On use of subsampling of the non-respondents for estimation of distribution function

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Abstract. In this study, we propose a general class of estimators of the finite population distribution function (DF) using two auxiliary variables under subsampling of non-respondents. We use the Hansen and Hurwitz pioneered model in our subsampling technique. Layout of response and non-response classes are discussed in various tables in detail. Expressions for the biases and mean square errors (MSEs) of the estimators are obtained up to first order of approximation. We also obtain the conditions by comparing the proposed estimator with existing estimators. Three real data sets are used to support the theoretical findings. In our findings, it is observed that the proposed class of estimators is more efficient as compared to all other existing estimators including the usual mean estimator, ratio estimator, exponential-ratio estimator, traditional difference estimator, and many well-known difference type estimators by using the criterion of MSE.

KEYWORDS: Distribution function (DF); Nonresponse; Bias; MSE; Efficiency.

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1. Introduction

The problem of non-response is common in sample survey due to many reasons such as non-availability at home or unwilling to respond due to social desirability concerns or the fear of catching some contagious disease such as Covid-19 virus by having a contact with interviewer. Hansen and Hurwitz [1] were the first who floated the indigenous idea of nonresponse. Much work has been done since then to deal the non-response by constructing composite types of estimators. The ratio, product, exponential-ratio and regression type estimators are commonly in this context. Some notable work is due to Gupta and Shabbir [2], Khan and Shabbir [3], Verma et al. [4], Bhushan and Kumar [5], Kumar et al. [6], Kumar and Bhoughal [7], Saleem et al. [8], Ahmed et al. [9], Waseem et al. [10] and Yaqub and Shabbir [11, 12]. .

Most of this work is based on estimation of finite population mean, total and variance but very little attention has been paid to estimating the distribution function (DF). Some works on estimating the DF can be found in Ahmad and Abu-Dayyah [13], Wang and Dorfman [14], Singh et al. [15] and Munoz et al. [16]. Some other useful references are, Irfan et al. [17], Abid et al. [18], Abid et al. [19], Javed et al. [20], Naz et al. [21], Younis and Shabbir [22], Ahmed and Shabbir [23] and Nazir et al. [24].

In our study, we propose a general class of estimators for estimating the DF under subsampling of non-respondents when nonresponse exists on the study variable as well as on the auxiliary variables.

Consider a finite population $U=\{U_1, U_2, ..., U_N\}$ of N units portioned into two classes i.e. (i) response class with size N_1 and (ii) nonresponse class with size N_2. Using Hansen and Hurwitz [1] technique, a sample of size n is drawn from U by using simple random sampling without replacement (SRSWOR). We assume that n_1 of the sampled units respond and n_2 do not. Let a sub-sample of r units be drawn from the n_2 non-responding units by SRSWOR and we collect the information on these r units by the interviewing method as $r=\frac{n_2}{K}, (K>1)$. Let y_i and (x_i, z_i) $(i=1,2,......,n)$ be the values of the study variable (Y) and the auxiliary variables (X, Z) respectively. We are interested in estimating the DF defined as $F_{Y(t_y)}=\frac{1}{N}\sum_{i=1}^{N} I(y_i \le t_y)$,

$-\infty < t_y < \infty$, where $I(.)$ is the indicator function such that $I = (1,0)$. Similarly, we can define $F_{X(t_x)} = \frac{1}{N}\sum_{i=1}^{N} I(x_i \le t_x)$ and $F_{Z(t_z)} = \frac{1}{N}\sum_{i=1}^{N} I(z_i \le t_z)$. The DF under stratification is $F_{Y(t_y)} = G_1 F_{Y(t_y)}^{(1)} + G_2 F_{Y(t_y)}^{(2)}$, where $G_i = \frac{N_i}{N}$ $(i = 1,2)$, $F_{Y(t_y)}^{(1)} = \frac{1}{N_1}\sum_{i=1}^{N_1} I(y_i \le t_y)$ and $F_{Y(t_y)}^{(2)} = \frac{1}{N_2}\sum_{i=1}^{N_2} I(y_i \le t_y)$. Hansen and Hurwitz [1] estimator of DF under nonresponse is defined as $\hat{F}^{*}_{F_Y(t_y)} = g_1 \hat{F}_{Y(t_y)}^{(1)} + g_2 F_{Y(t_y)}^{(2r)}$, where $g_i = \frac{n_i}{n}$ $(i = 1,2)$, $\hat{F}_{Y(t_y)}^{(1)} = \frac{1}{n_1}\sum_{i=1}^{n_1} I(y_i \le t_y)$ and $\hat{F}_{Y(t_y)}^{(2r)} = \frac{1}{r}\sum_{i=1}^{r} I(y_i \le t_y)$. Similarly, we can define $\hat{F}^{*}_{F_X(t_x)} = g_1 \hat{F}_{X(t_x)}^{(1)} + g_2 \hat{F}_{X(t_x)}^{(2r)}$ and $\hat{F}^{*}_{F_Z(t_z)} = g_1 \hat{F}_{Z(t_z)}^{(1)} + g_2 F_{Z(t_z)}^{(2r)}$. Let $S^2_{F_Y(t_y)} = F_{Y(t_y)}\left(1 - F_{Y(t_y)}\right)$, $S^2_{F_X(t_x)} = F_{X(t_x)}\left(1 - F_{X(t_x)}\right)$ and $S^2_{F_Z(t_z)} = F_{Z(t_z)}\left(1 - F_{Z(t_z)}\right)$ be the finite population variances for Y, X and Z respectively for the response class. Similarly, the population variances for the non-response class are defined as: $S^{2(2)}_{F_{Y(t_y)}} = F^{(2)}_{Y(t_y)}\left(1 - F^{(2)}_{Y(t_y)}\right)$, $S^{2(2)}_{F_X(t_x)} = F^{(2)}_{X(t_x)}\left(1 - F^{(2)}_{X(t_x)}\right)$ and $S^{2(2)}_{F_{Z(t_z)}} = F^{(2)}_{Z(t_z)}\left(1 - F^{(2)}_{Z(t_z)}\right)$. Let $S_{F_{YX}(t_y,t_x)} = \frac{N_{110}N_{220} - N_{120}N_{210}}{N^2}$, $S_{F_{YZ}(t_y,t_z)} = \frac{N_{101}N_{202} - N_{102}N_{201}}{N^2}$, $S_{F_{XZ}(t_x,t_z)} = \frac{N_{011}N_{022} - N_{012}N_{021}}{N^2}$ be the population covariances for the response class in their respective subscripts and similarly the population covariances for the non-response class in their respective subscripts are $S^{(2)}_{F_{YX}(t_y,t_x)} = \frac{N^{(2)}_{110}N^{(2)}_{220} - N^{(2)}_{120}N^{(2)}_{210}}{N^{(2)2}}$, $S^{(2)}_{F_{YZ}(t_y,t_z)} = \frac{N^{(2)}_{101}N^{(2)}_{202} - N^{(2)}_{102}N^{(2)}_{201}}{N^{(2)2}}$, $S^{(2)}_{F_{XZ}(t_x,t_z)} = \frac{N^{(2)}_{011}N^{(2)}_{022} - N^{(2)}_{012}N^{(2)}_{021}}{N^{(2)2}}$.
The layout for response and non-response classes are given in Tables 1-6.

[Table 1 Here]

Here N_{110}, N_{120}, N_{210} and N_{220} are the number of units in the population and similarly n_{110}, n_{120}, n_{210} and n_{220} be the number of units in the sample in their respective cells of respondents.

