Scientia Iranica D (2023) 30(3), 1148–1157

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions D: Computer Science & Engineering and Electrical Engineering
http://scientiairanica.sharif.edu

Research Note

Robust state-feedback controller for linear parameter-varying systems with time-invariant uncertainties

H. Asrari, I. Mohammadzaman*, and F. Allahverdizadeh

Faculty of Electrical and Computer Engineering, Malek Ashtar University of Technology, Tehran, P.O. Box 15875-1774, Iran.

Received 10 October 2020; received in revised form 7 June 2021; accepted 6 September 2021

KEYWORDS
Linear parameter-varying;
Robust state-feedback;
Uncertain time-varying parameters;
Time-invariant parametric uncertainties;
Gain-scheduling.

Abstract. In this paper, the robust gain-scheduled state-feedback controller problem is studied for uncertain linear parameter-varying systems whose state-space representations are a linear combination of the uncertain time-varying parameters including time-invariant parametric uncertainties. It is supposed that these uncertainties are bounded by the given intervals and cannot be pulled out as an uncertain block. This is a serious challenge because the exact information about the plant dynamics cannot be extracted from the uncertain time-varying parameters, while the gain-scheduled controllers need to have the exact information about the plant dynamics in order to satisfy the desired control purposes. To handle this challenge, we introduce a state-feedback gain, which is formed by a set of new scheduling parameters and a secondary time-varying term. The stabilization conditions are obtained in terms of the linear matrix inequalities. The effectiveness of the proposed method is shown using an example.

1. Introduction

Linear Parameter-Varying (LPV) systems have emerged from the theory of nonlinear systems and have become one of the most popular approaches in the field of control engineering. Indeed, the LPV scheme has great potential to overcome the difficulties of the real-world processes [1–5]. The LPV approach has been introduced to model gain-scheduling in a wide variety of applications. The main rationale for the LPV approaches lies in the possibility of designing the gain-scheduled controllers in a very systematic manner [6,7]. Thenceforth, numerous studies have focused on the use of the LPV controllers. In this context, many researchers have similarly supposed that the exact scheduling parameters are available for calculating a gain-scheduled controller [8–10]. This is while the presence of uncertainties is a major problem in reality [11]. This means that the scheduling parameters are expected to include the uncertainties in practical applications. For example, measurement of the time-varying parameters is not exact in practical applications. This results in a set of inexact scheduling parameters as mentioned in the literature [12–16]. However, one of the most important challenges in designing the gain-scheduled controllers for the LPV systems is the uncertain time-varying

*. *Corresponding author. Tel.: +98 21 22945142*
E-mail addresses: asrarihossein@gmail.com (H. Asrari); mohammadzaman@mut.ac.ir (I. Mohammadzaman); Saeed.Allahverdizadeh@gmail.com (F. Allahverdizadeh)

doi: 10.24200/sci.2021.56871.4953

parameters including Time-Invariant Parametric Uncertainties (TIPU). In the cases where we cannot extract the TIPU explicitly, the exact information about the plant dynamics is not available for the gain-scheduled controllers.

The control design problem for the uncertain LPV systems has been studied in recent years and some remarkable results have been achieved [17–22]. However, these works are not useful to cope with the mentioned challenges in the current paper. Some previous studies have introduced schemes considering the concepts of TIPU and gain scheduling, simultaneously. In a study, a robust gain-scheduled control method was developed in the presence of external disturbance as well as the TIPU [23]. The major drawback of the mentioned study was that the input matrix was constant. Although the input matrix was influenced by the TIPU, it was supposed that the input matrix did not vary significantly with the scheduling vector. Therefore, the elements of the input matrix were substituted with their mean values. This was while the TIPU might lead the stable system towards instability. Rotondo et al. [24] analysed the same problem. However, the resulting idea suffered from a major limitation on the matrix structure. They rewrote the uncertain matrices using a linear combination of the exact scheduling parameters. In other words, the weighting matrices used in this combination were only involved in the TIPU, which might be impossible in some of the uncertain matrices. In another study, a state-feedback control strategy was presented to cope with the uncertain time-varying parameters by the use of secondary convex parameters [25]. These parameters were obtained online by maximizing and minimizing the uncertain time-varying parameters over the bounds of the TIPU at the moment "t." Obviously, finding the upper and lower bounds of the uncertain time-varying parameters in every second of the simulation time makes it a time-consuming process, imposing complex computations on the control system. Therefore, employing the secondary convex parameters is unreasonable and impractical.

Motivated by the above-stated challenge, we address the design problem of the robust gain-scheduled state-feedback controllers for uncertain LPV systems in which the state-space matrices are affine with respect to the uncertain time-varying parameters. It is assumed that the uncertainties in the model originate from the TIPU, which cannot be extracted explicitly from the uncertain time-varying parameters. In this case, the exact information about the plant dynamics is not available to the controller. Therefore, the uncertain time-varying parameters cannot be considered as the exact scheduling parameters. To handle this challenge, we seek to find a simple strategy in the state-feedback framework, which stabilizes the uncertain LPV systems for all the allowable values of the interval uncertainties. For this purpose, a straightforward approach is introduced in this paper in the following steps:

a) As mentioned previously, a proper control strategy has to be employed to stabilize the uncertain LPV systems for all the allowable values of the TIPU. Accordingly, we are free to choose arbitrary values for the TIPU from the given intervals. Thus, a set of the time-varying parameters will be obtained by substituting the arbitrary values of the TIPU into the uncertain time-varying parameters. It is evident that the arbitrary values of the TIPU are not necessarily equal to the true ones in the real LPV system. Hence, the obtained parameters do not exactly describe the plant dynamics and hence, cannot be used to construct the exact scheduling parameters. Therefore, we must find a way to compensate for the difference between the arbitrary and the true values of the TIPU. The solution is provided in the next step;

b) New scheduling parameters are hired to handle the mentioned challenge in the previous step. To this purpose, it is necessary to find offline the values of the TIPU that maximize and minimize the uncertain time-varying parameters over the variation ranges of the time-dependent expressions present in the uncertain time-varying parameters. After that, substituting the obtained values in the uncertain time-varying parameters results in a set of known parameters. Finally, the new scheduling parameters will be created by combining the known parameters. Then, we can employ the new scheduling parameters to construct the state-feedback gain. In this process, the known parameters will be scaled by a constant value. However, this can lead to the lack of information on the plant dynamics. To cope with the mentioned challenge, a new state-feedback gain is introduced in the following step;

c) To deal with the lack of information on the plant dynamics, two mathematical statements are employed to form the new state-feedback gain. The first one is obtained by the use of the new scheduling parameters. The second statement is a secondary time-varying term, which is updated by a proper adaptation law to compensate for the lack of information on the first mathematical statement. Given the steps above, the secondary time-varying term can help compensate for the difference between the arbitrary and the true values of the TIPU.

