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Practical stability analysis and switching controller synthesis for discrete-time switched affine systems via linear matrix inequalities

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Abstract. This paper considers the practical asymptotic stabilization of a desired equilibrium point in discrete-time switched affine systems. The main purpose is to design a state feedback switching rule for the discrete-time switched affine systems whose parameters can be extracted with less computational complexities. In this regard, using switched Lyapunov functions, a new set of sufficient conditions based on matrix inequalities, are developed to solve the practical stabilization problem. For any size of the switched affine system, the derived matrix inequalities contain only one bilinear term as a multiplication of a positive scalar and a positive definite matrix. It is shown that the practical stabilization problem can be solved via a few convex optimization problems, including Linear Matrix Inequalities (LMIs) through gridding of a scalar variable interval between zero and one. The numerical experiments on an academic example and a DC-DC buck-boost converter, as well as comparative studies with the existing works, prove the satisfactory operation of the proposed method in achieving better performances and more tractable numerical solutions.

1. Introduction

Switched systems are an important subclass of hybrid systems that consist of a finite number of continuous-variable dynamics containing only controlled switching phenomena and a switching rule that determines which subsystem should be activated at each time [1]. One of the most important subclasses of switched systems is the *switched affine systems* that are very common in practice, especially in switching power converters [2–7]. In this class of switched systems, the equilibrium point varies discontinuously during switching among subsystems. Therefore, to achieve the *asymptotic stability* in a desired equilibrium point, the switching frequency should be approached to infinity, which is not realizable in practice [8–10]. It should be noted that such arbitrarily high-frequency switchings may occur either during *steady-state chattering* when approaching the operating point or as a *transient chattering* due to the presence of sliding modes in the proposed controller. To handle the chattering problems, one can use space or time regularization techniques. The former enforces a dwell-time using space-based hysteresis logic,

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and the latter utilizes an explicit timer preventing switching up to a guaranteed dwell-time [10]. The time regulation method can be implemented either in the continuous-time as a sampled-data switching system [11,12] or in a discrete-time setting [8,9,13]. In either case, due to the limitation on the switching frequency and non-existence of a common equilibrium point among subsystems, stability with respect to a set rather than a particular point can be achieved. As a result, many theoretical findings on the stability of switched systems with a common equilibrium point cannot be directly applied to the stability analysis of switched affine systems without any common equilibrium point. In this regard, the notion of *practical stability* has been proposed in the literature to analyze the stability of switched systems without any common equilibria [13–15]. These remarks motivate us to study the discrete-time switched affine systems with a natural one-step sampling dwell-time.

A review of recent results on the practical stabilization of switched systems without common equilibria or in the particular case of switched affine systems can be found in [16,17]. The main differences are basically in the level of conservatism and level of computational complexities [18]. Unfortunately, these two objectives are opposite together. Employing complex Lyapunov functions yields more computational complexities, and using simple Lyapunov functions causes a high level of conservativeness. In this regard, the main purpose of this paper is to propose a stabilization method to achieve these objectives simultaneously as far as possible. It is noted that the issue of the computational complexity in systems and control is one of the most important research subjects in this discipline from the viewpoint of the proposed theories applicability, especially in the case of industrial and large-scale problems [19,20].

Stability analysis of the system equilibrium point and state-feedback continuous controller design via time-dependent switching rules are investigated for a class of discrete-time switched nonlinear systems in [21,22]. In these works, first, each nonlinear subsystem of the presented switched system is modeled as a Piecewise Affine (PWA) system. Next, the stability conditions are derived based on a set of Linear Matrix Inequalities (LMIs) using the reachability analysis for Mixed Logical Dynamical (MLD) systems. However, the reachability analysis of the MLD systems is an NP-hard problem [23], and the proposed methods are more convenient for small-scale problems [21,22].

In [9], global practical stability conditions are proposed as a set of Bilinear Matrix Inequalities (BMIs) for discrete-time switched affine systems via min-type multiple Lyapunov functions and Lyapunov-Metzler inequalities. The stabilization approach in [9] was adopted in [24] to design an observer-based fault estimator for discrete-time switched affine systems. A set of BMI conditions was proposed in [11] for the global practical stabilization of continuous-time switched affine systems in the framework of sampled-data systems and using switched Lyapunov functions. In this work, only the attractive property of the convergence set is guaranteed. In [8,10,11], the global practical stability conditions have been proposed for discrete-time and continuous-time switched affine systems, respectively, via a common quadratic Lyapunov function. In these works, it is necessary to pre-compute a stable matrix as a convex combination of subsystems that is an *NP-hard* problem [25]. Inspired by the research outline in [8–11] to construct various Lyapunov functions for the global practical stabilization of switched affine systems and achieve less conservative results, in [16], a set of BMI conditions are developed for the global practical stabilization of discrete-time switched affine systems using a centralized quadratic Lyapunov functions and a centralized ellipsoid. In this work, a single-stage design procedure was proposed instead of the existing double-stage design method in [8–11]. If there exists a global optimizer tool, finding the global optimal solution in the single-stage design is more straightforward than that of the two-stage one [16].

The approach in [16] was extended in [17] using switched centralized quadratic Lyapunov functions. Reference [26] extended the existing works [8,10,11,17] employing single Lyapunov functions via a common shifted quadratic Lyapunov function and a shifted ellipsoid parameterized independently. These previous studies based on the common Lyapunov functions imply the existence of a Schur stable convex combination of the subsystems matrices as a necessary condition. In the case of using switched Lyapunov functions [11,17], this requirement is relaxed; however, the resulting BMI problems still contain many bilinear terms and offer a high degree of computational complexity. Similar to [16], in the present work, we use the switched centralized quadratic Lyapunov functions and relax the limitation of the existing stable convex combination of the subsystems matrices. However, contrary to [16], the switched Lyapunov function in the present work is utilized differently such that the number of bilinear terms in the resulting BMI problem is limited to one, independent of the number of the switched affine system modes.

All the existing solution methods for the global practical stabilization of switched affine systems suffer from a drawback of leading to a non-convex optimization problem containing BMI constraints with many bilinear terms, including the multiplication of the scalar and matrix variables. On the other hand, the complexity of the BMI constrained problems drastically increases upon increasing the number of decision variables and constraints [27]. In this regard, the main objective of this paper is to present an efficient problem formulation for the practical stabilization of discrete-time switched affine systems that leads to an efficient computational method with an acceptable level of conservativeness. The main advantage of the proposed practical stabilization is that the number of bilinear terms is reduced to one, which is an important result from the viewpoint of the proposed method applicability in large-scale switched systems. The existing approaches can only be applied to small-scale examples due to computational issues. The present work provides a possibility for the solution of the industrial-scale problems.

Hence, the main contributions of this paper are summarized as follows. First, a state-dependent practical stabilization method for discrete-time switched affine systems is developed such that in contrast to the existing methods, neither needs a stable convex combination of the subsystems matrices nor generates many bilinear terms whose numbers increase with the number of the system modes. With this approach, many bilinear and nonconvex terms resulting from the multiplication of the scalars and matrix variables in the traditional schemes are eliminated in the proposed method. These goals are achieved by increasing or decreasing the switched Lyapunov function within active modes of the switched affine system and its decreasing at the switching instants. Second, the stability conditions are written only for the attractiveness property of the ultimate convergence set based on switched Lyapunov functions. As a result, bilinear terms associated with the invariant condition are eliminated, and the final invariant set of attraction is constructed analytically. Finally, a new set of matrix inequalities for the stabilization problem are proposed that contain only one bilinear term as a multiplication of a positive scalar variable in the interval $[0,1]$ and a positive definite matrix. The optimization problem based on these matrix inequalities is solved easily with a few LMI problems by gridding the scalar variable in a one-dimensional space. The motivation behind the proposed contributions in this paper relies on the fact that the majority of the approaches for the practical stabilization of discrete-time switched affine systems based on the BMI or LMI methods are limited to small-scale problems. A common property of the existing techniques is that their computational cost increases drastically with the problem size. In this regard, the present study can apply the proposed practical stabilization to industrial-scale problems.

The remainder of the paper is organized as follows. The theoretical foundations of the practical stability of the discrete-time switched systems without a common equilibrium point are presented in Section 2. Section 3 provides a procedure to construct the ultimate invariant set of attraction analytically through an available attractive set via Proposition 3. Moreover, state-dependent switching rules and stability conditions as a set of matrix inequalities are developed in this section. Theorems 1, 2, and 3 provide conditions for the global practical stability of discrete-time switched affine systems with all unstable, all stable, and partially-stable partially-unstable subsystems, respectively. In Section 4, the optimization problems corresponding to the stability conditions of Theorems 1, 2, and 3 are formulated to minimize the size of the invariant set of attraction. Some numerical aspects of the proposed optimization problems are discussed through Remarks 3, 4, and Proposition 4 in this section. Moreover, the computational complexity of the proposed BMI conditions is compared to some existing works in this section. Section 5 discusses the applicability of the proposed stabilization method on a DC-DC buck-boost converter with and without parasitic elements. An academic example is also provided to illustrate the computational effectiveness and less conservativeness of the proposed method compared to the recent results available in the literature. Finally, concluding remarks are made in Section 6.

Notation: $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{\geq 0}$ denote the set of real, nonnegative real, and nonnegative integer numbers, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote real-valued n-dimensional column vectors and $m\times n$ matrices, respectively. We use I_n and $0_{m\times n}$ to denote the $n\times n$ identity matrix and the $m\times n$ zero matrix, respectively. For matrix $M\in\mathbb{R}^{m\times n}$, M^T denotes its transpose, and for a square matrix $M\in\mathbb{R}^{n\times n}$, M^{-1}, $tr(M)$, and $\lambda_i(M)$ are inverse, trace, and i^{th} eigenvalue of M, respectively. Moreover, $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the largest and smallest eigenvalue of M, respectively. For $x\in\mathbb{R}^n$, $||x||$ denotes the Euclidean vector norm, i.e., $||x|| = (x^Tx)^{1/2}$. $M\prec 0 (M\preceq 0)$ denotes that M is a negative definite(semi-definite) matrix. For the set $\mathcal{I}$, $|\mathcal{I}|$ denotes its number of elements (cardinality). In symmetric matrices $*$ denotes each of their symmetric blocks. Assuming the set $\mathbb{K}=\{1,\ldots,N\}$ as a collection of the N first positive integer numbers, the convex combination of matrices $\{M_1,\ldots,M_N\}$ is denoted by $M_\lambda=\sum_{i\in\mathbb{K}}\lambda_i M_i$ with $\lambda\in\Lambda$ where $\Lambda := \{\lambda\in\mathbb{R}^N|\lambda_i\geq 0, \sum_{i\in\mathbb{K}}\lambda_i = 1\}$ is the unitary simplex.

