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Combined type generalized classes of efficient estimators of mean in joint presence of measurement error and non-response using stratified two-phase sampling

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Combined regression estimator.

Abstract. In this study, three classes of generalized and more efficient combined regression-cum-ratio estimators are presented to estimate the population mean of the study variable in stratified two-phase sampling considering non-response and measurement error are present jointly. The expressions for the bias and mean square error of the three proposed generalized combined regression-cum-ratio estimators have been obtained. Optimal conditions which make the proposed generalized regression-cum-ratio estimators more efficient than modified combined regression estimator are discussed. The performance of the proposed generalized combined regression-cum-ratio estimators has been compared theoretically as well as empirically with various combined type estimators in stratified two-phase sampling including usual combined ratio estimator, usual combined exponential ratio estimator, usual combined regression estimator, and modified combined regression estimator. An empirical study shows that the proposed generalized combined regression-cum-ratio estimators perform more efficiently than all combined type ratio, exponential ratio, and regression estimators discussed in the study.

1. Introduction

In survey sampling, usually, it is presumed that all the observations of the variables under study are adequately measured and all units in the sample give a response. But in reality, such assumption infringes, because not all units respond and also measurement errors may arise due to the difference between the recorded and true values. Hence with these reasons the statistics are not error-free. In practice, it is therefore, researchers may need to deal with the problem of non-response and measurement errors if present jointly. Generally, non-response and measurement error are debated separately using supportive information.

Sanaullah et al. [1] proposed the "generalized

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exponential-type ratio-cum-ratio and product-cum-product estimators in the presence of non-response under stratified two-phase sampling. Sanaullah et al. [2] taking inspiration from Samiuddin and Hanif [3] and Sanaullah et al. [4], proposed the generalized exponential-type estimators in the presence of non-response under stratified random sampling by using two auxiliary variables. Saleem et al. [5] taking inspiration from Koyuncu and Kadilar [6], proposed the generalized estimators in stratified sampling using two auxiliary variables in the presence of non-response. Saleem et al. [7] suggested the general class of estimators in the presence of non-response using two auxiliary variables. Shabbir et al. [8] extended Grover and Kaur [9] difference type estimator and suggested a generalized class of estimators for finite population mean in two-phase sampling using two auxiliary variables in the presence of non-response. Khare and Jha [10] advised six classes of different ratio-type estimator of mean in stratified sampling assuming the existence of the non-response. Some more studies are available for estimation of mean considering the presence of the non-response. One can see for example, Singh and Usman [11], Unal and Kadilar [12], Sanaullah and Hanif [13], Ehsan and Sanaullah [14], Sanaullah et al. [15], Wu et al. [16], and Varshney and Mradula [17] among many other.

Cochran [18] is supposed to be the first who suggested an unbiased estimator assuming the occurrence of measurement error only. Many researchers following Cochran [18] have studied the problem of mean estimation considering the measurement errors only. Singh and Karpe [19] provided different separate-type and combined-type ratio and product estimators in stratified sampling assuming the existence of the measurement error. Shukla et al. [20] taking motivation from Manisha and Singh [21], proposed a dual to ratio estimator of mean in the presence of measurement error. Masood and Shabbir [22] suggested a generalized regression type estimators for estimation of finite population variance of study variable using multivariate supportive information under multi-phase sampling scheme taking measurement error on the study variable. Khalil et al. [23] suggested a generalized combined regression-cum-ratio estimator in stratified sampling using scrambled responses in the presence of measurement error. Shalabh and Tsai [24] presented ratio and product estimation procedures keeping the correlated measurement error in their consideration. Khalil et al. [25] highlighted the issues when measurement error can be present in the survey, and then provided a generalized estimator of mean using auxiliary variable. Keeping the presence of the measurement error in view as one component of survey error, there exist several studies for estimation of mean using auxiliary variable in simple random sampling and stratified random sampling as well. For more detail, one can see for example Singh et al. [26], Yaqub and Shabbir [27], and Singh et al. [28] including many other.

After having a careful review of the existing studies for the estimation of mean, it can be noted that the individual components of the survey error have been well documented in the literature, however relatively little is known about the intersection of these components of survey error. The researchers who have studied the measurement errors as individual component of error for estimation mean, have ignored the non-response as another possible component of survey error similarly those who have studied the non-response, have ignored the possibility of existence of the measurement errors, whereas in many real-situations both components of the survey errors may be present. One may have to deal with while estimation of mean if these both types of errors, non-response and measurement errors are existing jointly. Consequently, ignoring the existence of any component of error will yield the estimate(s) with a relatively larger amount of the bias. A few numbers of researchers have debated the estimation of mean in simple random sampling assuming the joint existence of the non-response and the measurement error, such as, Azeem [29] studied the problem of mean estimation considering the joint influence of the non-response and measurement error; Kumar et al. [30] proposed the exponential ratio-type estimator in the presence of non-response and measurement error; Kumar [31] extended the work of Azeem [29] and provided a class of more efficient estimators to estimate the population mean; Azeem and Hanif [32] suggested different types of estimators including dual to chain ratio estimator, a ratio-cum-dual to ratio-type exponential estimator, and ratio-cum dual to exponential ratio estimator; Irfan et al. [33] provided an optimum class of estimators for mean in simple random sampling. Sabir and Sanaullah [34] revisited Kumar [31] estimator and, hence provided a note on correct usage of Kumar's [31] for estimation of mean if the wo components of errors are present simultaneously.

A few more studies have been presented the studies in stratified random sampling. Zahid and Shabbir [35] suggested a class of estimators for mean estimation whereas Kumar et al. [36] suggested a ratio-cum-product exponential type estimator of the population mean in the joint existence of non-response and measurement error using two auxiliary variables. However in both of the studies seprate type estimator in stratified random sampling are advised for mean estimation when the two components of error are existing simultaneously.

After having a very careful review, it is felt that only a few research studies have discussed the estimation of mean in stratified random sampling but these studies provide only separate-type estimators of mean. It is also observed that mean estimation is discussed under simple random sampling whereas estimation in some other sampling designs such as stratified sampling and multi-stage sampling design is completely ignored. Another thing can also be felt that most of the existing estimators in simple random sampling and in stratified random sampling as well, are proposed under the assumption if population mean of the auxiliary variable is readily availble. However in many real situations such auxiliary information may not be readily avaiable and use of two-phase sampling is one of the possible alternates in such situations. Otherwise existing estimators can not be made useful for mean estimation unless they are modified accordingly. Hence many gaps are found to work on. Therefore assuming the situation when population mean of the auxiliary variable is not readily available, this study is motivated to present some combined-type estimators for mean estimation in stratified two-phase sampling.

Now in order to fill some of the gaps as stated in previous text, the objective of this study is to provide some generalized classes of more efficient combined type estimators for estimating the population mean of study variable following few assumptions such as:

(i) The two components of survey errors, the non-response, and the measurement error are simultaneously present;

(ii) Population mean of the auxiliary variable is not ready available in prior of the survey;

(iii) Units of the population under observation are not homogeneous;

(iv) Relationship between the study variable and the auxiliary variable is same in each stratum;

(v) Ratio between the means of the study variable and the auxiliary variable in each stratum is approximately equal to the ratio of the stratified means so as to get combined type estimators.

The study is then prompted to get such conditions which make each of the proposed generalized classes of the combined type estimators more efficient than combined-regression estimator. Furthermore, the study is motivated to evaluate the proposed class of efficient combined type estimators for its performance with the combined ratio estimator, combined exponential-type ratio estimator and combined regression estimator in stratified two-phase sampling. In the present study it is discussed that the proposed combined estimators can be molded into three different situations of real-life which are given separately as three remarks. In the following sections, some notations and sampling procedure for estimation of mean with the assumption of simultaneous existence of the non-response and the measurement error are discussed along with some results which will be helpful for observing the properties of an estimator. An attempt has also been made to compare the Mean Square Errors ($MSEs$) of the proposed estimators with the $MSEs$ of the existing estimators. A simulation study is performed to compute $MSEs$ of all the estimators discussed in the text. The simulation results are also demonstrated through the graphs to have a quick understanding with the performance of all the estimators by changing the non-response proportion.

2. Notations and stratified two-phase sampling

Before to present stratified two-phase sampling and estimation procedures, some basic notations to be used in this text are defined. Let a population of size N be divided into L homogeneous strata with N_h units $(h = 1, 2, \cdots, L)$ such that $\sum_{h=1}^{L} N_h = N$.

Notations

N	Population size
N_h	Population size of hth stratum
$Y\backslash X$	Study variable\Auxiliary variable
$\mu_Y\backslash\mu_X$	Population mean of Y\Population of X
$\mu_{Yh}\backslash\mu_{Xh}$	Population means in hth stratum
$\mu_{Yh(1)}, \mu_{Xh(1)}$	Population means of respondents group in hth stratum
$\mu_{Yh(2)}, \mu_{Xh(2)}$	Population means of group of non-respondents in hth stratum
$\sigma^2_{Yh}, \sigma^2_{Xh}$	Population variances of Y&X respectively in hth stratum
$\sigma^2_{Yh(1)}, \sigma^2_{Xh(1)}$	Population variances from group of respondents in hth stratum
$\sigma^2_{Yh(2)}, \sigma^2_{Xh(2)}$	Population variances from group of non-respondents in hth stratum
$C_{Yh(1)}, C_{Xh(1)}$	Coefficient of variations for Y&X from group of respondents in hth stratum
$C_{Yh(2)}, C_{Xh(2)}$	Coefficient of variation from group of non-respondents in hth stratum
$y_{hi}\backslash x_{hi}$	Reported values on Y and X for ith unit in hth stratum
$Y_{hi}\backslash X_{hi}$	True values on Y and X for ith unit in hth stratum
$U_{hi} = y_{hi} - Y_{hi}$	Measurement error on the study variable associated with ith unit in hth stratum

$V_{hi}=x_{hi}-X_{hi}$	Measurement error on the auxiliary variable associated with ith unit in hth stratum
$U^*_{hi}=y^*_{hi}-Y^*_{hi}$	Measurement error and non-response on Y associated with ith unit in hth stratum
$V^*_{hi}=x^*_{hi}-X^*_{hi}$	Measurement error and non-response on X associated with ith unit in hth stratum
$\sigma^2_{Uh(2)}$ & $\sigma^2_{Vh(2)}$	Population variances of U and V respectively from the group of non-respondents
$\rho_{YXh(1)}$& $\rho_{YXh(2)}$	Coefficients of correlation between the study and auxiliary variables for the respondent and non-respondent parts of the population respectively
$P_h=\dfrac{N_h}{N}$	Weight of hth stratum
n'_h	First-phase sample size in hth stratum
n''_h	2nd-phase sample size in hth stratum
$n'=\sum_{h=1}^{L} n'_h$	First-phase stratified sample size
$n''=\sum_{h=1}^{L} n''_h$	2nd-phase stratified sample size
$\mu'_{x(st)}$	Sample mean estimator based on first-phase sample
$\tilde{\mu}''^*_{y(st)} \backslash \tilde{\mu}''^*_{x(st)}$	Sample mean estimator (for y and x) based on 2nd-phase sample with non-response and measurement error.

Now consider,

$$\mu_Y=\sum_{i=1}^{L} P_h\mu_{Yh} \qquad \mu_X=\sum_{i=1}^{L} P_h\mu_{Xh},$$

where:

$$\mu_{Yh}=\frac{1}{N_h}\sum_{i=1}^{N_h} y_{hi}, \mu_{Xh}=\frac{1}{N_h}\sum_{i=1}^{N_h} x_{hi} \quad \text{and} \quad P_h=\frac{N_h}{N}.$$

The measurement errors $U^*_{hi}=y^*_{hi}-Y^*_{hi}$ and $V^*_{hi}=x^*_{hi}-X^*_{hi}$ in the presence of non-response associated are assumed to have their means zero and the variances $\sigma^2_{Uh(2)}$ and $\sigma^2_{Vh(2)}$ for the non-respondent part of the population.

Unknown population mean of the auxiliary variable is estimated using stratified two-phase sampling. Let (y_{hi}, x_{hi}) be the observed values instead of true values (Y_{hi}, X_{hi}) of the two characteristics (Y, X) respectively associated with ith sample unit of hth stratum where, ith $(i=1,2,\cdots,n'_h)$. Now we take a first-phase sample comparatively of large size say n'_h from each hth stratum such that $\sum_{h=1}^{L} n'_h=n'$ and information on the variable X is obtained. Now a usual unbiased mean estimator based on first-phase sample information in stratified sampling is defined by:

$$t'_u=\mu'_{x(st)}=\sum_{h=1}^{L} P_h\mu'_{xh} \quad \text{where} \quad \mu'_{xh}=\frac{1}{n'_h}\sum_{i=1}^{n'_h} x_{hi}, \tag{1}$$

with variances given by:

$$\operatorname{var}(t'_u)=\mu_Y^2\sum_{h=1}^{L} P_h^2\lambda'_h C^2_{Xh}. \tag{2}$$

A sub-sample of size $n''_h(\subset n'_h)$ from each stratum is taken as 2nd-phase sample such that $\sum_{h=1}^{L} n''_h=n''$ by simple random sampling without replacement and information on variables Y is taken. Here it is assumed that measurement error and non-response are jointly present. It is also assumed that only $n''_{h(1)}$ sample units respond and $n''_{h(2)}=\left(n''_h-n''_{h(1)}\right)$ do not. Following Hansen and Hurwitz [37] technique, let $r_h\left(=\frac{n''_{h(2)}}{k_h}; k_h>1\right)$ be a sub-sample of the individuals who do not respond to the survey question(s) but respond when they are contacted again for their personal interviews, where k_h is the inverse sampling ratio. It is further assumed that all r_h units respond while interviewing them for the study variable.

Following Hansen and Hurwitz [37] an unbiased estimator of means is reproduced for Y and X variables in stratified sampling as:

$$\tilde{t}^*_u=\tilde{\mu}''^*_{y(st)}=\sum_{i=1}^{L} P_h\tilde{\mu}''^*_{yh}, l, \tag{3}$$

where $\tilde{\mu}''^*_{yh}=w_1(\mu''_{yh(1)}+\mu''_{Uh(1)})+w_2(\mu''_{y(2)k_h}+\mu''_{U(2)k_h})$, $w_1=\frac{n''_{h(1)}}{n''_h}$, $w_2=\frac{n''_{h(2)}}{n''_h}$, $\mu''_{yh(1)}=\frac{1}{n''_{h(1)}}\sum_{i=1}^{n''_{h(1)}} y_{hi}$, $\mu''_{Uh(1)}=\frac{1}{n''_{h(1)}}\sum_{i=1}^{n''_{h(1)}} U_{hi}$, $\mu''_{y(2)k_h}=\frac{1}{r_h}\sum_{i=1}^{r_h} y''^*_{hi}$, $\mu''_{yh(2)k_h}=\frac{1}{k_h}\sum_{i=1}^{k_h} y''^*_{hi}$, and $\mu''_{Uh(2)k_h}=\frac{1}{k_h}\sum_{i=1}^{k_h} U^*_{hi}$.

The expression of the variance $\tilde{t}^*_u$ may be defined as,

$$\operatorname{var}\left(\tilde{t}^*_u\right)=\mu_Y^2\sum_{h=1}^{L} P_h^2\left(\lambda_{2h}\left(C^2_{Yh}+\frac{\sigma^2_{Uh}}{\mu_Y^2}\right)\right.$$
$$\left.+\theta_{2h}\left(C^2_{Yh(2)}+\frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right)\right), \tag{4}$$

where, $\lambda_{2h} = \left(\frac{1}{n''_h} - \frac{1}{N_h}\right)$, $\lambda_h = \left(\frac{1}{n'_h} - \frac{1}{N_h}\right)$, $W_{h(2)} = \frac{N_{h(2)}}{N_h}$, $\theta_{2h} = \frac{W_{h(2)}(r_h-1)}{n''_h}$.

Similarly, for the auxiliary variable, sample mean estimator is:
$\tilde{\mu}''^{*}_{x(st)} = \sum_{i=1}^{L} P_h \, \tilde{\mu}''^{*}_{xh}$, where
$\tilde{\mu}''^{*}_{xh} = w_1 \left(\mu''_{xh(1)} + \mu''_{Vh(1)}\right) + w_2 \left(\mu''_{x(2)k_h} + \mu''_{V(2)k_h}\right)$, $w_1 = \frac{n''_{h(1)}}{n''_h}$, $w_2 = \frac{n''_{h(2)}}{n''_h}$, $\mu''_{xh(1)} = \frac{1}{n''_{h(1)}} \sum_{i=1}^{n''_{h(1)}} x_{hi}$, $\mu''_{Vh(1)} = \frac{1}{n''_{h(1)}} \sum_{i=1}^{n''_{h(1)}} V_{hi}$, $\mu''_{x(2)k_h} = \frac{1}{r_h} \sum_{i=1}^{r_h} x''^{*}_{hi}$, $\mu''_{xh(2)k_h} = \frac{1}{k_h} \sum_{i=1}^{k_h} x''^{*}_{hi}$, and $\mu''_{\tilde{V}h(2)k_h} = \frac{1}{k_h} \sum_{i=1}^{k_h} V_{hi}$.

3. Method of mean estimation in stratified two-phase sampling

In this section, method for estimating population mean of the study variable is presented under the assumption, the non-response and the measurement error are jointly occurring on both variables, the study variable and the auxiliary variable in stratified two-phase sampling. It is also assumed that population of the auxiliary variable is not known in prior of survey; relationship between the study variable and the auxiliary variable is same in stratum; ratio between the means in each stratum is approximately equal to the ratio of the stratified means. Method of mean estimation in stratified two-phase sampling is proposed under three different procedures separately in the three following sub-sections.

3.1. Procedure I: Proposed class of usual combined type estimators

Now, for estimation of population mean, three modified combined type of estimators named as, usual combined ratio estimator, usual combined exponential ratio estimator, and usual combined regression estimator following the assumptions in stratified two-phase sampling are given respectively by:

$$\tilde{t}^{*}_{ra} = \frac{\tilde{\mu}''^{*}_{y(st)}}{\tilde{\mu}''^{*}_{x(st)}} \mu'_{x(st)}, \tag{5}$$

[modified combined ratio estimator]

$$\tilde{t}^{*}_{er} = \tilde{\mu}''^{*}_{y(st)} \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^{*}_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^{*}_{x(st)}}\right), \tag{6}$$

[modified combined exponential ratio estimator]
and

$$\tilde{t}^{*}_{reg} = \tilde{\mu}''^{*}_{y(st)} + w\left(\mu'_{x(st)} - \tilde{\mu}''^{*}_{x(st)}\right), \tag{7}$$

[modified combined regression estimator]
where w is an optimizing constant.