The rest of the paper is organised as follows: Section 2 presents the preliminaries for the present study; Section 3 provides the main results; Section 4 demonstrates the validity and applicability of the proposed technique for an inverted pendulum; finally, conclusions are given in Section 5.

The following notation is used in this paper:

$$[C_i]_p := \begin{bmatrix} C_1 & C_2 & \cdots & C_p \end{bmatrix}, \tag{1}$$

$$\left[C_{(i,j)}\right]_{p\times p} := \begin{bmatrix} C_{(1,1)} & C_{(1,2)} & \cdots & C_{(1,p)} \\ C_{(2,1)} & C_{(2,2)} & \cdots & C_{(2,p)} \\ \vdots & \vdots & \ddots & \vdots \\ C_{(p,1)} & C_{(p,2)} & \cdots & C_{(p,p)} \end{bmatrix}. \tag{2}$$

Also, the symbol $(*)$ denotes the elements below the main diagonal of a symmetric block matrix.

2. Preliminaries and statement of the problem

Suppose that an uncertain LPV system with p uncertain time-varying parameters $\Theta(t,\mu) = \begin{bmatrix}\theta_1(t,\mu) & \cdots & \theta_p(t,\mu)\end{bmatrix}^T$ is given in the following form:

$$\dot{X}(t) = A(\Theta(t,\mu))X(t) + B(\Theta(t,\mu))u(t). \tag{3}$$

The plant model can be expressed as the convex combination of the vertex systems:

$$A(\Theta(t,\mu)) = \sum_{i=1}^{p} A_i\theta_i(t,\mu),$$

$$B(\Theta(t,\mu)) = \sum_{i=1}^{p} B_i\theta_i(t,\mu), \tag{4}$$

where the combinations are given with the uncertain time-varying parameters $\theta_1(t,\mu), \theta_2(t,\mu), \cdots, \theta_p(t,\mu)$ satisfying:

$$\sum_{i=1}^{p}\theta_i(t,\mu)=1, \qquad 0\le\theta_i(t,\mu)\le 1, \qquad i=1,2,\cdots,p, \tag{5}$$

where $\mu \in R^{q\times 1}$ is the vector of the TIPU, $\Theta(t,\mu) \in R^{p\times 1}$ is the vector of the uncertain time-varying parameters, A_i, B_i denote the constant matrices with appropriate dimensions, and $X(t) \in R^{n\times 1}$ introduces the state vector assumed to be available for measurements. Also, $u(t) \in R^{m\times 1}$ is the control input.

Now, we define the state-feedback control law as follows:

$$u(t) = K\left(\Theta_{New}(t), J_{ag}(t)\right)X(t), \tag{6}$$

with:

$$K(\Theta_{New}(t), J_{ag}(t)) = \sum_{i=1}^{p}(K_i\theta_{iNew}(t)) + J_{ag}(t), \tag{7}$$

where $J_{ag}(t)$ is the secondary time-varying term and $K_i, i = 1, \cdots, p$, are the parameter-independent matrices to be calculated. Also, $\Theta_{New}(t) = \begin{bmatrix}\theta_{1New}(t) & \cdots & \theta_{pNew}(t)\end{bmatrix}^T$ is the vector of the new scheduling parameters. According to the steps presented in the section on Introduction, we are free to select the arbitrary values of the TIPU from the given intervals. Also, we stated that $\sum_{i=1}^{p}(K_i\theta_{iNew}(t))$ could not stabilize the uncertain LPV system for all the allowable values of the TIPU. Therefore, the secondary time-varying term, denoted by $J_{ag}(t)$, is added to $\sum_{i=1}^{p}(K_i\theta_{iNew}(t))$ to compensate for the difference between the arbitrary values of the TIPU and the true ones. In other words, the task of $J_{ag}(t)$ is to compensate for the lack of information on $\sum_{i=1}^{p}(K_i\theta_{iNew}(t))$.

Substituting Eq. (6) into Eq. (3) yields a closed-loop system in the following form:

$$\dot{X}(t) = A_c(\Theta(t,\mu), \Theta_{New}(t), J_{ag}(t))X(t), \tag{8}$$

where:

$$A_c(\Theta(t,\mu), \Theta_{New}(t), J_{ag}(t)) = A(\Theta(t,\mu)) + B(\Theta(t,\mu))K(\Theta_{New}(t), J_{ag}(t)). \tag{9}$$

Remark 1. The vector μ is unknown. However, the elements of μ are supposed to be bounded and their bounds are given a priori. On the other hand, it is supposed that we cannot explicitly extract the TIPU. Therefore, the arbitrary vector μ_c is selected for the elements of μ to simulate the uncertain LPV system. As mentioned earlier, the arbitrary values for the elements of μ_c are not necessarily equal to the true ones in the real LPV system. Therefore, the state-feedback control law in Eq. (6) is proposed to compensate for the difference between the arbitrary and true values of the TIPU.