2. Problem statement and preliminaries

We consider the discrete-time switched affine system as:

$$x(k+1)=A_{\sigma(x(k))}x(k)+b_{\sigma(x(k))}, \quad x(0)=x_0, \tag{1}$$

where $k\in\mathbb{Z}_{\geq 0}$ is the discrete-time instant, $x(k)\in\mathbb{R}^n$ is the state and $\sigma(x(k)):\mathbb{R}^n\to\mathbb{K}$ is a state-dependent switching function that selects one of the N available subsystems $(A_i,b_i), i\in\mathbb{K}$ at any time $k\in\mathbb{Z}_{\geq 0}$. It is intended to design the state-dependent switching function $\sigma(x(k))$ such that to impose finite-time convergence of the state trajectories $x(k), k \in \mathbb{Z}_{\geq 0}$, to a *neighborhood* of a desired equilibrium point, for all initial conditions $x_0 \in \mathbb{R}^n$. In general, such an equilibrium point does not coincide with the isolated subsystems equilibrium points, namely, $x_{ei} = (I_n - A_i)^{-1} b_i$. In some literature [2,8,10,28–30] based on the common Lyapunov function, it is assumed that this point belongs to a specific set of attainable equilibrium points. For instance, in [8] this set is defined as $X_e = \{x_e \in \mathbb{R}^n | x_e = (I_n - A_\lambda)^{-1} b_\lambda, \lambda \in S\}$ with $S \subseteq \Lambda$ such that A_λ is Schur stable. In the present paper, in contrast to the existing literature based on the single or multiple Lyapunov functions, a desired equilibrium point $x_e \notin X_e$ may be chosen. Given an equilibrium point x_e, we can always reformulate the stabilization problem around the null equilibrium point with defining the error state vector $e(k) = x(k) - x_e, \forall k \in \mathbb{Z}_{\geq 0}$ that follows the error dynamics as:

$$e(k+1) = A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))}, \quad e(0) = e_0, \qquad (2)$$

with $\sigma(e(k)) = \sigma(x(k) - x_e)$, $l_i = (A_i - I_n) x_e + b_i, \forall i \in \mathbb{K}$. We plan to design the switching function $\sigma(e(k))$ via a switched quadratic Lyapunov function $v(e(k)) = V_{\sigma(e(k))} = e(k)^T P_i e(k)$, $P_i = P_i^T \succ 0$ such that the ellipsoid defined as:

$$\mathcal{E}(P,1) = \{e \in \mathbb{R}^n | e^T P e \leq 1, P = P^T \succ 0\}, \qquad (3)$$

be an *invariant set of attraction* for the switched affine system (2) according to the following definitions.

Definition 1 ***(Invariant set).*** A compact set $\mathcal{V} \subseteq D$ is an invariant set on a given domain $D \subset \mathbb{R}^n$ for system (2) by the switching function $\sigma(e(k))$ if the following conditions are simultaneously satisfied:

(a) $0_{n\times 1} \in \mathcal{V}$

(b) if $e(k_0) \in \mathcal{V}$, $k_0 \in \mathbb{Z}_{\geq 0}$, then $e(k) \in \mathcal{V}$, $\forall k \geq k_0$, where $k \in \mathbb{Z}_{\geq 0}$.

Definition 1 is a modified version of the available definition of the invariant set in the literature [31–33] in the sense that the conditions (a) and set compactness have been added in our definition. These modifications cause that an analogy is obtained between the set invariance and local practical stability definitions in the literature [13–15,34]. Indeed, based on condition (b), the trajectories starting within $\mathcal{V}$, never escape from it, and therefore, according to (a) and compactness of $\mathcal{V}$, they will remain around the null set point.

Definition 2 ***(Attractive set).*** A compact set $\mathcal{V} \subseteq D$ containing a ball $B_r = \{e \in \mathbb{R}^n | \|e\| \leq r\}$, with given $r > 0$ is said to be an attractive set on a given domain $D \subset \mathbb{R}^n$ for system (2) by the switching function $\sigma(e(k))$ if the following conditions are simultaneously satisfied:

(a) $0_{n\times 1} \in \mathcal{V}$

(b) if $e(0) \in D - \mathcal{V}$ then there is a $T = T(e(0)) \geq 0$ such that $e(k) \in \mathcal{V}$, $\forall k \geq T$.

Definition 3 ***(Invariant set of attraction).*** A compact set $\mathcal{V} \subseteq D$ is an invariant set of attraction on a given domain $D \subset \mathbb{R}^n$ for system (2) by the switching function $\sigma(e(k))$ if it is both invariant and attractive based on Definitions 1 and 2.

Remark 1. Ellipsoid $\mathcal{E}(P,1)$ in Relation (3) is compact and containing a ball $B_r = \{e \in \mathbb{R}^n | \|e\| \leq r\}$, with given $r > 0$ whenever $\lambda_{\min}(P) \leq \frac{1}{r^2}$. This comes from the fact that $\lambda_{\min}(P) \|e\|^2 \leq e^T P e \leq 1$.

It should be noted that the notion of the set attractiveness in Definition 2 is a bit different from those available in the literature. In [8,31–33], a set M is said to be an attractive set with respect to system (2) if any solution $e(k)$ of (2) starting from $e(0) \notin M$ approaches to the set M when k approaches to infinity, i.e.:

$$e(0) \notin M \Rightarrow \lim_{k\to\infty} dist(e(k), M) = 0, \qquad (4)$$

where $dist(e(k), M) = \inf_{\rho \in M} \|\rho - e(k)\|$. According to [13–15,34], the asymptotic practical stability of a set M implies that for the state trajectories starting outside M, there is a finite time $T > 0$ such that the trajectory $e(k)$ is ultimately inside M. However, the existing definition of the set attractiveness in Relation (4) does not imply such property. It only guarantees that the distance between the state trajectory $e(k)$ and the set M approaches zero without ensuring that $e(k)$ enters M eventually. On the other hand, Definition 2 falls into the context of practical stability by which it is meant that the trajectories enter the ellipsoid $\mathcal{E}(P,1)$ in a finite time. Definitions 4, 5, and 6 clarify these notions.

Definition 4. System (2) is locally practically stable with respect to a set $\mathcal{V} \subseteq D$ on the domain $D \subset \mathbb{R}^n$ under switching function $\sigma(e(k))$ if the set $\mathcal{V}$ is an invariant set based on Definition 1.

Definition 5. System (2) is practically asymptotically stable in the large with respect to a set $\mathcal{V} \subset D$ on the domain $D \subset \mathbb{R}^n$ under switching function $\sigma(e(k))$ if the set $\mathcal{V}$ is an invariant set of attraction based on Definition 3.

Definition 6. System (2) is practically asymptotically stable in the whole or globally practically asymptotically stable if it is practically asymptotically stable in the large and $D = \mathbb{R}^n$.

Definition 7 In Definitions 4–6, the set $D - \mathcal{V}$ is called the domain of attraction of system (2) under switching rule $\sigma(e(k))$.

Various versions of the practical stability in Definitions 4–6 have been developed based on the general compact set $\mathcal{V}$. Hereafter, throughout this paper, we assume that the set $\mathcal{V}$ is the ellipsoid $\mathcal{E}(P, 1)$ as defined in Relation (3).

Lemma 1 states under what conditions the autonomous switched nonlinear system (5) is practically asymptotically stable in the large in the sense of Definition 5:

$$e(k+1) = f_{\sigma(e(k))}(e(k)), f_{\sigma(e(k))} : D \to D, D \subseteq \mathbb{R}^n, \quad (5)$$

where $\sigma(e(k)) \in \mathbb{K}$ is the switching function. We do not require $0_{n\times 1} = f_i(0_{n\times 1})$, $\forall i \in \mathbb{K}$. It is also assumed that the vector fields f_i satisfy suitable conditions guaranteeing the existence and uniqueness of solutions of system (5) starting any $e(0) \in D$. Clearly, switched affine system (2) is a special case of the nonlinear switched system in (5).

Lemma 1. System (5) is practically asymptotically stable in the large in a given domain $D \subset \mathbb{R}^n$ containing the origin in the sense of Definition 5 if there exist an ellipsoid $\mathcal{E}(P, 1) \subset D$, a scalar function $v(e(k)) : \mathbb{R}^n \to \mathbb{R}$ and a switching rule$\sigma(e(k))$ such that:

(a) if $e(k) \in \mathcal{E}(P, 1)$ then $e(k+1) = f_{\sigma(e(k))}(e(k)) \in \mathcal{E}(P, 1)$,

(b) if $e(k) \in D - \mathcal{E}(P, 1)$ then $v(e(k+1)) - v(e(k)) = \Delta v(e(k)) \leq -\gamma < 0$ where $\gamma > 0$,

(c) $v(e(k))$ is positive definite in $D - \mathcal{E}(P, 1)$, i.e., $v(e(k)) > 0$ when $e(k) \in D - \mathcal{E}(P, 1)$.

Proof. The proof of this lemma is similar to the proof of Lemma 1 in [26,35] and is omitted for the brevity. □

Remark 2 In Lemma 1, if $D = \mathbb{R}^n$, then according to Definition 6, system (2) is globally practically asymptotically stable.

3. Stability analysis and switching controller synthesis

In this section, the main results of this paper are presented. As discussed in Section 2, we are planning to design a switching rule $\sigma(e(k))$ via a switched Lyapunov function $v(e(k)) = V_{\sigma(e(k))} = e(k)^T P_i e(k)$, $P_i = P_i^T \succ 0$ for system (2) to make it globally practically asymptotically stable. In this regard, we first introduce the min-type state-feedback switching function as:

$$(i^*, j^*) = arg \min_{i,j\in\mathbb{K}, i\neq j} [(A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k)], \sigma(e(k)) = j^*. \quad (6)$$

Proposition 1 provides sufficient conditions that make the ellipsoid $\mathcal{E}(P, 1)$ in Relation (3) be an *attractive set* for the switched affine system (2) under switching function (6) in the sense of Definition 2.