In order to obtain the expressions for the bias and the *MSEs* of Eqs. (5)–(7), let us consider, $W^{*}_{Yh} = \sum_{i=1}^{n''_h} \left(y''^{*}_{hi} - \mu_{Yh}\right)$, $W^{*}_{Xh} = \sum_{i=1}^{n''_h} \left(x''^{*}_{hi} - \mu_{Xh}\right)$, $W^{*}_{Uh} = \sum_{i=1}^{n''_h} U^{*}_{hi}$, $W^{*}_{Vh} = \sum_{i=1}^{n''_h} V^{*}_{hi}$ and $W_{Xh} = \sum_{i=1}^{n'_h} (x'_{ih} - \mu_{Xh})$.
Now the errors due to sampling are defined by:

$$\tilde{e}''^{*}_{y(st)} = \frac{1}{\mu_Y} \sum_{h=1}^{L} \frac{P_h}{n''_h} \left(W^{*}_{Yh} + W^{*}_{Uh}\right),$$

$$\tilde{e}''^{*}_{x(st)} = \frac{1}{\mu_X} \sum_{h=1}^{L} \frac{P_h}{n''_h} \left(W^{*}_{Xh} + W^{*}_{Vh}\right),$$

and

$$e'_{x(st)} = \frac{1}{\mu_X} \sum_{h=1}^{L} \frac{P_h}{n'_h} \left(x'_{hi} - \mu_{Xh}\right),$$

and the sample means associated with the sampling errors assuming the joint presence of non-response and measurement error are defined by: $\tilde{\mu}''^{*}_{y(st)} = \mu_Y(1 + \tilde{e}''^{*}_{y(st)})$, $\tilde{\mu}''^{*}_{x(st)} = \mu_X(1 + \tilde{e}''^{*}_{x(st)})$, and $\mu'_{x(st)} = \mu_X(1 + e'_{x(st)})$, such that $E(\tilde{e}''^{*}_{y(st)}) = E(\tilde{e}''^{*}_{x(st)}) = E(e'_{x(st)}) = 0$,

$$E\left(\tilde{e}''^{*}_{y(st)}\right)^2 = \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h}\left(C^2_{Yh} + \frac{\sigma^2_{Uh}}{\mu^2_Y}\right) + \theta_{2h}\left(C^2_{Yh(2)} + \frac{\sigma^2_{Uh(2)}}{\mu^2_Y}\right)\right) = \tilde{A}''^{*}_{y(st)},$$

$$E\left(\tilde{e}''^{*}_{x(st)}\right)^2 = \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h}\left(C^2_{Xh} + \frac{\sigma^2_{Vh}}{\mu^2_X}\right) + \theta_{2h}\left(C^2_{Xh(2)} + \frac{\sigma^2_{Vh(2)}}{\mu^2_X}\right)\right) = \tilde{A}''^{*}_{x(st)},$$

$$E(\tilde{e}''^{*}_{y(st)}\tilde{e}''^{*}_{x(st)}) = \sum_{h=1}^{L} P_h^2 (\lambda_{2h}\, \rho_{YXh} C_{Yh} C_{Xh} + \theta_{2h}\, \rho_{YXh(2)} C_{Yh(2)} C_{Xh(2)}) = \tilde{C}''^{*}_{xy(st)},$$

$$E\left(e'_{x(st)}\right)^2 = E\left(\tilde{e}''^{*}_{x(st)} e'_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \lambda_h C^2_{Xh} = A'_{x(st)},$$

and

$$E\left(\tilde{e}''^{*}_{y(st)}e'_{x(st)}\right)=\sum_{h=1}^{L}P_h^2\left(\lambda_h\rho_{YXh}C_{Yh}C_{Xh}\right)$$

$$=C'_{xy(st)},$$

where $\tilde{A}^{*}_{x(st)}=\tilde{A}''^{*}_{x(st)}-A'_{x(st)}$ and $\tilde{C}^{*}_{xy(st)}=\tilde{C}''^{*}_{xy(st)}-C'_{xy(st)}$.

3.1.1. *Derivation of the biases and MSEs expressions of the estimators*

Now $\tilde{t}^{*}_{ra}$, $\tilde{t}^{*}_{er}$ and $\tilde{t}^{*}_{reg}$ are given alternatively in terms of e, s by:

$$\tilde{t}^{*}_{ra}=\frac{\mu_Y\left(1+\tilde{e}''^{*}_{y(st)}\right)}{\mu_X\left(1+\tilde{e}''^{*}_{x(st)}\right)}\mu_X\left(1+e'_{x(st)}\right), \quad (8)$$

$$\tilde{t}^{*}_{er}=\mu_Y\left(1+\tilde{e}''^{*}_{y(st)}\right)$$

$$\exp\left(\frac{\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^{*}_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)+\mu_X\left(1+\tilde{e}''^{*}_{x(st)}\right)}\right), \quad (9)$$

and

$$\tilde{t}^{*}_{reg}=\mu_Y\left(1+\tilde{e}''^{*}_{y(st)}\right)$$

$$+w\left(\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^{*}_{x(st)}\right)\right), \quad (10)$$

or alternatively by:

$$\tilde{t}^{*}_{ra}-\mu_Y=\mu_Y\left(\tilde{e}''^{*}_{y(st)}+e'_{x(st)}-\tilde{e}''^{*}_{x(st)}+\tilde{e}''^{*2}_{x(st)}\right.$$

$$\left.-\tilde{e}''^{*}_{x(st)}e'_{x(st)}-\tilde{e}''^{*}_{y(st)}\tilde{e}''^{*}_{x(st)}+\tilde{e}''^{*}_{y(st)}e'_{x(st)}\right), (11)$$

$$\tilde{t}^{*}_{er}-\mu_Y=\mu_Y\left(\tilde{e}''^{*}_{y(st)}+\frac{e'_{x(st)}}{2}-\frac{\tilde{e}''^{*}_{x(st)}}{2}+\frac{3}{8}\tilde{e}''^{*2}_{x(st)}\right.$$

$$-\frac{e'^{2}_{x(st)}}{8}-\frac{\tilde{e}''^{*}_{x(st)}e'_{x(st)}}{4}-\frac{\tilde{e}''^{*}_{y(st)}\tilde{e}''^{*}_{x(st)}}{2}$$

$$\left.+\frac{\tilde{e}''^{*}_{y(st)}e'_{x(st)}}{2}\right), \quad (12)$$

and,

$$\tilde{t}^{*}_{reg}-\mu_Y=\mu_Y\tilde{e}''^{*}_{y(st)}+w\mu_X\left(e'_{x(st)}-\tilde{e}''^{*}_{x(st)}\right). \quad (13)$$

The expressions of the biases for each of $\tilde{t}^{*}_{ra}$, $\tilde{t}^{*}_{er}$ and $\tilde{t}^{*}_{reg}$ up to the order $O\left(n^{-1}\right)$ are given by:

$$Bias\left(\tilde{t}^{*}_{ra}\right)=\mu_Y\left(\tilde{A}^{*}_{x(st)}-\tilde{C}^{*}_{xy(st)}\right), \quad (14)$$

$$Bias\left(\tilde{t}^{*}_{er}\right)=\mu_Y\left(\frac{3}{8}\tilde{A}^{*}_{x(st)}-\frac{1}{2}\tilde{C}^{*}_{xy(st)}\right), \quad (15)$$

and

$$Bias\left(\tilde{t}^{*}_{reg}\right)=0. \quad (16)$$

From Eq. (16), we get that the regression estimator $\tilde{t}^{*}_{reg}$ is an unbiased estimator while ratio estimator $\tilde{t}^{*}_{ra}$ and exponential estimator $\tilde{t}^{*}_{er}$ are biased estimators, see the Eqs. (14) and (15). Further expressions of the $MSEs$ for each of $\tilde{t}^{*}_{ra}$, $\tilde{t}^{*}_{er}$ and $\tilde{t}^{*}_{reg}$, are obtained up to the order $O\left(n^{-1}\right)$ and are given by:

$$MSE\left(\tilde{t}^{*}_{ra}\right)=\mu_Y^2\left(\tilde{A}''^{*}_{y(st)}+\tilde{A}^{*}_{x(st)}-2\tilde{C}^{*}_{xy(st)}\right), \quad (17)$$

$$MSE\left(\tilde{t}^{*}_{er}\right)=\mu_Y^2\left(\tilde{A}''^{*}_{y(st)}+\frac{1}{4}\tilde{A}^{*}_{x(st)}-\tilde{C}^{*}_{xy(st)}\right), \quad (18)$$

and

$$MSE\left(\tilde{t}^{*}_{reg}\right)=\left(\mu_Y^2\tilde{A}''^{*}_{y(st)}+w^2\mu_X^2\tilde{A}^{*}_{x(st)}\right.$$

$$\left.-2\mu_Y\mu_X w\tilde{C}^{*}_{xy(st)}\right). \quad (19)$$

In order to achieve the optimum value of w, Eq. (19) is differentiated partially with respect to w and then equating the first derivative with zero. The optimum value of w which gives the expression of the minimum $MSE\tilde{t}^{*}_{reg}$ is given by $w_{opt}=\frac{\mu_Y\tilde{C}^{*}_{xy(st)}}{\mu_X\tilde{A}^{*}_{x(st)}}$. Substituting the optimum value of w in Eq. (19), the expression of the minimum MSE of $\tilde{t}^{*}_{reg}$ is given by:

$$\min MSE\left(\tilde{t}^{*}_{reg}\right)=\mu_Y^2\tilde{A}''^{*}_{y(st)}\left(1-\frac{\tilde{C}^{*2}_{xy(st)}}{\tilde{A}^{*}_{x(st)}\tilde{A}''^{*}_{y(st)}}\right), \quad (20)$$

or after more simplification Eq. (20) is given by:

$$\min MSE\left(t_{reg}\right)=\mu_Y^2\tilde{A}''^{*}_{y(st)}\left(1-\rho^2_{xy(st)}\right),$$

where $$\rho^2_{xy(st)}=\frac{\tilde{C}^{*2}_{xy(st)}}{\tilde{A}^{*}_{x(st)}\tilde{A}''^{*}_{y(st)}}. \quad (21)$$

3.1.2. *Theoretical comparisons among the usual combined type estimators*

Now in this section, efficiency of combined regression estimator $\tilde{t}^{*}_{reg}$ is compared with the efficiencies of unbiased $\tilde{t}^{*}_{u}$ estimator, combined ratio estimator $\tilde{t}^{*}_{ra}$ and combined exponential estimator $\tilde{t}^{*}_{er}$.

(i) From equation Eqs. (4) and (21) we have:

$$MSE\left(\tilde{t}^*_{reg}\right) - MSE\left(\tilde{t}^*_{u}\right) > 0,$$

$$\mu_Y^2 \sum_{h=1}^{L} P_h^2 \theta_{2h} C^2_{Yh(2)} \left(1-\rho^2_{YXh(2)}\right) + \mu_Y^2 \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(\frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right)$$

$$\left(\mu_Y^2 \sum_{h=1}^{L} P_h^2 \lambda_{2h} C^2_{Yh} \left(1-\rho^2_{YXh}\right) + \mu_Y^2 \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right)\right)$$

$$- \mu_Y^2 \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h} \left(C^2_{Yh} + \frac{\sigma^2_{Uh}}{\mu_Y^2}\right) + \theta_{2h} \left(C^2_{Yh(2)} + \frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right)\right) > 0,$$

and

$$\mu_Y^2 \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(C^2_{Yh} + \frac{\sigma^2_{Uh}}{\mu_Y^2}\right) \rho^2_{YXh} + \mu_Y^2 \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(C^2_{Yh(2)} + \frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right) \rho^2_{YXh(2)} > 0. \tag{22}$$

(ii) From Eqs. (17) and (21) we have:

$$MSE\left(\tilde{t}^*_{reg}\right) - MSE\left(\tilde{t}^*_{ra}\right) > 0$$

$$\mu_Y^2 \left(\sum_{h=1}^{L} P_h^2 \lambda_{2h} C^2_{Yh} \left(1-\rho^2_{YXh}\right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} C^2_{Yh(2)} (1-\rho^2_{YXh(2)})\right.$$

$$\left. + \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(\frac{\sigma^2_{Uh}}{\mu_Y^2}\right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right)\right)$$

$$- \mu_Y^2 \left(\sum_{h=1}^{L} P_h^2 \lambda_{2h} C^2_{Yh} + \sum_{h=1}^{L} P_h^2 \lambda_{3h} \left(C^2_{Xh} - 2\rho_{YXh} C_{Yh} C_{Xh}\right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} (C^2_{Yh(2)} + C^2_{Xh(2)} - 2\rho_{YXh(2)} C_{Yh(2)} C_{Xh(2)})\right.$$

$$\left. + \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(\frac{\sigma^2_{Uh}}{\mu_Y^2} + \frac{\sigma^2_{Vh}}{\mu_X^2}\right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{\sigma^2_{Uh(2)}}{\mu_Y^2} + \frac{\sigma^2_{Vh(2)}}{\mu_X^2}\right)\right) > 0,$$

and

$$\mu_Y^2 \left(\sum_{h=1}^{L} P_h^2 \lambda_{3h} (C_{Xh} - \rho_{YXh} C_{Yh})^2 + \sum_{h=1}^{L} P_h^2 \lambda_h C^2_{Yh} \rho^2_{YXh} + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(C_{Xh(2)} - \rho_{YXh(2)} C_{Yh(2)}\right)^2\right.$$

$$+ \sum_{h=1}^{L} P_h^2 \lambda_{2h} \rho^2_{YXh} \left(\frac{\sigma^2_{Uh}}{\mu_Y^2}\right) + \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(\frac{\sigma^2_{Vh}}{\mu_X^2}\right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} \rho^2_{YXh(2)} \left(\frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right)$$

$$\left. + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{\sigma^2_{Vh(2)}}{\mu_X^2}\right)\right) > 0. \tag{23}$$

(iii) From Eq. (18) and Eq. (21) we have:

$$MSE\left(\tilde{t}^*_{reg}\right) - MSE\left(\tilde{t}^*_{er}\right) > 0,$$

$$\mu_Y^2 \left(\sum_{h=1}^{L} P_h^2 \lambda_{2h} C^2_{Yh} \left(1-\rho^2_{YXh}\right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} C^2_{Yh(2)} \left(1-\rho^2_{YXh(2)}\right)\right.$$

$$\left. + \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(\frac{\sigma^2_{Uh}}{\mu_Y^2}\right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{\sigma^2_{Uh(2)}}{\mu_Y^2}\right)\right)$$

$$- \mu_Y^2 \left(\sum_{h=1}^{L} P_h^2 \lambda_{2h} C_{Yh}^2 + \sum_{h=1}^{L} P_h^2 \lambda_{3h} \left(\frac{1}{4} C_{Xh}^2 - \rho_{YXh} C_{Yh} C_{Xh} \right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(C_{Yh(2)}^2 + \frac{1}{4} C_{Xh(2)}^2 - \rho_{YXh(2)} C_{Yh(2)} C_{Xh(2)} \right) + \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(\frac{\sigma_{Uh}^2}{\mu_Y^2} + \frac{1}{4} \frac{\sigma_{Vh}^2}{\mu_X^2} \right) \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{\sigma_{Uh(2)}^2}{\mu_Y^2} + \frac{1}{4} \frac{\sigma_{Vh(2)}^2}{\mu_X^2} \right) \right) > 0,$$

and

$$\mu_Y^2 \left(\sum_{h=1}^{L} P_h^2 \lambda_{3h} \left(\frac{1}{2} C_{Xh} - \rho_{YXh} C_{Yh} \right)^2 + \sum_{h=1}^{L} P_h^2 \lambda_h C_{Yh}^2 \rho_{YXh}^2 + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{1}{2} C_{Xh(2)} - \rho_{YXh(2)} C_{Yh(2)} \right)^2 + \sum_{h=1}^{L} P_h^2 \lambda_{2h} \rho_{YXh}^2 \left(\frac{\sigma_{Uh}^2}{\mu_Y^2} \right) + \sum_{h=1}^{L} P_h^2 \lambda_{2h} \left(\frac{1}{4} \frac{\sigma_{Vh}^2}{\mu_X^2} \right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} \rho_{YXh(2)}^2 \left(\frac{\sigma_{Uh(2)}^2}{\mu_Y^2} \right) + \sum_{h=1}^{L} P_h^2 \theta_{2h} \left(\frac{1}{4} \frac{\sigma_{Vh(2)}^2}{\mu_X^2} \right) \right) > 0. \quad (24)$$

From Eqs. (22)–(24), it can be easily seen that the combined regression estimator $\tilde{t}_{reg}^*$ is more efficient than combined ratio estimator $\tilde{t}_{ra}^*$, and combined exponential ratio estimator $\tilde{t}_{er}^*$.

3.2. Procedure II: Proposed class of modified combined regression-type estimators

In this section another procedure of mean estimation in stratified two-phase sampling is presented as a class of modified combined regression-type estimators. The proposed class of generalized combined regression-type estimators is due to some modifications in the form of usual combined regression estimator. The procedure includes to get a class of modified combined-type regression $t_{g(i)}$ by replacing $\tilde{\mu}''^{*}_{y(st)}$ with $J_{(i)}$ in Eq. (7). The proposed estimator $t_{g(i)}$ in the form of a general estimator is given by:

$$t_{g(i)} = \left(J_{(i)} + w_{i1} \left(\mu'_{x(st)} - \tilde{\mu}''^{*}_{x(st)} \right) \right) \quad \text{for} \quad i = 1, 2, 3, \quad (25)$$

where

$$J_{(1)} = \frac{\tilde{\mu}''^{*}_{y(st)}}{2} \left(\exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^{*}_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^{*}_{x(st)}} \right) + \exp\left(\frac{\tilde{\mu}''^{*}_{x(st)} - \mu'_{x(st)}}{\tilde{\mu}''^{*}_{x(st)} + \mu'_{x(st)}} \right) \right),$$

$$J_{(2)} = \frac{\tilde{\mu}''^{*}_{y(st)}}{2} \left(\frac{\mu'_{x(st)}}{\tilde{\mu}''^{*}_{x(st)}} + \frac{\tilde{\mu}''^{*}_{x(st)}}{\mu'_{x(st)}} \right),$$

and

$$J_{(3)} = \frac{\tilde{\mu}''^{*}_{y(st)}}{2} \left(\exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^{*}_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^{*}_{x(st)}} \right) + \frac{\mu'_{x(st)}}{\tilde{\mu}''^{*}_{x(st)}} \right).$$

