.194
	t_{pi2}'	12.520	418.37	16.847	310.92	4.327
	t_{pi}'	11.041	474.39	15.817	331.16	4.775

Table 2. The MSE and PRE (with and without ME) of different estimators for Population II.

Sample size	Estimator	Without ME		With ME		Amount of ME
		MSE	PRE	MSE	PRE	MSE
50	t_1	477.972	207.67	583.918	169.99	105.945
	t_2	467.370	212.38	566.401	175.25	99.031
	t_3	473.727	209.53	509.240	194.92	35.514
	${t_{pi1}}'$	385.912	257.21	531.197	186.86	145.286
	${t_{pi2}}'$	410.978	241.52	543.840	182.52	132.862
	${t_{pi}}'$	271.322	365.84	376.590	263.58	105.268
100	t_1	258.044	192.39	312.217	159.01	54.172
	t_2	252.729	196.44	303.598	163.52	50.869
	t_3	248.851	199.5	267.039	185.91	18.188
	${t_{pi1}}'$	193.878	256.06	262.217	189.33	68.339
	${t_{pi2}}'$	201.501	246.38	263.138	188.66	61.637
	${t_{pi}}'$	134.665	368.66	183.967	269.86	49.302
300	t_1	69.784	216.61	83.186	181.71	13.402
	t_2	68.303	221.31	80.898	186.85	12.595
	t_3	70.440	214.59	75.407	200.46	4.967
	${t_{pi1}}'$	54.210	278.84	71.319	211.95	17.109
	${t_{pi2}}'$	60.141	251.34	76.014	198.86	15.872
	${t_{pi}}'$	38.298	394.69	50.695	298.17	12.397
500	t_1	45.042	219.76	53.527	184.92	8.486
	t_2	44.335	223.26	52.391	188.93	8.056
	t_3	48.331	204.8	51.734	191.33	3.403
	${t_{pi1}}'$	32.321	306.25	42.532	232.73	10.211
	${t_{pi2}}'$	34.747	284.86	43.827	225.85	9.080
	${t_{pi}}'$	22.365	442.58	29.675	333.56	7.310

Table 3. The MSE and PRE (with and without ME) of different estimators for Population III.

Sample size	Estimator	Without ME		With ME		Amount of ME
		MSE	PRE	MSE	PRE	MSE
50	t_1	644.807	151.46	823.500	118.60	178.693
	t_2	629.157	155.23	794.612	122.91	165.455
	t_3	560.262	174.32	610.179	160.06	49.918
	${t_{pi1}}'$	468.299	208.55	692.795	140.97	224.496
	${t_{pi2}}'$	405.328	240.95	590.052	165.52	184.724
	${t_{pi}}'$	293.703	332.53	442.153	220.88	148.450
100	t_1	293.146	153.07	380.159	118.04	87.013
	t_2	286.550	156.59	367.643	122.05	81.093
	t_3	254.039	176.64	277.923	161.46	23.884
	${t_{pi1}}'$	210.265	213.41	317.218	141.46	106.953
	${t_{pi2}}'$	172.485	260.15	254.692	176.18	82.207
	${t_{pi}}'$	130.666	343.41	200.531	223.77	69.864
300	t_1	86.309	148.27	109.388	116.99	23.079
	t_2	84.556	151.34	106.247	120.45	21.690
	t_3	74.340	172.14	81.350	157.31	7.010
	${t_{pi1}}'$	59.570	214.82	86.192	148.47	26.622
	${t_{pi2}}'$	45.187	283.2	64.062	199.76	18.875
	${t_{pi}}'$	36.440	351.18	53.556	238.95	17.116
500	t_1	53.067	137.3	66.874	108.96	13.807
	t_2	51.876	140.46	64.833	112.39	12.957
	t_3	42.665	170.78	46.625	156.28	3.960
	${t_{pi1}}'$	37.711	193.21	53.772	135.50	16.060
	${t_{pi2}}'$	28.617	254.61	40.190	181.30	11.572
	${t_{pi}}'$	23.230	313.66	33.596	216.88	10.366

Table 4. Hypothetical population data.