Proposition 1 If there exist matrices $P^T = P \succ 0$, $P_i^T = P_i \succ 0$, and nonnegative numbers $\tau_j \geq 0$, $i, j \in \mathbb{K}$, satisfying the system of inequalities:

$$\begin{bmatrix} \tau_j P + A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j - \tau_j \end{bmatrix} \prec 0 \quad \forall i, j \in \mathbb{K}, i \neq j, \quad (7)$$

then the switching strategy in Relation (6) assures that the ellipsoid $\mathcal{E}(P, 1)$ in Relation (3) is an attractive set for system (2) on the domain $D = \mathbb{R}^n$ in the sense of Definition 2.

Proof. Please see Appendix A. □

According to the Schur complements, the matrix inequalities in Relation (7) yield $\tau_j P + A_j^T P_i A_j - P_j \prec 0$, $i \neq j$ as a necessary condition. Since $\tau_j P \succeq 0$, one can conclude $A_j^T P_i A_j - P_j \prec 0, i \neq j$. Since, in general, for $i \neq j$, $P_i \neq P_j$, it is not necessary $A_j, j \in \mathbb{K}$ to be Schur stable. It is also noted that the famous condition of a stable convex combination of subsystems matrices, i.e., $A_\lambda = \sum_{i\in\mathbb{K}} \lambda_i A_i$ with $\lambda \in \Lambda$ where $\Lambda := \{\lambda \in \mathbb{R}^N | \lambda_i \geq 0, \sum_{i\in\mathbb{K}} \lambda_i = 1\}$ does not appear in Relation (7). As a result, many bilinear and nonlinear terms $\lambda_i A_i$ are eliminated.

Lemma 2 provides sufficient conditions under which the ellipsoid $\mathcal{E}(P, 1)$ in Relation (3) is an attractive set for system (2) under the simple switching rule (8) instead of the more involved version of Relation (6). This lemma is used to derive the next results, such as Propsitions 2 and 3.

$$\sigma(e(k)) = arg \min_{i\in\mathbb{K}} (A_i e(k) + l_i)^T P (A_i e(k) + l_i) = arg \min_{i\in\mathbb{K}} v(A_i e(k) + l_i), \quad (8)$$

where $v(e(k)) = e(k)^T P e(k), P = P^T \succ 0$.

Lemma 2 Ellipsoid $\mathcal{E}(P, 1)$ in Relation (3) is an attractive set for system (2) according to Definition 2 and under the switching rule (8) if $\forall e(k) \notin \mathcal{E}(P, 1)$, $\exists P = P^T \succ 0$, $\exists i \in \mathbb{K}$ such $\exists \gamma > 0$ that:

$$(A_i e(k) + l_i)^T P (A_i e(k) + l_i) - e(k)^T P e(k) \leq \gamma. \quad (9)$$

Proof. The proof of this lemma is similar to the proof of Lemma 2 in [26] and is omitted for the brevity. □

Proposition 2 provides sufficient conditions under which the attractiveness conditions of Proposition 1 with the switching rule (6) imply the attractiveness property via switching law (8). Such a possibility is of particular importance since, as it will be shown later in Proposition 3, to construct an invariant set of attraction, instead of using two different switching rules, i.e., rule (8) inside of the ellipsoid and rule (6) outside of it, one can use a unified controller (8) in the whole state space. As a result, the controller logic is simplified, and it can be implemented more efficiently. Moreover, in some cases, due to numerical issues, the switching rule (6) does not result in convergence due to tiny components in the matrices P_i. However, the execution of rule (8) is successful.

Proposition 2 Consider matrix $P = P^T \succ 0$ that follows from a feasible solution of Proposition 1. If there exist nonnegative constants $\lambda_i \geq 0$ and $\tau' \geq 0$ satisfying the system of inequalities:

$$\begin{bmatrix} M_1 + \tau' P & * \\ M_2 & M_3 - \tau' \end{bmatrix} \prec 0, \tag{10}$$

$$\sum_{i \in \mathbb{K}} \lambda_i > 0, \tag{11}$$

where:

$$M_1 = \sum_{i \in \mathbb{K}} \lambda_i (A_i^T P A_i - P), \tag{12}$$

$$M_2 = \sum_{i \in \mathbb{K}} \lambda_i (l_i^T P A_i), \tag{13}$$

$$M_3 = \sum_{i \in \mathbb{K}} \lambda_i (l_i^T P l_i), \tag{14}$$

then the switching strategy in Relation (8) assures that the ellipsoid $\mathcal{E}(P, 1)$ in Relation (3) is an attractive set for discrete-time switched affine system (2) in the sense of Definition 2.

Proof. The proof of this proposition is similar to the proof of Proposition 3 in [26,35] and is omitted for the brevity. □

It should be noted that given matrix P by Proposition 1, the matrix inequalities in Proposition 2 are of the pure LMI type, and therefore, are numerically tractable.

The following proposition states that the existence of an attractive ellipsoid implies the existence of an invariant set of attraction.

Proposition 3 Let the ellipsoid $\mathcal{E}(P, 1)$ in Relation (3) be an attractive set for system (2) according to Definition 2 and set R and R^* to be:

$$R = \lambda_{\max}(P) \left[\max_{i \in \mathbb{K}} \left(\frac{||A_i||}{\sqrt{\lambda_{\min}(P)}} + ||l_i|| \right) \right]^2, \tag{15}$$

$$R^* = \lambda_{\max}(P) \left[\min_{i \in \mathbb{K}} \left(\frac{||A_i||}{\sqrt{\lambda_{\min}(P)}} + ||l_i|| \right) \right]^2. \tag{16}$$

Then the following statements are correct.

(a) if $R \leq 1$, then $\mathcal{E}(P, 1)$ is an invariant set of attraction for system (2) independent of any switching strategy within $\mathcal{E}(P, 1)$,

(b) if $R > 1$ and $R^* \leq 1$, then $\mathcal{E}(P, 1)$ is an invariant set of attraction for system (2) under switching rule (8) within $\mathcal{E}(P, 1)$,

(c) Let the LMI conditions in Proposition 2 be satisfied and $R^* > 1$. Then the ellipsoid $\mathcal{E}(P, R^*)$ defined as:

$$\mathcal{E}(P, R^*) = \{e \in \mathbb{R}^n | e^T P e \leq R^*, P = P^T \succ 0\}, \tag{17}$$

is an invariant set of attraction for system (2) under switching strategy (8) within $\mathcal{E}(P, R^*)$.

Proof. The proof of this proposition is similar to the proof of Proposition 4 in [26] and is omitted for the brevity. □

The importance of the Proposition 3 relies on the fact that using an available attractive set, the invariant set of attraction can always be constructed without writing additional conditions on the set invariance property. With this approach, one can obtain the matrix inequalities with smaller dimensions, and a small number of scalar variables appeared as bilinear terms.

Theorem 1 is a significant result and states that one can establish an *invariant set of attraction* and conclude the global practical asymptotical stability of system (2) whenever an *attractive set* is provided through conditions of Propsitions 1 and 2.

Theorem 1. Assume that there exist matrices $P^T = P \succ 0$, $P_i^T = P_i \succ 0$, and nonnegative numbers $\tau_j \geq 0$, $i, j \in \mathbb{K}$, satisfying the system of inequalities (7). Let R and R^* be given by Relations (15) and (16), respectively. Then the following statements hold:

(a) If $R \leq 1$, then the ellipsoid $\mathcal{E}(P, 1)$ in Relation (3) is an invariant set of attraction for system (2) under switching rule (6) when $e(k) \notin \mathcal{E}(P, 1)$. Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6.

(b) The ellipsoid $\mathcal{E}(P, R^*)$ in Relation (17) with $R^* \leq 1$ is an invariant set of attraction for system (2) under switching rule (8) when $e(k) \in \mathcal{E}(P, 1)$ and switching rule (6) when $e(k) \notin \mathcal{E}(Q, e_c, 1)$. Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6.

(c) Let the conditions of Proposition 2 be satisfied. Then, the ellipsoid $\mathcal{E}(P,R^*)$ in Relation (17) with $R^*>1$ is an invariant set of attraction for system (2) under switching rule (8). Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6.

Proof. The proof follows from the fact that the conditions of Proposition 1 are included within this theorem. As a result, the ellipsoid $\mathcal{E}(P,1)$ is an attractive set for system (2) under switching strategy (6) outside it. Now items (a)–(c) are concluded from items (a)–(c) of Proposition 3. □

It is worth noting that the finite-time convergence conditions of Relation (7) in Theorem 1 and Proposition 1 are developed using decreasing of multiple Lyapunov functions only during switching between modes and not within the modes. Therefore, the provided conditions in this theorem are suitable for switched affine systems with *all unstable* subsystem matrices A_i. If some or all of the subsystems are Schur stable, one can use this additional information to construct more efficient switching sequences. In this regard, the following result provides suitable conditions for switched affine systems with all stable subsystem matrices.

Theorem 2. Assume that there exist matrices $P^T = P \succ 0$, $P_i^T = P_i \succ 0$, and nonnegative numbers $\tau_j \geq 0$, $i,j \in \mathbb{K}$, satisfying the system of inequalities:

$$\begin{bmatrix} \tau_j P + A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j - \tau_j \end{bmatrix} \prec 0, \quad \forall i,j \in \mathbb{K}. \quad (18)$$

Let R and R^* be given by Relations (15) and (16), respectively. Then the following statements hold:

(a) If $R \leq 1$, then the ellipsoid $\mathcal{E}(P,1)$ in Relation (3) is an invariant set of attraction for system (2) under the switching rule (19) when $e(k) \notin \mathcal{E}(P,1)$. Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6,

$$(i^*,j^*) = arg \min_{i,j\in\mathbb{K}} [(A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k)], \quad \sigma(e(k)) = j^*, \quad (19)$$

(b) The ellipsoid $\mathcal{E}(P,R^*)$ in Relation (17) with $R^* \leq 1$ is an invariant set of attraction for system (2) under switching rule (8) when $e(k) \in \mathcal{E}(P,1)$ and switching rule (8) when $e(k) \notin \mathcal{E}(Q,e_c,1)$. Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6,

(c) Let the conditions of Proposition 2 be satisfied with the matrix $P = P^T \succ 0$ that follows from the feasible solution of Relation (18). Then, the ellipsoid $\mathcal{E}(P,R^*)$ in Relation (17) with $R^*>1$ is an invariant set of attraction for system (2) under switching rule (18). Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6.