3.2.1. Derivation of the biases and MSEs of the modified estimator

Now $t_{g(i)}$ can be given in terms of e, s by:

$$t_{g(1)} = \frac{\mu_Y \left(1 + \tilde{e}''^{*}_{y(st)} \right)}{2} \left(\exp\left(\frac{\mu_X \left(1 + e'_{x(st)} \right) - \mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right)}{\mu_X \left(1 + e'_{x(st)} \right) + \mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right)} \right) + w_{11} \left(\mu_X \left(1 + e'_{x(st)} \right) - \mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right) \right) \right), \quad (26)$$

$$t_{g(2)} = \frac{\mu_Y \left(1 + \tilde{e}''^{*}_{y(st)} \right)}{2} \left(\frac{\mu_X \left(1 + e'_{x(st)} \right)}{\mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right)} + \frac{\mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right)}{\mu_X \left(1 + e'_{x(st)} \right)} + \exp\left(\frac{\mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right) - \mu_X \left(1 + e'_{x(st)} \right)}{\mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right) + \mu_X \left(1 + e'_{x(st)} \right)} \right) \right) w_{21} \left(\mu_X \left(1 + e'_{x(st)} \right) - \mu_X \left(1 + \tilde{e}''^{*}_{x(st)} \right) \right), \quad (27)$$

and

$$t_{g(3)} = \frac{\mu_Y\left(1+\tilde{e}''^{*}_{y(st)}\right)}{2}\left(\exp\left(\frac{\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^{*}_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)+\mu_X\left(1+\tilde{e}''^{*}_{x(st)}\right)}\right)+\frac{\mu_X\left(1+e'_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)}\right)+w_{31}\left(\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^{*}_{x(st)}\right)\right), \quad (28)$$

or alternatively:

$$t_{g(1)}-\mu_Y=\mu_Y\left(\tilde{e}''^{*}_{y(st)}+\frac{e'^{2}_{x(st)}}{8}+\frac{\tilde{e}''^{*2}_{x(st)}}{8}-\frac{\tilde{e}''^{*}_{x(st)}e'_{x(st)}}{4}\right)+w_{11}\mu_X\left(e'_{x(st)}-\tilde{e}''^{*}_{x(st)}\right), \quad (29)$$

$$t_{g(2)}-\mu_Y=\mu_Y\left(\tilde{e}''^{*}_{y(st)}+\frac{e'^{2}_{1st}}{2}+\frac{\tilde{e}''^{*}_{y(st)}}{2}-\tilde{e}''^{*}_{x(st)}e'_{1st}\right)+w_{21}\mu_X\left(e'_{x(st)}-\tilde{e}''^{*}_{x(st)}\right), \quad (30)$$

and

$$t_{g(3)}-\mu_Y=\mu_Y\left(\tilde{e}''^{*}_{y(st)}+\frac{3}{4}e'_{x(st)}-\frac{3}{4}\tilde{e}''^{*}_{x(st)}-\frac{e'^{2}_{x(st)}}{16}+\frac{11}{16}\tilde{e}''^{*2}_{x(st)}-\frac{5}{8}\tilde{e}''^{*}_{x(st)}e'_{x(st)}+\frac{3}{4}\tilde{e}''^{*}_{y(st)}e'_{x(st)}-\frac{3}{4}\tilde{e}''^{*}_{y(st)}\tilde{e}''^{*}_{x(st)}\right)+w_{31}\mu_X\left(e'_{x(st)}-\tilde{e}''^{*}_{x(st)}\right). \quad (31)$$

The expressions for the biases and *MSEs* of $t_{g(i)}$ up to the order $O\left(n^{-1}\right)$ are respectively given by:

$$Bias\left(t_{g(1)}\right)=\mu_Y\frac{\tilde{A}^{*}_{x(st)}}{8}, \quad (32)$$

$$Bias\left(t_{g(2)}\right)=\mu_Y\frac{\tilde{A}^{*}_{x(st)}}{2}, \quad (33)$$

$$Bias\left(t_{g(3)}\right)=\mu_Y\left(\frac{11}{16}\tilde{A}^{*}_{x(st)}-\frac{3}{4}\tilde{C}^{*}_{xy(st)}\right), \quad (34)$$

and

$$MSE\left(t_{g(1)}\right)=MSE\left(t_{g(2)}\right)=(\mu_Y^2\tilde{A}''^{*}_{y(st)}+w_{11}^2\mu_X^2\tilde{A}^{*}_{x(st)}-2\mu_Y\mu_Xw_{11}\tilde{C}^{*}_{xy(st)}), \quad (35)$$

$$MSE\left(t_{g(3)}\right)=\mu_Y^2\left(\tilde{A}''^{*}_{y(st)}+\frac{9}{16}\tilde{A}^{*}_{x(st)}-\frac{3}{2}\tilde{C}^{*}_{xy(st)}\right)+w_{31}^2\mu_X^2\tilde{A}^{*}_{x(st)}+2\mu_Y\mu_Xw_{31}\left(\frac{3}{4}\tilde{A}^{*}_{x(st)}-\tilde{C}^{*}_{xy(st)}\right). \quad (36)$$

Now, to get the optimum values of w_{11}, w_{21}, and w_{31}, Eqs. (35) and (36) are differentiated partially with respect to w_{11}, w_{21}, and w_{31}, and then equating each of the first derivatives with zero. This gives three normal equations which are then solved simultaneously for the optimum values of w_{11}, w_{21}, and w_{31}. Finally, the optimum values are shown by:

$$w_{11}^{opt}=w_{21}^{opt}=\frac{\mu_Y\tilde{C}^{*}_{xy(st)}}{\mu_X\tilde{A}^{*}_{x(st)}}$$

and

$$w_{31}^{opt}=-\frac{\mu_Y\left(\frac{3}{4}\tilde{A}^{*}_{x(st)}-\tilde{C}^{*}_{xy(st)}\right)}{\mu_X\tilde{A}^{*}_{x(st)}}. \quad (37)$$

Substituting the optimum values of w_{11}, w_{21} and w_{31} in Eqs. (35) and (36), the expression of the minimum *MSE* is obtained. However, it is to mention that the minimum *MSE* expression is same for each of the three estimators, and it is given by:

$$\min MSE\left(t_{g(i)}\right)=\mu_Y^2\tilde{A}''^{*}_{y(st)}\left(1-\frac{\tilde{C}^{*2}_{xy(st)}}{\tilde{A}^{*}_{x(st)}\tilde{A}''^{*}_{y(st)}}\right). \quad (38)$$

3.3. Procedure III: Proposed efficient and generalized combined regression-cum-ratio type estimators

The proposed modified regression estimator $t_{g(i)}$ presented in the preceding section, can be taken as an alternate to the regression estimator. However, the proposed estimator $t_{g(i)}$ can further be molded into another form of combined regression-cum-ratio type estimator so as to get more efficient and more generalized estimators than usual combined regression estimator and the modified combined regression estimator presented in the preceding sections.

Therefore, now three new classes of more efficient and generalized combined regression-cum-ratio estimators $\tilde{t}^{*}_{s(i)(\alpha,\beta)}$, where $i=1,2,3$ are proposed for estimating the population mean, and form of the proposed estimator is given by:

$$\tilde{t}^*_{s(1)(\alpha,\beta)} = \Bigg(\tilde{\mu}''^*_{y(st)} \left(\alpha \frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}} + (1-\alpha)\frac{\tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)}}\right) + \tilde{w}^*_{1(1)}(\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}) + \tilde{w}^*_{2(1)}\tilde{\mu}''^*_{y(st)}\Bigg) \left(\beta \frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}} + (1-\beta)\frac{\tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)}}\right), \quad (39)$$

$$\tilde{t}^*_{s(2)(\alpha,\beta)} = \Bigg(\tilde{\mu}''^*_{y(st)} \left(\alpha \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\alpha)\exp\left(\frac{\tilde{\mu}''^*_{x(st)} - \mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)} + \mu'_{x(st)}}\right)\right) + \tilde{w}^*_{1(2)}(\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}) + \tilde{w}^*_{2(2)}\tilde{\mu}''^*_{y(st)}\Bigg) \left(\beta \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\beta)\exp\left(\frac{\tilde{\mu}''^*_{x(st)} - \mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)} + \mu'_{x(st)}}\right)\right), \quad (40)$$

and

$$\tilde{t}^*_{s(3)(\alpha,\beta)} = \Bigg(\tilde{\mu}''^*_{y(st)} \left(\alpha \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\alpha)\frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}}\right) + \tilde{w}^*_{1(3)}\left(\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}\right) + \tilde{w}^*_{2(3)}\tilde{\mu}''^*_{y(st)}\Bigg) \left(\beta \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\beta)\frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}}\right), \quad (41)$$

or alternatively:

$$\tilde{t}^*_{s(i)(\alpha,\beta)} = (\tilde{\mu}''^*_{y(st)}\tilde{H}^*_{(i)\alpha} + \tilde{w}^*_{1(i)}(\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}) + \tilde{w}^*_{2(i)}\tilde{\mu}''^*_{y(st)})\tilde{H}^*_{(i)\beta} \quad \text{for} \quad i=1,2,3, \quad (42)$$

where

$$\tilde{H}^*_{(1)\alpha} = \alpha\frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}} + (1-\alpha)\frac{\tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)}},$$

$$\tilde{H}^*_{(1)\beta} = \beta\frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}} + (1-\beta)\frac{\tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)}},$$

$$\tilde{H}^*_{(2)\alpha} = \alpha \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\alpha)\exp\left(\frac{\tilde{\mu}''^*_{x(st)} - \mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)} + \mu'_{x(st)}}\right),$$

$$\tilde{H}^*_{(2)\beta} = \beta \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\beta)\exp\left(\frac{\tilde{\mu}''^*_{x(st)} - \mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)} + \mu'_{x(st)}}\right),$$

$$\tilde{H}^*_{(3)\alpha} = \alpha \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\alpha)\frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}},$$

and

$$\tilde{H}^*_{(3)\beta} = \beta \exp\left(\frac{\mu'_{x(st)} - \tilde{\mu}''^*_{x(st)}}{\mu'_{x(st)} + \tilde{\mu}''^*_{x(st)}}\right) + (1-\beta)\frac{\mu'_{x(st)}}{\tilde{\mu}''^*_{x(st)}},$$

where $\alpha \in [0,1]$ and $\beta \in [0,1]$ are the generalizing constants whose values are suitably chosen, and $\tilde{w}^*_{1(i)}$ and $\tilde{w}^*_{2(i)}$ are the optimizing constants which are needed to be estimated such that the optimum values of $\tilde{w}^*_{1(i)}$ and $\tilde{w}^*_{2(i)}$ give the minimum MSE value to each estimator which belongs to the proposed class of regression-cum-ratio estimators.

3.3.1. Derivation of the biases and MSEs of the proposed combined regression-cum-ratio estimators

In order to obtain the expressions for the biases and the *MSEs* of the proposed regression-cum-ratio estimators $\tilde{t}^*_{s(1)(\alpha,\beta)}$, $\tilde{t}^*_{s(2)(\alpha,\beta)}$ and $\tilde{t}^*_{s(3)(\alpha,\beta)}$ are given in terms of e,s respectively by:

$$\tilde{t}^*_{s(1)(\alpha,\beta)} = \Bigg(\mu_Y\left(1+\tilde{e}''^*_{y(st)}\right)\left(\alpha\frac{\mu_X\left(1+e'_{x(st)}\right)}{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)} + (1-\alpha)\frac{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)}\right) + \tilde{w}^*_{1(1)}\left(\mu_X\left(1+e'_{x(st)}\right) - \mu_X\left(1+\tilde{e}''^*_{x(st)}\right)\right) + \tilde{w}^*_{2(1)}\mu_Y\left(1+\tilde{e}''^*_{y(st)}\right)\Bigg) \left(\beta\frac{\mu_X\left(1+e'_{x(st)}\right)}{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)} + (1-\beta)\frac{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)}\right), \quad (43)$$

$$\tilde{t}^*_{s(2)(\alpha,\beta)} = \Bigg(\mu_Y\left(1+\tilde{e}''^*_{y(st)}\right)$$

$$\left(\alpha\exp\left(\frac{\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)+\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}\right)\right.$$

$$\left.+(1-\alpha)\exp\left(\frac{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)-\mu_X\left(1+e'_{x(st)}\right)}{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)-\mu_X\left(1+e'_{x(st)}\right)}\right)\right)$$

$$\tilde{w}^*_{1(2)}\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)\Big)$$

$$+\tilde{w}^*_{2(2)}\mu_Y\left(1+\tilde{e}''^*_{y(st)}\right)\Big)$$

$$\left(\beta\exp\left(\frac{\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)+\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}\right)+\right.$$

$$\left.(1-\beta)\exp\left(\frac{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)-\mu_X\left(1+e'_{x(st)}\right)}{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)-\mu_X\left(1+e'_{x(st)}\right)}\right)\right)\Bigg), \quad (44)$$

and

$$\tilde{t}^*_{s(3)(\alpha,\beta)} = \Bigg(\mu_Y\left(1+\tilde{e}''^*_{y(st)}\right)$$

$$\left(\alpha\exp\left(\frac{\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)+\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}\right)\right.$$

$$\left.+(1-\alpha)\frac{\mu_X\left(1+e'_{x(st)}\right)}{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}\right)$$

$$+\tilde{w}^*_{1(3)}\left(\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)\right)$$

$$+\tilde{w}^*_{2(3)}\mu_Y\left(1+\tilde{e}''^*_{y(st)}\right)\Big)$$

$$\left(\beta\exp\left(\frac{\mu_X\left(1+e'_{x(st)}\right)-\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}{\mu_X\left(1+e'_{x(st)}\right)+\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}\right)\right.$$

$$\left.+(1-\beta)\frac{\mu_X\left(1+e'_{x(st)}\right)}{\mu_X\left(1+\tilde{e}''^*_{x(st)}\right)}\right). \quad (45)$$

Further simplification of the above expressions up to the first order of approximation $O(n^{-1})$ gives the expressions, as given by:

$$\tilde{t}^*_{s(1)(\alpha,\beta)}-\mu_Y=\mu_Y(\tilde{e}''^*_{y(st)}+2\tilde{e}''^*_{x(st)}-2e'_{x(st)}$$
$$+2\alpha e'_{x(st)}-2\alpha\tilde{e}''^*_{x(st)}-2\beta\tilde{e}''^*_{x(st)}+2\beta e'_{x(st)}$$
$$-\alpha\tilde{e}''^{*2}_{x(st)}-3\alpha e'^2_{x(st)}+4\alpha e'_{x(st)}\tilde{e}''^*_{x(st)}-\beta\tilde{e}''^{*2}_{x(st)}$$
$$-3\beta e'^2_{x(st)}+4\beta e'_{x(st)}\tilde{e}''^*_{x(st)}+\tilde{e}''^{*2}_{x(st)}+3e'^2_{x(st)}$$
$$-4e'_{x(st)}\tilde{e}''^*_{x(st)}+4\alpha\beta\tilde{e}''^{*2}_{x(st)}+4\alpha\beta e'^2_{x(st)}$$
$$+8\alpha\beta e'_{x(st)}\tilde{e}''^*_{x(st)}+2\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}-2\tilde{e}''^*_{y(st)}e'_{x(st)}$$
$$-2\alpha e'_{x(st)}\tilde{e}''^*_{x(st)}+2\alpha\tilde{e}''^*_{y(st)}e'_{x(st)}-2\beta\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}$$
$$+2\beta\tilde{e}''^*_{y(st)}e'_{x(st)}+\tilde{w}^*_{2(1)}(1+\tilde{e}''^*_{y(st)}+\tilde{e}''^*_{x(st)}$$
$$-e'_{x(st)}-2\beta\tilde{e}''^*_{x(st)}+2\beta e'_{x(st)}+\beta\tilde{e}''^{*2}_{x(st)}-\beta e'^2_{x(st)}$$
$$+\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}-\tilde{e}''^*_{y(st)}e'_{x(st)}-2\beta\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}$$
$$+2\beta\tilde{e}''^*_{y(st)}e'_{x(st)}+e'^2_{x(st)}-e'_{x(st)}\tilde{e}''^*_{x(st)}))$$
$$-\tilde{w}^*_{1(1)}\bar{X}(\tilde{e}''^*_{x(st)}-e'_{x(st)}+\tilde{e}''^{*2}_{x(st)}-2\beta e'^2_{x(st)}$$
$$+4\beta e'_{x(st)}\tilde{e}''^*_{x(st)}), \quad (46)$$

$$\tilde{t}_s(2)(\alpha,\beta)^*-\mu_Y=\mu_Y(\tilde{e}''^*_{y(st)}$$
$$+\tilde{e}''^*_{x(st)}-e'_{x(st)}+\alpha e'_{x(st)}-\alpha\tilde{e}''^*_{x(st)}-\beta\tilde{e}''^*_{x(st)}$$
$$+\alpha\tilde{e}''^*_{y(st)}e'_{x(st)}-\beta\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}+\beta\tilde{e}''^*_{y(st)}e'_{x(st)}$$
$$+\beta e'_{x(st)}-\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}+\tilde{e}''^*_{y(st)}e'_{x(st)}$$
$$-\alpha\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}\alpha\beta e''^{*2}_{x(st)}+\alpha\beta e'^2_{x(st)}-2\alpha\beta e'_{x(st)}\tilde{e}''^*_{x(st)}$$
$$+\tilde{w}^*_{2(2)}\Bigg(1+\tilde{e}''^*_{y(st)}+\frac{\tilde{e}''^*_{x(st)}}{2}-\frac{e'_{x(st)}}{2}$$
$$-\beta\tilde{e}''^*_{x(st)}-\frac{\tilde{e}''^*_{x(st)}}{8}-\frac{3e'^2_{x(st)}}{8}-\frac{e'_{x(st)}\tilde{e}''^*_{x(st)}}{4}+\frac{\beta e''^{*2}_{x(st)}}{2}$$
$$-\frac{\beta e'^2_{x(st)}}{2}-\beta\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}+\beta\tilde{e}''^*_{y(st)}e'_{x(st)}$$
$$-\frac{\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)}}{2}--\tilde{w}^*_{1(2)}\mu_X\left(\tilde{e}''^*_{x(st)}-e'_{x(st)}+\frac{\tilde{e}''^{*2}_{x(st)}}{2}\right.$$
$$-\frac{e'^2_{x(st)}}{2}+e'_{x(st)}\tilde{e}''^*_{x(st)}+\beta\tilde{e}''^{*2}_{x(st)}+\beta e'^2_{x(st)}$$
$$\left.-2\beta e'_{x(st)}\tilde{e}''^*_{x(st)}\right), \quad (47)$$

and

$$\tilde{t}^*_{s(3)(\alpha,\beta)} - \mu_Y = \mu_Y \Big(\tilde{e}''^*_{y(st)} + 2\tilde{e}''^*_{x(st)} - 2e'_{x(st)} + \frac{\alpha}{2}e'_{x(st)} - \frac{\alpha}{2}\tilde{e}''^*_{x(st)} - \frac{\beta}{2}e'_{x(st)} + \frac{\beta}{2}\tilde{e}''^*_{x(st)} + 3\tilde{e}''^{*2}_{x(st)} + e'^2_{x(st)} + 4e'_{x(st)}\tilde{e}''^*_{x(st)} - \frac{9}{8}\alpha\tilde{e}''^{*2}_{x(st)} - \frac{5}{8}\alpha e'^2_{x(st)} + \frac{7}{4}\alpha e'_{x(st)}\tilde{e}''^*_{x(st)} + \frac{9}{8}\beta\tilde{e}''^{*2}_{x(st)} - \frac{5}{8}\beta e'^2_{x(st)} + \frac{7}{4}\beta e'^*_{x(st)}\tilde{e}''^*_{x(st)} - 2\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)} + 2\tilde{e}''^*_{y(st)}e'_{x(st)} + \frac{\alpha}{2}\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)} - \frac{\alpha}{2}\tilde{e}''^*_{y(st)}e'_{x(st)} + \frac{\beta}{2}\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)} - \frac{\beta}{2}\tilde{e}''^*_{y(st)}e'_{x(st)}\frac{\alpha\beta}{4}\tilde{e}''^{*2}_{x(st)} + \frac{\alpha\beta}{4}e'^2_{x(st)} + \frac{\alpha\beta}{2}e'_{x(st)}\tilde{e}''^*_{x(st)} + \tilde{w}^*_{2(3)}\Big(1+\tilde{e}''^*_{y(st)} - \tilde{e}''^*_{x(st)} + e'_{x(st)} + \frac{\beta}{2}\tilde{e}''^*_{x(st)} - \frac{\beta}{2}e'_{x(st)} + \frac{\beta}{2}\tilde{e}''^*_{y(st)}\tilde{e}''^*_{x(st)} - \frac{\beta}{2}\tilde{e}''^*_{y(st)}e'_{x(st)} - \frac{5}{8}\beta\tilde{e}''^{*2}_{x(st)} - \frac{\beta e'^2_{x(st)}}{8} + \frac{3}{4}\beta e'_{x(st)}\tilde{e}''^*_{x(st)}\Big) - \tilde{w}^*_{1(3)}\mu_X\Big(\tilde{e}''^*_{x(st)} - e'_{x(st)} + \tilde{e}''^{*2}_{x(st)} + e'^2_{x(st)} - 2e'_{x(st)}\tilde{e}''^*_{x(st)} - \beta\frac{\tilde{e}''^{*2}_{x(st)}}{2} - \beta\frac{e'^2_{x(st)}}{2} + \beta e'_{x(st)}\tilde{e}''^*_{x(st)}\Big). \tag{48}$$