[Table 2 Here]

Here $N_{110}^{(2)}$, $N_{120}^{(2)}$, $N_{210}^{(2)}$ and $N_{220}^{(2)}$ are the number of units in the population and similarly $n_{110}^{(2)}$, $n_{120}^{(2)}$, $n_{210}^{(2)}$ and $n_{220}^{(2)}$ be the number of units in the sample in their respective cells of respondents.

[Table 3 Here]

Here N_{101}, N_{102}, N_{201} and N_{202} are the number of units in the population and similarly n_{101}, n_{102}, n_{201} and n_{202} be the number of units in the sample in their respective cells of respondents.

[Table 4 Here]

Here $N_{101}^{(2)}$, $N_{102}^{(2)}$, $N_{201}^{(2)}$ and $N_{202}^{(2)}$ are the number of units in the population and similarly $n_{101}^{(2)}$, $n_{102}^{(2)}$, $n_{201}^{(2)}$ and $n_{202}^{(2)}$ be the number of units in the sample in their respective cells of respondents.

[Table 5 Here]

Here N_{011}, N_{012}, N_{021} and N_{022} are the number of units in the population and similarly n_{011}, n_{012}, n_{021} and n_{022} be the number of units in the sample in their respective cells of respondents.

[Table 6 Here]

Here $N_{110}^{(2)}$, $N_{012}^{(2)}$, $N_{021}^{(2)}$ and $N_{022}^{(2)}$ are the number of units in the population and similarly $n_{011}^{(2)}$, $n_{012}^{(2)}$, $n_{021}^{(2)}$ and $n_{022}^{(2)}$ be the number of units in the sample in their respective cells of respondents.

Now we define some error terms to obtain the biases and MSEs up to first order of approximation.

$$\Delta_0^* = \frac{F_{Y(t_y)}^* - F_{Y(t_y)}}{F_{Y(t_y)}}, \; \Delta_1^* = \frac{F_{X(t_x)}^* - F_{X(t_x)}}{F_{X(t_x)}}, \; \Delta_2^* = \frac{F_{Z(t_z)}^* - F_{Z(t_z)}}{F_{Z(t_z)}} \text{ such that } E(\Delta_i^*) = 0 \text{ for } (i = 0,1,2)$$

and

$$E(\Delta_0^{*2}) = \frac{1}{F_Y^2(t_y)} \left\{ \lambda_1 S_{F_Y(t_y)}^2 + \lambda_2 S_{F_Y(t_y)}^{(2)2} \right\} = \Lambda_{200}^*, \; E(\Delta_1^{*2}) = \frac{1}{F_X^2(t_x)} \left\{ \lambda_1 S_{F_X(t_x)}^2 + \lambda_2 S_{F_X(t_x)}^{(2)2} \right\} = \Lambda_{020}^*$$

$$E(\Delta_2^{*2}) = \frac{1}{F_Z^2(t_z)} \left\{ \lambda_1 S_{F_Z(t_z)}^2 + \lambda_2 S_{F_Z(t_z)}^{(2)2} \right\} = \Lambda_{002}^*,$$

$$E(\Delta_0^* \Delta_1^*) = \frac{1}{F_Y(t_y) F_X(t_x)} \left\{ \lambda_1 S_{F_{YX}(t_y,t_x)} + \lambda_2 S_{F_{YX}(t_y,t_x)}^{(2)} \right\} = \Lambda_{110}^*,$$

$$E(\Delta_0^* \Delta_2^*) = \frac{1}{F_Y(t_y) F_Z(t_z)} \left\{ \lambda_1 S_{F_{YZ}(t_y,t_z)} + \lambda_2 S_{F_{YZ}(t_y,t_z)}^{(2)} \right\} = \Lambda_{101}^*,$$

$$E(\Delta_1^*\Delta_2^*)=\frac{1}{F_X(t_x)F_Z(t_z)}\left\{\lambda_1 S_{F_{XZ}(t_x,t_z)}+\lambda_2 S^{(2)}_{F_{XZ}(t_x,t_z)}\right\}=\Lambda^*_{101},$$

where $\lambda_1=\left(\frac{1}{n}-\frac{1}{N}\right)$, $\lambda_2=\frac{N_2(K-1)}{Nn}$,

$$\Lambda^*_{def}=\frac{E\left[\left\{F^*_{Y(t_y)}-F_Y(t_y)\right\}^d\left\{F^*_{X(t_x)}-F_X(t_x)\right\}^e\left\{F^*_{Z(t_z)}-F_Z(t_z)\right\}^f\right]}{\left\{F_Y(t_y)\right\}^d\left\{F_X(t_x)\right\}^e\left\{F_Z(t_z)\right\}^f}.$$

Now we discuss some estimators of DF using single auxiliary variable and two auxiliary variables.

2. Existing estimators

(i) The variance of the usual estimator $\hat{F}^*_{F_Y(t_y)}=\hat{F}^*_0$, is given by

$$Var(\hat{F}^*_0)=F^2_{Y(t_y)}\Lambda^*_{200}. \tag{1}$$

(ii) The traditional ratio estimator, is given by

$$\hat{F}^*_{R_1}=\hat{F}^*_{Y(t_y)}\left(\frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}}\right). \tag{2}$$

The bias and MSE respectively of $\hat{F}^*_{R_1}$, to first order of approximation, are given by

$$B(\hat{F}^*_{R_1})\cong F^2_{Y(t_y)}\left\{\Lambda^*_{020}-\Lambda^*_{110}\right\} \tag{3}$$

and

$$MSE(\hat{F}^*_{R_1})\cong F^2_{Y(t_y)}\left\{\Lambda^*_{200}+\Lambda^*_{020}-2\Lambda^*_{110}\right\}. \tag{4}$$

(iii) The traditional exponential-ratio type estimator, is given by

$$\hat{F}^*_{E_1}=\hat{F}^*_{Y(t_y)}\exp\left(\frac{F_{X(t_x)}-\hat{F}^*_{X(t_x)}}{F_{X(t_x)}+\hat{F}^*_{X(t_x)}}\right). \tag{5}$$

The bias and MSE respectively of $\hat{F}^*_{E_1}$, to first order of approximation, are given by

$$B(\hat{F}^*_{E_1})\cong F^2_{Y(t_y)}\left\{\frac{3\Lambda^*_{020}}{8}-\frac{\Lambda^*_{110}}{2}\right\} \tag{6}$$

and

$$MSE(\hat{F}^*_{E_1}) \cong F^2_{Y(t_y)} \left\{ \Lambda^*_{200} + \frac{\Lambda^*_{020}}{4} - \Lambda^*_{110} \right\}. \tag{7}$$

(iv) The usual difference estimator, is given by

$$\hat{F}^*_{D_1} = \hat{F}^*_{Y(t_y)} + d_0 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right), \tag{8}$$

where d_0 is the constant.

The minimum variance of $\hat{F}^*_{D_1}$ at the optimum value of $d_{0(opt)} = \frac{F_{Y(t_y)}\Lambda^*_{110}}{F_{X(t_x)}\Lambda^*_{020}}$, is given by

$$Var(\hat{F}^*_{D_1})_{\min} = MSE(\hat{F}^*_{D_1})_{\min} \cong F^2_{Y(t_y)} \Lambda^*_{200} \left(1 - \rho^{*2}_{110}\right), \tag{9}$$

where $\rho^*_{110} = \frac{\Lambda^*_{110}}{\sqrt{\Lambda^*_{200}}\sqrt{\Lambda^*_{020}}}$.

(v) Rao [25] difference type estimator, is given by

$$\hat{F}^*_{Rao} = d_1 \hat{F}^*_{Y(t_y)} + d_2 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right), \tag{10}$$

where $d_i \ (i = 1, 2)$ are the constants.

