A Variable Sampling Interval Multivariate Exponentially Weighted Moving Average Control Chart for Monitoring the Gumbel's Bivariate Exponential Data

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Abstract

The general assumption in designing a multivariate control chart is that the multiple variables are independent and normally distributed. This assumption may not be tenable in many practical situations, because multiple variables with dependency often need to be monitored simultaneously to ensure the process is in control. The Gumbel's bivariate exponential (GBE) distribution is considered to be a better model for skewed data with dependency in reliability analysis. In this paper, a multivariate exponentially weighted moving average (MEWMA) scheme with variable sampling interval (VSI) feature is developed to monitor the mean vector of GBE model. The Monte Carlo simulation is used to evaluate the average time to signal (ATS) performance of the proposed VSI MEWMA GBE scheme for three different types of shifts. Some tables are presented to show the ATS performance of the proposed scheme with different designed parameters. Additionally, both the zero-state and the steady-state ATS performance of the proposed scheme is compared with that of the conventional MEWMA chart with FSI (Fix Sampling Interval) feature. Comparative results show that the suggested scheme works better than its FSI counterpart in monitoring GBE data. Finally, a simulation example is provided to show that the VSI MEWMA GBE scheme performs well in monitoring GBE data.

Keywords: Average time to signal; Gumbel's bivariate exponential distribution; Multivariate EWMA control chart; Variable sampling interval; Zero-state and steady-state.

1 Introduction

Quality control plays a significant role in maintaining the reputation of a factory or company. The most effective way to improve product quality in the manufacturing processes is to use statistical process control (SPC). As one of the most important tools in SPC, control charts have been widely used in many quality control applications and developed with many variants, for example, Perry [1], Hassani et al. [2] and Khan et al. [3]. It is known that univariate control charts can only focus on a single quality characteristic of the process, for instance, the univariate Shewhart-type control charts are popular and efficient to detect large changes in a process, and the univariate memory-type control charts, for example, the exponentially weighted moving average (EWMA) chart and the cumulative sum (CUSUM) chart are much more effective for detecting small to moderate shifts (see Nazir et al. [4], Castagliola et al. [5] and Olawale Ajadi and Riaz [6]). However, in practice, there are many scenarios in which the simultaneous monitoring of several related quality characteristics of interest is a necessity. According to Cozzucoli and Marozzi [7] and Xie et al. [8], it is usually ineffective and misleading when several univariate schemes are used to monitor these related quality characteristics separately. In this context, the use of a multivariate control chart can be a good solution to the problem as it takes the natural correlation between the quality characteristics into account (see Montgomery [9]). Most of the traditional multivariate control charts are the generalizations of their univariate counterparts, such as the multivariate EWMA (MEWMA) control chart proposed by Lowry et al. [10] and the multivariate CUSUM (MCUSUM) control chart introduced by Crosier [11]. For more details on traditional multivariate control charts, readers can refer to Lowry and Montgomery [12] and Bersimis et al. [13].

Although multivariate control charts have received much attention in the literature, most of the works were based on the assumption that the data follow a multivariate normal distribution. But in practice, the multivariate data in many situations are usually non-normal and highly skewed, as the marginal distributions are usually based on exponential, Poisson or gamma distributions (see Xie et al. [8], Chen et al. [14], and Cheng et al. [15]). Up to now, there are many approaches have been proposed to construct multivariate control charts for those non-normal and highly skewed distributions, for instance, the double square-root transformation used in Xie et al. [8], and the weighted standard deviation method proposed by Chang [16]. Different from these transformation methods, both Stoumbos and Sullivan [17] and Testik et al. [18] pointed out that the MEWMA scheme using a small smoothing factor was fairly robust to the non-normality assumption. In this context, Xie et al. [8] investigated the ARL performance of the MEWMA scheme in detecting the mean shift vector of Gumbel's bivariate exponential distribution. Cheng et al. [15] further studied the ARL performance of the MEWMA scheme for simultaneously monitoring the frequency and magnitude of events. The results of these two studies implied that the MEWMA scheme with a small smoothing factor outperforms the other competing charts for monitoring the multivariate non-normal data in most scenarios.

The traditional practice of using a control chart for process monitoring is to take a fixed sample size from the process at a fixed sampling interval (FSI). Extensive research works have been shown that varying the sampling interval as a function of the process sample can make the process shift detection faster than the corresponding FSI strategy (see Tang et al. [19]). In addition, simulations in Reynolds et al. [20], Reynolds Jr [21], and Reynolds Jr and Arnold [22] showed that using two sampling intervals is sufficient to provide good performance in monitoring various magnitudes of shifts, and to keep the complexity of the variable sampling interval (VSI) charts at a reasonable level. It is known that the VSI charts can usually be partitioned into three regions, namely, the safe region, the warning region and the out-of-control region. The basic idea of the VSI charts is that the short sampling interval d_S can provide a quick detection when the current sample falls into the warning region, and a long sampling interval d_L is taken if the current sample falls into the safe region. Finally, if a sample falls outside the control limits, the sample belongs to the out-of-control region and the process is considered to be out-of-control, where corrective action(s) should be taken to remove the assignable cause(s). Some of the recent studies on VSI charts were made by Haq [23], Khoo et al. [24] and Shojaee et al. [25].

The time to signal of a scheme is not a constant multiple of its average run length (ARL) when the sampling interval is varied. Hence, the average time to signal (ATS), which is defined as the average time from the beginning of the process monitoring until the scheme generates an out-of-control signal, is often employed in VSI type charts (see Li et al. [26]). As defined in Saccucci et al. [27] and Chew et al. [28], if a shift in the parameter occurs at the beginning of the Phase II monitoring, the corresponding ATS is referred to as the *zero-state* ATS. Similarly, for the steady-state case, a mean shift is assumed to occur at an unknown random time after process monitoring has started, and the corresponding ATS can be referred to as the *adjusted* ATS (AATS) or the *steady-state* ATS (SATS) (see Haq [23]). In this paper, both the zero-state and the steady-state ATS performance is investigated to provide a comprehensive analysis of the GBE data monitoring.

According to Lee and Khoo [29] and Lee [30], the VSI feature can substantially improve the ATS properties of MEWMA scheme, this fact motivates us to study the effectiveness of the VSI feature on the MEWMA scheme in monitoring GBE data. Although monitoring the GBE data with a MEWMA scheme has already been developed by Xie et al. [8], there is still no research has been done on (1) proposing a MEWMA type scheme with VSI feature for monitoring the GBE data (hereafter denoted as the VSI MEWMA GBE scheme), and (2) evaluating both the zero-state and the steady-state ATS performance of the proposed VSI MEWMA GBE scheme for a direct comparison with its FSI counterpart. We address these research gaps in the current paper.

The outline of this study is organized as follows: In Section 2, the GBE model is first introduced, and then a VSI MEWMA type scheme for monitoring the GBE distributed data is developed. The Monte Carlo simulations for both the zero-state and the steady-state ATS computations of the proposed scheme are detailed in Section 3. Subsequently, in Section 4, numerical comparisons are performed between the proposed VSI MEWMA GBE scheme and its FSI counterpart in the case of downward, upward and hybrid shifts. Also, several guidelines for constructing the VSI MEWMA GBE scheme are offered. In Section 5, a simulation example is provided to illustrate the implementation of the proposed VSI MEWMA GBE scheme for monitoring the GBE data. Finally, some conclusions are made in Section 6.

2 The VSI MEWMA scheme for Gumbel's bivariate exponential distributed data

2.1 Gumbel's bivariate exponential model

The Gumbel's bivariate exponential model, which is known as the GBE model, was firstly introduced by Gumbel [31]. Let us assume that the random variables (X, Y) used in this paper follows a standard GBE distribution. The joint survival function $\bar{F}_{X,Y}(x, y)$ of X and Y is given as,

$$\bar{F}_{X,Y}(x,y) = \exp\left(-\left(\left(\frac{x}{\theta_x}\right)^{\frac{1}{\delta}} + \left(\frac{y}{\theta_y}\right)^{\frac{1}{\delta}}\right)^{\delta}\right),\ X \text{ and } Y \in \mathbb{R}^+ \tag{1}$$

where $\delta \in (0, 1]$ represents the dependence parameter, $\theta_x > 0$ and $\theta_y > 0$ are two scale parameters. Note that if $\delta \in (0, 1)$, X and Y are correlated. Otherwise, $\delta = 1$ corresponds to the case of independence. Furthermore, the probability density function (p.d.f.) of the standard GBE model is defined as,

$$f_{X,Y}(x,y) = (\theta_x\theta_y)^{-\frac{1}{\delta}}(xy)^{\frac{1}{\delta}-1}\left(\left(\left(\frac{x}{\theta_x}\right)^{\frac{1}{\delta}}+\left(\frac{y}{\theta_y}\right)^{\frac{1}{\delta}}\right)^{\delta}+\frac{1}{\delta}-1\right)$$
$$\times\left(\left(\frac{x}{\theta_x}\right)^{\frac{1}{\delta}}+\left(\frac{y}{\theta_y}\right)^{\frac{1}{\delta}}\right)^{\delta-2}\exp\left(-\left(\left(\frac{x}{\theta_x}\right)^{\frac{1}{\delta}}+\left(\frac{y}{\theta_y}\right)^{\frac{1}{\delta}}\right)^{\delta}\right). \tag{2}$$

According to Nadarajah and Kotz [32], the GBE models have been extensively used in various applications, for instance, competing risks modeling (see Lu and Bhattacharyya [33]), failure times (see Hougaard [34]), regional analyses of precipitation (see Bacchi et al. [35]), reliability or frailty modelling (see Pal and Murthy [36]). However, only a few studies have been conducted on the multivariate control charts to monitor the GBE distributed data due to its complexity. More importantly, the GBE model is a meaningful multivariate reliability model that the dependence can be explained by the random mixing effect of external stress (see Hougaard [37]). This makes it easier to be applied in realistic situations than other multivariate reliability models where the source of dependence needs to be specified, for example, the Marshall-Olkin's model and the Freund's model. All these factors motivate us to conduct further research on the monitoring of GBE data.