The expressions for the biases of $\tilde{t}^*_{s(1)(\alpha,\beta)}$, $\tilde{t}^*_{s(2)(\alpha,\beta)}$ and $\tilde{t}^*_{s(3)(\alpha,\beta)}$ are obtained up to the order of approximation $O(n^{-1})$, taking the expectation of Eqs. (46)–(48) respectively. The expressions of the biases are given respectively by:

$$Bias(\tilde{t}^*_{s(1)(\alpha,\beta)}) = \mu_Y\big(\tilde{A}^*_{x(st)} - \alpha\tilde{A}^*_{x(st)} - \beta\tilde{A}^*_{x(st)} + 4\alpha\beta\tilde{A}^*_{x(st)} + 2\tilde{C}^*_{xy(st)} - 2\alpha\tilde{C}^*_{xy(st)} - 2\beta\tilde{C}^*_{xy(st)} + \tilde{w}^*_{2(1)}\big(1 + \beta\tilde{A}^*_{x(st)} + \tilde{C}^*_{xy(st)} - 2\beta\tilde{C}^*_{xy(st)}\big)\big) - \tilde{w}^*_{1(1)}\mu_X(\tilde{A}^*_{x(st)} - 2\beta\tilde{A}^*_{x(st)}), \tag{49}$$

$$Bias(\tilde{t}^*_{s(2)(\alpha,\beta)}) = \mu_Y\Big(\alpha\beta\tilde{A}^*_{x(st)} + \tilde{C}^*_{xy(st)} - \alpha\tilde{C}^*_{xy(st)} - \beta\tilde{C}^*_{xy(st)} + \tilde{w}^*_{2(2)}\Big(\beta\frac{\tilde{A}^*_{x(st)}}{2} - \frac{\tilde{A}^*_{x(st)}}{8} + \frac{\tilde{C}^*_{xy(st)}}{2} - \beta\tilde{C}^*_{xy(st)}\Big)\Big) + \tilde{w}^*_{1(2)}\mu_X\Big(\beta\tilde{A}^*_{x(st)} - \frac{\tilde{A}^*_{x(st)}}{2}\Big), \tag{50}$$

and

$$Bias(\tilde{t}^*_{s(3)(\alpha,\beta)}) = \mu_Y\Big(3\tilde{A}^*_{x(st)} - \frac{9}{8}\alpha\tilde{A}^*_{x(st)} - \frac{9}{8}\beta\tilde{A}^*_{x(st)} + \frac{\alpha\beta}{4}\tilde{A}^*_{x(st)} - 2\tilde{C}^*_{xy(st)} + \frac{\alpha}{2}\tilde{C}^*_{xy(st)} + \frac{\beta}{2}\tilde{C}^*_{xy(st)} + \tilde{w}^*_{2(3)}\Big(1 + \frac{\beta}{2}\tilde{C}^*_{xy(st)} - \tilde{C}^*_{xy(st)} - \frac{5}{8}\beta\tilde{A}^*_{x(st)}\Big)\Big) + \tilde{w}^*_{1(3)}\mu_X\Big(\tilde{A}^*_{x(st)} - \frac{\beta}{2}\tilde{A}^*_{x(st)}\Big). \tag{51}$$

The expressions for the $MSEs$ of $\tilde{t}^*_{s(1)(\alpha,\beta)}$, $\tilde{t}^*_{s(2)(\alpha,\beta)}$ and $\tilde{t}^*_{s(3)(\alpha,\beta)}$ are obtained taking the square of Eqs. (46)–(48) respectively, retaining the terms up to the order of approximation $O(n^{-1})$, and then taking expectation. Finally, general expressions of the $MSEs$ of $\tilde{t}^*_{s(1)(\alpha,\beta)}$, $\tilde{t}^*_{s(2)(\alpha,\beta)}$ and $\tilde{t}^*_{s(3)(\alpha,\beta)}$ are given respectively by:

$$MSE\left(\tilde{t}^*_{s(1)(\alpha,\beta)}\right) = \mu_Y^2\Big(\tilde{A}''^*_{y(st)} + 4\tilde{A}^*_{x(st)} - 8\alpha\tilde{A}^*_{x(st)} - 8\beta\tilde{A}^*_{x(st)} + 4\alpha^2\tilde{A}^*_{x(st)} + 4\beta^2\tilde{A}^*_{x(st)} + 8\alpha\beta\tilde{A}^*_{x(st)} + 4\tilde{C}^*_{xy(st)} - 4\alpha\tilde{C}^*_{xy(st)} - 4\beta\tilde{C}^*_{xy(st)} + \tilde{w}^*_{2(1)}\Big(2\tilde{A}''^*_{y(st)} + 10\tilde{C}^*_{xy(st)} - 12\beta\tilde{C}^*_{xy(st)} + 6\tilde{A}^*_{x(st)} - 6\alpha\tilde{A}^*_{x(st)} - 14\beta\tilde{A}^*_{x(st)} + 16\alpha\beta\tilde{A}^*_{x(st)} + 8\beta^2\tilde{A}^*_{x(st)} - 8\alpha\tilde{C}^*_{xy(st)}\Big) + \tilde{w}^{*2}_{2(1)}\Big(1 + \tilde{A}''^*_{y(st)} + \tilde{A}^*_{x(st)} + 2\beta\tilde{A}^*_{x(st)} + 4\tilde{C}^*_{xy(st)} - 8\beta\tilde{C}^*_{xy(st)} + 4\beta^2\tilde{A}^*_{x(st)}\Big)\Big) + \tilde{w}^*_{1(1)}\mu_X^2\tilde{A}^*_{x(st)} + 2\mu_Y\mu_X\tilde{w}^*_{1(1)}$$

$$\left(2\alpha\tilde{A}^*_{x(st)}+2\beta\tilde{A}^*_{x(st)}-2\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}+\tilde{w}^*_{2(1)}\left(4\beta\tilde{A}^*_{x(st)}-2\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}\right)\right), \quad (52)$$

$$MSE(\tilde{t}^*_{s(2)(\alpha,\beta)})=\mu_Y^2\Big(\tilde{A}''^*_{y(st)}+\tilde{A}^*_{x(st)}-2\alpha\tilde{A}^*_{x(st)}-2\beta\tilde{A}^*_{x(st)}+\alpha^2\tilde{A}^*_{x(st)}+\beta^2\tilde{A}^*_{x(st)}+2\alpha\beta\tilde{A}^*_{x(st)}+2\tilde{C}^*_{xy(st)}-2\alpha\tilde{C}^*_{xy(st)}-2\beta\tilde{C}^*_{xy(st)}+\tilde{w}^*_{2(2)}(2A''_{y(st)}+\tilde{A}^*_{x(st)}+5\tilde{C}^*_{xy(st)}-4\alpha\tilde{C}^*_{xy(st)}-6\beta\tilde{C}^*_{xy(st)}-\alpha\tilde{A}^*_{x(st)}+4\alpha\beta\tilde{A}^*_{x(st)}+2\beta^2\tilde{A}^*_{x(st)}-3\beta\tilde{A}^*_{x(st)}\Big)+\tilde{w}^{*2}_{2(2)}(1+\tilde{A}''^*_{y(st)}+2\tilde{C}^*_{xy(st)}-4\beta\tilde{C}^*_{xy(st)}+4\beta^2\tilde{A}^*_{x(st)}\Big)\Big)+\tilde{w}^*_{1(2)}\mu_X^2\tilde{A}^*_{x(st)}+2\mu_Y\mu_X\tilde{w}^*_{1(2)}\Big(\alpha\tilde{A}^*_{x(st)}+\beta\tilde{A}^*_{x(st)}-\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}+\tilde{w}^*_{2(2)}\left(2\beta\tilde{A}^*_{x(st)}-\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}\right)\Big), \quad (53)$$

and

$$MSE\left(\tilde{t}^*_{s(3)(\alpha,\beta)}\right)=\mu_Y^2\Big(\tilde{A}''^*_{y(st)}+4\tilde{A}^*_{x(st)}-2\alpha\tilde{A}^*_{x(st)}-2\beta\tilde{A}^*_{x(st)}+\frac{\alpha^2}{4}\tilde{A}^*_{x(st)}+\frac{\beta^2}{4}\tilde{A}^*_{x(st)}+\frac{\alpha\beta}{2}\tilde{A}^*_{x(st)}-4\tilde{C}^*_{xy(st)}+\alpha\tilde{C}^*_{xy(st)}+\beta\tilde{C}^*_{xy(st)}+\tilde{w}^*_{2(3)}\Big(2\tilde{A}''^*_{y(st)}+10\tilde{C}^*_{xy(st)}+3\beta\tilde{C}^*_{xy(st)}+10\tilde{A}^*_{x(st)}-6\alpha\tilde{A}^*_{x(st)}-\frac{21}{4}\beta\tilde{A}^*_{x(st)}-\frac{13}{4}\alpha\tilde{A}^*_{x(st)}+\frac{\beta^2}{2}\tilde{A}^*_{x(st)}+\alpha\beta\tilde{A}^*_{x(st)}+2\alpha\tilde{C}^*_{xy(st)}\Big)+\tilde{w}^{*2}_{2(3)}\Big(1+\tilde{A}''^*_{y(st)}+\tilde{A}^*_{x(st)}-\frac{9}{4}\beta\tilde{A}^*_{x(st)}-4\tilde{C}^*_{xy(st)}+2\beta\tilde{C}^*_{xy(st)}+\frac{\beta^2}{4}\tilde{A}^*_{x(st)}\Big)\Big)+\tilde{w}^{*2}_{1(3)}\mu_X^2\tilde{A}^*_{x(st)}+2\mu_Y\mu_X\tilde{w}^*_{1(3)}\Big(2\tilde{A}^*_{x(st)}-\frac{\alpha}{2}.\tilde{A}^*_{x(st)}-\frac{\beta}{2}\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}+\tilde{w}^*_{2(3)}\left(2\tilde{A}^*_{x(st)}-\beta\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}\right)\Big). \quad (54)$$

Alternatively, the expressions $MSEs$ of $\tilde{t}^*_{s(1)(\alpha,\beta)}$, $\tilde{t}^*_{s(2)(\alpha,\beta)}$ and $\tilde{t}^*_{s(3)(\alpha,\beta)}$ can be given by the expression of MSE of $\tilde{t}^*_{s(i)(\alpha,\beta)}$, and it is expressed by:

$$MSE\left(\tilde{t}^*_{s(i)(\alpha,\beta)}\right)=\tilde{\Psi}^*_{0(i)}+\tilde{w}^{*2}_{2(i)}\tilde{\Psi}^*_{1(i)}+\tilde{w}^*_{2(i)}\tilde{\Psi}^*_{2(i)}+\tilde{w}^{*2}_{1(i)}\tilde{\Psi}^*_{3(i)}+\tilde{w}^*_{1(i)}\tilde{w}^*_{2(i)}\tilde{\Psi}^*_{4(i)}+\tilde{w}^*_{1(i)}\tilde{\Psi}^*_{5(i)}, \quad (55)$$

where

$$\tilde{\Psi}^*_{0(1)}=\mu_Y^2\Big(\tilde{A}''^*_{y(st)}+4\tilde{A}^*_{x(st)}-8\alpha\tilde{A}^*_{x(st)}-8\beta\tilde{A}^*_{x(st)}+4\alpha^2\tilde{A}^*_{x(st)}+4\beta^2\tilde{A}^*_{x(st)}+8\alpha\beta\tilde{A}^*_{x(st)}+4\tilde{C}^*_{xy(st)}-4\alpha\tilde{C}^*_{xy(st)}-4\beta\tilde{C}^*_{xy(st)}\Big),$$

$$\tilde{\Psi}^*_{1(1)}=\mu_Y^2\Big(1+\tilde{A}''^*_{y(st)}+\tilde{A}^*_{x(st)}+2\beta\tilde{A}^*_{x(st)}+4\tilde{C}^*_{xy(st)}-8\beta\tilde{C}^*_{xy(st)}+4\beta^2\tilde{A}^*_{x(st)}\Big),$$

$$\tilde{\Psi}^*_{2(1)}=\mu_Y^2\Big(2\tilde{A}''^*_{y(st)}+10\tilde{C}^*_{xy(st)}-12\beta\tilde{C}^*_{xy(st)}+6\tilde{A}^*_{x(st)}-6\alpha\tilde{A}^*_{x(st)}-14\beta\tilde{A}^*_{x(st)}+16\alpha\beta\tilde{A}^*_{x(st)}+8\beta^2\tilde{A}^*_{x(st)}-8\alpha\tilde{C}^*_{xy(st)}\Big),$$

$$\tilde{\Psi}^*_{3(1)}=\mu_X^2\tilde{A}^*_{x(st)},$$

$$\tilde{\Psi}^*_{4(1)}=2\mu_Y\mu_X\left(4\beta\tilde{A}^*_{x(st)}-2\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}\right),$$

$$\tilde{\Psi}^*_{5(1)}=2\mu_Y\mu_X\left(2\alpha\tilde{A}^*_{x(st)}+2\beta\tilde{A}^*_{x(st)}-2\tilde{A}^*_{x(st)}-\tilde{C}^*_{xy(st)}\right),$$

$$\tilde{\Psi}^*_{0(2)}=\mu_Y^2\Big(\tilde{A}''^*_{y(st)}+\tilde{A}^*_{x(st)}-2\alpha\tilde{A}^*_{x(st)}-2\beta\tilde{A}^*_{x(st)}+\alpha^2\tilde{A}^*_{x(st)}+\beta^2\tilde{A}^*_{x(st)}+2\alpha\beta\tilde{A}^*_{x(st)}+2\tilde{C}^*_{xy(st)}-2\alpha\tilde{C}^*_{xy(st)}-2\beta\tilde{C}^*_{xy(st)}\Big),$$

$$\tilde{\Psi}^*_{1(2)}=\mu_Y^2\Big(1+\tilde{A}''^*_{y(st)}+2\tilde{C}^*_{xy(st)}-4\beta\tilde{C}^*_{xy(st)}+4\beta^2\tilde{A}^*_{x(st)}\Big),$$

$$\tilde{\Psi}^*_{2(2)} = \mu_Y^2 \Big(2A''_{y(st)} + \tilde{A}^*_{x(st)} + 5\tilde{C}^*_{xy(st)} - 4\alpha\tilde{C}^*_{xy(st)} - 6\beta\tilde{C}^*_{xy(st)} - \alpha\tilde{A}^*_{x(st)} + 4\alpha\beta\tilde{A}^*_{x(st)} + 2\beta^2\tilde{A}^*_{x(st)} - 3\beta\tilde{A}^*_{x(st)} \Big),$$

$$\tilde{\Psi}^*_{3(2)} = \mu_X^2 \tilde{A}^*_{x(st)},$$

$$\tilde{\Psi}^*_{4(2)} = 2\mu_Y\mu_X \left(2\beta\tilde{A}^*_{x(st)} - \tilde{A}^*_{x(st)} - \tilde{C}^*_{xy(st)} \right),$$

$$\tilde{\Psi}^*_{5(2)} = 2\mu_Y\mu_X \left(\alpha\tilde{A}^*_{x(st)} + \beta\tilde{A}^*_{x(st)} - \tilde{A}^*_{x(st)} - \tilde{C}^*_{xy(st)} \right),$$

$$\tilde{\Psi}^*_{0(3)} = \mu_Y^2 \Big(A''_{y(st)} + 4B_{(st)} - 2\alpha B_{(st)} - 2\beta B_{(st)} + \frac{\alpha^2}{4}B_{(st)} + \frac{\beta^2}{4}B_{(st)} + \frac{\alpha\beta}{2}B_{(st)} - 4C_{(st)} + \alpha C_{(st)} + \beta C_{(st)} \Big),$$

$$\tilde{\Psi}^*_{1(3)} = \mu_Y^2 \Big(1 + A''_{y(st)} + B_{(st)} - \frac{9}{4}\beta B_{(st)} - 4v_{11} + 2\beta v_{11} + \frac{\beta^2}{4}B_{(st)} \Big),$$

$$\tilde{\Psi}^*_{2(3)} = \mu_Y^2 \Big(2A''_{y(st)} + 10C_{(st)} + 3\beta C_{(st)} + 10B_{(st)} - 6\alpha B_{(st)} - \frac{21}{4}\beta B_{(st)} - \frac{13}{4}\alpha B_{(st)} + \frac{\beta^2}{2}B_{(st)} + \alpha\beta B_{(st)} + 2\alpha C_{(st)} \Big),$$

$$\tilde{\Psi}^*_{3(3)} = \mu_X^2 B_{(st)},$$

$$\tilde{\Psi}^*_{4(3)} = 2\mu_Y\mu_X \left(2B_{(st)} - \beta B_{(st)} - C_{(st)} \right),$$

and

$$\tilde{\Psi}^*_{5(3)} = 2\mu_Y\mu_X \left(2B_{(st)} - \frac{\alpha}{2} B_{(st)} - \frac{\beta}{2}B_{(st)} - C_{(st)} \right).$$

Now, to get the optimum values, Eq. (55) is differentiated partially with respect to $\tilde{w}^*_{1(i)}$ and $\tilde{w}^*_{2(i)}$, and then equating each of the first derivatives with zero. This gives three systems of normal equations, including two normal equations in each system. For each system normal equations are then solved simultaneously to get the optimum values of $\tilde{w}^*_{1(i)}$ and $\tilde{w}^*_{2(i)}$. Finally, the optimum values are shown by:

$$\tilde{w}^{*\,opt}_{1(i)} = \frac{\tilde{\Psi}^*_{2(i)}\tilde{\Psi}^*_{4(i)} - 2\tilde{\Psi}^*_{1(i)}\tilde{\Psi}^*_{5(i)}}{4\tilde{\Psi}^*_{1(i)}\tilde{\Psi}^*_{3(i)} - \tilde{\Psi}^{*\,2}_{4(i)}},$$

and

$$\tilde{w}^{*\,opt}_{2(i)} = \frac{\tilde{\Psi}^*_{4(i)}\tilde{\Psi}^*_{5(i)} - 2\tilde{\Psi}^*_{2(i)}\tilde{\Psi}^*_{3(i)}}{4\tilde{\Psi}^*_{1(i)}\tilde{\Psi}^*_{3(i)} - \tilde{\Psi}^{*\,2}_{4(i)}}. \tag{56}$$