As compared to Relation (7), the corresponding conditions in Relation (18) need monotonically decreasing Lyapunov function within the modes.

The following result provides internal monotonically decreasing requirement only for the Schur stable subsystems of a switched affine system with partially-stable partially-unstable subsystems. Without loss of generality, we assume that $\mathbb{K} = \mathbb{S} \cup \mathbb{U}$, where $\mathbb{S} = \{1,2,\ldots,s\}$ and $\mathbb{U} = \{s+1,\ldots,N\}$, i.e., there are s stable subsystems and $N-s$ unstable subsystems.

Theorem 3. Assume that there exist matrices $P^T = P \succ 0$, $P_i^T = P_i \succ 0$, and nonnegative numbers $\tau_j \geq 0$, $i,j \in \mathbb{K}$, satisfying the system of inequalities:

$$\begin{bmatrix} \tau_j P + A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j - \tau_j \end{bmatrix} \prec 0, \quad \forall i,j \in \mathbb{S}, \quad (20)$$

$$\begin{bmatrix} \tau_j P + A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j - \tau_j \end{bmatrix} \prec 0, \quad \forall i,j \in \mathbb{U}, i \neq j. \quad (21)$$

Let R and R^* be given by Relations (15) and (16). Then the following statements hold:

(a) If $R \leq 1$, then the ellipsoid $\mathcal{E}(P,1)$ in Relation (3) is an invariant set of attraction for system (2) under the switching rule (22) when $e(k) \notin \mathcal{E}(P,1)$. Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6.

$$(i^*,j^*) = arg \min_{\substack{\forall i,j \in \mathbb{U}, j \neq i \\ \forall i,j \in \mathbb{S}}} [(A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k)], \quad \sigma(e(k)) = j^*, \quad (22)$$

(b) The ellipsoid $\mathcal{E}(P,R^*)$ in Relation (17) with $R^* \leq 1$ is an invariant set of attraction for system (2) under switching rule (8) when $e(k) \in \mathcal{E}(P,1)$ and switching rule (22) when $e(k) \notin \mathcal{E}(Q,e_c,1)$. Thus, system (2) is globally practically asymptotically stable in the sense of Definition 6.

(c) Let the conditions of Proposition 2 be satisfied with the matrix $P = P^T \succ 0$ that follows from the feasible solutions of Relations (20) and (21). Then, the ellipsoid $\mathcal{E}(P,R^*)$ in Relation (17) with

$R^* > 1$ is an invariant set of attraction for system (2) under switching rule (8). Thus, system (2) is globally practically asymptotically stable in the sense of Defintion 6.

The proofs of Theorems 2 and 3 follow from the similar arguments in the proofs of Proposition 1 and Theorem 1, and are omitted for brevity.

4. Minimization of the invariant set of attraction and numerical aspects

To achieve better performances with minimum deviations from the setpoint, we are interested in selecting the invariant set of attraction with a small size as far as possible. In this paper, first, the attractive ellipsoid $\mathcal{E}(P,1)$ in Relation (3) is constructed with the smallest size through an optimization problem, and then the ultimate invariant set of attraction is built via the proposed technique in Proposition 3.

In the sequel, we first consider a suitable criterion about the *size* of the ellipsoid $\mathcal{E}(P,1) = \{e \in \mathbb{R}^n | e^T Pe \leq 1, P = P^T \succ 0\}$. One technique is to minimize the ellipsoid volume through the minimization of $det(P^{-1})$ [5,8,36]. In [31], it is discussed since:

$$det(P^{-1}) = \prod_{i=1}^{n} \lambda_i(P^{-1}) = \prod_{i=1}^{n} \rho_i^2(P^{-1}),$$

where $\rho_i(P^{-1})$ is the distance from the center to each semiaxis of the ellipsoid $\mathcal{E}(P,1)$, the product $\prod_{i=1}^{n} \rho_i(P^{-1}) \propto Vol(\mathcal{E}(P,1))$ may admit a very large value of one semiaxis, say $\rho_j(P^{-1})$, while all others may be very small. As a result, the minimum volume criterion through the minimization of $det(P^{-1})$ may not perform well in all directions. Another approach is to use the trace of the matrix P^{-1} that defines the sum of the squares of the ellipsoid $\mathcal{E}(P,1)$ semiaxes [31]. Since the tool YALMIP/MATLAB [37] could not handle the objective function $tr(P^{-1})$, in what follows, we have modified it such that the software limitation is relaxed while the main property of the size minimization is preserved. Since we have:

$$tr(P^{-1}) = \sum_{i=1}^{i=n} \lambda_i(P^{-1}) \leq n \max_{i=1,\ldots,n} \lambda_i(P^{-1}) = n\lambda_{\max}(P^{-1}),$$

the minimization of $\lambda_{\max}(P^{-1})$, guarantees the minimization of the corresponding maximal semiaxis $\rho_{\max}(P^{-1}) = \sqrt{\lambda_{\max}(P^{-1})}$. Moreover, it minimizes an upper bound for $tr(P^{-1})$. Therefore, the ellipsoid size minimization problem is formulated as:

$$\inf_{P=P^T\succ 0, P_i=P_i^T\succ 0, \tau_i\geq 0, i\in\mathbb{K},} \lambda_{\max}(P^{-1}), \tag{23}$$

subject to Eqs. (7) or (18) or (20)–(21).

On the other hand, by introducing a slack variable $t > 0$, we have [38]:

$$\lambda_{\max}(P^{-1}) \leq t \Leftrightarrow P^{-1} - tI_n \preceq 0. \tag{24}$$

Using Relation (24) and Schur complements, the optimization problem in Relation (23) can be equivalently rewritten as:

$$\inf_{P=P^T\succ 0, P_i=P_i^T\succ 0, \tau_i\geq 0, i\in\mathbb{K}, , t>0} t,$$

subject to Relations [7] or (18) or (20)–(21).

$$\begin{bmatrix} tI_n & I_n \\ I_n & P \end{bmatrix} \succeq 0. \tag{25}$$

It can be concluded that the source of the non-convexity in the equivalent optimization problems (23) and (25) is related to the bilinear terms $\tau_i P$ in the matrix inequalities (7), (18), or (20)-(21). Indeed, these matrix inequalities are BMI with respect to the variables τ_i and P. The common practice in the literature to handle such problems is to grid up the unknown scalars. Next, for fixed values of the scalar variables, the BMI problem reduces to an LMI for which efficient solution techniques and solvers have been developed [3,8,39–40]. Such an approach is numerically tractable only in the case of a small number of scalar variables. Moreover, the interval length over which the gridding is made should be bounded with a small length. To reduce the number of scalar variables, we use a simplifying method in the following remark.

Remark 3. To obtain simpler-to-solve but more conservative results, we assume $\tau_i = \tau \geq 0, \forall i \in \mathbb{K}$, in optimization problems (23) and (25), and as a result in matrix inequalities (7), (8) or (20)-(21).

With the approach in Remark 3, independent of the number of subsystems N, there is only one scalar variable in the BMI problems (23) and (25). As it will be shown in the numerical examples, the imposed simplifying conditions in Remark 3 do not seem to be too restrictive. However, the range over which the scalar variable $\tau \geq 0$ varies, is too large to be searched thoroughly via gridding techniques. The following note states that the exploration of the unbounded region of $\tau \geq 0$ can be replaced by searching of a bounded interval of [0,1].

Remark 4. Let $\tau_{\max}$ be an arbitrarily large number such that $0 \leq \tau \leq \tau_{\max} < \infty$. Then, based on Remark 3, the matrix inequalities in Relation (7) with the nonnegative scalar $\tau \geq 0$ can be replaced with the matrix inequalities with $0 \leq \lambda \leq 1$. In other words, we have:

$$(7) \Leftrightarrow \begin{bmatrix} \lambda P + A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j - \lambda \end{bmatrix} \prec 0$$

$$\forall i, j \in \mathbb{K}, i \neq j. \tag{26}$$

The reason is that Relation (26) always holds with $\lambda = \frac{\tau}{\tau_{\max}}$.

It is noted that similar equivalency relations can be established for the matrix inequalities in Relations (18) and (20)-(21).

The following proposition states that if the feasibility of the matrix inequality (7) is fulfilled when τ approaches zero, then $\inf t$ in the optimization problem (23) is an increasing function with respect to τ.

Proposition 4. Let there exist feasible solutions of the optimization problem (23) when $\tau \to 0^+$. Moreover, $\exists \tilde{j} \in \mathbb{K}$ such that $\left|\left|l_{\tilde{j}}\right|\right| \neq 0$ and $|\mathbb{K}| > 2$. Then there exists a neighborhood around 0 in which $\inf t$ in (25) is an increasing function with respect to scalar parameter τ. In other words, $\inf t$ gets smaller values when τ becomes smaller.