The bias and minimum MSE respectively of $\hat{F}^*_{Rao}$ at optimum values of

$d_{1(opt)} = \frac{1}{1 + \Lambda^*_{020}(1 - \rho^{*2}_{110})}$ and $d_{2(opt)} = \frac{F_{Y(t_y)}\Lambda^*_{110}}{F_{X(t_x)}\Lambda^*_{020}\left\{1 + \Lambda^*_{020}(1 - \rho^{*2}_{110})\right\}}$, are given by

$$Bias(\hat{F}^*_{Rao}) \cong (d_1 - 1) F_{Y(t_y)} \tag{11}$$

and

$$MSE(\hat{F}^*_{Rao})_{\min} \cong F^2_{Y(t_y)} \frac{\Lambda^*_{200}\left(1 - \rho^{*2}_{110}\right)}{1 + \Lambda^*_{200}\left(1 - \rho^{*2}_{110}\right)}. \tag{12}$$

(vi) Gupta and Shabbir [2] estimator using two auxiliary variables, is given by

$$\hat{F}^*_{GS} = \left\{ J_1 \hat{F}^*_{Y(t_y)} + J_2 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right) \right\} \left\{ \frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}} \right\}, \tag{13}$$

where $J_i \ (i = 1, 2)$ are the constants.

The bias and minimum MSE respectively of $\hat{F}^*_{GS}$ at optimum values of

$J_{1(opt)} = \dfrac{B_jC_j - D_jE_j + B_j}{A_jB_j - E_j^2 + B_j}$ and $J_{2(opt)} = \dfrac{F_{Y(t_y)}(A_jD_j - C_jE_j + D_j - E_j)}{F_{X(t_x)}(A_jB_j - E_j^2 + B_j)}$, are given by

$$Bias(\hat{F}^*_{GS}) \cong (J_1 - 1)F_{Y(t_y)} + J_1F_{Y(t_y)}C_j + J_2F_{X(t_x)}D_j \tag{14}$$

and

$$MSE(\hat{F}^*_{GS})_{\min} \cong F^2_{Y(t_y)}\left\{1 - \frac{A_jD_j^2 + B_jC_j^2 - 2C_jD_jE_j + 2B_jC_j - 2D_jE_j + B_j}{(A_jB_j - E_j^2 + B_j)}\right\}, \tag{15}$$

where $A_j = \Lambda^*_{200} + \Lambda^*_{020} - 2\Lambda^*_{110}$, $B_j = \Lambda^*_{020}$, $C_j = \dfrac{3\Lambda^*_{020}}{8} - \dfrac{\Lambda^*_{110}}{2}$, $D_j = \dfrac{\Lambda^*_{020}}{2}$, $E_j = \Lambda^*_{020} - \Lambda^*_{110}$.

(vii) The traditional ratio estimator using two auxiliary variables, is given by

$$\hat{F}^*_{R_2} = \hat{F}^*_{Y(t_y)}\left(\frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}}\right)\left(\frac{F_{Z(t_z)}}{\hat{F}^*_{Z(t_z)}}\right). \tag{16}$$

The bias and MSE respectively of $\hat{F}^*_{R_2}$ to first order of approximation are given by

$$B(\hat{F}^*_{R_2}) \cong F_{Y(t_y)}\left\{\Lambda^*_{020} + \Lambda^*_{002} + \Lambda^*_{011} - \Lambda^*_{110} - \Lambda^*_{101}\right\} \tag{17}$$

and

$$MSE(\hat{F}^*_{R_2}) \cong F^2_{Y(t_y)}\left\{\Lambda^*_{200} + \Lambda^*_{020} + \Lambda^*_{002} - 2\Lambda^*_{110} - 2\Lambda^*_{101} + 2\Lambda^*_{011}\right\}. \tag{18}$$

(viii) The traditional exponential ratio estimator using two auxiliary variables, is given by

$$\hat{F}^*_{E_2} = \hat{F}^*_{Y(t_y)}\exp\left(\frac{F_{X(t_x)} - \hat{F}^*_{X(t_x)}}{F_{X(t_x)} + \hat{F}^*_{X(t_x)}}\right)\exp\left(\frac{F_{Z(t_z)} + \hat{F}^*_{Z(t_z)}}{F_{Z(t_z)} + \hat{F}^*_{Z(t_z)}}\right). \tag{19}$$

The bias and MSE respectively of $\hat{F}^*_{E_2}$ to first order of approximation, are given by

$$B(\hat{F}^*_{E_2}) \cong F_{Y(t_y)}\left\{\frac{3}{8}\left(\Lambda^*_{020} + \Lambda^*_{002}\right) - \frac{1}{2}\left(\Lambda^*_{110} - \Lambda^*_{101}\right) + \frac{1}{4}\Lambda^*_{011}\right\} \tag{20}$$

and

$$MSE(\hat{F}^*_{E_2}) \cong F^2_{Y(t_y)}\left\{\Lambda^*_{200} + \frac{1}{4}\left(\Lambda^*_{020} + \Lambda^*_{002}\right) - \left(\Lambda^*_{110} + \Lambda^*_{101}\right) + \frac{1}{2}\Lambda^*_{011}\right\}. \tag{21}$$

(ix) The usual difference estimator using two auxiliary variables, is given by

$$\hat{F}^*_{D_2} = \hat{F}^*_{Y(t_y)} + d_1\left(F_{X(t_x)} - \hat{F}^*_{X(t_x)}\right) + d_2\left(F_{Z(t_z)} - \hat{F}^*_{Z(t_z)}\right), \tag{22}$$

where $d_i \ (i=1,2)$ are constants.

The minimum variance or MSE of $\hat{F}^*_{D_2}$ at the optimum values of $d_i \ (i=1,2)$ i.e.

$d_{1(opt)} = \dfrac{F_{Y(t_y)}\left(\Lambda^*_{101}\Lambda^*_{011} - \Lambda^*_{002}\Lambda^*_{110}\right)}{F_{X(t_x)}\left(\Lambda^{*2}_{011} - \Lambda^*_{020}\Lambda^*_{022}\right)}$ and $d_{2(opt)} = \dfrac{F_{Y(t_y)}\left(\Lambda^*_{011}\Lambda^*_{110} - \Lambda^*_{020}\Lambda^*_{101}\right)}{F_{Z(t_x)}\left(\Lambda^{*2}_{011} - \Lambda^*_{020}\Lambda^*_{022}\right)}$, is given by

$$MSE(\hat{F}^*_{D_2})_{\min} \cong F^2_{Y(t_y)}\left\{\frac{\Lambda^{*2}_{101}\Lambda^*_{020} - 2\Lambda^*_{101}\Lambda^*_{011}\Lambda^*_{110} + \Lambda^{*2}_{011}\Lambda^*_{200} - \Lambda^*_{020}\Lambda^*_{002}\Lambda^*_{200} + \Lambda^{*2}_{110}\Lambda^*_{002}}{\Lambda^{*2}_{011} - \Lambda^*_{020}\Lambda^*_{002}}\right\},$$

or

$$MSE(\hat{F}^*_{D_2})_{\min} \cong F^2_{Y(t_y)}\Lambda^*_{200}\left\{1 - \frac{\rho^{*2}_{110} + \rho^{*2}_{101} - 2\rho^*_{110}\rho^*_{101}\rho^*_{011}}{1-\rho^{*2}_{011}}\right\}, \tag{23}$$

where $\rho^*_{110} = \dfrac{\Lambda^*_{110}}{\sqrt{\Lambda^*_{200}}\sqrt{\Lambda^*_{020}}}$, $\rho^*_{101} = \dfrac{\Lambda^*_{101}}{\sqrt{\Lambda^*_{200}}\sqrt{\Lambda^*_{002}}}$, $\rho^*_{011} = \dfrac{\Lambda^*_{011}}{\sqrt{\Lambda^*_{020}}\sqrt{\Lambda^*_{002}}}$.