Substituting the optimum values of $\tilde{w}^*_{1(i)}$ and $\tilde{w}^*_{2(i)}$ in Eq. (55), the expression of the minimum *MSE* of $\tilde{t}^*_{s(i)(\alpha,\beta)}$ is obtained as:

$$\min MSE\left(\tilde{t}^*_{s(i)(\alpha,\beta)}\right) = \tilde{\Psi}^*_{0(i)} - \frac{\left(\tilde{\Psi}^*_{2(i)}\tilde{\Psi}^*_{4(i)}\tilde{\Psi}^*_{5(i)} - \tilde{\Psi}^{*2}_{2(i)}\tilde{\Psi}^*_{3(i)} - \tilde{\Psi}^*_{1(i)}\tilde{\Psi}^{*2}_{5(i)}\right)}{\Psi^2_{4(i)} - 4\Psi_{1(i)}\Psi_{3(i)}},$$

$$\text{for} \quad i = 1, 2, 3. \tag{57}$$

3.3.2. Theoretical comparisons between the modified combined regression estimator and the generalized regression-cum-ratio estimators

$$MSE\left(\tilde{t}^*_{reg}\right) - MSE\left(\tilde{t}^*_{s(i)(\alpha,\beta)}\right) > 0,$$

$$\mu_Y^2\tilde{A}''^*_{y(st)}\left(1 - \rho^2_{xy(st)}\right) > MSE\left(\tilde{t}^*_{s(i)(\alpha,\beta)}\right),$$

$$\left(1 - \rho^2_{xy(st)}\right) > \frac{MSE\left(\tilde{t}^*_{s(i)(\alpha,\beta)}\right)}{\mu_Y^2\tilde{A}''^*_{y(st)}},$$

$$\rho^2_{xy(st)} < 1 - \frac{MSE\left(\tilde{t}^*_{s(i)(\alpha,\beta)}\right)}{\mathrm{var}\left(\tilde{t}^*_u\right)}. \tag{58}$$

Remark 1

When only the measurement error is present on the study and auxiliary variables, and complete response is available on both of the variables, then modifications to the estimation procedures presented in the preceding sections are followed by:

$$\tilde{\mu}''^*_{y(st)} \to \tilde{\mu}''_{y(st)} = \frac{\sum_{h=1}^{L}\tilde{y}''_{hi}}{n''_h},$$

and then the expression of the variance is given by:

$$\mathrm{var}\left(\tilde{\mu}''_{y(st)}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h}\left(C^2_{Yh} + \frac{\sigma^2_{Uh}}{\mu_Y^2}\right)\right) = \tilde{\tau}''_{y(st)}.$$

Similarly, sample mean estimator along with the variance expression can take the forms respectively as given by:

$$\tilde{\mu}''^*_{x(st)} \to \tilde{\mu}''_{x(st)} = \frac{\sum_{h=1}^{L}\tilde{x}''_{hi}}{n''_h},$$

and

$$\text{var}\left(\tilde{\mu}''_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h}\left(C_{Xh}^2 + \frac{\sigma_{Vh}^2}{\mu_X^2}\right)\right) = \tilde{\tau}''_{x(st)}.$$

Expressions for the different covariance terms are reproduced, and given by:

$$\text{cov}\left(\tilde{\mu}''_{y(st)}, \tilde{\mu}''_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h}\, \rho_{YXh} C_{Yh} C_{Xh}\right) = \tilde{\pi}_{xy(st)},$$

$$\text{cov}\left(\tilde{\mu}''_{x(st)}, {\mu'}_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \lambda_h C_{Xh}^2 = A'_{x(st)},$$

and

$$\text{cov}\left(\tilde{\mu}''_{y(st)}, {\mu'}_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_h\, \rho_{YXh} C_{Yh} C_{Xh}\right) = C'_{xy(st)},$$

where $\tilde{\tau}_{x(st)} = \tilde{\tau}''_{x(st)} - A'_{x(st)}$ and $\tilde{\pi}_{xy(st)} = \tilde{\pi}''_{xy(st)} - C'_{xy(st)}$.

The proposed estimators of Eq. (42) are reproduced under Remark 1, and can be given by:

$$\tilde{t}_{s(i)(\alpha,\beta)} = (\tilde{\mu}''_{y(st)} \tilde{H}_{(i)\alpha} + \tilde{w}_{1(i)}({\mu'}_{x(st)} - \tilde{\mu}''_{x(st)}) + \tilde{w}_{2(i)} \tilde{\mu}''_{y(st)}) \tilde{H}_{(i)\beta} \quad \text{for} \quad i = 1, 2, 3, \quad (59)$$

where

$$\tilde{H}_{(1)\alpha} = \alpha \frac{{\mu'}_{x(st)}}{\tilde{\mu}''_{x(st)}} + (1-\alpha) \frac{\tilde{\mu}''_{x(st)}}{{\mu'}_{x(st)}},$$

$$\tilde{H}_{(1)\beta} = \beta \frac{{\mu'}_{x(st)}}{\tilde{\mu}''_{x(st)}} + (1-\beta) \frac{\tilde{\mu}''_{x(st)}}{{\mu'}_{x(st)}},$$

$$\tilde{H}_{(2)\alpha} = \alpha \exp\left(\frac{{\mu'}_{x(st)} - \tilde{\mu}''_{x(st)}}{{\mu'}_{x(st)} + \tilde{\mu}''_{x(st)}}\right) + (1-\alpha) \exp\left(\frac{\tilde{\mu}''_{x(st)} - {\mu'}_{x(st)}}{\tilde{\mu}''_{x(st)} + {\mu'}_{x(st)}}\right),$$

$$\tilde{H}_{(2)\beta} = \beta \exp\left(\frac{{\mu'}_{x(st)} - \tilde{\mu}''_{x(st)}}{{\mu'}_{x(st)} + \tilde{\mu}''_{x(st)}}\right) + (1-\beta) \exp\left(\frac{\tilde{\mu}''_{x(st)} - {\mu'}_{x(st)}}{\tilde{\mu}''_{x(st)} + {\mu'}_{x(st)}}\right),$$

$$\tilde{H}_{(3)\alpha} = \alpha \exp\left(\frac{{\mu'}_{x(st)} - \tilde{\mu}''_{x(st)}}{{\mu'}_{x(st)} + \tilde{\mu}''_{x(st)}}\right) + (1-\alpha) \frac{{\mu'}_{x(st)}}{\tilde{\mu}''_{x(st)}},$$

and

$$\tilde{H}_{(3)\beta} = \beta \exp\left(\frac{{\mu'}_{x(st)} - \tilde{\mu}''_{x(st)}}{{\mu'}_{x(st)} + \tilde{\mu}''_{x(st)}}\right) + (1-\beta) \frac{{\mu'}_{x(st)}}{\tilde{\mu}''_{x(st)}}.$$

The expressions of the biases are reproduced for the estimators, and given respectively by:

$$\begin{aligned} Bias\left(\tilde{t}_{s(1)(\alpha,\beta)}\right) &= \mu_Y (\tilde{\tau}_{x(st)} - \alpha \tilde{\tau}_{x(st)} - \beta \tilde{\tau}_{x(st)} \\ &+ 4\alpha\beta \tilde{\tau}_{x(st)} + 2\tilde{\pi}_{xy(st)} - 2\alpha \tilde{\pi}_{xy(st)} - 2\beta \tilde{\pi}_{xy(st)} \\ &+ \tilde{w}_{2(1)} \left(1 + \beta\, \tilde{\tau}_{x(st)} + \tilde{\pi}_{xy(st)} - 2\beta\, \tilde{\pi}_{xy(st)}\right)) \\ &- \tilde{w}_{1(1)} \mu_X \left(\tilde{\tau}_{x(st)} - 2\beta \tilde{\tau}_{x(st)}\right), \end{aligned} \quad (60)$$

$$\begin{aligned} &Bias\left(\tilde{t}_{s(2)(\alpha,\beta)}\right) = \\ &\mu_Y \Bigg(\alpha\beta \tilde{\tau}_{x(st)} + \tilde{\pi}_{xy(st)} - \alpha \tilde{\pi}_{xy(st)} - \beta \tilde{\pi}_{xy(st)} \\ &+ \tilde{w}_{2(2)} \left(\beta \frac{\tilde{\tau}_{x(st)}}{2} - \frac{\tilde{\tau}_{x(st)}}{8} + \frac{\tilde{\pi}_{xy(st)}}{2} - \beta \tilde{\pi}_{xy(st)}\right)\Bigg) \\ &\tilde{w}_{1(2)} \mu_X \left(\beta \tilde{\tau}_{x(st)} - \frac{\tilde{A} \tilde{\tau}_{x(st)}}{2}\right), \end{aligned} \quad (61)$$

and

$$\begin{aligned} Bias\left(\tilde{t}_{s(3)(\alpha,\beta)}\right) &= \mu_Y \Bigg(3\tilde{\tau}_{x(st)} - \frac{9}{8} \alpha \tilde{\tau}_{x(st)} - \frac{9}{8} \beta \tilde{\tau}_{x(st)} \\ &+ \frac{\alpha\beta}{4} \tilde{\tau}_{x(st)} - 2\tilde{\pi}_{xy(st)} + \frac{\alpha}{2} \tilde{\pi}_{xy(st)} + \frac{\beta}{2} \tilde{\pi}_{xy(st)} \\ &+ \tilde{w}_{2(3)} \left(1 + \frac{\beta}{2}\, \tilde{\pi}_{xy(st)} - \tilde{\pi}_{xy(st)} - \frac{5}{8} \beta\, \tilde{\tau}_{x(st)}\right)\Bigg) \\ &+ \tilde{w}_{1(3)} \mu_X \left(\tilde{\tau}_{x(st)} - \frac{\beta}{2} \tilde{\tau}_{x(st)}\right). \end{aligned} \quad (62)$$

The *MSEs* of class of three estimator $\tilde{t}_{s(i)(\alpha,\beta)}$ are expressed by:

$$MSE\left(\tilde{t}_{s(i)(\alpha,\beta)}\right) = \tilde{\Theta}_{0(i)} + \tilde{w}_{2(i)}^2 \tilde{\Theta}_{1(i)} + \tilde{w}_{2(i)} \tilde{\Theta}_{2(i)} + \tilde{w}_{1(i)}^2 \tilde{\Theta}_{3(i)} + \tilde{w}_{1(i)} \tilde{w}_{2(i)} \tilde{\Theta}_{4(i)} + \tilde{w}_{1(i)} \tilde{\Theta}_{5(i)}, \quad (63)$$

where

$$\begin{aligned} \tilde{\Theta}_{0(1)} &= \mu_Y^2 \Big(\tilde{\tau}''_{y(st)} + 4\tilde{\tau}_{x(st)} - 8\alpha \tilde{\tau}_{x(st)} - 8\beta \tilde{\tau}_{x(st)} \\ &+ 4\alpha^2 \tilde{\tau}_{x(st)} + 4\beta^2 \tilde{\tau}_{x(st)} + 8\alpha\beta \tilde{\tau}_{x(st)} \\ &+ 4\tilde{\pi}_{xy(st)} - 4\alpha \tilde{\pi}_{xy(st)} - 4\beta \tilde{\pi}_{xy(st)}\Big), \end{aligned}$$

$$\tilde{\Theta}_{1(1)} = \mu_Y^2 (1+\tilde{\tau}''_{y(st)} + \tilde{\tau}_{x(st)} + 2\beta\tilde{\tau}_{x(st)} + 4\tilde{\pi}_{xy(st)} - 8\beta\tilde{\pi}_{xy(st)} + 4\beta^2 \tilde{\tau}_{x(st)}),$$

$$\tilde{\Theta}_{2(1)} = \mu_Y^2 \left(2\tilde{\tau}''_{y(st)} + 10\tilde{\pi}_{xy(st)} - 12\beta\tilde{\pi}_{xy(st)} + 6\tilde{\tau}_{x(st)} - 6\alpha\tilde{\tau}_{x(st)} - 14\beta\tilde{\tau}_{x(st)} + 16\alpha\beta\tilde{\tau}_{x(st)} + 8\beta^2\tilde{\tau}_{x(st)} - 8\alpha\tilde{\pi}_{xy(st)}\right),$$

$$\tilde{\Theta}_{3(1)} = \mu_X^2 \tilde{\tau}_{x(st)},$$

$$\tilde{\Theta}_{4(1)} = 2\mu_Y\mu_X \left(4\beta\tilde{\tau}_{x(st)} - 2\tilde{\tau}_{x(st)} - \tilde{\pi}_{xy(st)}\right),$$

$$\tilde{\Theta}_{5(1)} = 2\mu_Y\mu_X \left(2\alpha\, \tilde{\tau}_{x(st)} + 2\beta\tilde{\tau}_{x(st)} - 2\tilde{\tau}_{x(st)} - \tilde{\pi}_{xy(st)}\right),$$

$$\tilde{\Theta}_{0(2)} = \mu_Y^2 \left(\tilde{\tau}''_{y(st)} + \tilde{\tau}_{x(st)} - 2\alpha\tilde{\tau}_{x(st)} - 2\beta\tilde{\tau}_{x(st)} + \alpha^2\tilde{\tau}_{x(st)} + \beta^2\tilde{\tau}_{x(st)} + 2\alpha\beta\tilde{\tau}_{x(st)} + 2\tilde{\pi}_{xy(st)} - 2\alpha\tilde{\pi}_{xy(st)} - 2\beta\tilde{\pi}_{xy(st)}\right),$$

$$\tilde{\Theta}_{1(2)} = \mu_Y^2 (1 + \tilde{\tau}''_{y(st)} + 2\tilde{\pi}_{xy(st)} - 4\beta\tilde{\pi}_{xy(st)} + 4\beta^2\tilde{\tau}_{x(st)}),$$

$$\tilde{\Theta}_{2(2)} = \mu_Y^2 (2\tilde{\tau}''_{y(st)} + \tilde{\tau}_{x(st)} + 5\tilde{\pi}_{xy(st)} - 4\alpha\tilde{\pi}_{xy(st)} - 6\beta\tilde{\pi}_{xy(st)} - \alpha\tilde{\tau}_{x(st)} + 4\alpha\beta\tilde{\tau}_{x(st)} + 2\beta^2\tilde{\tau}_{x(st)} - 3\beta\tilde{\tau}_{x(st)}),$$

$$\tilde{\Theta}_{3(2)} = \mu_X^2 \tilde{\tau}_{x(st)},$$

$$\tilde{\Theta}_{4(2)} = 2\mu_Y\mu_X \left(2\beta\tilde{\tau}_{x(st)} - \tilde{\tau}_{x(st)} - \tilde{\pi}_{xy(st)}\right),$$

$$\tilde{\Theta}_{5(2)} = 2\mu_Y\mu_X (\alpha\, \tilde{\tau}_{x(st)} + \beta\, \tilde{\tau}_{x(st)} - \tilde{\tau}_{x(st)} - \tilde{\pi}_{xy(st)}),$$

$$\tilde{\Theta}_{0(3)} = \mu_Y^2 \left(\tilde{\tau}''_{y(st)} + 4\tilde{\tau}_{x(st)} - 2\alpha\tilde{\tau}_{x(st)} - 2\beta\tilde{\tau}_{x(st)} + \frac{\alpha^2}{4}\tilde{\tau}_{x(st)} + \frac{\beta^2}{4}\tilde{\tau}_{x(st)} + \frac{\alpha\beta}{2}\tilde{\tau}_{x(st)} - 4\tilde{\pi}_{xy(st)} + \alpha\tilde{\pi}_{xy(st)} + \beta\tilde{\pi}_{xy(st)}\right),$$

$$\tilde{\Theta}_{1(3)} = \mu_Y^2 \left(1+\tilde{\tau}''_{y(st)} + \tilde{\tau}_{x(st)} - \frac{9}{4}\beta\tilde{\tau}_{x(st)} - 4\tilde{\pi}_{xy(st)} + 2\beta\tilde{\pi}_{xy(st)} + \frac{\beta^2}{4}\tilde{\tau}_{x(st)}\right),$$

$$\tilde{\Theta}_{2(3)} = \mu_Y^2 \left(2\,\tilde{\tau}''_{y(st)} + 10\tilde{\pi}_{xy(st)} + 3\beta\tilde{\pi}_{xy(st)} + 10\tilde{\tau}_{x(st)} - 6\alpha\tilde{\tau}_{x(st)} - \frac{21}{4}\beta\tilde{\tau}_{x(st)} - \frac{13}{4}\alpha\tilde{\tau}_{x(st)} + \frac{\beta^2}{2}\tilde{\tau}_{x(st)} + \alpha\beta\,\tilde{\tau}_{x(st)} + 2\alpha\tilde{\pi}_{xy(st)}\right),$$

$$\tilde{\Theta}_{3(3)} = \mu_X^2 \tilde{\tau}_{x(st)},$$

$$\tilde{\Theta}_{4(3)} = 2\mu_Y\mu_X \left(2\,\tilde{\tau}_{x(st)} - \beta\,\tilde{\tau}_{x(st)} - \tilde{\pi}_{xy(st)}\right),$$

and

$$\tilde{\Theta}_{5(3)} = 2\mu_Y\mu_X \left(2\tilde{\tau}_{x(st)} - \frac{\alpha}{2}\,\tilde{\tau}_{x(st)} - \frac{\beta}{2}\tilde{\tau}_{x(st)} - \tilde{\pi}_{xy(st)}\right).$$

The optimum values of $\tilde{w}_{1(i)}$ and $\tilde{w}_{2(i)}$ from Eq. (63) are reproduced under Remark 1, and given by:

$$\tilde{w}_{1(i)}^{opt} = \frac{\tilde{\Theta}_{2(i)}\tilde{\Theta}_{4(i)} - 2\tilde{\Theta}_{1(i)}\tilde{\Theta}_{5(i)}}{4\tilde{\Theta}_{1(i)}\tilde{\Theta}_{3(i)} - \tilde{\Theta}_{4(i)}^2},$$

and

$$\tilde{w}_{2(i)}^{opt} = \frac{\tilde{\Theta}_{4(i)}\tilde{\Theta}_{5(i)} - 2\tilde{\Theta}_{2(i)}\tilde{\Theta}_{3(i)}}{4\tilde{\Theta}_{1(i)}\tilde{\Theta}_{3(i)} - \tilde{\Theta}_{4(i)}^2}. \quad (64)$$

Substituting the optimum values of $\tilde{w}_{1(i)}$ and $\tilde{w}_{2(i)}$ in Eq. (63), the expression of the minimum MSE of $\tilde{t}_{s(i)(\alpha,\beta)}$ is obtained as:

$$\min MSE\left(\tilde{t}_{s(i)(\alpha,\beta)}\right) = \tilde{\Theta}_{0(i)} - \frac{\left(\tilde{\Theta}_{2(i)}\tilde{\Theta}_{4(i)}\tilde{\Theta}_{5(i)} - \tilde{\Theta}_{2(i)}^2\tilde{\Theta}_{3(i)} - \tilde{\Theta}_{1(i)}\tilde{\Theta}_{5(i)}^2\right)}{\tilde{\Theta}_{4(i)}^2 - 4\tilde{\Theta}_{1(i)}\tilde{\Theta}_{3(i)}}. \quad (65)$$

Remark 2

When it is assumed that only non-response is present on the study and auxiliary variables, but no measurement error exists on the study and auxiliary variables, then modifications in the estimation procedure are followed by: $\tilde{\mu}''^{*}_{y(st)} \rightarrow \mu''^{*}_{y(st)} = \frac{\sum_{h=1}^{L} y^*_{hi}}{n''_h}$ and then the expression of the variance becomes var $\left(\mu''^{*}_{y(st)}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h} C_{Yh}^2 + \theta_{2h} C_{Yh(2)}^2\right) = \Omega''^{*}_{y(st)}$.