Accordingly, Eq. (8) will hold and can be rewritten in the following form:

$$\dot{X}(t) = A_c(\Theta(t,\mu_c), \Theta_{New}(t), J_{ag}(t))X(t). \tag{10}$$

The new scheduling parameters are defined in the following form:

$$\theta_{iNew}(t) = \frac{M_i(t)}{2p}, \qquad i = 1,2,\cdots,p, \tag{11}$$

with:

$$M_i(t) = \sum_{\substack{j=1\\ j\neq i}}^{p}\left[\theta_i(t,\mu)|_{\mu=\mu_{j\max}} + \theta_i(t,\mu)|_{\mu=\mu_{j\min}}\right] + \theta_i(t,\mu)|_{\mu=\mu_{i\max}} + \theta_i(t,\mu)|_{\mu=\mu_{i\min}}, \tag{12}$$

where $\mu_{i\max}$, $\mu_{i\min}$, $i = 1,2,\cdots,p$, consist of unique values, which can maximize and minimize the uncertain time-varying parameters over the bounds of the TIPU and the variation ranges of the time-varying expressions present in $\theta_i(t,\mu)$, $i = 1,2,\cdots,p$, respectively. In addition, $\mu_{i\max}$ and $\mu_{i\min}$, $i = 1,2,\cdots,p$, are determined offline by a proper software. p is the number of the new scheduling parameters. Also, $\theta_i(t,\mu)|_{\mu=\mu_{j\max}}$, $\theta_i(t,\mu)|_{\mu=\mu_{j\min}}$, $\theta_i(t,\mu)|_{\mu=\mu_{i\max}}$, and $\theta_i(t,\mu)|_{\mu=\mu_{i\min}}$

are obtained by substituting $\mu_{j\max}$, $\mu_{j\min}$, $\mu_{i\max}$, and $\mu_{i\min}$ into μ, respectively.

Substituting the obtained values in the uncertain time-varying parameters and combining them innovatively results in the development of the new scheduling parameters.

Now, let us prove that the time-varying parameters in the form of Eq. (11) satisfy the conditions mentioned in Eq. (5). Obviously, the upper and lower bounds of $\theta_i(t,\mu)$, $i = 1,2,\cdots,p$, are equal to one and zero, respectively. Therefore, we can obtain the following inequalities:

$$0 \le M_i(t) \le 2p. \tag{13}$$

Now, we divide Eq. (13) into $2p$. Then, we have the following inequality:

$$0 \le \frac{M_i(t)}{2p} \le 1, \Rightarrow 0 \le \theta_{iNew}(t) \le 1. \tag{14}$$

Therefore, we showed that the new scheduling parameters lied in the range of $\begin{bmatrix}0 & 1\end{bmatrix}$. Based on the equality $\sum_{i=1}^{p}\theta_i(t,\mu) = 1$, we can obtain that $\sum_{i=1}^{p}\theta_{iNew}(t)$ is equal to one. Then, the proof is completed.

Remark 2. To clarify the procedure presented in this paper, it is necessary to provide the following description. Suppose that $\theta_i(t_0,\mu)$ denotes the ith uncertain time-varying parameter at the moment "$t = t_0$." Evidently, we must know the variation range of $\theta_i(t_0,\mu)$ for all the allowable values of the TIPU in order to design an effective controller. This leads to a set of secondary scheduling parameters, which are obtained by maximizing and minimizing the uncertain time-varying parameters over the bounds of the TIPU at the moment "t". Such a procedure is used in [25]. However, it is very difficult, time-consuming, unreasonable, and impractical. Therefore, we need to look for a way to handle this challenge. In this regard, we propose the state-feedback gain in the form of Eq. (7). The first term of the state-feedback gain is scaled by $2p$. This means that the first term of Eq. (7) cannot cover the exact information about $\theta_i(t,\mu)$ for all the allowable values of the elements of μ. This is while we mentioned that the designers were free to select the arbitrary values for the elements of the vector μ. Accordingly, the task of $J_{ag}(t)$ in Eq. (7) is to compensate for the difference between the arbitrary values for the elements of the vector μ and the true ones. This means that $J_{ag}(t)$ compensates for the lack of information on the first term of Eq. (7). It should be noted that $J_{ag}(t)$ will be updated by a proper adaptation law.

3. Main results

The following theorem proposes a robust gain-scheduled state-feedback controller, which can stabilize the uncertain LPV systems in the presence of the TIPU.

Theorem 1. If there exist matrices L_0, W_0, ω_i, Y_i, R_i, L_i, H_i, S_i, E_i, Γ_{ij}, G_{ij}, Q_{ij}, $J_{ag}(t)$, P, and $\zeta(t)$ satisfying the following Linear Matrix Inequalities (LMIs) for $i,j = 1,2,\cdots,p$:

$$(\omega_i + \omega_i^T) > 0, \qquad (Y_i + Y_i^T) > 0,$$

$$(Q_{ij} + Q_{ij}^T) > 0, \qquad (G_{ij} + G_{ij}^T) > 0, \tag{15}$$

$$(\Gamma_{ij} + \Gamma_{ij}^T) > 0, \qquad P > 0, \qquad \Omega < 0, \tag{16}$$

$$\dot{\zeta}(t) = -\zeta(t)A(\Theta(t,\mu_c)), \tag{17}$$

$$\begin{aligned} J_{ag}(t) = &-\left(B^T(\Theta(t,\mu_c))B(\Theta(t,\mu_c))\right)^{-1} \\ &\times B^T(\Theta(t,\mu_c))\left(P^{-1} + \zeta^T(t)\zeta(t)\right)^{-1} \\ &\times \left(\zeta^T(t)\zeta(t)B(\Theta(t,\mu_c))\tilde{K}(\Theta_{New}(t))\right), \end{aligned} \tag{18}$$

where:

$$\Omega = \begin{bmatrix} U_0 & \left[U_{(0,i)} + \phi_{(0,i)}\right]_p & \left[U_{(0,iNew)}\right]_p \\ (*) & \left[U_{(i,j)}\right]_{p\times p} & \left[U_{(i,jNew)} + \phi_{(i,jNew)}\right]_{p\times p} \\ (*) & (*) & \left[U_{(iNew,jNew)}\right]_{p\times p} \end{bmatrix},$$

$$U_0 = 2(L_0 + L_0^T) - \sum_{i=1}^{p}(\omega_i + \omega_i^T) - \sum_{i=1}^{p}(Y_i + Y_i^T),$$