Y	X	Y	X
75.4666	80	67.6011	80.094
74.9801	100	75.4438	91.5721
102.8242	120	109.6956	112.1406
125.7651	140	129.4159	145.5969
106.5035	160	104.2388	168.5579
131.4318	180	125.8319	171.4793
149.3693	200	153.9926	203.5366
143.8628	220	152.9208	222.8533
177.5218	240	176.3344	232.9879
182.2748	260	174.5252	261.1813

The results of MSE and PRE of different estimators in the presence and absence of ME for populations I to III are summarized in Tables 2 to 4 and they reveal that the MSEs of the mentioned estimators reduce the PREs upon increase in the sample sizes. Bold values indicate the least MSE and high PRE of the estimators under study. It was also observed that for all the three populations, the suggested estimator t'_{pi} would have minimum MSE and maximum PRE compared to the estimators t_o, t_1, t_2, t_3, t'_{pi1}, and t'_{pi2} with and without ME.

6. An application to real population

This section considers a hypothetical data taken from [28]. In the data set, the study variable Y is the true consumption expenditure and the auxiliary variable X is the true income. Further, the variables are contaminated with ME and are taken as measured consumption (y) expenditure and measured income (x), as shown in Table 4.

The parameters of the real data set are presented in Table 5.

A sample of size 4 is taken for the computation of MSEs and PREs of different variance estimators in the presence and absence of ME. The results are presented in Table 6.

The results of MSE and PRE based on the real data presented in Table 6 show that the efficiency of the proposed estimator is higher than those of other existing estimators in the presence and absence of ME. The value of MSE of the proposed estimator is the least among all the competing estimators. It is also evident that the proposed estimator has the least ME value.

7. Conclusions

A new estimator was proposed in this paper for the estimation of population variance in the presence and absence of ME. Many special cases were obtained among various options of optimized and generalized constants of the proposed estimator including [7] and many other product-type, ratio-type, and exponential ratio-type estimators. The expressions of approximate bias and MSE were derived for the proposed estimator. The conditions in which the proposed estimator could be mathematically more efficient than the competing estimators were also determined. The role of sample size, which is significant in the simulation study, was considered to evaluate the performance of the proposed estimator. As the sample size increased from 50 to 500, the MSEs decreased and PRE increased for each sample size in the presence and absence of ME. From numerical findings, it was mathematically and numerically found that the proposed estimator performed better than the other mentioned estimators in the presence and absence of ME for all the artificial and real population data sets.

Further expansion of the present effort deserves further consideration. In this present study, one auxiliary variable is considered for variance estimation in the presence of ME under SRS design. A fruitful area for future research is to incorporate multi-auxiliary variable during the multi-estimation phases under dif-

Table 5. The parameters of real data.

Parameters	N	$\overline{Y}$	$\bar{X}$	S_Y^2	S_X^2	S_U^2	S_V^2	ρ_{xy}
Values	10	127	170	1420	3666.667	36	41.24	0.964

Table 6. The MSE and PRE (with and without ME) of different estimators.

Estimator	With ME		Without ME		Amount of ME
	MSE	PRE	MSE	PRE	MSE
t_1	159542	320.34	155341	329.00	4201
t_2	159461	320.50	155261	329.17	4200
t_3	210428	242.87	203026	251.73	7402
t_{pi1}'	152056	336.11	151744	336.80	312
t_{pi2}'	147175	347.26	146889	347.94	286
t_{pi}'	135669	376.71	135398	377.46	271

ferent sampling designs, such as stratified sampling, systematic sampling, etc. This could, in principle, be carried out through multivariable extension of the proposed generalized estimator.

Acknowledgement

The authors are thankful to the Editor-in-Chief, Prof. S.T.A. Niaki, and anonymous referees for their valuable suggestions that helped improve the article.