Similarly, sample mean estimator for the auxiliary variable can take the form $\tilde{\mu}''^{*}_{x(st)} \to \mu''^{*}_{x(st)} = \frac{\sum_{h=1}^{L} x''^{*}_{hi}}{n''_h}$, and the expression of the variance can be reproduced as:

$$\mathrm{var}\left(\tilde{\mu}''^{*}_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_{2h} C_{Xh}^2 + \theta_{2h} C_{Xh(2)}^2\right) = \Omega''^{*}_{x(st)},$$

and the expressions of the covariance by:

$$\mathrm{cov}\left(\tilde{\mu}''^{*}_{y(st)}, \tilde{\mu}''^{*}_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 (\lambda_{2h} \rho_{YXh} C_{Yh} C_{Xh} + \theta_{2h} \rho_{YXh(2)} C_{Yh(2)} C_{Xh(2)}) = \vartheta''^{*}_{xy(st)}, \mathrm{cov}\left(\tilde{\mu}''^{*}_{x(st)}, \mu'_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \lambda_h C_{Xh}^2 = A'_{x(st)},$$

and

$$\mathrm{cov}\left(\mu''^{*}_{y(st)}, \mu'_{x(st)}\right) = \sum_{h=1}^{L} P_h^2 \left(\lambda_h \rho_{YXh} C_{Yh} C_{Xh}\right) = C'_{xy(st)}$$

where $\Omega^{*}_{x(st)} = \Omega''^{*}_{x(st)} - A'_{x(st)}$ and $\vartheta^{*}_{xy(st)} = \vartheta''^{*}_{xy(st)} - C'_{xy(st)}$.

Following the assumption stated in Remark 2, the proposed estimators of Eq. (42) are reduced to the form given by:

$$t^{*}_{s(i)(\alpha,\beta)} = (\mu''^{*}_{y(st)} H^{*}_{(i)\alpha} + w^{*}_{1(i)}(\mu'_{x(st)} - \mu''^{*}_{x(st)}) + w^{*}_{2(i)} \mu''^{*}_{y(st)}) H^{*}_{(i)\beta} \quad \text{for} \quad i = 1,2,3, \tag{66}$$

where,

$$H^{*}_{(1)\alpha} = \alpha \frac{\mu'_{x(st)}}{\mu''^{*}_{x(st)}} + (1-\alpha) \frac{\mu''^{*}_{x(st)}}{\mu'_{x(st)}},$$

$$H^{*}_{(1)\beta} = \beta \frac{\mu'_{x(st)}}{\mu''^{*}_{x(st)}} + (1-\beta) \frac{\mu''^{*}_{x(st)}}{\mu'_{x(st)}},$$

$$H^{*}_{(2)\alpha} = \alpha \exp\left(\frac{\mu'_{x(st)} - \mu''^{*}_{x(st)}}{\mu'_{x(st)} + \mu''^{*}_{x(st)}}\right) + (1-\alpha) \exp\left(\frac{\mu''^{*}_{x(st)} - \mu'_{x(st)}}{\mu''^{*}_{x(st)} + \mu'_{x(st)}}\right),$$

$$H^{*}_{(2)\beta} = \beta \exp\left(\frac{\mu'_{x(st)} - \mu''^{*}_{x(st)}}{\mu'_{x(st)} + \mu''^{*}_{x(st)}}\right) + (1-\beta) \exp\left(\frac{\mu''^{*}_{x(st)} - \mu'_{x(st)}}{\mu''^{*}_{x(st)} + \mu'_{x(st)}}\right),$$

$$H^{*}_{(3)\alpha} = \alpha \exp\left(\frac{\mu'_{x(st)} - \mu''^{*}_{x(st)}}{\mu'_{x(st)} + \mu''^{*}_{x(st)}}\right) + (1-\alpha) \frac{\mu'_{x(st)}}{\mu''^{*}_{x(st)}},$$

and

$$H^{*}_{(3)\beta} = \beta \exp\left(\frac{\mu'_{x(st)} - \mu''^{*}_{x(st)}}{\mu'_{x(st)} + \mu''^{*}_{x(st)}}\right) + (1-\beta) \frac{\mu'_{x(st)}}{\mu''^{*}_{x(st)}},$$

$$Bias\left(t^{*}_{s(1)(\alpha,\beta)}\right) = \mu_Y \Big(\Omega^{*}_{x(st)} - \alpha\Omega^{*}_{x(st)} - \beta\Omega^{*}_{x(st)} + 4\alpha\beta\Omega^{*}_{x(st)} + 2\vartheta^{*}_{xy(st)} - 2\alpha\vartheta^{*}_{xy(st)} - 2\beta\vartheta^{*}_{xy(st)} + w^{*}_{2(1)} \left(1 + \beta\,\Omega^{*}_{x(st)} + \vartheta^{*}_{xy(st)} - 2\beta\vartheta^{*}_{xy(st)}\right)\Big) - w_{1(1)}\mu_X \left(\Omega^{*}_{x(st)} - 2\beta\Omega^{*}_{x(st)}\right), \tag{67}$$

$$Bias\left(t^{*}_{s(2)(\alpha,\beta)}\right) = \mu_Y \Big(\alpha\beta\,\Omega^{*}_{x(st)} + \vartheta^{*}_{xy(st)} - \alpha\vartheta^{*}_{xy(st)} - \beta\vartheta^{*}_{xy(st)} + w^{*}_{2(2)} \left(\beta\frac{\Omega^{*}_{x(st)}}{2} - \frac{\Omega^{*}_{x(st)}}{8} + \frac{\vartheta^{*}_{xy(st)}}{2} - \beta\vartheta^{*}_{xy(st)}\right)\Big) w^{*}_{1(2)}\mu_X \left(\beta\,\Omega^{*}_{x(st)} - \frac{\Omega^{*}_{x(st)}}{2}\right), \tag{68}$$

$$Bias\left(t^{*}_{s(3)(\alpha,\beta)}\right) = \mu_Y \Big(3\Omega^{*}_{x(st)} - \frac{9}{8}\alpha\Omega^{*}_{x(st)} - \frac{9}{8}\beta\Omega^{*}_{x(st)} + \frac{\alpha\beta}{4}\Omega^{*}_{x(st)} - 2\,\vartheta^{*}_{xy(st)} + \frac{\alpha}{2}\vartheta^{*}_{xy(st)} + \frac{\beta}{2}\vartheta^{*}_{xy(st)} + w^{*}_{2(3)} \left(1 + \frac{\beta}{2}\,\vartheta^{*}_{xy(st)} - \vartheta^{*}_{xy(st)} - \frac{5}{8}\beta\,\Omega^{*}_{x(st)}\right)\Big) + w^{*}_{1(3)}\mu_X \left(\Omega^{*}_{x(st)} - \frac{\beta}{2}\Omega^{*}_{x(st)}\right). \tag{69}$$

The *MSEs* of $t^{*}_{s(i)(\alpha,\beta)}$ is expressed by:

$$MSE\left(t^{*}_{s(i)(\alpha,\beta)}\right) = \mathrm{K}^{*}_{0(i)} + w^{*2}_{2(i)}\mathrm{K}^{*}_{1(i)} + w^{*}_{2(i)}\mathrm{K}^{*}_{2(i)} + w^{*}_{1(i)}\mathrm{K}^{*}_{3(i)} + w^{*}_{1(i)} w^{*}_{2(i)}\mathrm{K}^{*}_{4(i)} + w^{*}_{1(i)}\mathrm{K}^{*}_{5(i)}, \tag{70}$$

where

$$\mathrm{K}^*_{0(1)} = \mu_Y^2 \Big(\Omega''^*_{y(st)} + 4\,\Omega^*_{x(st)} - 8\alpha\,\Omega^*_{x(st)} - 8\beta\,\Omega^*_{x(st)} + 4\alpha^2\,\Omega^*_{x(st)} + 4\beta^2\,\Omega^*_{x(st)} + 8\alpha\beta\,\Omega^*_{x(st)} + 4\,\vartheta^*_{xy(st)} - 4\alpha\vartheta^*_{xy(st)} - 4\beta\vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{1(1)} = \mu_Y^2 \Big(1 + \Omega''^*_{y(st)} + \Omega^*_{x(st)} + 2\beta\Omega^*_{x(st)} + 4\,\vartheta^*_{xy(st)} - 8\beta\vartheta^*_{xy(st)} + 4\beta^2\Omega^*_{x(st)} \Big),$$

$$\mathrm{K}^*_{2(1)} = \mu_Y^2 \Big(2\Omega''^*_{y(st)} + 10C^*_{xy(st)} - 12\beta C^*_{xy(st)} + 6\,\Omega^*_{x(st)} - 6\alpha\Omega^*_{x(st)} - 14\beta\Omega^*_{x(st)} + 16\alpha\beta\Omega^*_{x(st)} + 8\beta^2\Omega^*_{x(st)} - 8\alpha\vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{3(1)} = \mu_X^2\,\Omega^*_{x(st)},$$

$$\mathrm{K}^*_{4(1)} = 2\mu_Y\mu_X \Big(4\beta\,\Omega^*_{x(st)} - 2\,\Omega^*_{x(st)} - \vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{5(1)} = 2\mu_Y\mu_X \Big(2\alpha\Omega^*_{x(st)} + 2\beta\Omega^*_{x(st)} - 2\Omega^*_{x(st)} - \vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{0(2)} = \mu_Y^2 \Big(\Omega''^*_{y(st)} + \Omega^*_{x(st)} - 2\alpha\,\Omega^*_{x(st)} - 2\beta\,\Omega^*_{x(st)} + \alpha^2\,\Omega^*_{x(st)} + \beta^2\,\Omega^*_{x(st)} + 2\alpha\beta\Omega^*_{x(st)} + 2\vartheta^*_{xy(st)} - 2\alpha\vartheta^*_{xy(st)} - 2\beta\vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{1(2)} = \mu_Y^2 \Big(1 + \Omega''^*_{y(st)} + 2\,\vartheta^*_{xy(st)} - 4\beta\,\vartheta^*_{xy(st)} + 4\beta^2\,\Omega^*_{x(st)} \Big),$$

$$\mathrm{K}^*_{2(2)} = \mu_Y^2 \Big(2\Omega''^*_{y(st)} + \Omega^*_{x(st)} + 5\,\vartheta^*_{xy(st)} - 4\alpha\vartheta^*_{xy(st)} - 6\beta\,\vartheta^*_{xy(st)} - \alpha\,\Omega^*_{x(st)} + 4\alpha\beta\,\Omega^*_{x(st)} + 2\beta^2\,\Omega^*_{x(st)} - 3\beta\,\Omega^*_{x(st)} \Big),$$

$$\mathrm{K}^*_{3(2)} = \mu_X^2\Omega^*_{x(st)},$$

$$\mathrm{K}^*_{4(2)} = 2\mu_Y\mu_X \Big(2\beta\,\Omega^*_{x(st)} - \Omega^*_{x(st)} - \vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{5(2)} = 2\mu_Y\mu_X \Big(\alpha\Omega^*_{x(st)} + \beta\,\Omega^*_{x(st)} - \Omega^*_{x(st)} - \vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{0(3)} = \mu_Y^2 \Big(\Omega''^*_{y(st)} + 4\,\Omega^*_{x(st)} - 2\alpha\,\Omega^*_{x(st)} - 2\beta\Omega^*_{x(st)} + \frac{\alpha^2}{4}\Omega^*_{x(st)} + \frac{\beta^2}{4}\Omega^*_{x(st)} + \frac{\alpha\beta}{2}\Omega^*_{x(st)} - 4\vartheta^*_{xy(st)} + \alpha\vartheta^*_{xy(st)} + \beta\,\vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{1(3)} = \mu_Y^2 \Big(1 + \Omega''^*_{y(st)} + \Omega^*_{x(st)} - \frac{9}{4}\beta\Omega^*_{x(st)} - 4\vartheta^*_{xy(st)} + 2\beta\vartheta^*_{xy(st)} + \frac{\beta^2}{4}\Omega^*_{x(st)} \Big),$$

$$\mathrm{K}^*_{2(3)} = \mu_Y^2 \Big(2\,\Omega''^*_{y(st)} + 10\,\vartheta^*_{xy(st)} + 3\beta\,\vartheta^*_{xy(st)} + 10\,\Omega^*_{x(st)} - 6\alpha\Omega^*_{x(st)} - \frac{21}{4}\beta\Omega^*_{x(st)} - \frac{13}{4}\alpha\,\Omega^*_{x(st)} + \frac{\beta^2}{2}\,\Omega^*_{x(st)} + \alpha\beta\,\Omega^*_{x(st)} + 2\alpha\vartheta^*_{xy(st)} \Big),$$

$$\mathrm{K}^*_{3(3)} = \mu_X^2\,\Omega^*_{x(st)},$$

$$\mathrm{K}^*_{4(3)} = 2\mu_Y\mu_X \Big(2\,\Omega^*_{x(st)} - \beta\,\Omega^*_{x(st)} - \vartheta^*_{xy(st)} \Big),$$

and

$$\mathrm{K}^*_{5(3)} = 2\mu_Y\mu_X \Big(2\Omega^*_{x(st)} - \frac{\alpha}{2}\Omega^*_{x(st)} - \frac{\beta}{2}\Omega^*_{x(st)} - \vartheta^*_{xy(st)} \Big).$$

The optimum values of $w^*_{1(i)}$ and $w^*_{2(i)}$ from Eq. (70) are reproduced under Remark 2, and given by:

$$w^{*\,opt}_{1(i)} = \frac{\mathrm{K}^*_{2(i)}\mathrm{K}^*_{4(i)} - 2\mathrm{K}^*_{1(i)}\mathrm{K}^*_{5(i)}}{4\mathrm{K}^*_{1(i)}\mathrm{K}^*_{3(i)} - \mathrm{K}^{*2}_{4(i)}}$$

and

$$w_{2(i)}^{*\,opt}=\frac{K^*_{4(i)}K^*_{5(i)}-2K^*_{2(i)}K^*_{3(i)}}{4K^*_{1(i)}K^*_{3(i)}-K^{*2}_{4(i)}} \quad (71)$$

Substituting the optimum values of $w^*_{1(i)}$ and $w^*_{2(i)}$ in Eq. (70), the expression of the minimum MSE of $t^*_{s(i)(\alpha,\beta)}$ is obtained as:

$$\min MSE\left(t^*_{s(i)(\alpha,\beta)}\right)=K^*_{0(i)}-\frac{\left(K^*_{2(i)}K^*_{4(i)}K^*_{5(i)}-K^{*2}_{2(i)}K^*_{3(i)}-K^*_{1(i)}K^{*2}_{5(i)}\right)}{K^{*2}_{4(i)}-4K^*_{1(i)}K^*_{3(i)}}. \quad (72)$$

Remark 3

When there is no non-response and no measurement error in both the study and auxiliary variables, the modifications to estimation procedures presented in the preceding sections can be modified accordingly. For example, a sample mean estimator of the study variable along with the expression of the variance can take the form as given by:

$\tilde{\mu}''^*_{y(st)} \rightarrow \mu''_{y(st)}=\frac{\sum_{h=1}^{L} y_{hi}}{n''_h}$ and $\mathrm{var}\left(\mu''_{y(st)}\right)=\sum_{h=1}^{L}P_h^2\left(\lambda_{2h}C_{Yh}^2\right)=\eta''_{y(st)}$.

Similarly, for the auxiliary variable, sample mean estimator is stated as: $\tilde{\mu}''^*_{x(st)} \rightarrow \mu''_{x(st)}=\frac{\sum_{h=1}^{L} x''_{hi}}{n''_h}$, and an expression of its variance is given by: $\mathrm{var}\left(\mu''_{x(st)}\right)=\sum_{h=1}^{L}P_h^2\left(\lambda_{2h}C_{Xh}^2\right)=\eta''_{x(st)}$. The expressions of the different covariances are reduced to the forms $\mathrm{cov}\left(\mu''_{y(st)}\mu''_{x(st)}\right)=\sum_{h=1}^{L}P_h^2\left(\lambda_{2h}\,\rho_{YXh}C_{Yh}C_{Xh}\right)=\phi''_{xy(st)}$, $\mathrm{cov}\left(\mu''_{x(st)}\mu'_{x(st)}\right)=\sum_{h=1}^{L}P_h^2\lambda_hC_{Xh}^2=A'_{x(st)}$, and $\mathrm{cov}\left(\mu''_{y(st)}\mu'_{x(st)}\right)=\sum_{h=1}^{L}P_h^2\left(\lambda_h\,\rho_{YXh}C_{Yh}C_{Xh}\right)=C'_{xy(st)}$, where $\eta_{x(st)}=\eta''_{x(st)}-A'_{x(st)}$ and $\phi_{xy(st)}=\phi''_{xy(st)}-C'_{xy(st)}$.