(x) Kumar et al. [6] estimator using two auxiliary variables, is given by

$$\hat{F}^*_{KU} = \hat{F}^*_{Y(t_y)}\left(\frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}}\right)\left\{\alpha_0 \exp\left(\frac{F_{Z(t_x)} - \hat{F}^*_{Z(t_x)}}{F_{Z(t_x)} + \hat{F}^*_{Z(t_x)}}\right) + (1-\alpha_0)\exp\left(\frac{\hat{F}^*_{Z(t_x)} - F_{Z(t_x)}}{\hat{F}^*_{Z(t_x)} + F_{Z(t_x)}}\right)\right\}, \tag{24}$$

where α_0 is the constant.

The bias and minimum MSE respectively of $\hat{F}^*_{KU}$ to first order of approximation at optimum value of $\alpha_{0(opt)} = \dfrac{1}{2} - \dfrac{\left(\Lambda^*_{011} - \Lambda^*_{101}\right)}{\Lambda^*_{002}}$, are given by

$$B(\hat{F}^*_{KU}) \cong F_{Y(t_y)}\left\{\Lambda^*_{020} + \left(\frac{1}{2} - \alpha_0\right)\left(\Lambda^*_{101} - \Lambda^*_{011}\right) - \left(\frac{1}{8} - \frac{1}{2}\alpha_0\right)\Lambda^*_{002}\right\} \tag{25}$$

and

$$MSE(\hat{F}^*_{KU})_{\min} \cong F^2_{Y(t_y)}\left\{\left(\Lambda^*_{200} + \Lambda^*_{020} - 2\Lambda^*_{110}\right) - \frac{\left(\Lambda^*_{011} - \Lambda^*_{101}\right)^2}{\Lambda^*_{002}}\right\}. \tag{26}$$

(xi) On the lines of Chami et al. [25], Guha and Chandra [26] and Singh and Usman [27] estimators using two auxiliary variables, we have

$$\hat{F}^*_{Ch} = \hat{F}^*_{Y(t_y)}\left\{\frac{\alpha_1\hat{F}^*_{X(t_x)} + (1-\alpha_1)F_{X(t_x)}}{(1-\alpha_1)\hat{F}^*_{X(t_x)} + \alpha_1 F_{X(t_x)}}\right\}\left\{\frac{\alpha_2\hat{F}^*_{Z(t_z)} + (1-\alpha_2)F_{Z(t_z)}}{(1-\alpha_2)\hat{F}^*_{Z(t_z)} + \alpha_2 F_{Z(t_z)}}\right\}, \tag{27}$$

where α_i $(i=1,2)$ are the constants.

The bias and minimum MSE respectively of $\hat{F}^*_{Ch}$ at the optimum values of α_i $(i=1,2)$ i.e.

$$\alpha_{1(opt)}=\frac{1}{2}\left\{1+\frac{\left(\Lambda^*_{101}\Lambda^*_{011}-\Lambda^*_{002}\Lambda^*_{110}\right)}{\left(\Lambda^{*2}_{011}-\Lambda^*_{020}\Lambda^*_{022}\right)}\right\}\text{and }\alpha_{2(opt)}=\frac{1}{2}\left\{1-\frac{\left(\Lambda^*_{020}\Lambda^*_{101}-\Lambda^*_{011}\Lambda^*_{110}\right)}{\left(\Lambda^{*2}_{011}-\Lambda^*_{020}\Lambda^*_{022}\right)}\right\}, \quad (28)$$

are given by

$$Bias(\hat{F}^*_{Ch})\cong F_{Y(t_y)}\left\{\left(2\alpha_1-1\right)\Lambda^*_{110}+\left(2\alpha_2-1\right)\Lambda^*_{101}+\left(2\alpha_1-1\right)\left(2\alpha_2-1\right)\Lambda^*_{011}\right.$$
$$\left.+\left(1-\alpha_1\right)\left(1-2\alpha_1\right)\Lambda^*_{020}+\left(1-\alpha_2\right)\left(1-2\alpha_2\right)\Lambda^*_{002}\right\} \quad (29)$$

and

$$MSE(\hat{F}^*_{Ch})_{\min}\cong F^2_{Y(t_y)}\left\{\frac{\Lambda^{*2}_{101}\Lambda^*_{020}-2\Lambda^*_{101}\Lambda^*_{011}\Lambda^*_{110}+\Lambda^{*2}_{011}\Lambda^*_{200}-\Lambda^*_{020}\Lambda^*_{002}\Lambda^*_{200}+\Lambda^{*2}_{110}\Lambda^*_{002}}{\Lambda^{*2}_{011}-\Lambda^*_{020}\Lambda^*_{002}}\right\}. \quad (30)$$

The minimum MSE of $\hat{F}^*_{Ch}$ is equal to minimum MSE of the difference estimator $\hat{F}^*_{D_2}$.

(ix) Singh and Usman [27] estimator using two auxiliary variables, is given by

$$\hat{F}^*_{SU}=\left\{\hat{F}^*_{Y(t_y)}+\hat{\beta}^*_{110}\left(F_{X(t_x)}-\hat{F}^*_{X(t_x)}\right)\right\}\left\{\frac{\gamma_1\hat{F}^*_{X(t_x)}+(1-\gamma_1)F_{X(t_x)}}{(1-\gamma_1)\hat{F}^*_{X(t_x)}+\gamma_1F_{X(t_x)}}\right\}\left\{\frac{\gamma_2\hat{F}^*_{Z(t_z)}+(1-\gamma_2)F_{Z(t_z)}}{(1-\gamma_2)\hat{F}^*_{Z(t_z)}+\gamma_2F_{Z(t_z)}}\right\}, \quad (31)$$

where γ_i $(i=1,2)$ are constants and $\hat{\beta}^*_{110}=\dfrac{\hat{F}_{Y(t_y)}\hat{\Lambda}^*_{110}}{\hat{F}_{X(t_x)}\hat{\Lambda}^*_{020}}$ is the sample regression coefficient with the corresponding population regression coefficient $\beta^*_{110}=\dfrac{F_{Y(t_y)}\Lambda^*_{110}}{F_{X(t_x)}\Lambda^*_{020}}$.

It is observed that $MSE(\hat{F}^*_{SU})_{\min}=MSE(\hat{F}^*_{D_2})_{\min}=MSE(\hat{F}^*_{Ch})_{\min}$.

3. Proposed estimator

We propose the following general class of difference type estimators of DF using two auxiliary variables. This estimator is constructed by using the ratio and exponential-ratio type estimators with the difference type estimator as:

$$\hat{F}^*_{P(\delta_1,\delta_2)}=\left\{\omega_1\hat{F}^*_{Y(t_y)}+\omega_2\left(F_{X(t_x)}-\hat{F}^*_{X(t_x)}\right)+\omega_3\left(F_{Z(t_z)}-\hat{F}^*_{Z(t_z)}\right)\right\}\times$$

$$\left\{\left(\frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}}\right)^{\delta_1} \exp\delta_2\left(\frac{F_{X(t_x)}-\hat{F}^*_{X(t_x)}}{F_{X(t_x)}+\hat{F}^*_{X(t_x)}}\right)\right\}, \tag{32}$$

where ω_i $(i=1,2,3)$ are the constants and $(0\le\delta_i\le 1)$ $(i=1,2)$ are known scaler values.