The mean vector and the corresponding covariance matrix are usually required in developing a multivariate control chart. From the joint survival function $\bar{F}_{X,Y}(x,y)$ in Equation (1), it is easy to prove that the marginal distributions of X and Y are, respectively, the exponential distributions of parameters θ_x and θ_y. Based on this condition, the mean vector $\boldsymbol{\mu}$ of $(X,Y)^\intercal$ is given as,

$$\boldsymbol{\mu} = \begin{pmatrix} \theta_x \\ \theta_y \end{pmatrix}, \tag{3}$$

and the corresponding covariance matrix $\boldsymbol{\Sigma}$ is defined as,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \theta_x^2 & \rho\theta_x\theta_y \\ \rho\theta_y\theta_x & \theta_y^2 \end{pmatrix}, \tag{4}$$

where the coefficient of correlation ρ is (see Lu and Bhattacharyya [38]),

$$\rho = \frac{2\Gamma^2(\delta+1)}{\Gamma(2\delta+1)} - 1. \tag{5}$$

Note that $\Gamma(\cdot)$ is the gamma function. For more details about the properties of GBE model, readers can refer to Lu and Bhattacharyya [33] and Lu and Bhattacharyya [38].

2.2 The proposed VSI MEWMA GBE scheme

The standard MEWMA scheme was firstly developed by Lowry et al. [10]. Similar to the univariate EMWA control charts, MEWMA schemes also take both the current and the past samples information of the process into account, which makes the schemes more effective in detecting small to moderate shifts. Furthermore, based on the fact that the MEWMA scheme with a small smoothing parameter was considered to be fairly robust to the non-normality assumption, the VSI MEWMA GBE scheme recommended in this study is robust to the GBE distributed data.

For the proposed VSI MEWMA GBE scheme, suppose that the vectors $\mathbf{M}_t = (X_t, Y_t)^\intercal$, $t = 1, 2, \ldots$ are the GBE vectors collected at regular sampling points. Similar to Lowry et al. [10], the MEWMA statistic $\mathbf{W}_t$ of the recommended scheme can be written as,

$$\mathbf{W}_t = \mathbf{R}(\mathbf{M}_t - \boldsymbol{\mu}_0) + (\mathbf{I} - \mathbf{R})\mathbf{W}_{t-1}, \tag{6}$$

where $\boldsymbol{\mu}_0 = (\theta_x, \theta_y)^\intercal$ is the in-control mean vector, $\mathbf{I}$ is the $(2, 2)$ identity matrix, the process initial statistic $\mathbf{W}_0 = \mathbf{0}$, and $\mathbf{R}$ =diag(r_1, r_2), where $r_i \in (0, 1]$ for $i = 1, 2$. Furthermore, the charting statistic Q_t^2 of the proposed VSI MEWMA GBE scheme is defined as,

$$Q_t^2 = \mathbf{W}_t^\intercal \boldsymbol{\Sigma}_{\mathrm{W}_t}^{-1} \mathbf{W}_t, \tag{7}$$

where $\boldsymbol{\Sigma}_{\mathrm{W}_t}$ is the in-control covariance matrix of the MEWMA statistic $\mathbf{W}_t$. In general, when $r_1 = r_2 = r$, the MEWMA statistic $\mathbf{W}_t$ is re-stated as,

$$\mathbf{W}_t = r(\mathbf{M}_t - \boldsymbol{\mu}_0) + (1 - r)\mathbf{W}_{t-1}. \tag{8}$$

Since the asymptotic in-control covariance matrix $\boldsymbol{\Sigma}_{\mathrm{W}_t}$ is,

$$\boldsymbol{\Sigma}_{\mathrm{W}_t} = \left(\frac{r}{2 - r}\right)\boldsymbol{\Sigma}, \tag{9}$$

where $\boldsymbol{\Sigma}$ is defined in Equation (4). The charting statistic Q_t^2 can be written as follows,

$$Q_t^2 = \frac{2 - r}{r}\mathbf{W}_t^\intercal \boldsymbol{\Sigma}^{-1} \mathbf{W}_t. \tag{10}$$

For the proposed VSI MEWMA GBE scheme, the safety region, the warning region, and the out-of-control region are divided by the upper control limit H_U and the warning control limit H_W. Furthermore, the VSI strategy of the scheme is given as follows:

- If $Q_t^2 \in [0, H_W]$, i.e., the current sample belongs to the safety region, the process is considered as in-control, and the next sample is taken after a long sampling interval $d_L > 1$.
- Otherwise, if $Q_t^2 \in (H_W, H_U]$, the current sample belongs to the warning region, the process is also considered as in-control, but a short sampling interval $d_S \in (0, 1)$ is used for the next sample.
- Finally, if $Q_t^2 \in (H_U, +\infty)$, the current sample belongs to the out-of-control region, the process is deemed to be out-of-control, the proposed VSI MEWMA GBE scheme signals and corresponding corrective action(s) should be taken to remove the assignable cause(s).

3 Average time to signal of the VSI MEWMA GBE scheme

For the VSI type schemes, it is necessary to directly measure the time required to signal. According to Tang et al. [19], the ATS of the FSI type scheme is just a multiple of its ARL. Without loss of generality, we have,

$$\mathrm{ATS}^{\mathrm{FSI}} = \mathrm{ARL}^{\mathrm{FSI}} \times d^{\mathrm{FSI}}, \tag{11}$$

where d^{FSI} represents the fixed sampling interval used in the FSI type scheme. But the ATS of the VSI type scheme depends on both the ARL value and the predetermined sampling intervals, say,

$$\mathrm{ATS}^{\mathrm{VSI}} = \mathrm{ARL}^{\mathrm{VSI}} \times E(d), \tag{12}$$

where $E(d)$ represents the average of sampling intervals (ASI). As shown in Reynolds et al. [39], Saccucci et al. [27], Castagliola et al. [40], and Tang et al. [19], the ASI is usually taken to be $E(d) = 1$ time unit.

In this study, the same in-control ATS value is used to provide a fair comparison between the VSI MEWMA GBE scheme and its FSI counterpart. Based on this assumption, we have,

$$E(d) = p_S \times d_S + p_L \times d_L = d^{FSI} = 1, \tag{13}$$

where $p_S + p_L = 1$, and p_S (p_L) represents the probability of adopting the short (long) sampling interval. It is easy to see that Equation (13) can keep the in-control ARL (hereafter denoted as ARL_0) and the in-control ATS (hereafter denoted as ATS_0) of the scheme at the same value (i.e., $\mathrm{ATS}_0^{\mathrm{FSI}} = \mathrm{ATS}_0^{\mathrm{VSI}} = \mathrm{ARL}_0$). With an acceptable ATS_0, the smaller the out-of-control ATS (hereafter denoted as ATS_1), the better the performance of the control chart. In this paper, the in-control and out-of-control GBE processes are, respectively, modeled by GBE$(\theta_x, \theta_y, \delta)$ and GBE$(\theta'_x, \theta'_y, \delta)$. In what follows, both the zero-state and the steady-state ATS performance of the suggested VSI MEWMA GBE scheme is studied.

3.1 Zero-state case

The *zero-state* ATS is defined as the average time to signal when the process operates with the mean off target from the start (see Lee and Khoo [41]). As recommended by Chen et al. [14], Tang et al. [19], and Guo and Wang [42], the short sampling interval d_S should be used in the proposed VSI MEWMA GBE scheme as a safeguard to provide additional protection against problems that may occur during the start-up (i.e., $d_0 = d_S$, where d_0 is the sampling interval used before the first sample is taken). The Monte Carlo simulation for computing the zero-state ATS value of the VSI MEWMA GBE scheme is given as follows:

Step 1: Specify the scale parameters θ_x and θ_y, the smoothing parameter r, and the dependence parameter δ. Meanwhile, determine the upper control limit H_U and the warning control limit H_W of the suggested VSI MEWMA GBE scheme. Set the cumulative number of short sampling interval $N_S = 1$ (i.e., $d_0 = d_S$) and the cumulative number of long sampling interval $N_L = 0$. In addition, let the initial sampling point $t = 1$.