$$U_{(0,i)} = W_0 + 2R_i + \omega_i - L_0,$$

$$U_{(0,iNew)} = -W_0 + 2L_i + Y_i - L_0,$$

$$U_{(i,j)} = \begin{cases} -(\omega_i + \omega_i^T) - (H_i + H_i^T) - (R_i + R_i^T), \\ \quad (i = j) \\ G_{ij} + (H_i + H_j) - (R_i + R_j), \\ \quad (i < j) \end{cases}$$

$$U_{(iNew,jNew)} = \begin{cases} -(S_i + S_i^T) - (L_i + L_i^T) - (Y_i + Y_i^T), \\ \quad (i = j) \\ \Gamma_{ij} - (S_i + S_j) - (L_i + L_j), \\ \quad (i < j) \end{cases}$$

$$U_{(i,jNew)} = Q_{ij} + S_j - H_i - R_i - L_j,$$

$$\phi_{(0,i)} = A_i P,$$

$$\phi_{(i,jNew)} = B_i E_{jNew},$$

$$\tilde{K}(\Theta_{New}(t)) = \sum_{i=1}^{p}(K_i\theta_{iNew}(t)).$$

Then, the closed-loop system, which is presented in Eq. (8), is asymptotically stable. In this case, the feedback gains in the vertices can be obtained in the following form:

$$K_i = E_i P^{-1}, \qquad i = 1, 2, \cdots, p. \tag{19}$$

Also, $\zeta(t)$ is a matrix updated by an adaptation law in Eq. (17).

Proof. Let us consider a candidate Lyapunov function in the following form:

$$V(t) = X^T(t)P^{-1}X(t) + (\zeta(t)X(t))^T(\zeta(t)X(t)). \tag{20}$$

Differentiating $V(t)$ with respect to "t" yields:

$$\begin{aligned}\dot{V}(t) =& \dot{X}^T(t)P^{-1}X(t) + X^T(t)P^{-1}\dot{X}(t)\\ &+ 2(\zeta(t)X(t))^T\left(\dot{\zeta}(t)X(t) + \zeta(t)\dot{X}(t)\right).\end{aligned} \tag{21}$$

Now, we substitute Eqs. (3), (6), and (7) into Eq. (21) to get:

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t), \tag{22}$$

where:

$$\begin{aligned}\dot{V}_1(t) =& X^T(t)\left(A^T(\Theta(t,\mu_c))P^{-1} + P^{-1}A(\Theta(t,\mu_c))\right.\\ &+ \tilde{K}^T(\Theta_{New}(t)) \times B^T(\Theta(t,\mu_c))P^{-1}\\ &\left.+P^{-1}B(\Theta(t,\mu_c)) \times \tilde{K}(\Theta_{New}(t))\right)X(t),\end{aligned} \tag{23}$$

$$\dot{V}_2(t) = 2X^T(t)\Big(\zeta^T(t)\dot{\zeta}(t) + \zeta^T(t)\zeta(t)A(\Theta(t,\mu_c))\Big)X(t), \tag{24}$$

$$\begin{aligned}\dot{V}_3(t) =& 2X^T(t)\left(P^{-1}B(\Theta(t,\mu_c))J_{ag}(t) + \zeta^T(t)\zeta(t)\right.\\ &\times B(\Theta(t,\mu_c))\tilde{K}(\Theta_{New}(t)) + \zeta^T(t)\zeta(t)\\ &\left.\times B(\Theta(t,\mu_c))J_{ag}(t)\right)X(t).\end{aligned} \tag{25}$$

We can achieve $\dot{V}(t) < 0$ by obtaining $\dot{\zeta}(t)$ and $J_{ag}(t)$ in the following form:

$$\dot{V}_2(t) = 0 \Rightarrow \dot{\zeta}(t) = -\zeta(t)A(\Theta(t,\mu_c)), \tag{26}$$

$$\begin{aligned}&\dot{V}_3(t) = 0 \Rightarrow\\ &J_{ag}(t) = -\left(B^T(\Theta(t,\mu_c))B(\Theta(t,\mu_c))\right)^{-1}\\ &\qquad\times B^T(\Theta(t,\mu_c))\left(P^{-1} + \zeta^T(t)\zeta(t)\right)^{-1}\\ &\qquad\times\left(\zeta^T(t)\zeta(t)B(\Theta(t,\mu_c))\tilde{K}(\Theta_{New}(t))\right).\end{aligned} \tag{27}$$

Finally, the stability criterion is in the following form:

$$\dot{V}(t) = \dot{V}_1(t) < 0. \tag{28}$$

For this purpose, the following inequality is valid:

$$\begin{aligned}G(t) =& \left(PA^T(\Theta(t,\mu_c)) + A(\Theta(t,\mu_c))P\right.\\ &+ P\tilde{K}^T(\Theta_{New}(t))B^T(\Theta(t,\mu_c))\\ &\left.+B(\Theta(t,\mu_c))\tilde{K}(\Theta_{New}(t))P\right) < 0,\end{aligned} \tag{29}$$

Eq. (29) can be rewritten in the following form:

$$\begin{aligned}G(t) =& \sum_{i=1}^{p}\theta_i(t,\mu_c)(\phi_{(0,i)} + \phi^T_{(0,i)})\\ &+\sum_{i=1}^{p}\sum_{j=1}^{p}\theta_i(t,\mu_c)\theta_{jNew}(t)\Big(\phi_{(i,jNew)} + \phi^T_{(i,jNew)}\Big) < 0,\end{aligned} \tag{30}$$

where $\phi_{(i,jNew)} = B_i E_{jNew}$ and $\phi_{(0,i)} = A_i P$.