Rewriting $\hat{F}^*_{P(\delta_1,\delta_2)}$ in terms of errors terms, we have

$$\hat{F}^*_{P(\delta_1,\delta_2)}-F_{Y(t_y)}\cong(\omega_1-1)F_{Y(t_y)}+\omega_1F_{Y(t_y)}\left[\Delta_0^*-\delta_1^*\Delta_1^*+\delta_2^*\Delta_1^{*2}-\delta_1^*\Delta_0^*\Delta_1^*\right]$$

$$-\omega_2F_{X(t_x)}\left[\Delta_1^*-\delta_1^*\Delta_1^{*2}\right]-\omega_3F_{Z(t_z)}\left[\Delta_2^*-\delta_1^*\Delta_1^*\Delta_2^*\right], \tag{33}$$

where $\delta_1^*=\left(\delta_1+\frac{\delta_2}{2}\right)$ and $\delta_2^*=\left\{\frac{\delta_1\delta_2}{2}+\frac{\delta_1(\delta_1+1)}{2}+\frac{\delta_2(\delta_2+2)}{8}\right\}$.

From (33), the bias of $\hat{F}^*_{P(\delta_1,\delta_2)}$, is given by

$$Bias(\hat{F}^*_{P(\delta_1,\delta_2)})\cong(\omega_1-1)F_{Y(t_y)}+\omega_1F_{Y(t_y)}\left\{\delta_2^*\Lambda_{020}^*-\delta_1^*\Lambda_{110}^*\right\}.$$

$$+F_{Z(t_z)}\delta_1^*\left(\omega_2\Lambda_{020}^*+\omega_3\Lambda_{011}^*\right). \tag{34}$$

Squaring and then taking expectation on (33), we get MSE of $\hat{F}^*_{P(\delta_1,\delta_2)}$, which is given by

$$MSE(\hat{F}^*_{P(\delta_1,\delta_2)})\cong(\omega_1-1)^2F^2_{Y(t_y)}+\omega_1^2F^2_{Y(t_y)}A+\omega_2^2F^2_{X(t_x)}B+\omega_3^2F^2_{Z(t_z)}C-2\omega_1F^2_{Y(t_y)}D,$$

$$-2\omega_2F_{Y(t_y)}F_{X(t_x)}E-2\omega_3F_{Y(t_y)}F_{Z(t_z)}F+2\omega_1\omega_2F_{Y(t_y)}F_{X(t_x)}G,$$

$$+2\omega_1\omega_3F_{Y(t_y)}F_{Z(t_z)}H+2\omega_2\omega_3F_{X(t_x)}F_{Z(t_z)}I,$$

where

$A=\Lambda_{200}^*+\left(\delta_1^{*2}+2\delta_2^*\right)\Lambda_{020}^*-4\delta_1^*\Lambda_{110}^*$, $B=\Lambda_{020}^*$, $C=\Lambda_{002}^*$, $D=\delta_2^*\Lambda_{020}^*-\delta_1^*\Lambda_{110}^*$, $E=\delta_1^*\Lambda_{020}^*$,

$F=\delta_1^*\Lambda_{011}^*$, $G=2\delta_1^*\Lambda_{020}^*-\Lambda_{110}^*$, $H=2\delta_1^*\Lambda_{011}^*-\Lambda_{101}^*$, $I=\Lambda_{011}^*$.

The minimum MSE of $\hat{F}^*_{P(\delta_1,\delta_2)}$ at optimum values of ω_i $(i=1,2,3)$ i.e.,

$\omega_{1(opt)}=\frac{l_5}{l_1}$, $\omega_{2(opt)}=\frac{F_{Y(t_y)}l_6}{F_{X(t_x)}l_1}$ and $\omega_{3(opt)}=\frac{F_{Y(t_y)}l_7}{F_{Z(t_z)}l_1}$, is given by

$$MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min}\cong F^2_{Y(t_y)}\left(\frac{l_2+l_3+l_4}{l_1}\right), \tag{35}$$

where

$l_1=ABC-BH^2-AI^2+BC-CG^2+2GHI-I^2$,

$l_2=-ABF^2-BCD^2+2BDFH+ABC-ACE^2+2AEFI-2BCD-BF^2+2BFH-BH^2$

$l_3 = 2CDEG + D^2I^2 - 2DEHI - 2DFGI + E^2H^2 - 2EFGH + F^2G^2 - AI^2 - CE^2$,
$l_4 = 2CEG - CG^2 + 2DI^2 + 2EFI - 2EHI - 2FGI + 2GHI$,
$l_5 = BCD - BFH + BC - CEG - DI^2 + EHJ + FGI - I^2$,
$l_6 = ACE - AFI - CDG + DHI - EH^2 + FGH + CE - CG - FI + HI$,
$l_7 = ABF - BDH - AEI + BF - BH + DGI + EGH - FG^2 - EI + GI$.

We can generate many estimators from this proposed class of estimators as follows:

(i) Putting $\delta_1 = 0$ and $\delta_2 = 0$ in (32), we get

$$\hat{F}^*_{P(0,0)} = \omega_1 \hat{F}^*_{Y(t_y)} + \omega_2 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right) + \omega_3 \left(F_{Z(t_z)} - \hat{F}^*_{Z(t_z)} \right). \tag{36}$$

(ii) Putting $\delta_1 = 1$ and $\delta_2 = 0$ in (32), we get

$$\hat{F}^*_{P(1,0)} = \left\{ \omega_1 \hat{F}^*_{Y(t_y)} + \omega_2 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right) + \omega_3 \left(F_{Z(t_z)} - \hat{F}^*_{Z(t_z)} \right) \right\} \left(\frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}} \right). \tag{37}$$

(iii) Putting $\delta_1 = 1$ and $\delta_2 = 1$ in (32), we get

$$\hat{F}^*_{P(1,1)} = \left\{ \omega_1 \hat{F}^*_{Y(t_y)} + \omega_2 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right) + \omega_3 \left(F_{Z(t_z)} - \hat{F}^*_{Z(t_z)} \right) \right\}.$$
$$\left\{ \left(\frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}} \right) \exp\left(\frac{F_{X(t_x)} - \hat{F}^*_{X(t_x)}}{F_{X(t_x)} + \hat{F}^*_{X(t_x)}} \right) \right\}. \tag{38}$$

(iv) Putting $\delta_1 = 0.5$ and $\delta_2 = 0.5$ in (32), we get

$$\hat{F}^*_{P(0.5,0.5)} = \left\{ \omega_1 \hat{F}^*_{Y(t_y)} + \omega_2 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right) + \omega_3 \left(F_{Z(t_z)} - \hat{F}^*_{Z(t_z)} \right) \right\}$$
$$\left\{ \left(\frac{F_{X(t_x)}}{\hat{F}^*_{X(t_x)}} \right)^{0.5} \exp\left(0.5 \frac{F_{X(t_x)} - \hat{F}^*_{X(t_x)}}{F_{X(t_x)} + \hat{F}^*_{X(t_x)}} \right) \right\}. \tag{39}$$

(vi) Putting $\delta_1 = 0$ and $\delta_2 = 1$ in (32), we get

$$\hat{F}^*_{P(0,1)} = \left\{ \omega_1 \hat{F}^*_{Y(t_y)} + \omega_2 \left(F_{X(t_x)} - \hat{F}^*_{X(t_x)} \right) + \omega_3 \left(F_{Z(t_z)} - \hat{F}^*_{Z(t_z)} \right) \right\} \left\{ \exp\left(\frac{F_{X(t_x)} - \hat{F}^*_{X(t_x)}}{F_{X(t_x)} + \hat{F}^*_{X(t_x)}} \right) \right\}. \tag{40}$$

The biases and minimum MSEs of above estimators can be obtained by substituting the different values δ_i $(i = 1, 2)$ in (34) and (35). Also, we can generate many more estimators by substituting the different values of δ_i and ω_i $(i = 1, 2)$ in (32).