References

1. Isaki, C.T. "Variance estimation using auxiliary information", *Journal of the American Statistical Association*, **78**(381), pp. 117–123 (1983). https://doi.org/10.1080/01621459.1983.10477939
2. Kadilar, C. and Cingi, H. "Ratio estimators for the population variance in simple and stratified random sampling", *Applied Mathematics and Computation*, **173**(2), pp. 1047–1059 (2006). https://doi.org/10.1016/j.amc.2005.04.032
3. Shabbir, J. and Gupta, S.A.T. "On improvement in variance estimation", *Communications in Statistics-Theory and Methods*, **36**(12), pp. 2177–2185 (2007). https://doi.org/10.1080/03610920701215092
4. Singh, R., Chauhan, P., Sawan, N., et al. "Improved exponential estimator for population variance using two auxiliary variables", *Italian Journal of Pure and Applied Mathematics*, **28**, pp. 101–108 (2011).
5. Yadav, S.K., Mishra, S.S., and Gupta, S. "An efficient estimator for population variance using parameters of an auxiliary variable", *Journal of Statistics and Management Systems*, **22**(6), pp. 1005–1013 (2019a). https://doi.org/10.1080/09720510.2017.1406643
6. Adichwal, N.K., Kumar, J., and Singh, R. "An improved generalized class of estimators for population variance using auxiliary variables", *Cogent Mathematics and Statistics*, **5**(1), pp. 1–8 (2018). https://doi.org/10.1080/25742558.2018.1454579
7. Singh, H.P., Pal, S.K., and Yadav, A. "A study on the chain ratio-ratio-type exponential estimator for finite population variance", *Communications in Statistics - Theory and Methods*, **47**(6), pp. 1442–1458 (2017). https://doi.org/10.1080/03610926.2017.1321124
8. Ismail, M., Kanwal, N., and Shahbaz, M.Q. "Generalized ratio-product-type estimator for variance using auxiliary information in simple random sampling", *Kuwait Journal of Science*, **45**(1), pp. 79–88 (2018).
9. Bhat, M.A. "An improvement in variance estimator for the estimation of population variance, using known values of auxiliary information", *International Journal of Pure and Applied Bioscience*, **6**(5), pp. 135–138 (2018). https://doi.org/10.18782/2320-7051.6842
10. Javed, K., Jamal, N., Hanif, M., et al. "Improved estimator of finite population variance using coefficient of quartile deviation", *Asian Journal of Advanced Research and Reports*, **1**(3), pp. 1–6 (2018). https://doi.org/10.9734/ajarr/2018/v1i313065
11. Yadav, S.K., Sharma, D.K., and Mishra, S.S. "Searching efficient estimator of population variance using tri-mean and third quartile of auxiliary variable", *International Journal of Business and Data Analytics*, **1**(1), pp. 30–40 (2019b). https://doi.org/10.1504/ijbda.2019.098830
12. Hansen, M.H., Hurwitz, W.N., Marks, E.S., et al. "Response Errors in Survey", *JASA*, **46**, pp. 147 – 190 (1951).
13. Singh, H.P. and Karpe, N. "A class of estimators using auxiliary information for estimating finite population variance in presence of measurement errors", *Communications in Statistics - Theory and Methods*, **38**(5), pp. 734–741 (2009a). https://doi.org/10.1080/03610920802290713
14. Misra, S., Yadav, D.K., Dipika, A., et al. "On estimation of population coefficient of variation in presence of measurement errors", *International Journal of Mathematics Trends and Technology*, **51**(4), pp. 307–311 (2017). https://doi.org/10.14445/22315373/ijmtt-v51p540
15. Khalil, S., Noor-ul-Amin, M., and Hanif, M. "Estimation of population mean for a sensitive variable in the presence of measurement error", *Journal of Statistics and Management Systems*, **21**(1), pp. 81–91 (2018). https://doi.org/10.1080/09720510.2017.1367478
16. Zahid, E. and Shabbir, J. "Estimation of finite population mean for a sensitive variable using dual auxiliary information in the presence of measurement errors", *PLoS ONE*, **14**(2), pp. 1–17 (2019). https://doi.org/10.1371/journal.pone.0212111
17. Khalil, S., Gupta, S., and Hanif, M. "Estimation of finite population mean in stratified sampling using scrambled responses in the presence of measurement errors", *Communications in Statistics - Theory and Methods*, **48**(6), pp. 1553–1561 (2019). https://doi.org/10.1080/03610926.2018.1435817
18. Du Nguyen, H. and Phuc Tran, K. "Effect of the measurement errors on two one-sided Shewhart control charts for monitoring the ratio of two normal variables", *Quality and Reliability Engineering International*, **36**(5), pp. 1731–1750 (2020). https://doi.org/10.1002/qre.2656
19. Singh, R., Bouza, C., and Mishra, M. "Estimation in stratified random sampling in the presence of errors", *Revista Investigacion Operacional*, **41**(1), pp. 123–135 (2020).
20. Singh, H.P. and Karpe, N. "Estimation of population variance using auxiliary information in the presence of measurement errors", *Statistics in Transition*, **9**(3), pp. 443–470 (2008).
21. Diana, G. and Giordan, M. "Finite population variance estimation in presence of measurement errors", *Communications in Statistics - Theory and Methods*, **41**(23), pp. 4302–4314 (2012). https://doi.org/10.1080/03610926.2011.573165