Now, we use a relaxation technique introduced in [25] to derive LMI conditions. Applying the ***S***-procedure, the condition $\dot{V}(t) < 0$ for all the trajectories is equivalent to the existence of $N(t) \geq 0$ such that:

$$G(t) + N(t) < 0, \tag{31}$$

$N(t)$ is proposed in this paper in the following form:

$$\begin{aligned}N(t) =& U_0 + \sum_{i=1}^{p}\theta_i(t,\mu_c)(U_{(0,i)} + U^T_{(0,i)})\\ &+ \sum_{i=1}^{p}\theta_{iNew}(t)\left(U_{(0,iNew)} + U^T_{(0,iNew)}\right)\\ &+\sum_{i=1}^{p}\theta_i^2(t,\mu_c)U_{(i,i)} + \sum_{i=1}^{p}\theta^2_{iNew}(t)U_{(iNew,iNew)}\\ &+ \sum_{i=1}^{p}\sum_{j=1}^{p}\theta_i(t,\mu_c)\theta_{jNew}(t)\\ &\left(U_{(i,jNew)} + U^T_{(i,jNew)}\right)\\ &+ \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\theta_{iNew}(t)\theta_{jNew}(t)\\ &\left(U_{(iNew,jNew)} + U^T_{(iNew,jNew)}\right)\\ &+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\theta_i(t,\mu_c)\theta_j(t,\mu_c)(U_{(i,j)} + U^T_{(i,j)}),\end{aligned} \tag{32}$$

where U_0, $U_{(0,i)}$, $U_{(0,iNew)}$, $U_{(i,j)}$, $U_{(i,jNew)}$, and $U_{(iNew,jNew)}$ have been introduced in Theorem 1.

According to Eqs. (30) and (32), Inequality (31) can be rewritten as follows:

$$\Psi = \Lambda^T \Omega \Lambda < 0, \tag{33}$$

where:

$$\Lambda = \begin{bmatrix} I \\ \theta_1(t,\mu_c)I \\ \vdots \\ \theta_p(t,\mu_c)I \\ \theta_{1New}(t)I \\ \vdots \\ \theta_{pNew}(t)I \end{bmatrix}. \tag{34}$$

Therefore, the proof is completed.

Remark 3. As proven above, the proposed control law in Eq. (6) is useful to stabilize the uncertain LPV system, which is introduced in Eq. (3). However, any selection of the TIPU admits a new set of time-varying parameters $\theta_i(t,\mu_c)$, $i = 1,2,\cdots,p$. This means that the poles of $A(\Theta(t,\mu_c)) + B(\Theta(t,\mu_c))K(\Theta_{New}(t), J_{ag}(t))$ are dependent on the selection of μ_c. Therefore, any μ_c has a significant impact on the location of the closed-loop poles.

4. Simulation results

We aim to stabilize an inverted pendulum on a moving cart influenced by the TIPU to demonstrate the efficiency of the proposed technique. The model has been given by Park et al. [25] in the following form:

$$\ddot{x}(t) = \frac{g\sin(x(t))}{4L/3 - bmL\cos^2(x(t))} - \frac{\frac{bmL\dot{x}^2(t)\sin(2x(t))}{2} + b\cos(x(t))u(t)}{4L/3 - bmL\cos^2(x(t))}, \tag{35}$$

where $x(t)$ (rad), m (kg), and $L = 0.5$ (m) are the angular displacement, mass, and length of the pendulum, respectively. M (kg) is the mass of the cart, $g = 9.8$ m/s^2 is the gravitational acceleration, and b is a function of m and M. Also, $u(t)$ is the control effort applied to the cart. b, m, and M are considered as the TIPU in the following form:

$$b = \frac{1}{M+m}, \qquad 2 \le m \le 3, \qquad 8 \le M \le 16. \tag{36}$$

The state space model of Eq. (35) is represented by the following LPV model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f(X(t)) & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g(X(t)) \end{bmatrix} u(t), \tag{37}$$

where $X(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ and:

$$f(X(t)) = \frac{g - bmLx_2^2(t)\cos(x_1(t))}{4L/3 - bmL\cos^2(x_1(t))} \frac{\sin(x_1(t))}{x_1(t)}, \tag{38}$$

$$g(X(t)) = \frac{-b\cos(x_1(t))}{4L/3 - bmL\cos^2(x_1(t))}. \tag{39}$$

Also, it is supposed that $x_1(t)$ and $x_2(t)$ vary in the following ranges:

$$-5\pi/12 \le x_1(t) \le 5\pi/12, \qquad -5 \le x_2(t) \le 5. \tag{40}$$

Now, we can convert Eq. (37) into Eq. (3) using the following trend:

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ f_{\min} & 0 \end{bmatrix}, \qquad A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ f_{\max} & 0 \end{bmatrix},$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ g_{\min} \end{bmatrix} \qquad B_2 = B_4 = \begin{bmatrix} 0 \\ g_{\max} \end{bmatrix},$$

$$\theta_1(t) = \beta_1(t)\varphi_1(t), \qquad \theta_2(t) = \beta_1(t)\varphi_2(t),$$

$$\theta_3(t) = \beta_2(t)\varphi_1(t), \qquad \theta_4(t) = \beta_2(t)\varphi_2(t), \tag{41}$$

where:

$$\beta_1(t) = \frac{f_{\max} - f}{f_{\max} - f_{\min}}, \qquad \beta_2(t) = \frac{f - f_{\min}}{f_{\max} - f_{\min}},$$

$$\varphi_1(t) = \frac{g_{\max} - g}{g_{\max} - g_{\min}}, \qquad \varphi_2(t) = \frac{g - g_{\min}}{g_{\max} - g_{\min}}.$$

$f_{\min}$, $f_{\max}$, $g_{\min}$, $g_{\max}$ can be calculated as follows:

$$f_{\min} = 9.9225, \qquad f_{\max} = 18.9677,$$

$$g_{\min} = -0.1935, \qquad g_{\max} = -0.0205,$$

Solving the matrix Inequalities (15) and (16) yields:

$$E_1 = \begin{bmatrix} 154.0078 & 28.4551 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 154.8349 & 30.5422 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 154.8349 & 30.5422 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 154.4318 & 30.5186 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.9347 & -2.9676 \\ -2.9676 & 12.3033 \end{bmatrix}.$$