The proposed estimators of Eq. (42) are reproduced under Remark 3, and the estimator is given by:

$$t_{s(i)(\alpha,\beta)}=\left(\mu''_{y(st)}H_{(i)\alpha}+w_{1(i)}\left(\mu'_{x(st)}-\mu''_{x(st)}\right)+w_{2(i)}\mu''_{y(st)}\right)H_{(i)\beta} \quad \text{for} \quad i=1,2,3, \quad (73)$$

where

$$H_{(1)\alpha}=\alpha\frac{\mu'_{x(st)}}{\mu''_{x(st)}}+(1-\alpha)\frac{\mu''_{x(st)}}{\mu'_{x(st)}},$$

$$H_{(1)\beta}=\beta\frac{\mu'_{x(st)}}{\mu''_{x(st)}}+(1-\beta)\frac{\mu''_{x(st)}}{\mu'_{x(st)}},$$

$$H_{(2)\alpha}=\alpha\exp\left(\frac{\mu'_{x(st)}-\mu''_{x(st)}}{\mu'_{x(st)}+\mu''_{x(st)}}\right)+(1-\alpha)\exp\left(\frac{\mu''_{x(st)}-\mu'_{x(st)}}{\mu''_{x(st)}+\mu'_{x(st)}}\right),$$

$$H_{(2)\beta}=\beta\exp\left(\frac{\mu'_{x(st)}-\mu''_{x(st)}}{\mu'_{x(st)}+\mu''_{x(st)}}\right)+(1-\beta)\exp\left(\frac{\mu''_{x(st)}-\mu'_{x(st)}}{\mu''_{x(st)}+\mu'_{x(st)}}\right),$$

$$H_{(3)\alpha}=\alpha\exp\left(\frac{\mu'_{x(st)}-\mu''_{x(st)}}{\mu'_{x(st)}+\mu''_{x(st)}}\right)+(1-\alpha)\frac{\mu'_{x(st)}}{\mu''_{x(st)}},$$

and

$$H_{(3)\beta}=\beta\exp\left(\frac{\mu'_{x(st)}-\mu''_{x(st)}}{\mu'_{x(st)}+\mu''_{x(st)}}\right)+(1-\beta)\frac{\mu'_{x(st)}}{\mu''_{x(st)}},$$

$$\begin{aligned}Bias\left(t_{s(1)(\alpha,\beta)}\right)&=\mu_Y\big(\eta_{x(st)}-\alpha\eta_{x(st)}-\beta\eta_{x(st)}\\&+4\alpha\beta\eta_{x(st)}+2\phi_{xy(st)}-2\alpha\phi_{xy(st)}\\&-2\beta\phi_{xy(st)}+w_{2(1)}(1+\beta\,\eta_{x(st)}+\phi_{xy(st)}\\&-2\beta\phi_{xy(st)})\big)-w_{1(1)}\mu_X\left(\eta_{x(st)}-2\beta\eta_{x(st)}\right),\end{aligned} \quad (74)$$

$$\begin{aligned}Bias\left(t_{s(2)(\alpha,\beta)}\right)=\mu_Y\Big(&\alpha\beta\eta_{x(st)}\\&+\phi_{xy(st)}-\alpha\phi_{xy(st)}-\beta\phi_{xy(st)}+w_{2(2)}\\&\left(\beta\frac{\eta_{x(st)}}{2}-\frac{\eta_{x(st)}}{8}+\frac{\phi_{xy(st)}}{2}-\beta\,\phi_{xy(st)}\right)\Big)\\&w_{1(2)}\mu_X\left(\beta\eta_{x(st)}-\frac{\eta_{x(st)}}{2}\right)\end{aligned} \quad (75)$$

$$Bias\left(t_{s(3)(\alpha,\beta)}\right)=\mu_Y\left(3\eta_{x(st)}-\frac{9}{8}\alpha\eta_{x(st)}-\frac{9}{8}\beta\,\eta_{x(st)}\right.$$

$$+\frac{\alpha\beta}{4}\eta_{x(st)}-2\,\phi_{xy(st)}+\frac{\alpha}{2}\,\phi_{xy(st)}+\frac{\beta}{2}\,\phi_{xy(st)}$$

$$+w_{2(3)}\left(1+\frac{\beta}{2}\,\phi_{xy(st)}\,-\phi_{xy(st)}-\frac{5}{8}\beta\eta_{x(st)}\right)\Bigg)$$

$$+w_{1(3)}\mu_X\left(\eta_{x(st)}-\frac{\beta}{2}\eta_{x(st)}\right). \quad (76)$$

The *MSEs* of $t_{s(i)(\alpha,\beta)}$ is expressed by:

$$MSE\left(t_{s(i)(\alpha,\beta)}\right)=\nabla_{0(i)}+w_{2(i)}^2\nabla_{1(i)}+w_{2(i)}\nabla_{2(i)}+w_{1(i)}^2\nabla_{3(i)}+w_{1(i)}w_{2(i)}\nabla_{4(i)}+w_{1(i)}\nabla_{5(i)}, \quad (77)$$

where

$$\nabla_{0(1)}=\mu_Y^2\Big({\eta''}_{y(st)}+4\,\eta_{x(st)}-8\alpha\eta_{x(st)}-8\beta\,\eta_{x(st)}+4\alpha^2\eta_{x(st)}+4\beta^2\,\eta_{x(st)}+8\alpha\beta\eta_{x(st)}+4\,\phi_{xy(st)}-4\alpha\,\phi_{xy(st)}-4\beta\,\phi_{xy(st)}\Big),$$

$$\nabla_{1(1)}=\mu_Y^2\Big(1+{\eta''}_{y(st)}+\eta_{x(st)}+2\beta\,\eta_{x(st)}+4\,\phi_{xy(st)}-8\beta\phi_{xy(st)}+4\beta^2\eta_{x(st)}\Big),$$

$$\nabla_{2(1)}=\mu_Y^2\Big(2{\eta''}_{y(st)}+10\,\phi_{xy(st)}-12\beta\,\phi_{xy(st)}+6\,\eta_{x(st)}-6\alpha\eta_{x(st)}-14\beta\eta_{x(st)}+16\alpha\beta\eta_{x(st)}+8\beta^2\eta_{x(st)}-8\alpha\phi_{xy(st)}\Big),$$

$$\nabla_{3(1)}=\mu_X^2\,\eta_{x(st)},$$

$$\nabla_{4(1)}=2\mu_Y\mu_X\left(4\beta\,\eta_{x(st)}-2\,\eta_{x(st)}-\phi_{xy(st)}\right),$$

$$\nabla_{5(1)}=2\mu_Y\mu_X\Big(2\alpha\eta_{x(st)}+2\beta\,\eta_{x(st)}-2\,\eta_{x(st)}-\phi_{xy(st)}\Big),$$

$$\nabla_{0(2)}=\mu_Y^2\Big({\eta''}_{y(st)}+\eta_{x(st)}-2\alpha\,\eta_{x(st)}-2\beta\,\eta_{x(st)}+\alpha^2\eta_{x(st)}+\beta^2\eta_{x(st)}+2\alpha\beta\,\eta_{x(st)}+2\,\phi_{xy(st)}-2\alpha\,\phi_{xy(st)}-2\beta\,\phi_{xy(st)}\Big),$$

$$\nabla_{1(2)}=\mu_Y^2\Big(1+{\eta''}_{y(st)}+2\,\phi_{xy(st)}-4\beta\,\phi_{xy(st)}+4\beta^2\,\eta_{x(st)}\Big),$$

$$\nabla_{2(2)}=\mu_Y^2\Big(2{\eta''}_{y(st)}+\eta_{x(st)}+5\,\phi_{xy(st)}-4\alpha\,\phi_{xy(st)}-6\beta\,\phi_{xy(st)}-\alpha\,\eta_{x(st)}+4\alpha\beta\eta_{x(st)}+2\beta^2\eta_{x(st)}-3\beta\eta_{x(st)}\Big),$$

$$\nabla_{3(2)}=\mu_X^2\,\eta_{x(st)},$$

$$\nabla_{4(2)}=2\mu_Y\mu_X\left(2\beta\eta_{x(st)}-\eta_{x(st)}-\phi_{xy(st)}\right),$$

$$\nabla_{5(2)}=2\mu_Y\mu_X\left(\alpha\,\eta_{x(st)}+\beta\eta_{x(st)}-\eta_{x(st)}-\phi_{xy(st)}\right),$$

$$\nabla_{0(3)}=\mu_Y^2\,(\,{\eta''}_{y(st)}+4\,\eta_{x(st)}-2\alpha\,\eta_{x(st)}-2\beta\,\eta_{x(st)}+\frac{\alpha^2}{4}\,\eta_{x(st)}+\frac{\beta^2}{4}\,\eta_{x(st)}+\frac{\alpha\beta}{2}\,\eta_{x(st)}-4\,\phi_{xy(st)}+\alpha\,\phi_{xy(st)}+\beta\,\phi_{xy(st)}),$$

$$\nabla_{1(3)}=\mu_Y^2\,(\,1+{\eta''}_{y(st)}+\eta_{x(st)}-\frac{9}{4}\beta\eta_{x(st)}-4\,\phi_{xy(st)}+2\beta\,\phi_{xy(st)}+\frac{\beta^2}{4}\eta_{x(st)}\,),$$

$$\nabla_{2(3)}=\mu_Y^2\Big(2{\eta''}_{y(st)}+10\,\phi_{xy(st)}+3\beta\,\phi_{xy(st)}+10\eta_{x(st)}-6\alpha\eta_{x(st)}-\frac{21}{4}\beta\eta_{x(st)}-\frac{13}{4}\alpha\eta_{x(st)}+\frac{\beta^2}{2}\eta_{x(st)}+\alpha\beta\eta_{x(st)}+2\alpha\phi_{xy(st)}\Big),$$

$$\nabla_{3(3)}=\mu_X^2\eta_{x(st)},$$

$$\nabla_{4(3)}=2\mu_Y\mu_X\left(2\,\eta_{x(st)}-\beta\,\eta_{x(st)}-\phi_{xy(st)}\right),$$

and

$$\nabla_{5(3)}=2\mu_Y\mu_X\left(2\eta_{x(st)}-\frac{\alpha}{2}\,\eta_{x(st)}-\frac{\beta}{2}\eta_{x(st)}-\phi_{xy(st)}\right).$$

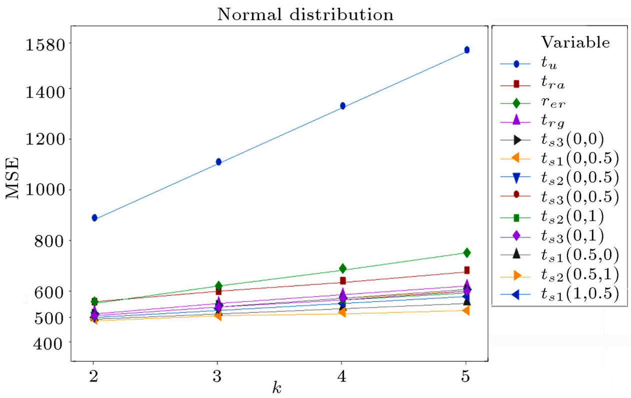

Figure 1. Mean Squared Error (MSEs) of the all estimators at different values of k with presence of the non-response and measurement error.

The optimum values of $w_{1(i)}$ and $w_{2(i)}$ from Eq. (77) are given by:

$$w_{1(i)}^{opt} = \frac{\nabla_{2(i)}\nabla_{4(i)} - 2\nabla_{1(i)}\nabla_{5(i)}}{4\nabla_{1(i)}\nabla_{3(i)} - \nabla_{4(i)}^2},$$

and

$$w_{2(i)}^{opt} = \frac{\nabla_{4(i)}\nabla_{5(i)} - 2\nabla_{2(i)}\nabla_{3(i)}}{4\nabla_{1(i)}\nabla_{3(i)} - \nabla_{4(i)}^2}. \quad (78)$$

Substituting the optimum values of $w_{1(i)}$ and $w_{2(i)}$ in Eq. (77), the expression of the minimum *MSE* of $t_{s(i)(\alpha,\beta)}$ is obtained as:

$$\min MSE\left(t_{s(i)(\alpha,\beta)}\right) = \nabla_{0(i)} - \frac{\left(\nabla_{2(i)}\nabla_{4(i)}\nabla_{5(i)} - \nabla_{2(i)}^2\nabla_{3(i)} - \nabla_{1(i)}\nabla_{5(i)}^2\right)}{\nabla_{4(i)}^2 - 4\nabla_{1(i)}\nabla_{3(i)}}. \quad (79)$$

4. Results and discussion

In this section, all of the proposed combined regression estimators are compared for their efficiency using the criterion of absolute *MSE*. The *MSEs* of all estimators are computed by changing the value of k following the four different situations: (i) when the non-response and the measurement error are simultaneously present; (ii) when only the non-response is present; (iii) when only the measurement error is present; (iv) when neither the non-response nor the measurement error is present; and results are presented in Tables 1–4. *MSEs* of all the estimators are also expressed by Figures 1 and 2 by changing the value of k. The caption of each table shows the situation under which the *MSEs* are computed in the given table. The efficiency comparisons of the unbiased sample mean estimator, combined ratio estimator, and combined exponential ratio estimator with usual combined regression estimator are computed

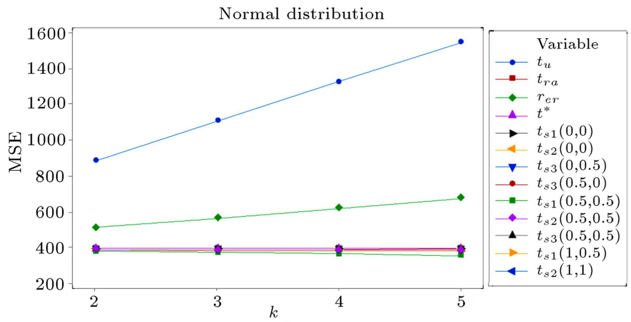

Figure 2. Mean Squared Error (MSEs) of the all estimators with presence of the only non-response.

numerically following the conditions expressed by Eqs. (22) to (24), and results are presented in Table 5. In Tables 6–8, the efficiency comparison of the proposed generalized combined regression-cum-ratio estimators with the proposed modified combined regression estimator is given using the condition expressed by Eq. (57).

To compute the *MSE* of the estimator, an artificial stratified populated is generated using the *R*-language. The stratified population is generated with arbitrary parameters of normal distribution. Further description on parameters, population size and sample size are shown in Table 9.

$$stratum - 1 \rightarrow X_1 = N\left(5000, 4, 15\right);$$

$$z_1 = N\left(5000, 0, 1\right);\; Y_1 = 50X_1 + 15z_1;$$

$$y_1 = Y_1 + N\left(1, 3\right);\; x_1 = X_1 + N\left(1, 3\right);$$

$$stratum - 2 \rightarrow X_2 = N\left(5000, 5, 15\right);$$

$$z_2 = N\left(5000, 0, 1\right);\; Y_2 = 50X_2 + 15z_2;$$

$$y_2 = Y_2 + N\left(1, 3\right);\; x_2 = X_2 + N\left(1, 3\right);$$

$$stratum - 3 \rightarrow X_3 = N\left(5000, 6, 15\right);$$

$$z_3 = N\left(5000, 0, 1\right); Y_3 = 50X_3 + 15z_3;$$

$$y_3 = Y_3 + N\left(1, 3\right);\; x_3 = X_3 + N\left(1, 3\right).$$

The *MSEs* of the estimators are computed, and results are presented in Tables 1–4.

From Tables 1–4, this can be noted that the two proposed estimators, usual combined regression estimator $\tilde{t}_{reg}^*$, and modified combined regression estimator $\tilde{t}_{g(i)}^*$ are equally efficient as these are achieving same *MSE* values whereas usual combined ratio estimator $\tilde{t}_{ra}^*$, and usual combined exponential ratio estimator $\tilde{t}_{er}^*$ have their *MSEs* larger than the *MSEs* of $\tilde{t}_{reg}^*$ and

Table 1. Mean Squared Errors (MSEs) of all estimators with presence of the non-response and measurement error.

	(α, β)	Estimators	Proposition of revisit sample (1/k)			
			1/2	**1/3**	**1/4**	**1/5**
Existing estimators		$\tilde{t}_u^*$	885.23	1105.39	1325.54	1545.70
		$\tilde{t}_{ra}^*$	464.50	481.27	498.05	514.83
		$\tilde{t}_{er}^*$	528.46	587.66	646.87	706.08
		$\tilde{t}^{*\otimes}$	452.89	469.29	485.33	501.19
Proposed estimators	(0,0)	$\tilde{t}_{s(1)(0,0)}^*$	450.55	466.93	482.95	498.80
		$\tilde{t}_{s(2)(0,0)}^*$	450.05	466.33	482.25	497.99
		$\tilde{t}_{s(3)(0,0)}^*$	447.99	463.70	479.00	494.08
	(0,0.5)	$\tilde{t}_{s(1)(0,0.5)}^*$	**427.76**	**428.32**	**424.93**	**417.90**
		$\tilde{t}_{s(2)(0,0.5)}^*$	**444.68**	**457.89**	**470.21**	**481.87**
		$\tilde{t}_{s(3)(0,0.5)}^*$	**446.61**	**461.58**	**476.02**	**490.15**
	(0,1)	$\tilde{t}_{s(1)(0,1)}^*$	**438.17**	**446.64**	**453.18**	**458.05**
		$\tilde{t}_{s(2)(0,1)}^*$	**444.66**	**457.85**	**470.14**	**481.76**
		$\tilde{t}_{s(3)(0,1)}^*$	**446.63**	**461.61**	**476.07**	**490.22**
	(0.5,0)	$\tilde{t}_{s(1)(0.5,0)}^*$	452.45	469.24	485.30	500.82
		$\tilde{t}_{s(2)(0.5,0)}^*$	451.32	468.13	484.51	500.66
		$\tilde{t}_{s(3)(0.5,0)}^*$	448.98	465.10	480.85	496.41
	(0.5,0.5)	$\tilde{t}_{s(1)(0.5,0.5)}^*$	**435.52**	**442.71**	**447.75**	**450.89**
		$\tilde{t}_{s(2)(0.5,0.5)}^*$	446.95	461.77	476.04	489.97
		$\tilde{t}_{s(3)(0.5,0.5)}^*$	447.73	463.23	478.28	493.09
	(0.5,1)	$\tilde{t}_{s(1)(0.5,1)}^*$	**443.93**	**456.95**	**469.10**	**480.63**
		$\tilde{t}_{s(2)(0.5,1)}^*$	446.94	461.74	476.00	489.91
		$\tilde{t}_{s(3)(0.5,1)}^*$	447.75	463.25	478.32	493.15
	(1,0)	$\tilde{t}_{s(1)(1,0)}^*$	452.85	468.18	481.66	493.42
		$\tilde{t}_{s(2)(1,0)}^*$	452.21	469.10	485.33	501.09
		$\tilde{t}_{s(3)(1,0)}^*$	449.87	466.30	482.37	498.29
	(1,0.5)	$\tilde{t}_{s(1)(1,0.5)}^*$	**441.84**	**454.00**	**465.18**	**475.60**
		$\tilde{t}_{s(2)(1,0.5)}^*$	448.85	464.84	480.46	495.89
		$\tilde{t}_{s(3)(1,0.5)}^*$	448.75	464.68	480.22	495.58
	(1,1)	$\tilde{t}_{s(1)(1,1)}^*$	448.27	464.15	479.63	494.94
		$\tilde{t}_{s(2)(1,1)}^*$	448.84	464.83	480.44	495.86
		$\tilde{t}_{s(3)(1,1)}^*$	448.76	464.70	480.25	495.62

* Note: $\tilde{t}^{*\otimes} = \tilde{t}_{reg}^* = \tilde{t}_{g(1)}^* = \tilde{t}_{g(2)}^* = \tilde{t}_{g(3)}^*$.

Table 2. Mean Squared Errors (MSEs) of the all estimators with presence of the only measurement error.