22. Sharma, P. and Singh, R. "A generalized class of estimators for finite population variance in presence of measurement errors", *Journal of Modern Applied Statistical Methods*, **12**(2), pp. 231–241 (2013). https://doi.org/10.22237/jmasm/1383279120

23. Singh, H.P. and Pal, S.K. "Improved estimation of finite population variance using auxiliary information in presence of measurement errors", *Investigacion Operacional*, **37**(2), pp. 147–162 (2016).

24. Misra, S., Kumari, D., and Yadav, D.K. "Some improved estimators for estimating population variance in the presence of measurement errors", *Journal of Statistics Applications and Probability*, **5**(2), pp. 311–320 (2016). https://doi.org/10.18576/jsap/050212

25. Masood, S. and Shabbirz, J. "Generalized multiphase regression-type estimators under the effect of measurement error to estimate the population variance", *Hacettepe Journal of Mathematics and Statistics*, **45**(4), pp. 1297–1306 (2016). https://doi.org/10.15672/HJMS.201612116352

26. Singh, H.P. and Karpe, N. "A general procedure for estimating the general parameter using auxiliary information in presence of measurement errors", *Communications for Statistical Applications and Methods*, **16**(5), pp. 821–840 (2009b). https://doi.org/10.5351/ckss.2009.16.5.821

27. R Core Team., *R: A language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria (2020). https://www.R-project.org/.

28. Gujarati, D.N., *Basic Econometrics* (Fourth Ed., The McGraw-Hill Companies (2004).

29. Subramani, J. and Kumarapandiyan, G. "Variance estimation using quartiles and their functions of an auxiliary variable", *International Journal of Statistics and Applications*, **2**(5), pp. 67–72 (2012). https://doi.org/10.5923/j.statistics.20120205.04

30. Singh, R. and Malik, S. "Improved estimation of population variance using information on auxiliary attribute in simple random sampling", *Applied Mathematics and Computation*, **235**, pp. 43–49 (2014). https://doi.org/10.1016/j.amc.2014.03.002

31. Swain, A. "Generalized estimator of finite population variance", *J Stat Theory Appl.*, **14**(1), pp. 45–51 (2015).

32. Muili, J.O., Audu, A., Singh, R.V.K., et al. "Improved variance estimator using linear combination of tri-mean and quartile average", *Annals. Computer Science Series*, **XVII**(1), pp. 142–147 (2019).

Appendix

Many special cases of the proposed estimators are obtained and presented in Table A.1.

Table A.1. Some known cases of the proposed generalized estimator.