We consider $\mu = \begin{bmatrix} b & m \end{bmatrix}$ as the vector of the TIPU. The new scheduling parameters are proposed according to Eqs. (11) and (12). In this regard, $\mu_{i\max}$, $\mu_{i\min}$, $i = 1,2,3,4$, and p are as follows:

$$\mu_{1\max} = \begin{bmatrix} 0.1 & 3 \end{bmatrix}, \qquad \mu_{1\min} = \begin{bmatrix} 0.052 & 2 \end{bmatrix},$$

$$\mu_{2\max} = \begin{bmatrix} 0.052 & 3 \end{bmatrix}, \qquad \mu_{2\min} = \begin{bmatrix} 0.1 & 3 \end{bmatrix},$$

$$\mu_{3\max} = \begin{bmatrix} 0.1 & 3 \end{bmatrix}, \qquad \mu_{3\min} = \begin{bmatrix} 0.052 & 2 \end{bmatrix},$$

$$\mu_{4\max} = \begin{bmatrix} 0.052 & 3 \end{bmatrix}, \qquad \mu_{4\min} = \begin{bmatrix} 0.1 & 3 \end{bmatrix},$$

$$p = 4.$$

$\begin{bmatrix}1.2 & 0\end{bmatrix}^T$ and $\begin{bmatrix}1 & -0.1\\ 1 & 0.3\end{bmatrix}$ are the initial conditions for the states and matrix $\zeta(t)$. Now, we simulate the closed-loop system by taking $M = 16$ (kg), $m = 3$ (kg) and $b = 1/19$ (kg^{-1}) as the arbitrary values of the TIPU. Figures 1, 2, 3, and 4 illustrate the behaviour of the states, the new scheduling parameters, the scheduling parameters used to simulate the LPV system, and the matrix $\zeta(t)$, respectively. Note that the dimensions of the matrix $\zeta(t)$ employed by $J_{ag}(t)$ are 2×2.

As can be seen, the time-varying parameters in

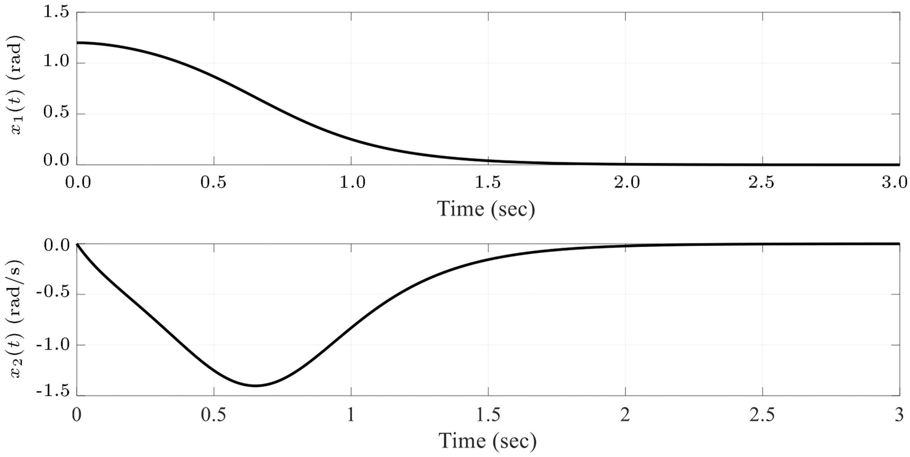

Figure 1. $x_1(t)$ and $x_2(t)$ under the proposed method.

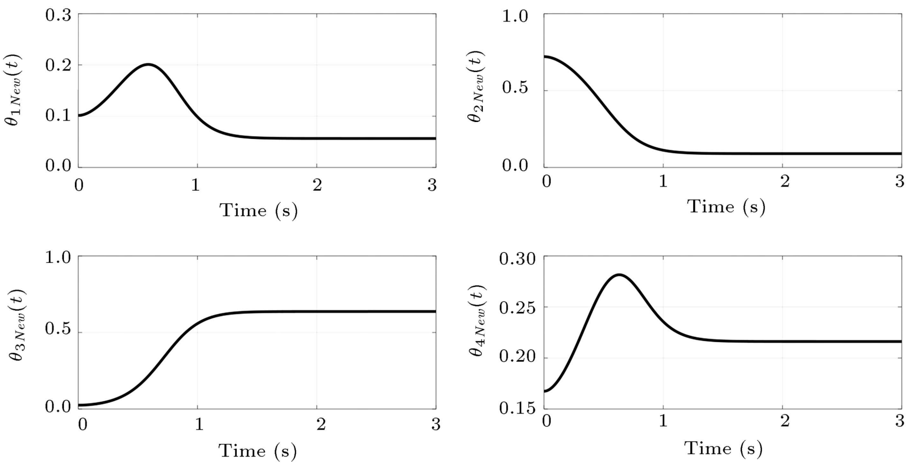

Figure 2. The new scheduling parameters.

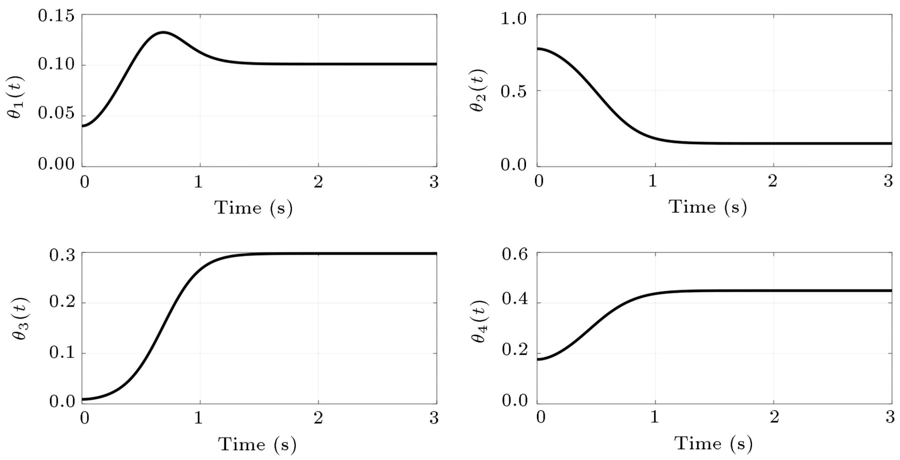

Figure 3. The scheduling parameters used to simulate the LPV system.