Step 2: Generate an out-of-control GBE vector $\mathrm{M}_t^{\mathrm{OC}} = (X_t, Y_t)^\intercal$ of the GBE $(\theta'_1, \theta'_2, \delta)$ model at sampling point t using the following equations (see Xie et al. [43]),

$$E = E_1 + VE_2, \tag{14}$$

$$X_t = \theta_x U^\delta E, \tag{15}$$

$$Y_t = \theta_y (1 - U)^\delta E, \tag{16}$$

where U, V, E_1 and E_2 are four independent random variables such that U is a uniform $(0, 1)$ random variable, $V \in \{0, 1\}$ is a Bernoulli random variable with parameter δ (the dependence parameter of the GBE model), say, $P(V = 0) = 1 - \delta$ and $P(V = 1) = \delta$, and E_1 and E_2 are two exponential random variables both with scale parameter $\theta_E = 1$.

Step 3: Compute the charting statistic Q_t^2 at sampling point t using Equation (10). Then,

- if $Q_t^2 \in [0, H_W]$, the process is considered as in-control, let $t = t + 1$, and then move to Step 2 to generate a new $\mathrm{M}_t^{\mathrm{OC}}$ after a long sampling interval d_L (i.e., $N_L = N_L + 1$).

- If $Q_t^2 \in (H_W, H_U]$, the process is also considered as in-control, let $t = t + 1$, and then go to Step 2 to obtain a new $\mathrm{M}_t^{\mathrm{OC}}$ after a short sampling interval d_S (i.e., $N_S = N_S + 1$).
- Otherwise, if $Q_t^2 > H_U$, the process is deemed to be out-of-control, the TS (Time to Signal) value can be calculated using $\mathrm{TS} = d_L \times N_L + d_S \times N_S$. Reset $t = 1$, $N_S = 1$, and $N_L = 0$, and then move to the next step.

Step 4: Repeat Steps 2 and 3 to obtain 10^5 TS values, and then the zero-state ATS of the proposed VSI MEWMA GBE scheme is estimated.

Furthermore, with the constraint on the desired ATS_0, a two-stage procedure for searching the upper control limit H_U and the warning control limit H_W of the suggested VSI MEWMA GBE scheme is given as follows:

Step 1: Choose an acceptable in-control ATS value, say, $\mathrm{ATS}_0 = A$, and specify the smoothing parameter r, the short (long) sampling interval d_S (d_L), and the probability p_S (p_L) of adopting the short (long) sampling interval.

Step 2: With the constraint on $\mathrm{ARL}_0 = \mathrm{ATS}_0/E(d) = A$, search the upper control limit H_U of the recommended scheme first.

Step 3: Based on the specified d_S and p_S (or, d_L and p_L), the corresponding long (short) sampling interval d_L (d_S) can be computed using Equation (13). For instance, if $d_S = 0.1$ and $p_S = 0.4$, it is easy to known that $p_L = 1 - p_S = 0.6$ and $d_L = 1.6$.

Step 4: For each combination of (d_S, d_L), the warning control limit H_W will then be determined by the fixed upper control limit H_U and $\mathrm{ATS}_0 = A$.

According to the two-stage procedure introduced above, when $\delta = 0.5$, $r \in$ {0.01, 0.02, 0.05, 0.07, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1}, $d_S \in \{0.1, 0.3, 0.5\}$, and $p_S \in \{0.2, 0.5, 0.8\}$, the H_U and H_W values of the proposed VSI MEWMA GBE scheme leading to the desired $\mathrm{ATS}_0 = 200$ are presented in Table 1.

(Please insert Table 1 here)

3.2 Steady-state case

The steady-state case is commonly based on a more realistic assumption that some random shifts occur after a period of in-control time, the ATS obtained in this scenario is named the *steady-state* ATS (see Lee and Khoo [29]). The Monte Carlo simulation for computing steady-state ATS is similar to the zero-state ATS case, except that the in-control "warm-up period" should be implemented first (see Xu and Jeske [44]). The steps are given as:

Step 1: Take the scale parameters θ_x and θ_y, the smoothing parameter r, and the dependence parameter δ. Meanwhile, determine the upper control limit H_U and warning control limit H_W. Set the cumulative number of the short (long) sampling interval $N_S = 1$ ($N_L = 0$), and the in-control warm-up period T_w. Additionally, let the initial sampling point $t = 1$.

Step 2: Generate an in-control GBE vector $\mathrm{M}_t^{\mathrm{IC}} = (X_t, Y_t)^\intercal$ of the GBE$(\theta_1, \theta_2, \delta)$ model at sampling point t using Equations (14) to (16).

Step 3: Compute the charting statistic Q_t^2 at sampling point t using Equation (10).

- If $Q_t^2 \in [0, H_W]$ and the corresponding TS $< T_w$ (i.e., set $N_L = N_L + 1$ and then $\text{TS} = d_L \times N_L + d_S \times N_S < T_w$), let $t = t+1$, and then go to Step 2 to get a new $\mathbf{M}_t^{\text{IC}}$. Otherwise, if $Q_t^2 \in [0, H_W]$ but the corresponding TS $\geqslant T_w$ (i.e., set $N_L = N_L + 1$ and then $\text{TS} = d_L \times N_L + d_S \times N_S \geqslant T_w$), let $t = t+1$, and then move to Step 4 to generate a new $\mathbf{M}_t^{\text{OC}}$.
- If $Q_t^2 \in (H_W, H_U]$ and the corresponding TS $< T_w$ (i.e., set $N_S = N_S + 1$ and then $\text{TS} = d_L \times N_L + d_S \times N_S < T_w$), let $t = t+1$, and then go to Step 2 to get a new $\mathbf{M}_t^{\text{IC}}$. Otherwise, if $Q_t^2 \in (H_W, H_U]$ but the corresponding TS $\geqslant T_w$ (i.e., set $N_S = N_S + 1$ and then $\text{TS} = d_L \times N_L + d_S \times N_S \geqslant T_w$), let $t = t + 1$, and then move to Step 4 to generate a new $\mathbf{M}_t^{\text{OC}}$.
- If $Q_t^2 > H_U$, we discard the current in-control random vector $\mathbf{M}_t^{\text{IC}}$, and move back to Step 2 to generate a new $\mathbf{M}_t^{\text{IC}}$.

Step 4: Generate an out-of-control GBE vector $\mathbf{M}_t^{\text{OC}} = (X_t, Y_t)^\intercal$ of the GBE$(\theta_1', \theta_2', \delta)$ model at sampling point t using Equations (14) to (16).

Step 5: Compute the charting statistic Q_t^2 at sampling point t using Equation (10).

- If $Q_t^2 \in [0, H_W]$, let $t = t + 1$, and then go to Step 4 to get a new $\mathbf{M}_t^{\text{OC}}$ after a long sampling interval d_L (i.e., let $N_L = N_L + 1$).
- If $Q_t^2 \in (H_W, H_U]$, let $t = t + 1$, and then go to Step 4 to get a new $\mathbf{M}_t^{\text{OC}}$ after a short sampling interval d_S (i.e., let $N_S = N_S + 1$).
- Otherwise, if $Q_t^2 > H_U$, the process is considered as out-of-control, and the corresponding steady-state TS value, which equals to $d_L \times N_L + d_S \times N_S - T_w$, can be obtained. Then reset $t = 1$, $N_S = 1$, and $N_L = 0$, move to the next step.

Step 6: Repeat Steps 2 to 5 to get 10^5 TS values, then the steady-state ATS value is computed by averaging these TS values.

In the steady-state case, the two-stage procedure for searching the upper control limit H_U and the warning control limit H_W of the suggested VSI MEWMA GBE scheme is similar to the one in the zero-state case, except that the in-control warm-up period T_w has to be run first. With the constraint on $\text{ATS}_0 = 200$, the H_U and H_W values of the proposed scheme are listed in Table 2 for $\delta = 0.5$, $r \in \{0.01, 0.02, 0.05, 0.07, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$, $d_S \in \{0.1, 0.3, 0.5\}$, $p_S \in \{0.2, 0.5, 0.8\}$, and the in-control warm-up period $T_w = 50$.

(Please insert Table 2 here)

4 Performance Comparisons

Since Xie et al. [8] have systematically compared the ARL performance between the MEWMA scheme and the other comparative charts, this section will compare both the zero-state and the steady-state ATS performance between the proposed VSI MEWMA GBE scheme and its FSI counterpart in three different types of shifts (namely, the downward shift, the upward shift, and the hybrid shift). For both the zero-state and the steady-state cases, the mean shift vector $(\varepsilon_x, \varepsilon_y)$ of the GBE model can be defined as,

$$(\varepsilon_x, \varepsilon_y) = \left(\frac{\theta_x'}{\theta_x}, \frac{\theta_y'}{\theta_y} \right), \tag{17}$$

where ε_x and ε_y are the variables used to quantify the mean shift. In this paper, the mean shift vector $(\varepsilon_x, \varepsilon_y) = (1, 1)$ is used to denote the process is in control, and then the ATS_1 properties of the recommended scheme in three different types of shifts are investigated, respectively.