Estimator	M_1	M_2	α	β	a	b
$t_0 = \hat{s}_y^2$	1	0	0	0	1	1
$t_1 = \hat{s}_y^2\left(\frac{S_x^2}{\hat{s}_x^2}\right)$. [1]	1	0	1	0	1	1
$t_2 = \hat{s}_y^2\left(\frac{S_x^2+C_x}{\hat{s}_x^2+C_x}\right)$. [2]	1	0	1	0	1	C_x
$t_3 = \hat{s}_y^2 exp\left(\frac{S_x^2-\hat{s}_x^2}{S_x^2+\hat{s}_x^2}\right)$. [4]	1	0	0	1	1	0
$t_4 = \hat{s}_y^2 exp\left(\frac{\hat{s}_x^2-S_x^2}{S_x^2+\hat{s}_x^2}\right)$. [4]	1	0	0	-1	1	1
$t_5 = \hat{s}_y^2\left(\frac{S_x^2+Med}{\hat{s}_x^2+Med}\right)$. [29]	1	0	1	0	1	Med
$t_6 = \hat{s}_y^2\left[k_1 + k_2\left(S_x^2 - \hat{s}_x^2\right)\right] \exp\left(g\frac{a\left(S_x^2-\hat{s}_x^2\right)}{a\left(S_x^2+\hat{s}_x^2\right)+2b}\right)$. [30]	k_1	k_2	1	g	a	b
$t_7 = \hat{s}_y^2\left(\frac{S_x^2}{\hat{s}_x^2}\right)^{1/2}$. [31]	1	0	1/2	0	1	1
$t_8 = \hat{s}_y^2\left(\frac{S_x^2}{\hat{s}_x^2}\right)^{2}$. [7]	1	0	2	0	1	1
$t_9 = \hat{s}_y^2\left(\frac{S_x^2+D}{\hat{s}_x^2+D}\right)$. [9]	1	0	1	0	1	D
$t_{10} = \hat{s}_y^2\left(\frac{S_x^2Q_c^2+Q_d}{\hat{s}_x^2Q_c^2+Q_d}\right)$. [10]	1	0	1	0	Q_c^2	Q_d
$t_{11} = \hat{s}_y^2\left(\frac{S_x^2+D_{AM}b_{2x}}{\hat{s}_x^2+D_{AM}b_{2x}}\right)$. [9]	1	0	1	0	1	$D_{AM}b_{2x}$
$t_{12} = \hat{s}_y^2\left(\frac{S_x^2Q_c^2+Q_1}{\hat{s}_x^2Q_c^2+Q_1}\right)$. [32]	1	0	1	0	Q_c^2	Q_1

Table A.1. Some known cases of the proposed generalized estimator (continued).

Estimator	M_1	M_2	α	β	a	b
$t_{13}=\hat{s}_y^2\left(\frac{S_x^2Q_c^2+Q_3}{\hat{s}_x^2Q_c^2+Q_3}\right)$. [32]	1	0	1	0	Q_c^2	Q_3
$t_{14}=\hat{s}_y^2\left(\frac{S_x^2Q_c^2+Q_r}{\hat{s}_x^2Q_c^2+Q_r}\right)$. [32]	1	0	1	0	Q_c^2	Q_r
$t_{15}=\hat{s}_y^2\left(\frac{S_x^2(TM+Q_3)}{\hat{s}_x^2(TM+Q_3)}\right)$. [11]	1	0	1	0	TM	Q_3
$t_{16}=\hat{s}_y^2\left(\frac{S_x^2\rho+Q_r}{\hat{s}_x^2\rho+Q_r}\right)$. [11]	1	0	1	0	P	Q_r

Biographies

Muhammad Umair Tariq obtained his PhD degree in Statistics from National College of Business Administration and Economics, Lahore, Pakistan. He has been working as an Assistant Professor at the Department of Statistics, Government Graduate College Sahiwal. His research interests include sampling theory, quality control, and biostatistics.

Muhammad Nouman Qureshi holds the position of Assistant Professor of Statistics from National College of Business Administration and Economics, Lahore, Pakistan. He has more than 20 research publications in well-reputed journals. His areas of interest are survey sampling, probability distributions, and time series analysis.

Muhammad Hanif received his MPhil from New South Wales Australia, and PhD in Statistics from the Department of Statistics, University of the Punjab, Pakistan. He is currently working as Pro Rector and Professor of Statistics at National College of Business Administration and Economics, Lahore, Pakistan. His research interests include advanced survey sampling, quality control, probability distribution, biostatistics, and regression analysis.