4. **Comparison of estimators**

We compare the proposed generalized class of estimators with some other competing estimators.

(i) By (1) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < Var(\hat{F}^*_0)$ if

$$\left[\Lambda^*_{200} - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(ii) By (4) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{R_1})$ if

$$\left[\left\{\Lambda^*_{200} + \Lambda^*_{020} - 2\Lambda^*_{110}\right\} - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(ii) By (7) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{E_1})$ if

$$\left[\left\{\Lambda^*_{200} + \frac{\Lambda^*_{020}}{4} - \Lambda^*_{110}\right\} - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(iii) By (9) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{D_1})_{\min}$ if

$$\left[\Lambda^*_{200}\left(1 - \rho^{*2}_{110}\right) - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(iv) By (12) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{Rao})_{\min}$ if

$$\left[\frac{\Lambda^*_{200}\left(1 - \rho^{*2}_{110}\right)}{1 + \Lambda^*_{200}\left(1 - \rho^{*2}_{110}\right)} - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(v) By (15) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{GS})_{\min}$ if

$$\left[\left\{1 - \frac{A_j D_j^2 + B_j C_j^2 - 2C_j D_j E_j + 2B_j C_j - 2D_j E_j + B_j}{(A_j B_j - E_j^2 + B_j)}\right\} - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(vi) By (18) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{R_2})$ if

$$\left[\left\{\Lambda^*_{200} + \Lambda^*_{020} + \Lambda^*_{002} - 2\left(\Lambda^*_{110} + \Lambda^*_{101} - \Lambda^*_{011}\right)\right\} - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(vii) By (21) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{E_2})$ if

$$\left[\left\{\Lambda^*_{200} + \frac{1}{4}\left(\Lambda^*_{020} + \Lambda^*_{002}\right) - \left(\Lambda^*_{110} + \Lambda^*_{101}\right) + \frac{1}{2}\Lambda^*_{011}\right\} - \left(\frac{l_2 + l_3 + l_4}{l_1}\right)\right] > 0.$$

(vii) By (23) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{D_2})_{\min}$ if

$$\left[\Lambda^*_{200}\left\{1-\frac{\rho^{*2}_{110}+\rho^{*2}_{101}-2\rho^*_{110}\rho^*_{101}\rho^*_{011}}{1-\rho^{*2}_{011}}\right\}-\left(\frac{l_2+l_3+l_4}{l_1}\right)\right]>0.$$

(vii) By (26) and (35), $MSE(\hat{F}^*_{P(\delta_1,\delta_2)})_{\min} < MSE(\hat{F}^*_{KU})_{\min}$ if

$$\left[\left\{\left(\Lambda^*_{200}+\Lambda^*_{020}-2\Lambda^*_{110}\right)-\frac{\left(\Lambda^*_{011}-\Lambda^*_{101}\right)^2}{\Lambda^*_{002}}\right\}-\left(\frac{l_2+l_3+l_4}{l_1}\right)\right]>0.$$

5. Numerical study

We use the following three data sets for numerical study.

Population 1. Source: Singh [28]

Let *Y*, *X* and *Z* be the number of immigrants admitted in the USA during 1996, 1995 and 1994 respectively. Let $I(y_i \le t_y)=1$ for $t_y=17702.76$ and $I(y_i > t_y)=0$, otherwise; $I(x_i \le t_x)=1$ for $t_x=13903.24$ and $I(x_i > t_x)=0$, otherwise; $I(z_i \le t_z)=1$ for $t_z=15483.67$ and $I(z_i > t_z)=0$, otherwise. Last 25% observations i.e. 13 units are considered as non-responding units.

$N=51,\ n=20,\ F_{Y(t_y)}=0.8039,\ F_{X(t_x)}=0.7647,\ F_{Z(t_z)}=0.8039,\ S^2_{F_{Y(t_y)}}=0.1576,\ S^2_{F_{X(t_x)}}=0.1799,$

$S^2_{F_{Z(t_z)}}=0.1576,\ N_{110}=39,\ N_{120}=02,\ N_{210}=00,\ N_{220}=10,\ N_{101}=40,\ N_{102}=01,\ N_{201}=01,$

$N_{202}=09,\ N_{011}=39,\ N_{012}=00,\ N_{021}=02,\ N_{022}=10.$

For non-response, we have:

$N^{(2)}_2=13,\ F^{(2)}_{Y(t_y)}=0.7692,\ F^{(2)}_{X(t_x)}=0.6923,\ F^{(2)}_{Z(t_z)}=0.7692,\ S^{2(2)}_{F_{Y(t_y)}}=0.1775,\ S^{2(2)}_{F_{X(t_x)}}=0.2130,$

$S^{2(2)}_{F_{Z(t_z)}}=0.1775,\ N^{(2)}_{110}=09,\ N^{(2)}_{120}=01,\ N^{(2)}_{210}=00,\ N^{(2)}_{220}=03,\ N^{(2)}_{101}=09,\ N^{(2)}_{102}=01,\ N^{(2)}_{201}=01,$

$N^{(2)}_{202}=02,\ N^{(2)}_{011}=09,\ N^{(2)}_{012}=00,\ N^{(2)}_{021}=01,\ N^{(2)}_{022}=03.$

Population 2: Source: Gujarati and Porter [29]

Let *Y* , *X* and *Z* be the production of eggs in USA during 1992, 1991 and 1990 respectively. Let $I(y_i \le t_y)=1$ for $t_y=1377.854$ and $I(y_i > t_y)=0$, otherwise; $I(x_i \le t_x)=1$ for $t_x=75.872$ and $I(x_i > t_x)=0$, otherwise; $I(z_i \le t_z)=1$ for $t_z=78.276$ and $I(z_i > t_z)=0$, otherwise. Last 25% observations i.e. 13 units are considered as non-responding units.

$N = 50,\ n = 18,\ F_{Y(t_y)} = 0.6600,\ F_{X(t_x)} = 0.5800,\ F_{Z(t_z)} = 0.5800,\ S^2_{F_{Y(t_y)}} = 0.2244,\ S^2_{F_{X(t_x)}} = 0.2436,$

$S^2_{F_{Z(t_z)}} = 0.2436,\ N_{110} = 17,\ N_{120} = 16,\ N_{210} = 12,\ N_{220} = 05,\ N_{101} = 17,\ N_{102} = 16,\ N_{201} = 12,$

$N_{202} = 05,\ N_{011} = 28,\ N_{012} = 01,\ N_{021} = 01,\ N_{022} = 20$.

For nonresponse, we have:

$N_2^{(2)} = 13,\ F^{(2)}_{Y(t_y)} = 0.7692,\ F^{(2)}_{X(t_x)} = 0.5385,\ F^{(2)}_{Z(t_z)} = 0.6154,\ S^{2(2)}_{F_{Y(t_y)}} = 0.1775,\ S^{2(2)}_{F_{X(t_x)}} = 0.2485,$
$S^{2(2)}_{F_{Z(t_z)}} = 0.2366,\ N^{(2)}_{110} = 04,\ N^{(2)}_{120} = 06,\ N^{(2)}_{210} = 03,\ N^{(2)}_{220} = 00,\ N^{(2)}_{101} = 05,\ N^{(2)}_{102} = 05,\ N^{(2)}_{201} = 03,$
$N^{(2)}_{202} = 00,\ N^{(2)}_{011} = 07,\ N^{(2)}_{012} = 00,\ N^{(2)}_{021} = 01,\ N^{(2)}_{022} = 05$.