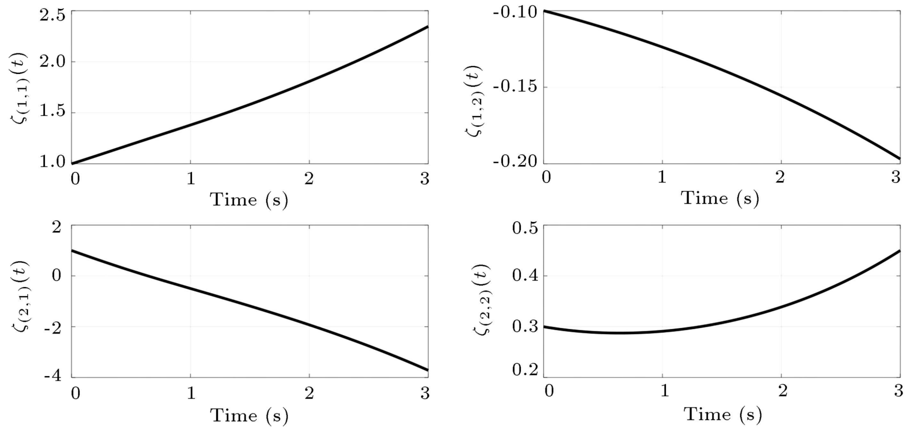

Figure 4. Time variations of the elements of the matrix $\zeta(t)$ under the proposed method.

Figures 2 and 3 are bounded by 0 and 1. The states exhibit a good performance and go to zero asymptotically. Acceptable behaviour of the states confirms that $\zeta(t)$ works well. The simulation results indicate the effectiveness of the proposed method in the presence of the TIPU.

Remark 4. Let us explain the presented idea from another viewpoint. Without loss of generality, it is assumed that $\theta_1(t,\mu)$, $\theta_2(t,\mu)$ are the uncertain time-varying parameters. μ is the vector of the TIPU. Obviously, we have no exact knowledge about μ. On the other hand, the scheduling parameters must be known to schedule the gain controller. Thus, we assign an arbitrary vector μ_c to μ in order to handle this challenge. Therefore, the polytope in Figure 5 will be built due to the variations of $\theta_1(t,\mu)|_{\mu=\mu_c}$ and $\theta_2(t,\mu)|_{\mu=\mu_c}$. However, this polytope is not exact, because μ_c is not the true vector. Different polytopes can be built for all the allowable μ_c. Therefore, there is a need to propose an approach in order to stabilize the closed-loop system for all the polytopes.

To this purpose, according to a previous study [25], the scheduling parameters, $\underline{\theta}_i(t)$, $\overline{\theta}_i(t)$ are introduced in the following form:

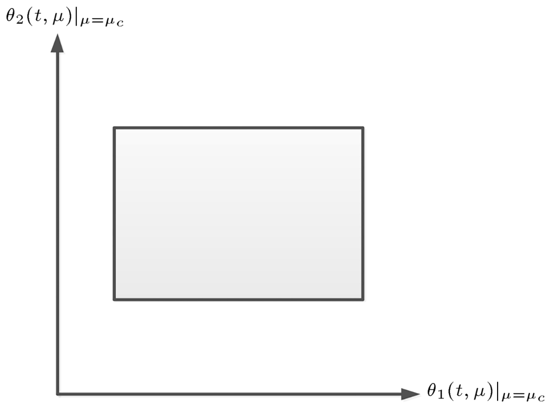

Figure 5. The polytope built due to the variations of $\theta_1(t,\mu)|_{\mu=\mu_c}$ and $\theta_2(t,\mu)|_{\mu=\mu_c}$.

$$\underline{\theta}_i(t)=\frac{\theta_{iL}(t)}{\sum_{i=1}^{p}\theta_{iL}(t)+\theta_{iU}(t)},$$

$$\overline{\theta}_i(t)=\frac{\theta_{iU}(t)}{\sum_{i=1}^{p}\theta_{iL}(t)+\theta_{iU}(t)},$$

$$i=1,2,\cdots,p, \tag{42}$$

where $\theta_{iL}(t)$ and $\theta_{iU}(t)$ are the lower and upper boundaries of the uncertain time-varying parameters, respectively. Now, we describe how to obtain these parameters. $\theta_{iL}(t_0)$ and $\theta_{iU}(t_0)$ will be determined by minimizing and maximizing $\theta_i(t_0,\mu)$ over the uncertain vector μ at the moment "$t = t_0$". As mentioned previously, the uncertain vector consists of the TIPU with the bounded intervals. Accordingly, one needs to search over the intervals of the TIPU by an iterative algorithm for obtaining the maximum and minimum $\theta_i(t_0,\mu)$ at the moment "$t = t_0$". Therefore, it is necessary to repeat cumbersome calculations at each moment of the simulation process. It is evident that such a procedure consumes an enormous amount of time and effort. Implementing such a procedure in practice can be difficult and may lead to a numerical challenge. Therefore, we must use a strong processor with a high clock speed. Given the price of the industrial products, this is not justifiable in some cases. Therefore, finding the lower and upper boundaries of the uncertain time-varying parameters leads to an online process imposing complex computations on the control system. This is while $\mu_{i\max}$ and $\mu_{i\min}$ are obtained offline in our method. Accordingly, many iterations and computations will be removed from the practical process. Furthermore, the number of scheduling parameters employed in our scheme to schedule the controller is half those used in [25]. This results in the simplicity and low cost in the field of industry. Therefore, our approach is more practical than the proposed method in [25] from the computational point of view.

5. Conclusion

In this paper, the design problem of robust gain-scheduled state-feedback controllers was addressed for uncertain Linear Parameter-Varying (LPV) systems, in which some of the state-space matrices were affine with respect to the uncertain time-varying parameters. It was supposed that the uncertainties in the plant model originated from the Time-Invariant Parametric Uncertainties (TIPU), which could be pulled out as an uncertain block. Therefore, the exact scheduling parameters were not available. The proposed controller used a state-feedback gain, which was constructed by a set of new scheduling parameters and a secondary time-varying term in order to stabilize the uncertain LPV systems in the presence of the TIPU. The proposed scheme guarantees closed-loop stability in terms of the Lyapunov method using the linear matrix inequalities and other mathematical techniques.