	(α,β)	Estimators	MSE	(α,β)	Estimators	MSE
Existing estimators		$\tilde{t}_{u}$	665.07	(0.5,0.5)	$\tilde{t}_{s(1)(0.5,0.5)}$	**407.61**
		$\tilde{t}_{ra}$	421.28		$\tilde{t}_{s(2)(0.5,0.5)}$	**412.42**
		t'_{er}	462.64		$\tilde{t}_{s(3)(0.5,0.5)}$	**412.87**
		$\tilde{t}^{\otimes}$	416.51	(0.5,1)	$\tilde{t}_{s(1)(0.5,1)}$	**411.07**
Proposed estimators	(0,0)	$\tilde{t}_{s(1)(0,0)}$	**414.11**	**(1,0)**	$\tilde{t}_{s(2)(0.5,1)}$	**412.42**
		$\tilde{t}_{s(2)(0,0)}$	**413.89**		$\tilde{t}_{s(3)(0.5,1)}$	**412.87**
		$\tilde{t}_{s(3)(0,0)}$	**413.01**		$\tilde{t}_{s(1)(1,0)}$	**416.21**
	(0,0.5)	$\tilde{t}_{s(1)(0,0.5)}$	**404.42**	**(1,0.5)**	$\tilde{t}_{s(2)(1,0)}$	**415.26**
		$\tilde{t}_{s(2)(0,0.5)}$	**411.36**		$\tilde{t}_{s(3)(1,0)}$	**413.87**
		$\tilde{t}_{s(3)(0,0.5)}$	**412.38**		$\tilde{t}_{s(1)(1,0.5)}$	**410.30**
	(0,1)	$\tilde{t}_{s(1)(0,1)}$	**408.53**	**(1,1)**	$\tilde{t}_{s(2)(1,0.5)}$	**413.36**
		$\tilde{t}_{s(2)(0,1)}$	**411.35**		$\tilde{t}_{s(3)(1,0.5)}$	**413.32**
		$\tilde{t}_{s(3)(0,1)}$	**412.39**		$\tilde{t}_{s(1)(1,1)}$	**413.12**
	(0.5,0)	$\tilde{t}_{s(1)(0.5,0)}$	**415.41**		$\tilde{t}_{s(2)(1,1)}$	**413.36**
		$\tilde{t}_{s(2)(0.5,0)}$	**414.63**		$\tilde{t}_{s(3)(1,1)}$	**413.33**
		$\tilde{t}_{s(3)(0.5,0)}$	**413.46**		–	–

$\tilde{t}^{*}_{g(i)}$. These results are also confirmed by the required conditions shown in Eqs. (12)–(14), and are also computed numerically in Table 5. Therefore, subsequently proposed generalized combined regression-cum-ratio estimators $\tilde{t}^{*}_{s(i)(\alpha,\beta)}$ for $i = 1, 2, 3$ are compared only with $\tilde{t}^{*}_{reg}$ and $\tilde{t}^{*}_{g(i)}$ based on their MSE values. Further from Tables 1-4, it is observed that the proposed estimators $\tilde{t}^{*}_{s(i)(\alpha,\beta)}$ for $i = 1, 2, 3$ are more efficient than $\tilde{t}^{*}_{reg}$ and $\tilde{t}^{*}_{g(i)}$, as the bold figures in Tables 1-4 indicate, the MSE values of $\tilde{t}^{*}_{s(i)(\alpha,\beta)}$ are smaller than the MSE values of $\tilde{t}^{*}_{reg}$ and $\tilde{t}^{*}_{g(i)}$. These results are also confirmed as the required conditions shown by Eq. (40) are met, and the required conditions are computed numerically in Tables 6–8.

Form Figures 1–2, it is much easier to understand that $MSEs$ of each estimator is increasing as sub-sample size of recontact is decreasing which is expected for each estimator. Sub-group sized is decreased as the value of k is increased. However, Figures 1–2 are also clearly indicating that the proposed generalized combined regression-cum-ratio estimators $\tilde{t}^{*}_{s(i)(\alpha,\beta)}$ for $i = 1, 2, 3$ are achieving smaller MSE values than the MSE of $\tilde{t}^{*}_{reg}$. Whereas MSE of $\tilde{t}^{*}_{reg}$ is smaller than usual combined ratio estimator, and usual combined exponential ratio estimator.

Table 3. Mean Squared Errors (MSE) values of all estimators with the only presence of the non-response.

	(α,β)	Estimators	Proposition of revisit sample (1/k)			
			1/2	**1/3**	**1/4**	**1/5**
Existing estimators		t_u^*	885.22	1105.37	1325.52	1545.67
		t_{ra}^*	393.91	393.99	394.07	394.15
		t_{er}^*	510.80	565.83	620.86	675.89
		$t^{*\otimes}$	393.63	393.79	393.92	394.02
Proposed estimators	(0,0)	$t_{s(1)(0,0)}^*$	391.15	391.28	391.38	391.45
		$t_{s(2)(0,0)}^*$	391.12	391.27	391.38	391.48
		$t_{s(3)(0,0)}^*$	390.83	390.96	391.04	391.11
	(0,0.5)	$t_{s(1)(0,0.5)}^*$	**372.07**	**358.49**	**341.70**	**321.83**
		$t_{s(2)(0,0.5)}^*$	**386.74**	**384.35**	**381.53**	**378.29**
		$t_{s(3)(0,0.5)}^*$	389.89	389.56	389.16	388.70
	(0,1)	$t_{s(1)(0,1)}^*$	**380.62**	**373.81**	**365.57**	**355.97**
		$t_{s(2)(0,1)}^*$	**386.72**	**384.32**	**381.48**	**378.22**
		$t_{s(3)(0,1)}^*$	389.90	389.58	389.18	388.74
	(0.5,0)	$t_{s(1)(0.5,0)}^*$	393.32	393.78	393.79	393.36
		$t_{s(2)(0.5,0)}^*$	392.46	393.05	393.51	393.85
		$t_{s(3)(0.5,0)}^*$	391.30	391.60	391.86	392.11
	(0.5,0.5)	$t_{s(1)(0.5,0.5)}^*$	**380.27**	**373.38**	**365.06**	**355.37**
		$t_{s(2)(0.5,0.5)}^*$	389.11	388.27	387.29	386.20
		$t_{s(3)(0.5,0.5)}^*$	390.43	390.35	390.23	390.07
	(0.5,1)	$t_{s(1)(0.5,1)}^*$	**386.77**	**384.51**	**381.87**	**378.90**
		$t_{s(2)(0.5,1)}^*$	389.10	388.25	387.27	386.17
		$t_{s(3)(0.5,1)}^*$	390.44	390.37	390.24	390.10
	(1,0)	$t_{s(1)(1,0)}^*$	393.43	391.92	388.65	383.58
		$t_{s(2)(1,0)}^*$	393.30	393.77	393.81	393.43
		$t_{s(3)(1,0)}^*$	391.72	392.16	392.54	392.89
	(1,0.5)	$t_{s(1)(1,0.5)}^*$	386.52	384.21	381.54	378.52
		$t_{s(2)(1,0.5)}^*$	390.99	391.15	391.27	391.38
		$t_{s(3)(1,0.5)}^*$	390.93	391.06	391.16	391.24
	(1,1)	$t_{s(1)(1,1)}^*$	390.97	391.16	391.30	391.43
		$t_{s(2)(1,1)}^*$	390.98	391.14	391.26	391.37
		$t_{s(3)(1,1)}^*$	390.94	391.07	391.17	391.26

Table 4. Mean Squared Error (MSE) values of all estimators without presence of the non-response and measurement error.

	(α,β)	Estimators	MSE	(α,β)	Estimators	MSE
Existing estimators		$\tilde{t}_u$	412.75	**(0.5,0.5)**	$\tilde{t}_{s(1)(0.5,0.5)}$	**231.39**
		$\tilde{t}_{ra}$	251.00		$\tilde{t}_{s(2)(0.5,0.5)}$	234.42
		t'_{er}	258.20		$\tilde{t}_{s(3)(0.5,0.5)}$	**234.45**
Proposed estimators		$\tilde{t}^{\otimes}$	236.01		$\tilde{t}_{s(1)(0.5,1)}$	233.60
	(0,0)	$\tilde{t}_{s(1)(0,0)}$	235.52	**(0.5,1)**	$\tilde{t}_{s(2)(0.5,1)}$	**234.41**
		$\tilde{t}_{s(2)(0,0)}$	235.32		$\tilde{t}_{s(3)(0.5,1)}$	**234.45**
		$\tilde{t}_{s(3)(0,0)}$	234.46		$\tilde{t}_{s(1)(1,0)}$	236.01
	(0,0.5)	$\tilde{t}_{s(1)(0,0.5)}$	**229.88**	**(1,0)**	$\tilde{t}_{s(2)(1,0)}$	235.76
		$\tilde{t}_{s(2)(0,0.5)}$	**233.97**		$\tilde{t}_{s(3)(1,0)}$	235.10
		$\tilde{t}_{s(3)(0,0.5)}$	**234.07**		$\tilde{t}_{s(1)(1,0.5)}$	232.69
	(0,1)	$\tilde{t}_{s(1)(0,1)}$	**232.48**	**(1,0.5)**	$\tilde{t}_{s(2)(1,0.5)}$	**234.80**
		$\tilde{t}_{s(2)(0,1)}$	233.97		$\tilde{t}_{s(3)(1,0.5)}$	234.79
		$\tilde{t}_{s(3)(0,1)}$	234.07		$\tilde{t}_{s(1)(1,1)}$	234.51
	(0.5,0)	$\tilde{t}_{s(1)(0.5,0)}$	235.88	**(1,1)**	$\tilde{t}_{s(2)(1,1)}$	234.80
		$\tilde{t}_{s(2)(0.5,0)}$	235.57		$\tilde{t}_{s(3)(1,1)}$	234.79
		$\tilde{t}_{s(3)(0.5,0)}$	234.80		–	–

Table 5. Efficiency comparisons of $\tilde{t}^*_{reg}$ with $\tilde{t}^*_u, \tilde{t}^*_{ra}, \tilde{t}^*_{er}$.

$\tilde{t}^*_{reg}$ vs $\tilde{t}^*_u$	$0.01457 > 0$
$\tilde{t}^*_{reg}$ vs $\tilde{t}^*_{ra}$	$0.00682 > 0$
$\tilde{t}^*_{reg} vs$ $\tilde{t}^*_{er}$	$0.00828 > 0$

5. Conclusion

In the present study, usual combined regression estimator $\tilde{t}^*_{reg}$ and modified combined regression estimator $\tilde{t}^*_{g(i)}$ for mean estimation in stratified two-phase sampling are concluded to be equally efficient, however both types of combined regression estimators are remained more efficient than usual combined ratio estimator $\tilde{t}^*_{ra}$ and usual combined exponential estimator $\tilde{t}^*_{er}$. Another proposed generalized combined regression-cum-ratio estimators $\tilde{t}^*_{s(i)(\alpha,\beta)}$ for $i = 1, 2, 3$ is found to be the most efficient class of estimators as all combined regression-cum-ratio estimators attain least Mean Squared Error (MSE) values than the $MSEs$ of all the estimators discussed in the text. Therefore it is concluded from Tables 1–8, that the proposed generalized combined regression-cum-ratio estimators

Table 6. Efficiency comparison of $\tilde{t}^*_{s(1)(\alpha,\beta)}$ with $\tilde{t}^*_{reg}$.

	$\rho^2_{xy(st)} = 0.48839$	$\rho^2_{xy(st)} = 0.57544$	$\rho^2_{xy(st)} = 0.63386$	$\rho^2_{xy(st)} = 0.67574$
Estimators	**1/2**	**1/3**	**1/4**	**1/5**
$\tilde{t}^*_{s(1)(0,0)}\ vs\ \tilde{t}^*_{reg}$	$0.49103 > \rho^2_{xy(st)}$	$0.57758 > \rho^2_{xy(st)}$	$0.63566 > \rho^2_{xy(st)}$	$0.67729 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(0,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.51677 > \rho^2_{xy(st)}$	$0.61251 > \rho^2_{xy(st)}$	$0.67942 > \rho^2_{xy(st)}$	$0.72961 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(0,1)}\ vs\ \tilde{t}^*_{reg}$	$0.50502 > \rho^2_{xy(st)}$	$0.59594 > \rho^2_{xy(st)}$	$0.65811 > \rho^2_{xy(st)}$	$0.70366 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(0.5,0)}\ vs\ \tilde{t}^*_{reg}$	$0.48888 > \rho^2_{xy(st)}$	$0.57549 > \rho^2_{xy(st)}$	$0.63388 > \rho^2_{xy(st)}$	$0.67599 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(0.5,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.50802 > \rho^2_{xy(st)}$	$0.59949 > \rho^2_{xy(st)}$	$0.66221 > \rho^2_{xy(st)}$	$0.70829 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(0.5,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49851 > \rho^2_{xy(st)}$	$0.58661 > \rho^2_{xy(st)}$	$0.64610 > \rho^2_{xy(st)}$	$0.68905 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(1,0)}\ vs\ \tilde{t}^*_{reg}$	$0.48844 > \rho^2_{xy(st)}$	$0.57645 > \rho^2_{xy(st)}$	$0.63663 > \rho^2_{xy(st)}$	$0.68077 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(1,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.50087 > \rho^2_{xy(st)}$	$0.58928 > \rho^2_{xy(st)}$	$0.64906 > \rho^2_{xy(st)}$	$0.69230 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(1)(1,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49360 > \rho^2_{xy(st)}$	$0.58010 > \rho^2_{xy(st)}$	$0.63815 > \rho^2_{xy(st)}$	$0.67979 > \rho^2_{xy(st)}$

Table 7. Efficiency comparison of $\tilde{t}^*_{s(2)(\alpha,\beta)}$ with $\tilde{t}^*_{reg}$.

	$\rho^2_{xy(st)} = 0.48839$	$\rho^2_{xy(st)} = 0.57544$	$\rho^2_{xy(st)} = 0.63386$	$\rho^2_{xy(st)} = 0.67574$
Estimators	**1/2**	**1/3**	**1/4**	**1/5**
$\tilde{t}^*_{s(2)(0,0)}\ vs\ \tilde{t}^*_{reg}$	$0.4916 > \rho^2_{xy(st)}$	$0.57812 > \rho^2_{xy(st)}$	$0.63619 > \rho^2_{xy(st)}$	$0.67782 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(0,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.49766 > \rho^2_{xy(st)}$	$0.58577 > \rho^2_{xy(st)}$	$0.64527 > \rho^2_{xy(st)}$	$0.68825 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(0,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49769 > \rho^2_{xy(st)}$	$0.5858 > \rho^2_{xy(st)}$	$0.64532 > \rho^2_{xy(st)}$	$0.68832 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(0.5,0)}\ vs\ \tilde{t}^*_{reg}$	$0.49017 > \rho^2_{xy(st)}$	$0.5765 > \rho^2_{xy(st)}$	$0.63448 > \rho^2_{xy(st)}$	$0.67609 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(0.5,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.4951 > \rho^2_{xy(st)}$	$0.58226 > \rho^2_{xy(st)}$	$0.64087 > \rho^2_{xy(st)}$	$0.68301 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(0.5,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49512 > \rho^2_{xy(st)}$	$0.58228 > \rho^2_{xy(st)}$	$0.64091 > \rho^2_{xy(st)}$	$0.68305 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(1,0)}\ vs\ \tilde{t}^*_{reg}$	$0.48916 > \rho^2_{xy(st)}$	$0.57562 > \rho^2_{xy(st)}$	$0.63387 > \rho^2_{xy(st)}$	$0.67582 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(1,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.49295 > \rho^2_{xy(st)}$	$0.57947 > \rho^2_{xy(st)}$	$0.63754 > \rho^2_{xy(st)}$	$0.67918 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(2)(1,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49296 > \rho^2_{xy(st)}$	$0.57949 > \rho^2_{xy(st)}$	$0.63755 > \rho^2_{xy(st)}$	$0.6792 > \rho^2_{xy(st)}$

Table 8. Efficiency comparison of s with $\tilde{t}^*_{reg}$.

	$\rho^2_{xy(st)} = 0.48839$	$\rho^2_{xy(st)} = 0.57544$	$\rho^2_{xy(st)} = 0.63386$	$\rho^2_{xy(st)} = 0.67574$
Estimators	**1/2**	**1/3**	**1/4**	**1/5**
$\tilde{t}^*_{s(3)(0,0)}\ vs\ \tilde{t}^*_{reg}$	$0.49392 > \rho^2_{xy(st)}$	$0.58051 > \rho^2_{xy(st)}$	$0.63864 > \rho^2_{xy(st)}$	$0.68035 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(0,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.49549 > \rho^2_{xy(st)}$	$0.58243 > \rho^2_{xy(st)}$	$0.64088 > \rho^2_{xy(st)}$	$0.6829 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(0,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49547 > \rho^2_{xy(st)}$	$0.58241 > \rho^2_{xy(st)}$	$0.64084 > \rho^2_{xy(st)}$	$0.68285 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(0.5,0)}\ vs\ \tilde{t}^*_{reg}$	$0.4928 > \rho^2_{xy(st)}$	$0.57924 > \rho^2_{xy(st)}$	$0.63724 > \rho^2_{xy(st)}$	$0.67884 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(0.5,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.49422 > \rho^2_{xy(st)}$	$0.58093 > \rho^2_{xy(st)}$	$0.63918 > \rho^2_{xy(st)}$	$0.68099 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(0.5,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49421 > \rho^2_{xy(st)}$	$0.58091 > \rho^2_{xy(st)}$	$0.63915 > \rho^2_{xy(st)}$	$0.68095 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(1,0)}\ vs\ \tilde{t}^*_{reg}$	$0.49181 > \rho^2_{xy(st)}$	$0.57815 > \rho^2_{xy(st)}$	$0.63609 > \rho^2_{xy(st)}$	$0.67763 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(1,0.5)}\ vs\ \tilde{t}^*_{reg}$	$0.49307 > \rho^2_{xy(st)}$	$0.57962 > \rho^2_{xy(st)}$	$0.63771 > \rho^2_{xy(st)}$	$0.67938 > \rho^2_{xy(st)}$
$\tilde{t}^*_{s(3)(1,1)}\ vs\ \tilde{t}^*_{reg}$	$0.49306 > \rho^2_{xy(st)}$	$0.57961 > \rho^2_{xy(st)}$	$0.63769 > \rho^2_{xy(st)}$	$0.67935 > \rho^2_{xy(st)}$

Table 9. Stratified population: Based on simulated normal distribution.

Stratum	N_h	ρ_{yxh}	μ_{Yh}	μ_{Xh}	σ^2_{Uh}	σ^2_{Vh}	σ^2_{yh}	σ^2_{xh}
1	5000	0.98	183.62	3.67	9.19	9.32	543610.50	227.16
2	5000	0.98	248.67	4.97	9.19	9.25	583685.70	242.30
3	5000	0.98	301.68	6.04	9.10	9.18	578114.50	240.50
Stratum	n'_h	n''_h	$\sigma^2_{yh(2)}$	$\sigma^2_{xh(2)}$	$\sigma^2_{Uh(2)}$	$\sigma^2_{Vh(2)}$	$\rho_{yxh(2)}$	
1	500	300	575428.90	230.66	9.15	9.75	0.99	
2	500	300	599476.60	239.94	9.14	8.80	0.99	
3	500	300	577044.10	230.86	9.32	8.63	0.99	

$\tilde{t}^*_{s(i)(\alpha,\beta)}$ is the most efficient and more generalized combined estimator of mean than $\tilde{t}^*_{g(i)}$, $\tilde{t}^*_{reg}$, $\tilde{t}^*_{ra}$, and $\tilde{t}^*_{er}$. Further, it is also concluded that $\tilde{t}^*_{s(i)(\alpha,\beta)}$ performs well in all of the four situations. Therefore, the proposed generalized combined regression-cum-ratio estimators $\tilde{t}^*_{s(i)(\alpha,\beta)}$ for $i = 1, 2, 3$ are recommended for their applications of mean estimation under stratified two-phase sampling when the two components of survey error, the non-response and the measurement error are present simultaneously.

This study may be extended for mean estimation assuming the simultaneous presence of non-response and measurement error in different sampling designs, such as multistage sampling, and ranked set sampling. For estimation of unknown parameter(s) under ranked set sampling schemes, one can find Zamanzade and Mahizadeh [38], Zamanzade and Wang [39], and Dumbgen and Zamanzade [40] worth reading and helpful for future work.

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