4.1 Detection of the downward shift

For the GBE model studied in this paper, a downward shift is defined, when

- X shifts downward but Y keeps in-control (i.e., $\varepsilon_x < 1, \varepsilon_y = 1$),
- X keeps in-control but Y shifts downward (i.e., $\varepsilon_x = 1, \varepsilon_y < 1$),
- or both X and Y shift downward (i.e., $\varepsilon_x < 1, \varepsilon_y < 1$).

By definition, the first two cases are denoted as "single downward shifts", and the third one is denoted as "double downward shift". Since two single downward shift cases $(\varepsilon_x < 1, \varepsilon_y = 1)$ and $(\varepsilon_x = 1, \varepsilon_y < 1)$ have similar properties, for simplicity, only the ATS performance of the single downward shift case $(\varepsilon_x < 1, \varepsilon_y = 1)$ and the double downward shift case $(\varepsilon_x < 1, \varepsilon_y < 1)$ are investigated in detail in this paper.

Without loss of generality, assume that the $\text{ATS}_0 = 200$, the dependency parameter $\rho = 0.5$, the scale parameters $\theta_x = \theta_y = 1$, and the sampling intervals $(d_S, d_L) = (0.1, 1.9)$. For the downward shift case, both the zero-state and the steady-state ATS performance of the suggested VSI MEWMA GBE scheme is, respectively, listed in Tables 3 and 4. For instance, if $(\varepsilon_x, \varepsilon_y) = (0.8, 1)$ and $r = 0.05$, the zero-state (or the steady-state) ATS_1 value of the proposed VSI MEWMA GBE scheme equals to 63.12 (60.56) (see Table 3 and Table 4). For comparison, the in-control ATS value of the FSI MEWMA scheme (denoted as ATS'_0) is set to be the same as that of the proposed scheme, i.e., $\text{ATS}'_0 = \text{ATS}_0 = 200$. Then, both the zero-state and the steady-state ATS performance of the FSI MEWMA scheme is also presented in Tables 3 and 4, respectively. For instance, the out-of-control ATS value of the FSI MEWMA scheme (denoted as ATS'_1) in the zero-state (or the steady-state) case is 82.65 (79.85) when $(\varepsilon_x, \varepsilon_y) = (0.8, 0.8)$ and $r = 0.02$ (see Table 3 and Table 4).

(Please insert Tables 3 and 4 here)

Note that the steady-state ATS values of these two schemes are obtained by simulations with the in-control warm-up period $T_w = 50$. For each predetermined downward shift vector $(\varepsilon_x, \varepsilon_y)$, the minimum out-of-control ATSs of the VSI MEWMA GBE scheme and its FSI counterpart (denoted as $\text{ATS}_{\min}$ and $\text{ATS}'_{\min}$) are respectively bolded in tables. Several conclusions for the downward shift detection can be drawn as follows:

(1) For both the zero-state and the steady-state cases, the proposed VSI MEWMA GBE scheme is effective for detecting the whole downward shift domain when a relatively small smoothing parameter $r \in (0, 0.1]$ is selected. For example, when the downward shift vector $(\varepsilon_x, \varepsilon_y) = (0.8, 1)$, the zero-state $\text{ATS}_{\min}$ of the proposed VSI MEWMA GBE scheme can be obtained when $r = 0.02$ is selected (see Table 3). Meanwhile, when the downward shift vector $(\varepsilon_x, \varepsilon_y) = (0.8, 0.8)$, the steady-state $\text{ATS}_{\min}$ of the suggested scheme is obtained when r equals to 0.01 (see Table 4). On the other hand, for both the zero-state and the steady-state cases, most ATS_1 values of the suggested VSI MEWMA GBE scheme are larger than the desired ATS_0 when r ranges from 0.2 to 1 (the phenomenon known as "ATS-biased"). This means that the proposed scheme may lose its effect on the downward shift detection if a relatively large smoothing parameter $r \in [0.2, 1]$ is considered.

(2) Irrespective of the zero-state or the steady-state cases, every ATS_{min} of the VSI MEWMA GBE scheme is smaller than the corresponding ATS'_{min} of the FSI MEWMA scheme. This indicates that the VSI MEWMA GBE scheme is effective than the FSI MEWMA scheme to detect a downward shift. For example, when $(\varepsilon_x, \varepsilon_y) = (0.2, 1)$, the zero-state ATS_{min} and ATS'_{min} are 5.08 and 11.86, respectively (see Table 3), and the corresponding steady-state ATS_{min} and ATS'_{min} are 5.14 and 11.26, respectively (see Table 4).

(3) Irrespective of the zero-state or the steady-state cases, with the same smoothing parameter r, the VSI MEWMA GBE scheme is more effective for detecting a single downward shift than the corresponding double downward shift. For instance, when the smoothing parameter $r = 0.05$, the zero-state ATS_1 values of the proposed VSI MEWMA GBE scheme for the single downward shift vector $(\varepsilon_x, \varepsilon_y) = (0.5, 1)$ and the corresponding double downward shift vector $(\varepsilon_x, \varepsilon_y) = (0.5, 0.5)$ are 11.91 and 14.52, respectively (see Table 3). Meanwhile, the corresponding steady-state ATS_1 values are 10.98 and 12.86, respectively (see Table 4).

(4) In order to compare the ATS performance from quantitative assessment, according to Wu et al. [45], the average of the ratio (AR) of ATS values is defined as follows,

$$\text{AR} = \frac{\sum_{j=1}^{m} \left(\text{ATS}_{\text{min}}(\varepsilon_{x,j}, \varepsilon_{y,j}) / \text{ATS}'_{\text{min}}(\varepsilon_{x,j}, \varepsilon_{y,j})\right)}{m}, \tag{18}$$

where m is the number of mean shift vectors included in the comparison, $\text{ATS}_{\text{min}}(\varepsilon_{x,j}, \varepsilon_{y,j})$ is the minimum ATS_1 value produced by the proposed VSI MEWMA GBE scheme at the jth mean shift vector $(\varepsilon_{x,j}, \varepsilon_{y,j})$, and $\text{ATS}'_{\text{min}}(\varepsilon_{x,j}, \varepsilon_{y,j})$ is the minimum ATS'_1 value of the FSI MEWMA scheme at the same mean shift vector. Obviously, if the AR value is smaller than one, the proposed scheme is generally more effective than the FSI MEWMA scheme in the whole shift domain and *vice versa*. For the downward shift detection, as it can be computed from Tables 3 and 4, the AR of the proposed VSI MEWMA GBE scheme for the zero-state case is 0.54. Meanwhile, for the steady-state case, the AR of the proposed scheme is 0.53. This fact indicates that the VSI MEWMA GBE scheme outperforms the FSI MEWMA scheme for detecting the whole downward shift domain, in average.

4.2 Detection of the upward shift

The upward shift in the GBE model can be defined, if

- X shifts upward but Y keeps in-control (i.e., $\varepsilon_x > 1, \varepsilon_y = 1$),
- X keeps in-control but Y shifts upward (i.e., $\varepsilon_x = 1, \varepsilon_y > 1$),
- or both X and Y shift upward (i.e., $\varepsilon_x > 1, \varepsilon_y > 1$).

Similar to the downward shift cases, only the ATS performance of the single upward shift case $(\varepsilon_x > 1, \varepsilon_y = 1)$ and the double upward shift case $(\varepsilon_x > 1, \varepsilon_y > 1)$ are investigated in detail in this section. In addition, the same settings, say, $\text{ATS}_0 = 200$, $\rho = 0.5$, $\theta_x = \theta_y = 1$, and $(d_S, d_L) = (0.1, 1.9)$ are used in this section. For the upward shift detection, both the zero-state and the steady-state ATS performance of the proposed VSI MEWMA GBE scheme is presented in Tables 5 and 6, respectively. For example, the zero-state (or the steady-state) ATS_1 of the proposed VSI MEWMA GBE scheme is 15.51 (15.39) when $(\varepsilon_x, \varepsilon_y) = (1.5, 1)$ and $r = 0.1$ are selected (see Table 5 and Table 6).

(Please insert Tables 5 and 6 here)

As a comparison, when $\text{ATS}'_0 = \text{ATS}_0 = 200$, both the zero-state and the steady-state ATS performance of the FSI MEWMA scheme for the upward shift domain is also given in Tables 5 and 6, respectively. For instance, when $(\varepsilon_x, \varepsilon_y) = (2, 1)$ and $r = 0.07$, the zero-state ATS'_1 of the FSI MEWMA scheme is 6.62 (see Table 5), and the corresponding steady-state ATS'_1 is 6.73 (see Table 6). It is worth noting that, for the upward shift detection, the steady-state ATS values of these two schemes are also obtained by using the in-control warm-up period $T_w = 50$. Several conclusions of the upward shift detection can be made as follows:

(1) For both the zero-state and the steady-state cases, the proposed VSI MEWMA GBE scheme with a relatively small (or large) smoothing parameter r is effective in detecting a small (large) upward shift. For instance, when the upward shift vectors are $(\varepsilon_x, \varepsilon_y) = (1.5, 1)$ and $(\varepsilon_x, \varepsilon_y) = (10, 1)$, the zero-state ATS_{min} of the proposed VSI MEWMA GBE scheme can be obtained when the smoothing parameters r are 0.07 and 0.9, respectively (see Table 5). Meanwhile, when the upward shift vectors are $(\varepsilon_x, \varepsilon_y) = (1.5, 1.5)$ and $(\varepsilon_x, \varepsilon_y) = (10, 10)$, the steady-state ATS_{min} of the suggested scheme can be obtained when the smoothing parameters r are 0.1 and 1, respectively (see Table 6).