References

1. Mohammadpour, J. and Scherer, C.W., *Control of Linear Parameter Varying Systems with Applications*, Springer, New York, USA (2012).
2. Polcz, P., Péni, T., Kulcsár, B., et al. "Induced ℓ_2-gain computation for rational LPV systems using Finsler's lemma and minimal generators", *Systems & Control Letters*, **142**, pp. 1–8 (2020).
3. Sadeghzadeh, A. "LMI relaxations for robust gain-scheduled control of uncertain linear parameter varying systems", *IET Control Theory & Applications*, **13**(4), pp. 486–495 (2019).
4. Nruyen, A.-T., Chevrel, Ph., and Chaveau, F. "LPV static output feedback for constrained direct tilt control of narrow tilting vehicles", *IEEE Trans. Control Systems Technology*, **28**(2), pp. 661–670 (2020).
5. Najeh, M., Cauet, S., and Coirault, P. "LPV control of ICE torque ripple in hybrid electric vehicles", *18th IFAC World Congress*, Milano, Italy, **44**(1), pp. 2931–2936 (2011).
6. Masubuchi, I., Kato, J., Saeki, M., et al. "Gain-scheduled controller design based on descriptor representation of LPV systems: application to flight vehicle control", *43th IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, DOI: 10.1109/CDC.2004.1428770, pp. 815–820 (2004).
7. Sadeghzadeh, A. "On exploiting inexact scheduling parameters for gain-scheduled control of linear parameter-varying discrete-time systems", *Systems & Control Letters*, **117**, pp. 1–10 (2018).
8. Behrouz, H., Mohammadzaman, I., and Mohammadi, A. "Robust static output feedback H_2/H_∞ control synthesis with pole placement constraints: An LMI approach", *Int. J. of Control, Automation and Systems*, **18**, pp. 1–14 (2020).
9. Nguyen, A.-T., Chevrel, P., and Claveau, F. "Gain-scheduled static output feedback control for saturated LPV systems with bounded parameter variations", *Automatica*, **89**, pp. 420–424 (2018).
10. He, T., Zhu, G., and Swei, S.S.-M. "Smooth switching LPV dynamic output-feedback control", *Int. J. of Control, Automation and Systems*, **18**, pp. 1367–1377 (2020).
11. Behrouz, H., Mohammadzaman, I., and Mohammadi, A. "Robust static output feedback design with pole placement constraints for linear systems with polytopic uncertainties", *Transactions of the Institute of Measurement and Control*, **00**, pp. 1–11 (2019). DOI: 10.1177/0142331218823857
12. Sato, M. "Gain-scheduled flight controller using bounded inexact scheduling parameters ", *IEEE Trans. Control Systems Technology*, **26**(3), pp. 1074–1082 (2018).
13. Daafouz, J., Bernussou, J., and Geromel, J.C. "On inexact LPV control design of continuous-time polytopic systems", *IEEE Trans. Autom. Control*, **53**(7), pp. 1674–1678 (2008).
14. Sato, M. "One-shot design of performance scaling matrices and observer-based gain-scheduled controllers depending on inexact scheduling parameters", *Systems & Control Letters*, **137**, pp. 1–10 (2020).
15. Chandra, K.P.B., Alwi, H., and Edwards, Ch. "Fault detection in uncertain LPV systems with imperfect scheduling parameter using sliding mode observers", *European Journal of Control*, **34**, pp. 1–15 (2017).
16. Sato, M. and Peaucelle, D. "Gain-scheduled output-feedback controllers using inexact scheduling parameters for continuous-time LPV systems", *Automatica*, **49**(4), pp. 1019–1025 (2013).
17. Moreira, H.R., Oliveira, R., and Peres, L.D. "Robust H_2 static output feedback design starting from a parameter-dependent state feedback controller for time-invariant discrete-time polytopic systems", *Optimal Control Applications and Methods*, **32**(1), pp. 1–13 (2011).
18. Dong, J. and Yang, G.-H. "Robust static output feedback control synthesis for linear continuous systems with polytopic uncertainties", *Automatica*, **49**(6), pp. 1821–1829 (2013).
19. Chang, X.-H., Park, H., and Zhou, J. "Robust static output feedback H_∞ control design for linear systems with polytopic uncertainties", *Systems & Control Letters*, **85**, pp. 23–32 (2015).
20. Chesi, G. "Robust static output feedback controllers via robust stabilizability functions", *IEEE Trans. Autom. Control*, **59**(6), pp. 1618–1623 (2014).
21. Kose, I. and Jabbari, F. "Control of LPV systems with partly measured parameters", *IEEE Trans. Autom. Control*, **44**(3), pp. 658–663 (1999).

22. Efimov, D., Raissi, T., and Zolghadri, A. "Control of nonlinear and LPV systems: interval observer-based framework", *IEEE Trans. Autom. Control*, **58**(3), pp. 773–778 (2013).

23. Jafar, A., Bhatti, A.I., and Ahmad, S.M. "Robust gain-scheduled linear parameter-varying control algorithm for a lab helicopter: A linear matrix inequality-based approach", *J. Systems and Control Engineering*, **232**(5), pp. 558–571 (2018).

24. Rotondo, D., Nejjari, F., and Puig, V. "Robust state-feedback control of uncertain LPV systems: An LMI-based approach", *J. Franklin Inst.*, **351**(5), pp. 2781–2803 (2014).

25. Park, P., Kwon, N.K., and Park, B.Y. "State-feedback control for LPV systems with interval uncertain parameters", *J. Franklin Inst.*, **352**(11), pp. 5214–5225 (2015).

Biographies

Hossein Asrari is currently pursuing the PhD degree in Control Engineering at Malek Ashtar University of Technology, Tehran, Iran. His research interests include control of LPV systems and LMI optimization.

Iman Mohammadzaman is an Assistant Professor with the Faculty of Electrical and Computer Engineering, Malek Ashtar University of Technology, Tehran, Iran. His research interests include robust control, nonlinear control, and LMI optimization.

Firouz Allahverdizadeh is an Assistant Professor with the Faculty of Electrical and Computer Engineering, Malek Ashtar University of Technology, Tehran, Iran. His research interests include navigation and control of LPV systems.