Proof. Without loss of generality, we consider the set of matrix inequality (7) in the equivalent optimization problems (23) and (25). Using Schur complements, the feasibility of matrix inequalities in Relation (7) implies the feasibility of the following inequalities:

$$\tau P + A_j^T P_i A_j - P_j \prec 0, \tag{27}$$

$$l_j^T P_i l_j - \tau < 0, \tag{28}$$

$\forall i, j \in \mathbb{K}$, $i \neq j$. Since $P_i = P_i^T \succ 0$, thus $P_i = P_i^{\frac{1}{2}} P_i^{\frac{1}{2}}$. Hence, from Relation (28) one can conclude:

$$l_j^T P_i l_j = l_j^T P_i^{\frac{1}{2}} P_i^{\frac{1}{2}} l_j = \left|\left|P_i^{\frac{1}{2}} l_j\right|\right|^2 < \tau. \tag{29}$$

Using Relation (29), and setting $l_j = l_{\tilde{j}}$, one can write:

$$0 < \lim_{\tau \to 0^+} \left|\left|P_i^{\frac{1}{2}} l_{\tilde{j}}\right|\right|^2 < \lim_{\tau \to 0^+} \tau = 0,$$

which results in $\lim_{\tau \to 0^+} \left|\left|P_i^{\frac{1}{2}} l_{\tilde{j}}\right|\right| = 0$. Since $P_i^{\frac{1}{2}} \succ 0$ and $\left|\left|l_{\tilde{j}}\right|\right| \neq 0$, thus $\lim_{\tau \to 0^+} \left|\left|P_i^{\frac{1}{2}}\right|\right| = 0$, $\forall i \neq \tilde{j}$. Finally, using $\sqrt{||P_i||} \leq \left|\left|P_i^{\frac{1}{2}}\right|\right|$, one can conclude:

$$\lim_{\tau \to 0^+} ||P_i|| = 0, \forall i \neq \tilde{j}. \tag{30}$$

Now, using Inequalities (27) and noting that $P \succ 0$ one can write:

$$0 < \tau \lambda_{\min}(P) < \lambda_{\max}(P_j - A_j^T P_i A_j). \tag{31}$$

Since $(P_j - A_j^T P_i A_j)^T = P_j - A_j^T P_i A_j$ and due to Relation (27), $P_j - A_j^T P_i A_j \succ \tau P \succ 0$, then $P_j - A_j^T P_i A_j$ is symmetric positive definite matrix and all its eigenvalues are positive real numbers. Thus:

$$\begin{aligned} |\lambda_{\max}(P_j - A_j^T P_i A_j)| &= \lambda_{\max}(P_j - A_j^T P_i A_j) \\ &\leq \left|\left|P_j - A_j^T P_i A_j\right|\right| \\ &\leq ||P_j|| + \left|\left|A_j^T P_i A_j\right|\right| \\ &\leq ||P_j|| + \left|\left|A_j^T\right|\right| \, ||P_i|| \, ||A_j|| \, . \end{aligned} \tag{32}$$

The first inequality in Relation (32) comes from the inequality $\rho(A) \leq ||A||$ where $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$ is the spectral radius of the matrix $A \in \mathbb{R}^{n \times n}$. Using Relations (31) and (32), one can write:

$$\begin{aligned} 0 < \lim_{\tau \to 0^+} \tau \lambda_{\min}(P) &< \lim_{\tau \to 0^+} ||P_j|| \\ &+ \left|\left|A_j^T\right|\right| \, ||P_i|| \, ||A_j|| \, . \end{aligned} \tag{33}$$

Now, since $|\mathbb{K}| > 2$, then according to Relation (30) one can conclude $\exists i \neq j \neq \tilde{j} \in \mathbb{K}$ such that $\lim_{\tau \to 0^+} ||P_j|| + \left|\left|A_j^T\right|\right| \, ||P_i|| \, ||A_j|| = 0$. Thus, using Relation (33) one can write:

$$\lim_{\tau \to 0^+} \tau \lambda_{\min}(P) = 0. \tag{34}$$

Relation (34) means that $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that:

$$0 < \tau < \delta(\epsilon) \Rightarrow \tau \lambda_{\min}(P) < \epsilon. \tag{35}$$

Since, $\lambda_{\max}(P^{-1}) = \frac{1}{\lambda_{\min}(P)}$, using the left term in Relation (24), one can reach $\lambda_{\min}(P) \geq \frac{1}{t}$. This, in conjunction with Relation (35), yields:

$$0 < \tau < \delta(\epsilon) \Rightarrow t > \frac{\tau}{\epsilon}. \tag{36}$$

Relation (36) means that invariably there exists an interval around 0, such that $\inf t$ is an increasing function of the scalar variable τ. This is because, according to the right-hand side of Relation (36), when τ decreases, the infimum of t is taken over larger sets. The proof is concluded. □

Remark 5. If $|\mathbb{K}| = 2$, then Proposition 4 is valid if $||l_j|| \neq 0$, $\forall j \in |\mathbb{K}|$.

The importance of Proposition 4 relies on the fact that it reveals a structural property of the non-convex BMI problems in Relations (23) and (25). Although it is a locally behavior, however, its efficiency depends on the size of the interval $0 < \tau < \delta(\epsilon)$ in Relation (36). Moreover, it denotes that the regions around 0 over the gridding interval $0 \leq \tau \leq 1$ are candidates containing the global optimum point.

In this paper, the optimization problem in Relation (25) is solved via solver PENBMI [41] interfaced by YALMIP [37].

Table 1. Number of scalar decision variables appeared as bilinear terms.

Type of Lyapunov function	Method	NB
Min-type	Theorems 1,2 [9]	N^2-1
Min-type	Corollary 1 and Theorem 2 [9]	N^2-N-1
Common shifted quadratic + Centralized ellipsoid	Theorem 3 [8]	$N+1$
Switched	Theorem 12 [17]	$2N^2+2N$
Switched	Theorem 12 & Remark 13, Theorem 14 [17]	N^2+2N
Common quadratic	Theorem 12 [16]	$2N+2$
Common shifted quadratic + Shifted ellipsoid	Theorem 2 [26]	$N+4$
Switched	Present work, Theorems 1, 2, and 3	N
Switched	Present work, Remark 3 and Theorems 1, 2, and 3	1

4.1. *Complexity analysis of the proposed method versus some existing methods*

BMI problems are non-convex and NP hard problems whose solution procedures can be classified in three large groups [27,40]:

(a) The methods or software packages based on local optimization approaches,

(b) The methods or software packages based on global optimization techniques,

(c) The methods based on the gridding of scalar parameters to find a local or global optimal solution.

The BMI solution methods in items (a), (b), and (c) have their advantages and disadvantages. For example, in the approaches (a) and (b), it is not necessary to fix the scalar variables beforehand. The local optimizer in (a) or global optimizer in (b) tries to find a local or global solution corresponding to all unknown variables (scalar and matrix variables) simultaneously. On the other hand, in approach (c), it is essential to fix the scalar variables beforehand on an arbitrary gridding point. The resulting LMI problem may be feasible or infeasible. If it is infeasible, we need to try another gridding point. There is no guarantee to reach a feasible local solution rapidly. Even more, if the global optimal solution is intended, the whole gridding points should be checked, and the resulting computational complexity increases drastically. The approach (b) based on the global optimization methods still suffers from increased computational complexity with increasing the number of decision variables [27]. Nevertheless, among the preceding approaches, the third one (item c) is more popular in the literature. The reason is that when the scalar parameters are fixed in the bilinear terms as the multiplication of the scalars and matrices, the BMI problem is reduced to an LMI one, which is convex and can be solved efficiently by the available numerical LMI tools and convex optimization methods.

To find the global optimal solution in approach (c), a typical gridding method is implemented to divide the intervals over which the scalar variables are defined. Therefore, it is clear that the complexity of the global optimization problem with BMI constraints is exponentially related to the number of bilinear terms. In this regard, suppose that a BMI problem contains NB bilinear terms with NB scalar variables, each of which gridded in M points. Thus, the total number of LMI problems that should be solved to cover all the gridding points is M^{NB}. Table 1 presents the complexity of the different problem formulation methods associated with the number of scalar variables that appeared in bilinear terms, i.e., NB. In this table, N is the number of subsystems in the discrete-time switched affine system (1). As can be seen, the proposed method in this paper provides less computational complexity in terms of the number of scalar variables that appeared in bilinear terms.

5. Examples

In this section, two examples are provided to illustrate the previous results. The first one, borrowed from [9], is an academic example with two unstable subsystems for which Theorem 1 is applied. The second one is devoted to applying the proposed theory for a DC-DC buck-boost converter output voltage control. In this case, the conditions of Theorems 2 and 3 are applied to design a suitable switching controller.

Example 1. We consider the discrete-time switched affine system consisting of 2 unstable systems defined by the following state-space matrices:

$$A_{c1} = \begin{bmatrix} 5.8 & -5.9 \\ -4.1 & -4.0 \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} 0.1 & -0.5 \\ -0.3 & -5.0 \end{bmatrix},$$

$$b_{c1} = [0 \ -2]^T, \qquad b_{c2} = [-2 \ 2]^T.$$

The state-space matrices A_i and l_i, $i \in \{1,2\}$, corresponding to the discrete-time system, can be obtained as [42]:

$$A_i = e^{A_{ci}T_s}, l_i = \int_0^{T_s} e^{A_{ci}(T_s-t)} b_{ci} dt$$

$$= \int_0^{T_s} e^{A_{ci}t} dt b_{ci},$$

where $T_s = 0.5$ [9]. The desired equilibrium point is chosen as $x_e = [0 \ 0]^T$ that does not belong to the set X_e. The design of the switching functions (6) and (8) follows from the solution of the optimization problem (25) with the conditions of Theorem 1. The solution of the optimization problem in Relation (25) corresponding to $\tau = 10^{-11}$ yields $t = 0.0155$, $R = 504.5836$, $R^* = 282.0199$ with the matrices P, P_1, and P_2 as:

$$P = \begin{bmatrix} 134.6462 & 98.8124 \\ 98.8124 & 245.6968 \end{bmatrix},$$

$$P_1 = 10^{-7} \begin{bmatrix} 0.0229 & 0.0723 \\ 0.0723 & 0.2730 \end{bmatrix},$$

$$P_2 = 10^{-7} \begin{bmatrix} 0.3682 & 0.3223 \\ 0.3223 & 0.2903 \end{bmatrix}.$$

The associated invariant set of attraction is calculated via $R^* = 282.0199$ with an area of 5.8021, which is 79.45% smaller than [9] with an area of 28.23. The proposed techniques in [16,17,26] based on local optimizer tool PENBMI did not yield feasible solutions for this example with $T_s = 0.5$. It is noted that in contrast to [9] and the present work, the solution procedure in [16,17,26] is not based on the gridding of scalar variables. Instead, in these references, the BMI problems are treated directly as a nonlinear optimization problem, and as a result, they are more difficult to handle numerically. These results illustrate that the proposed method in this work is not only computationally tractable but also it is less conservative compared to the existing works. Figure 1 shows the invariant set of attraction and the state trajectory $x(k)$ starting from various initial points. The time evolution of the state trajectory $x(kT_s)$ starting from $x(0) = [10 \ -10]^T$ and the corresponding switching function $\sigma(x(kT_s))$ are shown in Figure 2.