(2) Irrespective of the zero-state or the steady-state cases, when using the same smoothing parameter r, the VSI MEWMA GBE scheme seems more effective in detecting the double upward shift than the corresponding single upward shift. For instance, when the smoothing parameter $r = 0.2$, for the upward shift vectors $(\varepsilon_x, \varepsilon_y) = (5, 1)$ and $(\varepsilon_x, \varepsilon_y) = (5, 5)$, the zero-state ATS_1 values of the proposed scheme are, respectively, 1.13 and 1.03 (see Table 5). Meanwhile, the steady-state ATS_1 are 1.82 and 1.80, respectively (see Table 6).

(3) Irrespective of the zero-state or the steady-state cases, every ATS_{min} of the VSI MEWMA GBE scheme is smaller than the corresponding ATS'_{min} of the FSI MEWMA scheme. This indicates that the VSI MEWMA GBE scheme works better than the FSI MEWMA scheme in monitoring upward shifts. For instance, when the upward shift vector $(\varepsilon_x, \varepsilon_y) = (2, 2)$, the zero-state ATS_{min} and ATS'_{min} are 5.56 and 8.68, respectively (see Table 5), and the corresponding steady-state ATS_{min} and ATS'_{min} are 1.70 and 2.34, respectively (see Table 6).

(4) The zero-state and the steady-state AR values for the upward shift case are 0.47 and 0.72, respectively (see Tables 5 and 6). This means that, in average, the VSI MEWMA GBE scheme works better than the FSI MEWMA scheme in detecting upward shifts.

4.3 Detection of the hybrid shift

For the GBE model, a hybrid shift is defined, if

- X shifts downward but Y shifts upward (i.e., $\varepsilon_x < 1, \varepsilon_y > 1$),
- or X shifts upward but Y shifts downward (i.e., $\varepsilon_x > 1, \varepsilon_y < 1$).

The same settings as in the upward shift case are also used for the hybrid shift case, i.e., $\text{ATS}_0 = 200$, $\rho = 0.5$, $\theta_x = \theta_y = 1$, and $(d_S, d_L) = (0.1, 1.9)$. For the hybrid shift detection, both the zero-state and the steady-state ATS performance of the proposed VSI MEWMA GBE scheme is, respectively, given in Tables 7 and 8. For instance, when $(\varepsilon_x, \varepsilon_y) = (0.8, 1.5)$ and $r = 0.2$, the zero-state ATS_1 of the proposed VSI MEWMA GBE scheme is 11.44 (see Table 7), and the corresponding steady-state ATS_1 value equals to 11.55 (see Table 8). For comparison, both the zero-state and the steady-state ATS performance of the FSI MEWMA scheme based on $\text{ATS}'_0 = 200$ is also listed in Tables 7 and 8. For instance, when $(\varepsilon_x, \varepsilon_y) = (0.5, 2)$ and $r = 0.4$, the zero-state ATS'_1 of the FSI MEWMA scheme is 2.68 (see Table 7), and the corresponding steady-state ATS'_1 value is 3.24 (see Table 8).

(Please insert Tables 7 and 8 here)

From Tables 7 and 8, some conclusions can be made as follows:

(1) Similar to the upward shift case, when using a relatively small (or large) smoothing parameter r, the suggested VSI MEWMA GBE scheme is also effective for detecting small (large) hybrid shifts. For example, when the hybrid shift vectors $(\varepsilon_x, \varepsilon_y) = (0.8, 1.5)$ and $(\varepsilon_x, \varepsilon_y) = (0.2, 5)$, the zero-state $\mathrm{ATS}_{\mathrm{min}}$ of the proposed scheme can be obtained when the smoothing parameters are $r = 0.1$ and $r = 0.8$, respectively (see Table 7). In addition, when the hybrid shift vectors are $(\varepsilon_x, \varepsilon_y) = (0.5, 2)$ and $(\varepsilon_x, \varepsilon_y) = (0.1, 10)$, the steady-state $\mathrm{ATS}_{\mathrm{min}}$ of the suggested scheme can be obtained when smoothing parameters are $r = 0.3$ and $r = 1$, respectively (see Table 8).

(2) Every $\mathrm{ATS}_{\mathrm{min}}$ of the VSI MEWMA GBE scheme is smaller than the corresponding $\mathrm{ATS}'_{\mathrm{min}}$ of the FSI MEWMA scheme. This indicates that the VSI MEWMA GBE scheme performs better than the FSI MEWMA scheme for detecting hybrid shifts. For example, when the upward shift vector $(\varepsilon_x, \varepsilon_y) = (0.5, 2)$, the zero-state $\mathrm{ATS}_{\mathrm{min}}$ and $\mathrm{ATS}'_{\mathrm{min}}$ values are 2.61 and 6.95, respectively (see Table 7). Meanwhile, the steady-state $\mathrm{ATS}_{\mathrm{min}}$ and $\mathrm{ATS}'_{\mathrm{min}}$ values are 3.17 and 6.80, respectively (see Table 8).

(3) The zero-state and the steady-state AR values in the case of hybrid shift are 0.30 and 0.56, respectively (see Tables 7 and 8). This fact means that the suggested VSI MEWMA GBE scheme also outperforms the FSI MEWMA scheme in the whole hybrid shift detection.

5 A simulation example

Similar to the headache relief time dataset analyzed by Xie et al. [8], a simulation example of muscle strain relief time is used here to illustrate the implementation of the proposed VSI MEWMA GBE scheme for monitoring GBE data.

In the medical experiment, it was assumed that each of the first 10 volunteers had been treated separately for muscle strain with modality therapy and manual therapy, the corresponding relief time (in hours) from muscle strain are denoted as X and Y, respectively. All of these 10 paired GBE data (X, Y) are recorded in Table 9. According to Xie et al. [8], if the first 10 couples are considered as the in-control GBE data, the corresponding average relief time $\widehat{\boldsymbol{\mu}}$ can be estimated using $\widehat{\boldsymbol{\mu}} = \bar{\mathrm{M}} = \frac{1}{n}\sum_{t=1}^{n} \mathrm{M}_t$. Additionally, the dependency parameter is estimated by using $\widehat{\delta} = -\log_2\left(\frac{1}{n}\sum_{t=1}^{n} \min\left(X_t/\bar{X}, Y_t/\bar{Y}\right)\right)$, where $n = 10$ in this example (see Lu and Bhattacharyya [38]). Based on these two formulas, the estimated scale parameters and dependency parameter are $\widehat{\theta}_1 = 3.43$, $\widehat{\theta}_2 = 2.68$, and $\widehat{\delta} = 0.21$, respectively.

(Please insert Table 9 here)

We further assume that a new medicine will be used in combination with these two physical therapies in the subsequent medical experiment, and the pharmaceutical company claims that the new medicine could reduce the average relief time of these two treatments effectively. Based on this assumption, we use the proposed VSI MEWMA GBE scheme to monitor the relief time data to verify the effectiveness of the new medicine, and the medical experiment can be ended when the suggested VSI MEWMA GBE scheme signals. According to the pre-designed experimental guidelines, only after a specified waiting time to ensure that the new medicine is completely metabolized and has no other side effects, then a new volunteer can be invited to combine the new medicine for two physical therapies to obtain the next relief time data. If the statistic (in this example, it is Q_t^2) shows that the efficacy of the new medicine is not significant, we need to wait longer to ensure that the health of volunteers is not damaged by other unknown factors. But when the statistic (i.e., Q_t^2) shows that the therapeutic effect is obvious, we can shorten the waiting time to speed up the experiment on the basis that the medicine is completely metabolized. In this example, we assume that the desired experimental time is 4800 hours (i.e., $\mathrm{ATS}_0 = 4800$), the average waiting time interval $E(d) = 24$ hours, the short waiting time interval $d_S = 12$ hours, and the long waiting time interval $d_L = 36$ hours.

On the other hand, if we assume that the use of the new medicine can shorten the average relief time of two physical therapies to 80% and 50% of the defined one, say, $(\varepsilon_x, \varepsilon_y) = (0.8, 0.5)$, then the corresponding Phase II GBE data are listed in Table 9. According to the guideline mentioned in Section 4.1, the proposed VSI MEWMA GBE scheme is effective for detecting the whole downward shift domain when a relatively small smoothing parameter $r \in (0, 0.1]$ is selected. Hence, the smoothing parameter $r = 0.02$ is considered in this example. Moreover, based on the two-stage procedure introduced above, when $\widehat{\delta} = 0.21$, $d_S = 12$, and $d_L = 36$, the upper control limit H_U and the warning control limit H_W of the VSI MEWMA GBE scheme can be easily obtained to achieve the desired $\mathrm{ATS}_0 = 4800$, say, $H_U = 5.256$ and $H_W = 0.902$. With these designed parameters, the charting statistic Q_t^2 of the VSI MEWMA GBE scheme can be obtained, see Column 6 in Table 9. Note that the proposed VSI MEWMA GBE scheme signals at the $25th$ volunteer (in bold), and the corresponding waiting time is 276 hours. However, if the FSI MEWMA scheme is used in the process to monitor the relief time data, the waiting time will be 360 hours (15 volunteers $\times$ 24 hours). This fact indicates that the suggested VSI MEWMA GBE scheme works better than the FSI MEWMA scheme in monitoring the GBE data.