Population 3: Source: Singh [28]

Let *Y*, *X* and *Z* be the estimated number of fish caught by marine recreational fisherman by species group during 1995, 1994 and 1993 respectively.

Let $I(y_i \le t_y) = 1$ for $t_y = 4514.90$ and $I(y_i > t_y) = 0$, otherwise; $I(x_i \le t_x) = 1$ for $t_x = 4954.43$ and $I(x_i > t_x) = 0$, otherwise; $I(z_i \le t_z) = 1$ for $t_z = 4591.07$ and $I(z_i > t_z) = 0$, otherwise. Last 25% observations i.e. 17 units are considered as non-responding units.

$N = 69,\ n = 23,\ F_{Y(t_y)} = 0.7246,\ F_{X(t_x)} = 0.7681,\ F_{Z(t_z)} = 0.7391,\ S^2_{F_{Y(t_y)}} = 0.1995,\ S^2_{F_{X(t_x)}} = 0.1781,$

$S^2_{F_{Z(t_z)}} = 0.1928,\ N_{110} = 47,\ N_{120} = 03,\ N_{210} = 06,\ N_{220} = 13,\ N_{101} = 48,\ N_{102} = 02,\ N_{201} = 03,$

$N_{202} = 16,\ N_{011} = 49,\ N_{012} = 04,\ N_{021} = 02,\ N_{022} = 14$.

For nonresponse, we have:

$N_2^{(2)} = 17,\ F^{(2)}_{Y(t_y)} = 0.8824,\ F^{(2)}_{X(t_x)} = 0.8824,\ F^{(2)}_{Z(t_z)} = 0.8824,\ S^{2(2)}_{F_{Y(t_y)}} = 0.1038,\ S^{2(2)}_{F_{X(t_x)}} = 0.1038,$
$S^{2(2)}_{F_{Z(t_z)}} = 0.1038,\ N^{(2)}_{110} = 15,\ N^{(2)}_{120} = 00,\ N^{(2)}_{210} = 00,\ N^{(2)}_{220} = 02,\ N^{(2)}_{101} = 15,\ N^{(2)}_{102} = 00,\ N^{(2)}_{201} = 00,$
$N^{(2)}_{202} = 02,\ N^{(2)}_{011} = 15,\ N^{(2)}_{012} = 00,\ N^{(2)}_{021} = 00,\ N^{(2)}_{022} = 02$.

The MSE values of all estimators based on three populations are given in Tables 7-9.

[Table 7 Here]

[Table 8 Here]

[Table 9 Here]

From Tables 7-9, we observed that the proposed general class of estimators $\hat{F}^*_{P(\delta_1,\delta_2)}$ is performing better than all considered estimators at different choices of *K*.

5. Conclusion

We proposed a general class of DF estimators $\hat{F}^*_{P(\delta_1,\delta_2)}$ using two auxiliary variables under non-response in simple random sampling. It is clear from Tables 7-9, that the proposed general class of estimators $\hat{F}^*_{P(\delta_1,\delta_2)}$ for different values of *K*, is more efficient as compared to the estimators $\hat{F}^*_i \; (i = 0, R_1, E_1, D_1, Rao, GS, R_2, E_2, (D_2, Ch, SU), KU)$ when non-response exists on all the study variable (*Y*) and the auxiliary variables (*X*, *Z*) . It is also observed that the MSE values of all estimators increase with increase in the values of *K* from 1.5 to 3.5 in all Populations 1-3, which are expected results. The ratio estimator $\hat{F}^*_{R_2}$ shows poor performance in Tables 7 and 9 but in Table 8, the ratio, exponential-ratio and Kumar et al. [6] estimators i.e. $\hat{F}^*_i(i = R_1, R_2, E_1, E_2, K)$ perform poorly as compared to all other estimators. The difference estimator ($\hat{F}^*_{D_2}$), Chami [26] estimator ($\hat{F}^*_{Ch}$) and Singh and Usman [27] estimator ($\hat{F}^*_{SU}$) give the same MSE values. Among proposed general class of estimators $\hat{F}^*_{P(\delta_1,\delta_2)}$, the performance of the estimator $\hat{F}^*_{P(0,1)}$ is the best in terms of MSE.

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List of Table Captions

Table 1. Layout of the response class for Y and X.

	$X \le F_X(t_x)$	$X > F_X(t_x)$	Total
$Y \le F_Y(t_y)$	n_{110} / N_{110}	n_{120} / N_{120}	$N_{1\bullet\bullet}$
$Y > F_Y(t_y)$	n_{210} / N_{210}	n_{220} / N_{220}	$N_{2\bullet\bullet}$
Total	$N_{\bullet 1 \bullet}$	$N_{\bullet 2 \bullet}$	N

Table 2. Layout of the non-response class for Y and X.

	$X_2 \le F_X^{(2)}(t_x)$	$X_2 > F_X^{(2)}(t_x)$	Total
$Y_2 \le F_Y^{(2)}(t_y)$	$n_{110}^{(2)} / N_{110}^{(2)}$	$n_{120}^{(2)} / N_{120}^{(2)}$	$N_{1\bullet\bullet}^{(2)}$
$Y_2 > F_Y^{(2)}(t_y)$	$n_{210}^{(2)} / N_{210}^{(2)}$	$n_{220}^{(2)} / N_{220}^{(2)}$	$N_{2\bullet\bullet}^{(2)}$
Total	$N_{\bullet 1 \bullet}^{(2)}$	$N_{\bullet 2 \bullet}^{(2)}$	N

Table 3. Layout of the response class for Y and Z.

	$Z \le F_Z(t_z)$	$Z > F_Z(t_z)$	Total
$Y \le F_Y(t_y)$	n_{101} / N_{101}	n_{102} / N_{102}	$N_{1\bullet\bullet}$
$Y > F_Y(t_y)$	n_{201} / N_{201}	n_{202} / N_{202}	$N_{2\bullet\bullet}$
Total	$N_{\bullet\bullet 1}$	$N_{\bullet\bullet 2}$	N

Table 4. Layout of the non-response class for Y and Z.

	$Z_2 \le F_Z^{(2)}(t_z)$	$Z_2 > F_Z^{(2)}(t_z)$	Total
$Y_2 \le F_Y^{(2)}(t_y)$	$n_{101}^{(2)} / N_{101}^{(2)}$	$n_{102}^{(2)} / N_{102}^{(2)}$	$N_{1\bullet\bullet}^{(2)}$
$Y_2 > F_Y^{(2)}(t_y)$	$n_{201}^{(2)} / N_{201}^{(2)}$	$n_{202}^{(2)} / N_{202}^{(2)}$	$N_{2\bullet\bullet}^{(2)}$
Total	$N_{\bullet\bullet 1}^{(2)}$	$N_{\bullet\bullet 2}^{(2)}$	N

Table 5. Layout of the response class for *X* and *Z*.

	$Z \le F_Z(t_z)$	$Z > F_Z(t_z)$	Total
$X \le F_X(t_x)$	n_{011} / N_{011}	n_{012} / N_{012}	$N_{\bullet 1 \bullet}$
$X > F_X(t_x)$	n_{021} / N_{021}	n_{022} / N_{022}	$N_{\bullet 2 \bullet}$
Total	$N_{\bullet\bullet 1}$	$N_{\bullet\bullet 2}$	N

Table 6. Layout of the non-response class for *X* and *Z*.