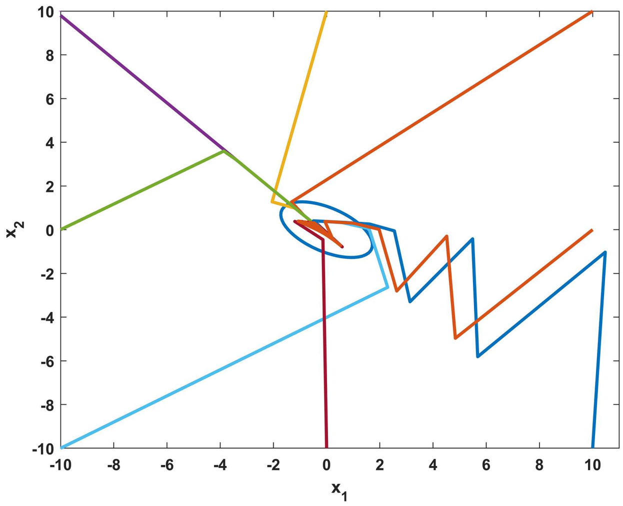

Figure 1. State trajectories corresponding to the conditions of Theorem 1, item (c).

Example 2. A conventional DC-DC buck-boost converter with parasitic elements is shown in Figure 3.

Let the continuous states of the system be defined as $x(t) = [i_L(t), v_o(t)]^T$, which are the inductor current and converter output voltage, respectively. There are two discrete modes of operation for a buck-boost converter in Continuous Conduction Mode (CCM). Mode 1 begins when the switch is turned on. The diode is reversed biased and is off in this mode. The state equation in this mode is:

$$\dot{x}(t) = A_{c1}x(t) + b_{c1},$$

where,

$$A_{c1} = \begin{bmatrix} -\frac{r_L + r_{on}}{L} & 0 \\ 0 & -\frac{1}{C}\frac{1}{(R+r_C)} \end{bmatrix}, \qquad b_{c1} = \begin{bmatrix} \frac{V_s}{L} \\ 0 \end{bmatrix},$$

Mode 2 begins when the switch is turned off, and the diode is turned on. The state equation is:

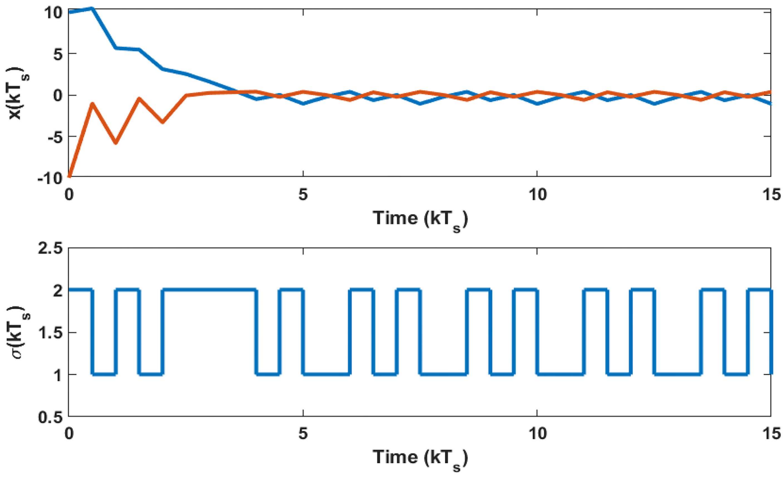

Figure 2. Time evolution of state trajectories $x(kT_s)$ and switching function $\sigma(x(kT_s))$ starting from $x(0) = [10 \ \ -10]^T$.

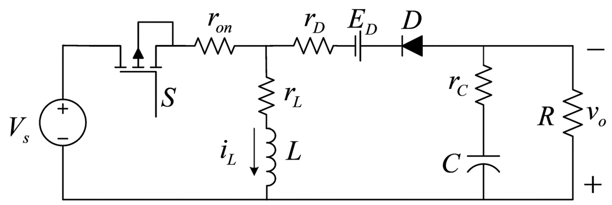

Figure 3. Buck-boost converter.

$$\dot{x}(t) = A_{c2}x(t) + b_{c2}$$

where,

$$A_{c2} = \begin{bmatrix} -\frac{r_L+r_D}{L} & -\frac{1}{L} \\ \frac{R}{R+r_C}\left(\frac{1}{C} - \frac{r_C(r_L+r_D)}{L}\right) & -\frac{R}{R+r_C}\left(\frac{r_C}{L} + \frac{1}{RC}\right) \end{bmatrix}$$

$$b_{c2} = -\begin{bmatrix} \frac{1}{L} \\ \frac{Rr_C}{(R+r_C)L} \end{bmatrix} E_D.$$

Circuit parameters are given in Table 2.

In the numerical simulations, to avoid ill-conditioned matrix inequalities and make the problem more amenable for numerical purposes, we use perunit parameters [11,29]. In this regard, the base values for the chosen perunit system are $v_{base}=V_s=28$ V, $R_{base}=R = 12\ \Omega$ and $T_{base}=5\ \mu$s. The sampling time is set to $T_s = 5\ \mu$s and, as a result, the maximum value of switching frequency is limited to $\frac{1}{2T_s} = 100$ kHz$=f_s$. Considering $x(k) = [i_L(k), v_o(k)]^T$ as the discrete-time and normalized state vector, and $\sigma(x(k)) \in \{1, 2\}$ as the discrete control input, the normalized discrete-time switched affine system is described as:

$$x(k+1) = \begin{cases} A_1x(k) + b_1 & \text{if } \sigma(x(k)) = 1 \\ A_2x(k) + b_2 & \text{if } \sigma(x(k)) = 2 \end{cases}$$

where $A_i = e^{A_i T_s}$, $b_i = \int_0^{T_s} e^{A_i \tau} d\tau\, b_i$, $i \in \{1, 2\}$. If the parasitic elements r_L, r_C, r_{on}, r_D, and E_D are considered to be non-zero during the modeling and controller design stage, then one can observe that all subsystems A_i, $i \in \{1, 2\}$ are stable, and therefore, one can use the conditions of Theorem 2 to achieve less conservative results. On the other hand, if some or all parasitic elements are set to zero, then subsystem A_1 will be unstable, and one can use the results of Theorem 3. Obviously, in all cases, independent of the values of the parasitic elements, one can use Theorem 1 established for the most general case of all unstable subsystems.

Table 2. The specifications of the DC-DC buck-boost converter [43].

Input voltage V_s	28 V
Converter inductor L	500 μH
Output capacitor C	2.2 mF
Load resistance R	12 Ω
Switching frequency f_s	100 kHz
Inductor series resistance r_L	100 mΩ
Capacitor series resistance r_C	6 mΩ
MOSFET turn-on resistance r_{on}	0.11 Ω
Diode turn-on resistance r_D	20 mΩ
Diode forward voltage drop E_D	0.7 V

5.1. Converter operation with non-zero parasitic elements

Some references investigate the operation of the switching converters in more realistic conditions considering the effect of the parasitic elements in their analyzes [3,44–48]. According to the specifications given

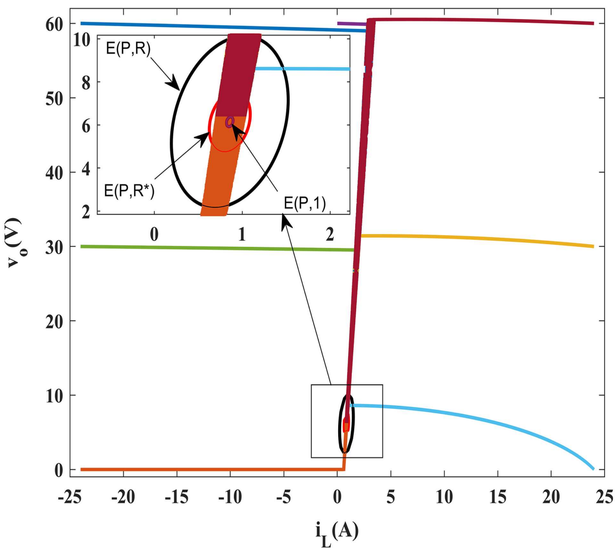

Figure 4. State trajectories corresponding to the conditions of Theorem 2 in buck operation.

in Table 2 with non-zero parasitic elements, a set of numerical experiments is performed in this section.

Although the equilibrium point x_e can be chosen arbitrarily at the expense of obtaining the invariant set of attraction with possibly greater size, in this section, the desired targets are selected as the equilibrium points of the averaged system as x_e=$(I_n - A_{\lambda_{ref}})^{-1}b_{\lambda_{ref}}$=$[0.86A, 6.15V]^T$ correspon ding to λ_{ref}=0.2 in buck operation and x_e=$[16.87A, 54.30V]^T$ corresponding to $\lambda_{ref} = 0.7$ in boost operation.

5.1.1. *Buck operation*

Using the conditions of Theorem 2, the solution of the optimization problem in Relation (25) corresponding to $\tau = 5 \cdot 10^{-14}$ yields $t = 3.05 \cdot 10^{-4}$, with the matrices P, P_1, and P_2 as:

$$P = 10^4 \begin{bmatrix} 0.3864 & -0.1962 \\ -0.1962 & 1.5559 \end{bmatrix},$$

$$P_1 = 10^{-5} \begin{bmatrix} 0.0001 & 0.0004 \\ 0.0004 & 0.4669 \end{bmatrix},$$

$$P_2 = 10^{-5} \begin{bmatrix} 0.0001 & 0.0004 \\ 0.0004 & 0.4668 \end{bmatrix}.$$

Proposition 3 yields $R = 293.25$ and $R^* = 35.97$, by which the ellipsoids $\mathcal{E}(P,R)$ and $\mathcal{E}(P,R^*)$ are constructed. These sets in conjunction with the state trajectories $x(k)$ corresponding to various initial conditions, are shown in Figure 4. Figure 5 demonstrates the time evolutions of state variables starting from $[24A\ 60V]^T$ and the corresponding switching function.

Figure 6 illustrates a comparison between the trajectories and the size of ellipsoids corresponding to the conditions of Theorem 1 (red lines) and Theorem 2 (green lines). As it can be seen, Theorem 1 yields more conservative results. The reason is that this theorem is established for switched affine systems with all unstable subsystems, while the converter subsystems are all stable when the parasitic elements are considered to be non-zero. In other words, Theorem 1 does not use the additional information available within the subsystems to achieve better performances.