6 Conclusion

In this paper, a VSI MEWMA type scheme is proposed to monitor the mean vector of GBE model. For each type of shift, the Monte Carlo approach is used to evaluate the properties of the proposed scheme in both the zero-state and the steady-state cases. Comparisons between the suggested VSI MEWMA GBE scheme and the FSI MEWMA scheme in detecting three types of shifts are conducted, and the simulation results show that the suggested VSI MEWMA GBE scheme works better than the FSI MEWMA scheme in monitoring the whole shift domain. Finally, a simulation example is provided to illustrate the implementation of the suggested VSI MEWMA GBE scheme for monitoring the data of muscle strain relief time.

It is sensible to note that the zero-state and the steady-state ATS can not represent the entire TS distribution of the VSI charts. The median time to signal (MTS) could also be used as a new performance measure for the VSI MEWMA GBE scheme in both the zero-state and steady-state cases. Hence, the current work can be extended to design a VSI MEWMA type scheme based on MTS for monitoring the GBE data.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Table captions

Table 1: Both H_U and H_W values of the suggested VSI MEWMA GBE scheme in the zero-state case, for $\mathrm{ATS}_0 = 200$, $\delta = 0.5$, $r \in \{$0.01, 0.02, 0.05, 0.07, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$\}$, $d_S \in \{0.1, 0.3, 0.5\}$, and $p_S \in \{0.2, 0.5, 0.8\}$.
Table 2: Both H_U and H_W values of the suggested VSI MEWMA GBE scheme in the steady-state case, for $\mathrm{ATS}_0 = 200$, $\delta = 0.5$, $r \in \{$0.01, 0.02, 0.05, 0.07, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$\}$, $d_S \in \{0.1, 0.3, 0.5\}$, $p_S \in \{0.2, 0.5, 0.8\}$, and $T_w = 50$.
Table 3: Zero-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the downward shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).
Table 4: Steady-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the downward shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).
Table 5: Zero-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the upward shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).
Table 6: Steady-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the upward shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).
Table 7: Zero-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the hybrid shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).
Table 8: Steady-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the hybrid shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).
Table 9: An example of using the proposed VSI MEWMA GBE chart on monitoring the relief time of volunteers after taking the new medicine (in hours).

Table 1: Both H_U and H_W values of the suggested VSI MEWMA GBE scheme in the zero-state case, for $\text{ATS}_0 = 200$, $\delta = 0.5$, $r \in \{0.01, 0.02, 0.05, 0.07, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$, $d_S \in \{0.1, 0.3, 0.5\}$, and $p_S \in \{0.2, 0.5, 0.8\}$.

			r													
p_S	(d_S, d_L)		0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
		H_U	3.78	5.26	7.52	8.67	10.34	14.67	18.17	21.15	23.65	25.72	27.32	28.47	29.21	29.53
0.5	(0.1,1.225)	H_W	1.64	2.11	2.62	2.67	2.656	2.73	2.58	2.334	2.27	2.11	2.01	2.04	1.95	1.946
	(0.3,1.175)	H_W	1.649	2.13	2.55	2.637	2.656	2.705	2.561	2.316	2.253	2.07	2.01	2.01	1.94	1.924
	(0.5,1.125)	H_W	1.647	2.129	2.533	2.637	2.629	2.705	2.561	2.325	2.24	2.04	2.01	2.009	1.94	1.924
	(0.1,1.9)	H_W	0.71	0.93	1.13	1.15	1.17	1.18	1.11	1.05	0.98	0.93	0.88	0.84	0.81	0.80
	(0.3,1.7)	H_W	0.71	0.93	1.13	1.17	1.17	1.18	1.12	1.06	0.984	0.93	0.88	0.84	0.814	0.801
	(0.5,1.5)	H_W	0.72	0.94	1.13	1.17	1.156	1.18	1.13	1.06	0.99	0.94	0.89	0.84	0.81	0.798
	(0.1,4.6)	H_W	0.214	0.288	0.36	0.377	0.387	0.394	0.389	0.376	0.365	0.352	0.344	0.34	0.339	0.346
	(0.3,3.8)	H_W	0.214	0.288	0.36	0.377	0.387	0.394	0.387	0.376	0.363	0.353	0.347	0.345	0.344	0.344
	(0.5,3.0)	H_W	0.214	0.292	0.36	0.377	0.387	0.394	0.387	0.376	0.363	0.352	0.344	0.345	0.344	0.348

Table 2: Both H_U and H_W values of the suggested VSI MEWMA GBE scheme in the steady-state case, for $\text{ATS}_0 = 200$, $\delta = 0.5$, $r \in \{0.01, 0.02, 0.05, 0.07, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$, $d_S \in \{0.1, 0.3, 0.5\}$, $p_S \in \{0.2, 0.5, 0.8\}$, and $T_w = 50$.

			r													
p_S	(d_S, d_L)		0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
		H_U	4.16	5.56	7.65	8.77	10.37	14.75	18.23	21.18	23.62	25.71	27.34	28.49	29.21	29.54
0.5	(0.1,1.225)	H_W	1.551	1.96	2.45	2.55	2.65	2.56	2.45	2.28	2.17	2.03	1.91	1.80	1.785	1.77
	(0.3,1.175)	H_W	1.611	1.96	2.48	2.58	2.65	2.56	2.45	2.28	2.20	2.03	1.91	1.83	1.865	1.77
	(0.5,1.125)	H_W	1.641	1.99	2.51	2.58	2.65	2.56	2.45	2.28	2.23	2.03	1.91	1.83	1.895	1.77
	(0.1,1.9)	H_W	0.70	0.89	1.11	1.15	1.18	1.16	1.11	1.04	0.99	0.93	0.88	0.83	0.81	0.796
	(0.3,1.7)	H_W	0.72	0.91	1.12	1.15	1.19	1.15	1.10	1.04	0.98	0.935	0.873	0.84	0.81	0.80
	(0.5,1.5)	H_W	0.75	0.93	1.11	1.16	1.19	1.16	1.09	1.04	0.98	0.93	0.87	0.84	0.82	0.80
	(0.1,4.6)	H_W	0.214	0.278	0.353	0.371	0.382	0.387	0.379	0.369	0.363	0.349	0.339	0.337	0.34	0.339
	(0.3,3.8)	H_W	0.224	0.289	0.359	0.374	0.387	0.387	0.379	0.369	0.36	0.349	0.34	0.337	0.34	0.34
	(0.5,3.0)	H_W	0.229	0.293	0.361	0.374	0.389	0.389	0.379	0.369	0.36	0.349	0.337	0.337	0.34	0.341

Table 3: Zero-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the downward shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).

VSI MEWMA GBE chart and FSI MEWMA chart															
	r	0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$(\varepsilon_x, \varepsilon_y)$	H_W	0.71	0.93	1.13	1.15	1.17	1.18	1.11	1.05	0.98	0.93	0.88	0.84	0.81	0.80
	H_U	3.78	5.26	7.52	8.67	10.34	14.67	18.17	21.15	23.65	25.72	27.32	28.47	29.21	29.53
(1,1)	ATS_0	199.78	200.12	200.58	199.84	200.59	200.17	199.85	200.22	199.64	199.64	199.41	200.38	199.80	200.44
	ATS'_0	199.94	200.12	200.40	200.29	200.20	199.62	199.86	199.67	200.01	200.13	200.17	200.18	199.89	199.83
(0.8,1)	ATS_1	53.28	**49.94**	63.12	86.78	127.17	*	*	*	*	*	*	*	*	*
	ATS'_1	**66.23**	67.92	98.17	136.99	185.20	*	*	*	*	*	*	*	*	*
(0.5,1)	ATS_1	18.96	15.72	11.91	**11.04**	11.56	23.09	46.77	75.93	104.93	133.28	159.83	186.27	*	*
	ATS'_1	24.97	22.76	**22.51**	25.82	39.35	96.77	141.81	176.64	*	*	*	*	*	*
(0.2,1)	ATS_1	11.46	9.32	6.62	5.77	**5.08**	5.72	7.46	9.90	12.83	16.87	22.78	33.08	50.71	84.69
	ATS'_1	15.34	13.53	**11.86**	11.94	13.38	30.53	56.10	82.05	104.94	124.73	139.88	153.82	165.39	176.77
(0.1,1)	ATS_1	10.17	8.28	5.65	5.07	**4.19**	4.43	5.46	7.42	9.29	10.85	12.19	14.03	18.34	32.51
	ATS'_1	13.64	11.94	10.27	**10.16**	10.90	21.38	40.08	61.27	81.20	99.40	114.06	127.36	137.75	149.23
(0.8,0.8)	ATS_1	60.66	**59.10**	104.19	*	*	*	*	*	*	*	*	*	*	*
	ATS'_1	**76.01**	82.65	175.11	*	*	*	*	*	*	*	*	*	*	*
(0.5,0.5)	ATS_1	21.89	18.46	14.52	**13.94**	21.41	*	*	*	*	*	*	*	*	*
	ATS'_1	28.07	**26.05**	27.67	35.32	112.56	*	*	*	*	*	*	*	*	*
(0.2,0.2)	ATS_1	13.07	10.76	7.73	6.93	**6.16**	*	*	*	*	*	*	*	*	*
	ATS'_1	16.99	15.10	**13.50**	13.90	16.55	*	*	*	*	*	*	*	*	*
(0.1,0.1)	ATS_1	11.40	9.22	6.62	5.90	**4.98**	*	*	*	*	*	*	*	*	*
	ATS'_1	15.03	13.24	**11.53**	11.57	12.82	*	*	*	*	*	*	*	*	*

The asterisk (*) represents the ATS value lager than 200.