	$Z_2 \le F_Z^{(2)}(t_z)$	$Z_2 > F_Z^{(2)}(t_z)$	Total
$X_2 \le F_X^{(2)}(t_x)$	$n_{011}^{(2)} / N_{011}^{(2)}$	$n_{012}^{(2)} / N_{012}^{(2)}$	$N_{\bullet 1 \bullet}^{(2)}$
$X_2 > F_X^{(2)}(t_x)$	$n_{021}^{(2)} / N_{021}^{(2)}$	$n_{022}^{(2)} / N_{022}^{(2)}$	$N_{\bullet 2 \bullet}^{(2)}$
Total	$N_{\bullet\bullet 1}^{(2)}$	$N_{\bullet\bullet 2}^{(2)}$	N

Table 7. MSE values of different estimators for different values of *K* in Population 1.

Estimator	$K = 1.5$	$K = 2.0$	$K = 2.5$	$K = 3.0$	$K = 3.5$
$\hat{F}_0^*$	0.005922	0.007054	0.008185	0.009316	0.010448
$\hat{F}_{R_1}^*$	0.001744	0.002235	0.002726	0.003217	0.003708
$\hat{F}_{E_1}^*$	0.001947	0.002383	0.002820	0.003256	0.003692
$\hat{F}_{D_1}$	0.001369	0.001742	0.002113	0.002484	0.002853
$\hat{F}_{Rao}^*$	0.001366	0.001737	0.002106	0.002474	0.002841
$\hat{F}_{GS}^*$	0.001361	0.001729	0.002095	0.002459	0.002822
$\hat{F}_{R_2}^*$	0.009717	0.012198	0.014679	0.017160	0.019640
$\hat{F}_{E_2}^*$	0.001523	0.002136	0.002749	0.003362	0.003975
$\hat{F}_{D_2}^*, \hat{F}_{CH}^*, \hat{F}_{SU}^*$	0.001311	0.001726	0.002112	0.002481	0.002840
$\hat{F}_{KU}^*$	0.001567	0.001935	0.002293	0.002643	0.002983
$\hat{F}_{P(0,0)}^*$	0.001308	0.001721	0.002105	0.002472	0.002828
$\hat{F}_{P(1,0)}^*$	0.001308	0.001721	0.002105	0.002472	0.002827
$\hat{F}_{P(1,1)}^*$	0.001307	0.001722	0.002107	0.002475	0.002832
$\hat{F}_{P(0.5,0.5)}^*$	0.001304	0.001716	0.002097	0.002461	0.002814

$\hat{F}^*_{P(0,1)}$	0.001303	0.001713	0.002094	0.002457	0.002809

Table 8: MSE values of different estimators for different values of K in Population 2.

Estimator	$K = 1.5$	$K = 2.0$	$K = 2.5$	$K = 3.0$	$K = 3.5$
$\hat{F}^*_0$	0.009261	0.010543	0.011825	0.013107	0.014390
$\hat{F}^*_{R_1}$	0.028014	0.033371	0.038728	0.044085	0.049442
$\hat{F}^*_{E_1}$	0.015253	0.017991	0.020730	0.023468	0.026207
$\hat{F}_{D_1}$	0.008759	0.009779	0.010781	0.011772	0.012756
$\hat{F}^*_{Rao}$	0.008586	0.009564	0.010521	0.011463	0.012393
$\hat{F}^*_{GS}$	0.008513	0.009468	0.010399	0.011313	0.012212
$\hat{F}^*_{R_2}$	0.070830	0.083730	0.096630	0.109540	0.122440
$\hat{F}^*_{E_2}$	0.027187	0.032177	0.037166	0.042156	0.047146
$\hat{F}^*_{D_2}, \hat{F}^*_{CH}, \hat{F}^*_{SU}$	0.008754	0.009776	0.010780	0.011772	0.012755
$\hat{F}^*_{KU}$	0.011939	0.014108	0.016271	0.018429	0.020586
$\hat{F}^*_{P(0,0)}$	0.008582	0.009562	0.010520	0.011462	0.012392
$\hat{F}^*_{P(1,1)}$	0.008576	0.009554	0.010509	0.011447	0.012373
$\hat{F}^*_{P(1,0)}$	0.008721	0.009740	0.010740	0.011729	0.012709
$\hat{F}^*_{P(0.5,0.5)}$	0.008525	0.009488	0.010425	0.011345	0.012251
$\hat{F}^*_{P(0,1)}$	0.008509	0.009488	0.010425	0.011345	0.012251

Table 9: MSE values of different estimators for different values of K in Population 3.

Estimator	$K=1.5$	$K=2.0$	$K=2.5$	$K=3.0$	$K=3.5$
$\hat{F}_0^*$	0.006340	0.006896	0.007451	0.008007	0.008563
$\hat{F}_{R_1}^*$	0.003569	0.003570	0.003572	0.003573	0.003574
$\hat{F}_{E_1}^*$	0.003682	0.003837	0.003992	0.004147	0.004302
$\hat{F}_{D_1}$	0.003305	0.003342	0.003373	0.003399	0.003421
$\hat{F}_{Rao}^*$	0.003284	0.003321	0.003351	0.003377	0.003399
$\hat{F}_{GS}^*$	0.003275	0.003311	0.003340	0.003365	0.003386
$\hat{F}_{R_2}^*$	0.007953	0.008426	0.008900	0.009374	0.009848
$\hat{F}_{E_2}^*$	0.002231	0.002232	0.002232	0.002233	0.002233
$\hat{F}_{D_2}^*, \hat{F}_{CH}^*, \hat{F}_{SU}^*$	0.001928	0.001936	0.001943	0.001949	0.001954
$\hat{F}_{KU}^*$	0.003471	0.003473	0.003474	0.003476	0.003477
$\hat{F}_{P(0,0)}^*$	0.001921	0.001929	0.001936	0.001942	0.001947
$\hat{F}_{P(1,1)}^*$	0.001921	0.001919	0.001936	0.001942	0.001947
$\hat{F}_{P(1,0)}^*$	0.001928	0.001936	0.001943	0.001949	0.001953
$\hat{F}_{P(0.5,0.5)}^*$	0.001917	0.001925	0.001931	0.001937	0.001941
$\hat{F}_{P(0,1)}^*$	0.001916	0.001923	0.001929	0.001935	0.001939

Biographies

Javid Shabbir is working as Tenured Professor of Statistics and Dean Faculty of Natural Sciences at Quaid-i-Azam University Islamabad, Pakistan. He completed his MS degree from the University of Southampton, UK and PhD degree from the University of Kent at Canterbury, UK. His field of interests are, Survey sampling, Nonresponse, Ranked set sampling and Randomized response techniques. He published more than 200 research papers in internationally reputed journals and produced 115 MPhil and 14 PhD students. He participated in many national and international conferences. He is an associated editor of the "Journal of Statistical Theory and Practice"-Springer-Verlag.

Sat Gupta is a Professor of Statistics and Head of the Department of Mathematics and Statistics at the University of North Carolina at Greensboro. He is a Fellow of the American Statistical Association and has bagged several other awards in his career. These include the Senior University-wide Research Excellence Award, and the Senior College of Arts and Sciences Teaching Excellence Award at University of North Carolina Greensboro; Outstanding Faculty Award and the Outstanding Teacher/Scholar Award from the University of Southern Maine; the Distinguished Service Award for the Cause of Statistics given by the North Carolina Chapter of the American Statistical Association; and the Sankhyiki Bhushan Award from The Indian Society of Agricultural Statistics. He has been the founding Editor-in-Chief since 2007 of the prestigious research journal '*Journal of Statistical Theory and Practice*' published by Springer. Professor Sat Gupta had his college education from Delhi University from where he earned a PhD degree in mathematics in 1977. After teaching assignments at Delhi University Professor Gupta came to Colorado State University, USA in 1982 from where he earned a PhD degree in statistics in 1987.