5.1.2. *Boost operation*

Considering the equilibrium point x_e=[16.87A,54.30 $V]^T$ corresponding to λ_{ref}=0.7, the conditions of Theorem 2 is employed, and the optimization problem in Relation (25) yields $t = 1.20 \cdot 10^{-4}$ corresponding to $\tau = 2 \cdot 10^{-14}$ with matrices P, P_1 and P_2 as:

$$P = 10^4 \begin{bmatrix} 0.9205 & -0.2650 \\ -0.2650 & 3.5416 \end{bmatrix},$$

$$P_1 = 10^{-5} \begin{bmatrix} 0.0000 & 0.0021 \\ 0.0021 & 0.6368 \end{bmatrix},$$

$$P_2 = 10^{-5} \begin{bmatrix} 0.0000 & 0.0021 \\ 0.0021 & 0.6365 \end{bmatrix}.$$

The quantities $R = 2329.8$ and $R^* = 474.17$ are calculated via Proposition 3. Figure 7 illustrates the state trajectories of the converter with corresponding ellipsoids $\mathcal{E}(P,R)$, $\mathcal{E}(P,R^*)$, and $\mathcal{E}(P,1)$. The time evolutions of the state trajectories starting from $x(0) = [0\ \ 0]^T$ and the corresponding switching profile are shown in Figure 8. The preceding examples illustrated the validity and effectiveness of the proposed practical stabilization method.

5.2. *Converter operation with zero parasitic elements*

Some literatures [10–12,29,49,50] investigate the converters' operation under ideal conditions without considering the effect of parasitic elements. In this case, some of the converter subsystems may appear as unstable systems. Thus, one cannot use the conditions of Theorem 2 for which it is required all subsystems to be Schur stable. Instead, we will use the conditions of Theorems 1 and 3. In the case of the DC-DC buck-boost converter, one can verify that the subsystem matrix A_1 is not Schur stable.

5.2.1. *Buck operation*

In this mode, the desired target is selected as the equilibrium point of the averaged system as $x_e = (I_n - A_{\lambda_{ref}})^{-1}b_{\lambda_{ref}} = [0.76, 7]^T$ corresponding to $\lambda_{ref} = 0.2$. Due to eliminating the parasitic elements, the equilibrium point is slightly different from that obtained in Section 5.1. Using the conditions of Theorem 3, the solution of the optimization problem in Relation (25)

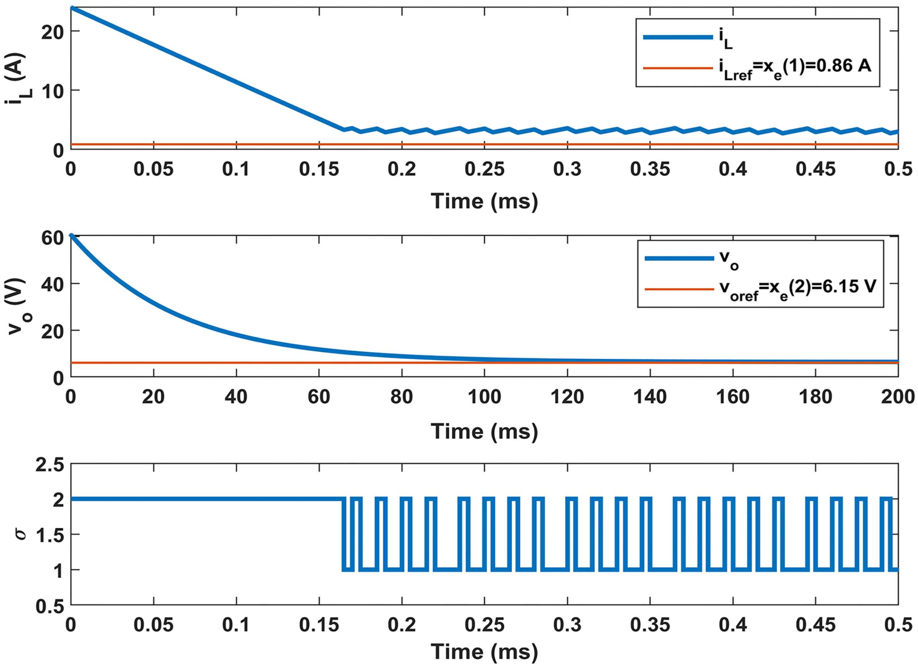

Figure 5. Time evolution of state trajectories $x(kT_s)$ and switching function $\sigma(x(kT_s))$ starting from $x(0) = [24A\ 60V]^T$ using conditions of Theorem 2 in buck operation.

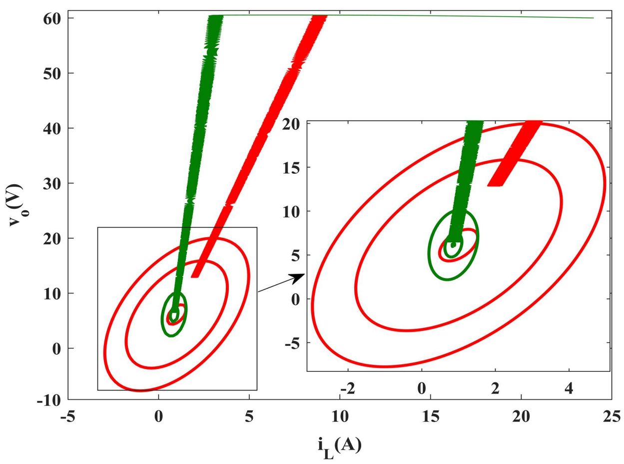

Figure 6. State trajectories corresponding to the conditions of Theorem 2 (green lines) and Theorem 1 (red lines).

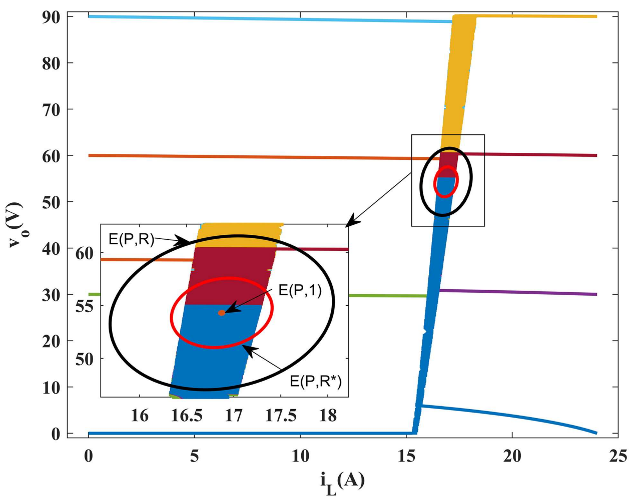

Figure 7. State trajectories corresponding to conditions of Theorem 2 in boost operation.

corresponding to $\tau = 10^{-13}$ yields $t = 6.1122 \cdot 10^{-4}$, with the matrices P, P_1, and P_2 as:

$$P = 10^4 \begin{bmatrix} 0.2219 & -0.1633 \\ -0.1633 & 1.0155 \end{bmatrix},$$

$$P_1 = 10^{-5} \begin{bmatrix} 0.0002 & 0.0004 \\ 0.0004 & 0.4687 \end{bmatrix},$$

$$P_2 = 10^{-5} \begin{bmatrix} 0.0002 & 0.0005 \\ 0.0005 & 0.4687 \end{bmatrix}.$$

Proposition 3 yields $R = 214.147$ and $R^* = 30.981$. Figure 9 shows the corresponding ellipsoids with the state trajectories $x(k)$ starting from various initial conditions.

A comparison between the trajectories and the size of various ellipsoids corresponding to the conditions of Theorem 1 (blue lines) and Theorem 3 (green lines) is made in Figure 10. As it can be seen, similar to Figure 10, Theorem 1 gives more conservative results because this theorem is established for the switched affine systems with all unstable subsystems, while the subsystem matrix A_2 in the buck-boost converter is Schur stable even though the parasitic elements have been set to zero.

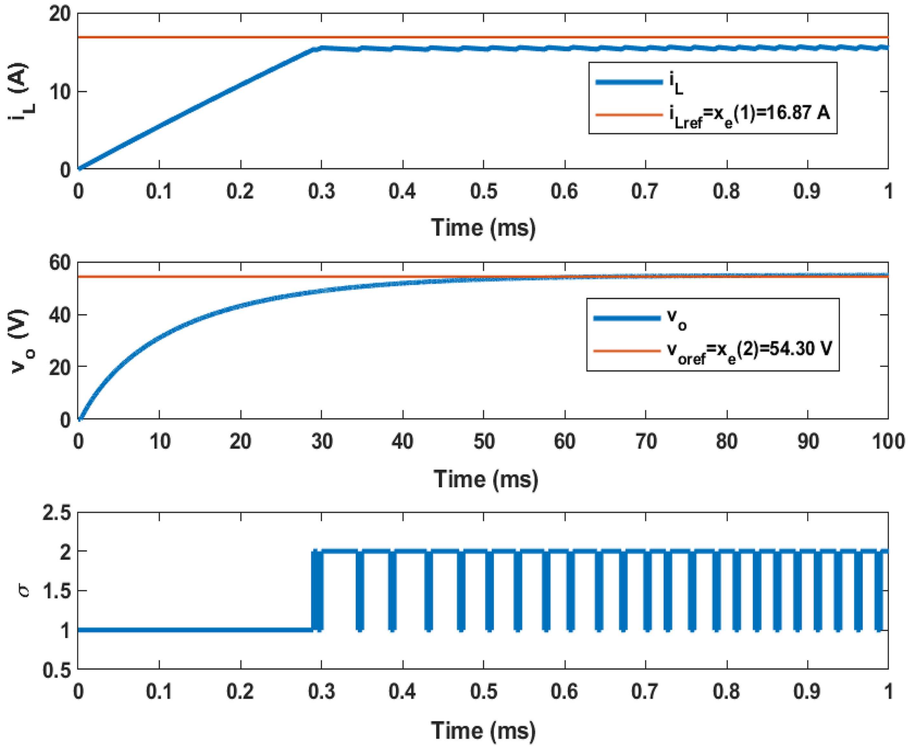

Figure 8. Time evolution of state trajectories $x(kT_s)$ and switching function $\sigma(x(kT_s))$ starting from $x(0) = [0A\ 0V]^T$ using conditions of Theorem 2 in boost operation.