Table 4: Steady-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the downward shift domain ($\mathrm{ATS}_0 = \mathrm{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).

VSI MEWMA GBE chart and FSI MEWMA chart															
	r	0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$(\varepsilon_x, \varepsilon_y)$	H_W	0.70	0.89	1.11	1.15	1.18	1.16	1.11	1.04	0.99	0.93	0.88	0.83	0.81	0.796
	H_U	4.16	5.56	7.65	8.77	10.37	14.75	18.23	21.18	23.62	25.71	27.34	28.49	29.21	29.54
(1,1)	ATS_0	199.93	200.70	200.64	200.04	200.51	199.47	199.83	200.40	200.06	200.62	199.71	199.20	200.40	200.51
	ATS'_0	200.25	199.82	200.14	200.17	199.80	199.72	199.73	200.11	199.48	200.59	199.62	199.90	199.65	200.18
(0.8,1)	ATS_1	45.86	**44.26**	60.56	85.90	126.91	*	*	*	*	*	*	*	*	*
	ATS'_1	**63.98**	64.94	97.13	137.02	184.41	*	*	*	*	*	*	*	*	*
(0.5,1)	ATS_1	16.48	13.80	10.98	**10.39**	11.15	22.85	47.05	75.91	106.75	134.07	161.06	185.07	*	*
	ATS'_1	24.30	21.71	**21.53**	25.07	38.82	97.19	143.22	176.97	*	*	*	*	*	*
(0.2,1)	ATS_1	10.32	8.47	6.34	5.66	**5.14**	5.87	7.95	10.41	13.21	17.18	23.53	33.14	51.49	84.71
	ATS'_1	14.94	12.93	**11.26**	11.46	13.00	30.68	56.44	81.90	104.50	124.52	140.81	153.48	164.69	175.97
(0.1,1)	ATS_1	9.23	7.57	5.61	4.98	**4.46**	4.59	6.04	7.80	9.64	11.31	12.96	14.92	19.12	32.78
	ATS'_1	13.26	11.42	9.75	**9.73**	10.52	21.49	40.38	61.48	81.00	98.36	114.41	126.68	138.49	148.34
(0.8,0.8)	ATS_1	**51.84**	51.98	103.64	*	*	*	*	*	*	*	*	*	*	*
	ATS'_1	**73.31**	79.85	178.53	*	*	*	*	*	*	*	*	*	*	*
(0.5,0.5)	ATS_1	18.53	15.64	12.86	**12.74**	20.74	*	*	*	*	*	*	*	*	*
	ATS'_1	27.30	**24.86**	26.53	34.70	113.58	*	*	*	*	*	*	*	*	*
(0.2,0.2)	ATS_1	11.39	9.37	7.11	6.38	**5.92**	*	*	*	*	*	*	*	*	*
	ATS'_1	16.45	14.32	**12.75**	13.27	16.05	*	*	*	*	*	*	*	*	*
(0.1,0.1)	ATS_1	10.08	8.27	6.18	5.50	**4.98**	*	*	*	*	*	*	*	*	*
	ATS'_1	14.52	12.52	**10.87**	11.03	12.37	*	*	*	*	*	*	*	*	*

The asterisk (*) represents the ATS value lager than 200.

Table 5: Zero-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the upward shift domain ($\text{ATS}_0 = \text{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).

VSI MEWMA GBE chart and FSI MEWMA chart															
	r	0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$(\varepsilon_x, \varepsilon_y)$	H_W	0.71	0.93	1.13	1.15	1.17	1.18	1.11	1.05	0.98	0.93	0.88	0.84	0.81	0.80
	H_U	3.78	5.26	7.52	8.67	10.34	14.67	18.17	21.15	23.65	25.72	27.32	28.47	29.21	29.53
(1,1)	ATS_0	199.78	200.12	200.58	199.84	200.59	200.17	199.85	200.22	199.64	199.64	199.41	200.38	199.80	200.44
	ATS'_0	199.94	200.12	200.40	200.29	200.20	199.62	199.86	199.67	200.01	200.13	200.17	200.18	199.89	199.83
(1.5,1)	ATS_1	22.90	19.71	16.35	**15.46**	15.51	18.49	21.96	25.74	28.93	32.17	34.69	37.24	38.86	40.06
	ATS'_1	26.15	23.65	**21.24**	21.38	22.28	26.37	30.63	34.26	37.28	39.75	41.71	43.58	45.07	46.29
(2,1)	ATS_1	11.32	9.59	7.34	6.62	6.07	**5.69**	6.06	6.88	7.80	8.87	9.97	11.03	12.01	12.77
	ATS'_1	13.35	11.78	10.01	9.69	**9.66**	10.32	11.39	12.52	13.62	14.60	15.56	16.27	17.08	17.72
(5,1)	ATS_1	2.90	2.42	1.80	1.60	1.42	1.13	0.97	0.90	0.86	0.81	**0.77**	0.78	0.82	0.91
	ATS'_1	4.02	3.58	3.04	2.89	2.78	2.65	**2.65**	2.67	2.71	2.75	2.82	2.87	2.95	3.02
(10,1)	ATS_1	1.33	1.12	0.85	0.76	0.68	0.54	0.46	0.42	0.39	0.35	0.31	0.27	**0.26**	0.27
	ATS'_1	2.34	2.13	1.88	1.81	1.77	1.69	1.66	**1.65**	1.66	1.66	1.66	1.68	1.68	1.71
(1.5,1.5)	ATS_1	23.01	19.89	16.37	15.41	**15.02**	16.18	17.82	19.59	21.12	22.68	23.84	24.69	25.47	25.17
	ATS'_1	26.01	23.00	19.63	**19.06**	19.19	20.67	22.59	24.52	26.00	27.33	28.25	29.30	30.05	30.81
(2,2)	ATS_1	10.96	9.19	7.07	6.41	5.88	**5.56**	5.74	6.15	6.66	7.17	7.73	8.20	8.37	8.34
	ATS'_1	13.15	11.42	9.43	8.93	**8.68**	8.68	9.13	9.62	10.16	10.55	10.98	11.45	11.78	12.15
(5,5)	ATS_1	2.53	2.11	1.58	1.39	1.24	1.03	0.93	0.89	0.87	**0.86**	0.87	0.88	0.90	0.87
	ATS'_1	3.78	3.32	2.78	2.64	2.53	2.37	2.32	**2.32**	2.32	2.33	2.35	2.38	2.41	2.43
(10,10)	ATS_1	1.11	0.93	0.71	0.64	0.57	0.48	0.44	0.41	0.39	0.38	0.37	0.36	0.36	**0.35**
	ATS'_1	2.14	1.96	1.73	1.67	1.62	1.54	1.52	1.50	**1.49**	1.50	1.50	1.50	1.51	1.51

Table 6: Steady-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the upward shift domain ($\text{ATS}_0 = \text{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).

VSI MEWMA GBE chart and FSI MEWMA chart															
	r	0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$(\varepsilon_x, \varepsilon_y)$	H_W	0.70	0.89	1.11	1.15	1.18	1.16	1.11	1.04	0.99	0.93	0.88	0.83	0.81	0.796
	H_U	4.16	5.56	7.65	8.77	10.37	14.75	18.23	21.18	23.62	25.71	27.34	28.49	29.21	29.54
(1,1)	ATS_0	199.93	200.70	200.64	200.04	200.51	199.47	199.83	200.40	200.06	200.62	199.71	199.20	200.40	200.51
	ATS'_0	200.25	199.82	200.14	200.17	199.80	199.72	199.73	200.11	199.48	200.59	199.62	199.90	199.65	200.18
(1.5,1)	ATS_1	20.38	17.73	15.48	**15.08**	15.39	18.54	22.62	26.32	29.84	32.78	35.29	37.39	39.85	40.64
	ATS'_1	25.42	22.79	**20.76**	20.99	21.89	26.40	30.55	34.14	37.15	39.59	41.94	43.50	45.14	46.54
(2,1)	ATS_1	10.53	9.00	7.30	6.73	6.31	**6.12**	6.61	7.46	8.50	9.57	10.74	11.71	12.90	13.66
	ATS'_1	13.04	11.46	9.80	9.52	**9.50**	10.31	11.40	12.53	13.61	14.54	15.54	16.38	17.17	17.70
(5,1)	ATS_1	3.41	2.98	2.45	2.26	2.09	1.82	1.71	1.65	1.61	**1.59**	1.60	1.62	1.71	1.79
	ATS'_1	3.96	3.50	2.97	2.85	2.74	2.63	**2.63**	2.66	2.71	2.76	2.82	2.88	2.94	3.01
(10,1)	ATS_1	2.10	1.90	1.62	1.52	1.44	1.31	1.26	1.22	1.20	1.18	1.17	**1.14**	1.15	1.16
	ATS'_1	2.30	2.09	1.86	1.80	1.75	1.68	1.65	**1.65**	1.65	1.66	1.67	1.68	1.68	1.70
(1.5,1.5)	ATS_1	21.06	18.41	15.87	15.32	**15.20**	16.56	18.52	20.16	22.01	23.40	24.78	25.46	26.21	25.79
	ATS'_1	25.47	22.30	19.33	**18.76**	18.86	20.57	22.70	24.38	25.87	27.09	28.31	29.38	29.91	30.70
(2,2)	ATS_1	10.55	9.00	7.27	6.77	6.39	**6.13**	6.45	6.88	7.44	7.99	8.57	8.91	9.24	9.14
	ATS'_1	12.88	11.16	9.27	8.80	**8.54**	8.65	9.08	9.60	10.07	10.55	11.07	11.37	11.76	12.08
(5,5)	ATS_1	3.21	2.79	2.32	2.15	2.00	1.80	1.74	**1.70**	1.71	1.71	1.74	1.75	1.78	1.76
	ATS'_1	3.77	3.29	2.77	2.63	2.51	2.36	2.32	**2.31**	2.31	2.35	2.37	2.38	2.40	2.44
(10,10)	ATS_1	1.97	1.77	1.54	1.46	1.39	1.30	1.28	1.26	1.25	1.25	1.25	1.25	1.25	**1.23**
	ATS'_1	2.14	1.95	1.72	1.66	1.61	1.54	1.51	1.50	**1.49**	1.50	1.50	1.50	1.50	1.51

Table 7: Zero-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the hybrid shift domain ($\text{ATS}_0 = \text{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).

		VSI MEWMA GBE chart and FSI MEWMA chart													
	r	0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$(\varepsilon_x, \varepsilon_y)$	H_W	0.71	0.93	1.13	1.15	1.17	1.18	1.11	1.05	0.98	0.93	0.88	0.84	0.81	0.80
	H_U	3.78	5.26	7.52	8.67	10.34	14.67	18.17	21.15	23.65	25.72	27.32	28.47	29.21	29.53
(1,1)	ATS_0	199.78	200.12	200.58	199.84	200.59	200.17	199.85	200.22	199.64	199.64	199.41	200.38	199.80	200.44
	ATS'_0	199.94	200.12	200.40	200.29	200.20	199.62	199.86	199.67	200.01	200.13	200.17	200.18	199.89	199.83
(0.8,1.5)	ATS_1	17.07	14.39	11.22	10.32	**9.83**	11.44	14.52	18.20	21.84	25.61	29.08	32.41	35.65	38.05
	ATS'_1	20.52	18.41	**16.50**	16.62	17.66	22.15	26.51	30.55	34.22	37.06	39.77	42.34	43.95	45.56
(0.5,2)	ATS_1	7.69	6.38	4.67	4.07	3.57	2.89	**2.61**	2.68	2.93	3.37	3.96	4.83	6.10	7.92
	ATS'_1	9.83	8.61	7.26	7.02	**6.95**	7.45	8.42	9.44	10.54	11.58	12.47	13.37	14.17	14.92
(0.2,5)	ATS_1	2.41	2.01	1.47	1.28	1.11	0.84	0.70	0.63	0.58	0.49	0.36	**0.31**	0.31	0.33
	ATS'_1	3.60	3.21	2.73	2.58	2.50	2.37	**2.36**	2.39	2.42	2.47	2.54	2.57	2.64	2.72
(0.1,10)	ATS_1	1.20	1.00	0.75	0.67	0.59	0.46	0.39	0.35	0.32	0.26	0.17	**0.16**	0.16	0.17
	ATS'_1	2.23	2.03	1.80	1.74	1.69	1.61	1.59	1.58	**1.58**	1.58	1.60	1.60	1.62	1.63

Table 8: Steady-state ATS values of the proposed VSI MEWMA GBE and FSI MEWMA schemes for the hybrid shift domain ($\text{ATS}_0 = \text{ATS}'_0 = 200, \delta = 0.5, d_S = 0.1, d_L = 1.9$).

		VSI MEWMA GBE chart and FSI MEWMA chart													
	r	0.01	0.02	0.05	0.07	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$(\varepsilon_x, \varepsilon_y)$	H_W	0.70	0.89	1.11	1.15	1.18	1.16	1.11	1.04	0.99	0.93	0.88	0.83	0.81	0.796
	H_U	4.16	5.56	7.65	8.77	10.37	14.75	18.23	21.18	23.62	25.71	27.34	28.49	29.21	29.54
(1,1)	ATS_0	199.93	200.70	200.64	200.04	200.51	199.47	199.83	200.40	200.06	200.62	199.71	199.20	200.40	200.51
	ATS'_0	200.25	199.82	200.14	200.17	199.80	199.72	199.73	200.11	199.48	200.59	199.62	199.90	199.65	200.18
(0.8,1.5)	ATS_1	15.17	12.96	10.61	10.01	**9.79**	11.55	14.94	18.65	22.57	26.26	29.75	32.78	36.36	38.91
	ATS'_1	19.97	17.70	**16.10**	16.31	17.31	22.00	26.44	30.46	34.19	37.08	39.61	42.24	43.86	45.66
(0.5,2)	ATS_1	7.37	6.18	4.79	4.31	3.90	3.29	**3.17**	3.24	3.53	4.01	4.71	5.55	6.96	8.65
	ATS'_1	9.61	8.30	7.02	6.81	**6.80**	7.40	8.39	9.42	10.49	11.55	12.50	13.36	14.22	14.90
(0.2,5)	ATS_1	3.00	2.60	2.11	1.94	1.79	1.54	1.43	1.36	1.31	1.27	1.24	1.20	**1.19**	1.22
	ATS'_1	3.53	3.13	2.66	2.55	2.45	2.37	**2.35**	2.38	2.42	2.47	2.53	2.59	2.64	2.71
(0.1,10)	ATS_1	2.00	1.78	1.53	1.43	1.35	1.23	1.18	1.15	1.12	1.10	1.08	1.06	1.05	**1.04**
	ATS'_1	2.20	2.00	1.77	1.72	1.67	1.61	1.58	**1.58**	1.58	1.58	1.59	1.60	1.62	1.63

Table 9: An example of using the proposed VSI MEWMA GBE chart on monitoring the relief time of volunteers after taking the new medicine (in hours).

VSI MEWMA GBE chart ($r = 0.02$, $\delta = 0.21$, $H_U = 5.256$, $H_W = 0.902$)							
t	$\mathbf{M}_t$		$\mathbf{W}_t$		Q_t^2	d_t	Σd_t
	X_t	Y_t	$W_{1,t}$	$W_{2,t}$			
0	-	-	0	0	-	-	-
1	3.400	1.900	-0.001	-0.016	0.015	-	-
2	2.700	1.800	-0.015	-0.033	0.035	-	-
3	5.100	5.300	0.019	0.020	0.006	-	-
4	4.600	4.400	0.042	0.054	0.057	-	-
5	4.300	3.000	0.058	0.059	0.053	-	-
6	4.100	3.800	0.070	0.081	0.109	-	-
7	4.000	1.100	0.080	0.047	0.059	-	-
8	2.700	2.400	0.064	0.041	0.036	-	-
9	3.100	3.000	0.056	0.047	0.030	-	-
10	0.300	0.100	-0.007	-0.006	0.001	-	-
11	0.161	0.122	-0.073	-0.057	0.047	36	36
12	0.985	0.410	-0.120	-0.101	0.142	36	72
13	9.933	3.391	0.012	-0.085	0.590	36	108
14	3.485	1.760	0.013	-0.102	0.828	36	144
15	0.342	0.142	-0.049	-0.150	0.931	36	156
16	3.529	1.162	-0.046	-0.178	1.447	12	168
17	5.199	2.431	-0.010	-0.179	2.002	12	180
18	6.325	2.459	0.048	-0.180	3.093	12	192
19	2.070	0.454	0.020	-0.221	3.719	12	204
20	3.116	0.878	0.014	-0.253	4.618	12	216
21	3.913	2.209	0.023	-0.257	5.013	12	228
22	2.475	1.995	0.003	-0.266	4.824	12	240
23	1.419	0.891	-0.037	-0.296	4.922	12	252
24	1.610	1.266	-0.072	-0.318	4.865	12	264
25	5.456	2.118	-0.031	-0.323	**6.135**	12	276

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