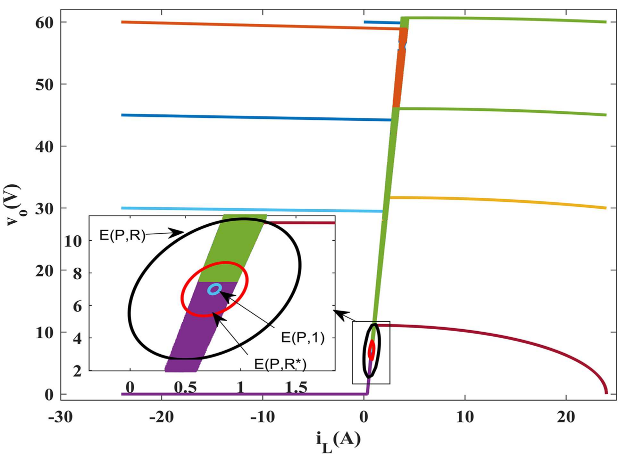

Figure 9. State trajectories corresponding to the conditions of Theorem 3 in buck operation and zero parasitic elements.

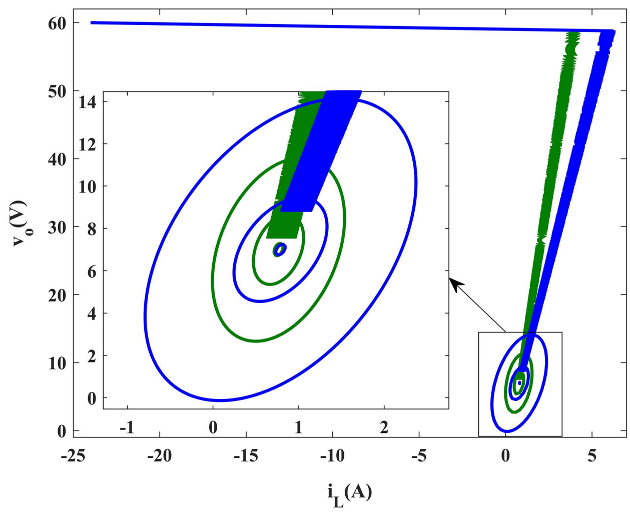

Figure 10. State trajectories corresponding to the conditions of Theorem 3 (green lines) and Theorem 1 (blue lines).

5.2.2. Boost operation

As in the case of Subsection 5.1.2., here we choose the equilibrium point of the averaged system as $x_e = (I_n - A_{\lambda_{ref}})^{-1} b_{\lambda_{ref}} = [18.47A, 65.32V]^T$ corresponding to $\lambda_{ref} = 0.7$. Since we have ignored the effect of parasitic elements, there is a difference between the equilibrium point in this section and that in Subsection 5.1.2. Using the conditions of Theorem 3, the solution of the optimization problem in Relation (25) corresponding to $\tau = 2 \cdot 10^{-14}$ yields $t = 1.3297 \cdot 10^{-4}$, with the matrices P, P_1, and P_2 as:

$$P = 10^4 \begin{bmatrix} 0.8632 & -0.3462 \\ -0.3462 & 3.6078 \end{bmatrix},$$

$$P_1 = 10^{-5} \begin{bmatrix} 0.0000 & 0.0019 \\ 0.0019 & 0.5608 \end{bmatrix},$$

$$P_2 = 10^{-5} \begin{bmatrix} 0.0000 & 0.0020 \\ 0.0020 & 0.5605 \end{bmatrix}.$$

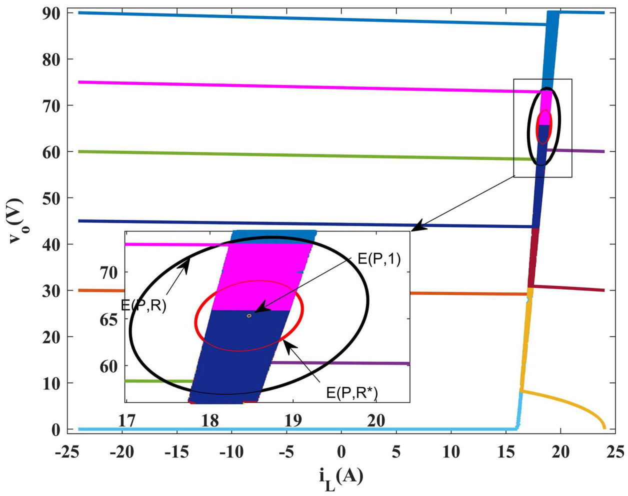

Figure 11. State trajectories corresponding to Theorem 3 in boost operation without parasitic elements.

The quantities $R = 3107.0$ and $R^* = 626.93$ are computed via Proposition 3. Figure 11 illustrates the state trajectories of the converter with corresponding ellipsoids $\mathcal{E}(P,R)$, $\mathcal{E}(P,R^*)$, and $\mathcal{E}(P,1)$.

6. Conclusion

In this paper, a new approach has been proposed to design state-dependent switching controllers for the discrete-time switched affine systems. This method is established through the analytical construction of the invariant set using a numerically computed attractive ellipsoid and requires the solution of a few convex optimization problems with LMI constraints. For any size of switched affine systems, the proposed approach provides sufficient conditions based on matrix inequalities with only one bilinear term as a multiplication of a positive scalar and a positive definite matrix. It has been shown that the infinite search space of the positive scalar variable can be reduced to the finite interval of $[0,1]$, for which a numerically tractable technique can be adopted via gridding of this interval. The proposed approach has been developed for a large class of discrete-time switched affine systems with all stable, all unstable, and partially-stable partially-unstable subsystems. The numerical examples and comparative studies with the existing works illustrate the effectiveness of the proposedstabilization methods from both conservativeness and computational complexity point of view.

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Appendix A.

Proof of Proposition 1: According to Relation (7), one can conclude $\exists\gamma > 0$ such that:

$$\begin{bmatrix} \tau_j P + A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j - \tau_j \end{bmatrix} \preceq -\gamma I_{n+1}, \ \forall i,\ j \in \mathbb{K}, \quad i \neq j, \tag{A.1}$$

Pre-multiplying matrix Inequality (A.1) by $[e(k)^T 1]$ and post-multiplying by $[e(k)^T 1]^T$ one can obtain:

$$\begin{bmatrix} e(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} \tau_j P + A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j - \tau_j \end{bmatrix} \leq -\gamma \begin{bmatrix} e(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} e(k) \\ 1 \end{bmatrix} \leq -\gamma, \tag{A.2}$$

$\forall i, j \in \mathbb{K}, i \neq j$. Inequalities in Relation (A.2) can be rewritten as in Relation (A.3):

$$-\tau_j \begin{bmatrix} e(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} -P & * \\ 0_{1\times n} & 1 \end{bmatrix} \begin{bmatrix} e(k) \\ 1 \end{bmatrix} + \begin{bmatrix} e(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j \end{bmatrix} \begin{bmatrix} e(k) \\ 1 \end{bmatrix} \leq -\gamma,$$
$$\forall i, j \in \mathbb{K}, i \neq j. \tag{A.3}$$

Using S-procedure, Relation (A.3) implies Relation (A.4), $\forall e(k) \in \mathbb{R}^n$:

$$\begin{bmatrix} e(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} -P & * \\ 0_{1\times n} & 1 \end{bmatrix} \begin{bmatrix} e(k) \\ 1 \end{bmatrix} < 0,$$
$$\Rightarrow \begin{bmatrix} e(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} A_j^T P_i A_j - P_j & * \\ l_j^T P_i A_j & l_j^T P_i l_j \end{bmatrix} \begin{bmatrix} e(k) \\ 1 \end{bmatrix} \leq -\gamma, \qquad \forall i, j \in \mathbb{K}, i \neq j. \tag{A.4}$$

After some algebra, Relation (A.4) can be rewritten as:

$$e(k)^T P e(k) > 1 \Rightarrow \forall i, j \in \mathbb{K}, i \neq j,$$
$$(A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) \leq -\gamma. \tag{A.5}$$

Since $e(k) \notin \mathcal{E}(P, 1)$, according to Relation (A.5) and the switching Rule (6), one can write:

$$e(k) \notin \mathcal{E}(P,1) \Rightarrow \forall i, \quad j \in \mathbb{K}, \quad i \neq j,$$
$$(A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))})^T P_i (A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))}) - e(k)^T P_{\sigma(e(k))} e(k)$$
$$\leq (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) \leq -\gamma. \tag{A.6}$$

Since $v(e(k)) = e(k)^T P_{\sigma(e(k)} e(k)$, from Relation (A.6) one can conclude that:

$$e(k) \notin \mathcal{E}(P,1) \Rightarrow v(e(k+1)) - v(e(k)) = \Delta v(e(k)) \leq -\gamma, \tag{A.7}$$

which confirms Condition (b) of Lemma 1. Since $P_i = P_i^T \succ 0$, $v(e(k)) = e(k)^T P_{\sigma(e(k))} e(k)$ is positive in the whole state space unless $e(k) = 0_{n\times 1}$ which satisfies Condition (c) of Lemma 1. Therefore Conditions (b) and (c) in Lemma 1 are satisfied, and the attractive property of the ellipsoid $\mathcal{E}(P,1)$ is concluded based on Definition 2.

Biography

Mohammad Hejri received his BSc degree from Tabriz University in 2000 and the MSc degree from Sharif University of Technology, Tehran, Iran in 2002 both in electrical engineering. He received his PhD degree in electrical engineering from Sharif University of Technology, Tehran, Iran and the University of Cagliari, Cagliari, Italy, in 2010 as a co-tutorship program. He has been with several industries and research centers such as Iran Tractor Foundry Company, Azerbaijan Regional Electric Company, Tabriz Oil and Refining Company and Iran's Niroo (Energy) Research Institute (NRI). From 2010 to 2012, he was a Postdoctoral Research Associate with the Department of Electric Power and Energy Systems, School of Electrical Engineering, Royal Institute of Technology (KTH), Stockholm, Sweden. Since 2012, he joined the Department of Electrical Engineering, Sahand University of Technology, Tabriz where he is now an Associate Professor. His research interests include control theory with applications in power electronics, renewable energy and power systems.