$T$-spherical Fuzzy Soft Matrices with Applications in Decision-Making and Selection Process

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**Abstract**

In the present communication, we have introduced the notion of $T$-spherical fuzzy soft matrix (TSFSM) and studied various types of associated binary operations and properties. In literature, it has been observed that the concept of soft matrix plays a vital role in many engineering applications as well as to cater different socio-economic and financial sector problems. As per the definition of $T$-spherical fuzzy set, the proposed notion would have an additional capability to address the impreciseness of the information close enough to human opinion mathematically. Further, on the basis of the structure of proposed TSFSM and using the concept of choice matrix along with its weighted form, a new algorithm for the decision-making process has been outlined. Next, utilizing the score/utility matrix, we present another algorithm for the selection process. For the sake of understanding of the proposed methodologies, illustrative examples have also been presented. Some comparative remarks for the proposed techniques in contrast with existing techniques have been listed for a better readability and understanding.

***Keywords:*** $T$-spherical fuzzy soft set; Choice matrix; Score matrix; Utility matrix; Decision-making; Selection process.

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# 1 Introduction

In the real world, it is difficult for the decision makers to achieve the finest alternative/attribute/object from the set of feasible ones due to the increasing complications in the system. However, it is hard to summarize but not incredible to achieve the best single objective. In the decision-making process, there are large number of multi-criteria decision-making problems where the criteria are found to be uncertain, ambiguous, imprecise and vague. Therefore, to handle this uncertainty and impreciseness in the information, the crisp set seems to be ineffective while it can easily be handled by using the fuzzy information. In order to handle such ambiguous and uncertain situations, Zadeh [1] presented the mathematical idea of fuzzy set (FS) which has been characterized by using the membership function of the element/object. In the past, various researchers have found the applicability of the fuzzy set in different fields viz., decision-making, medical diagnosis, engineering, socio-economic, finance problems etc.

Incorporating the idea of hesitancy/indeterminacy, Atanassov [2] extended the existing concept and introduced intuitionistic fuzzy set (IFS) on the basis of two characterized function, i.e., membership & non-membership function such that there sum is less than equal to 1. Next, Yager [3] introduced a new extension of fuzzy set called as Pythagorean fuzzy set (PyFS) based on the membership & non-membership function such that there squared sum is $\leq 1$. It may be observed that the PyFS effectively enlarged the span of information than IFS. For having a detailed discussion and for the sake of future directions in the field of PyFS, the article by Peng & Selvachandran [4] may be referred. Further, Cuong [5] revealed that the structure of FSs, IFSs and PyFS are not capable enough to represent the human opinion in complete sense and introduced the concept of picture fuzzy set (PFS). The definition of PFS has been illustrated and supported by the example of voting system where the concept of refusal has been additionally taken into consideration for an advantageous coverage of information. The Coung's picture fuzzy set captures the uncertainty/ambiguity sufficiently close to human nature/opinion in terms of membership, indeterminacy (neutral), nonmembership and refusal.

The structure of picture fuzzy set seems to have diverse dimensions, however, it also has the restriction that addition of the three parameters (membership, neutral-membership with the non-membership grade) must be $\leq 1$ which is similar to the intuitionistic fuzzy set. To overcome such restrictions/limitations, Mahmood et al. [6] presented the concept of $T$-spherical fuzzy set (TSFS) which further strengthened the structure of picture fuzzy set by broadening the span for the membership of all the essential parameters. Next, the geometrical comparative analysis of fuzzy set, IFS, Pythagorean and PFS with the $T$-spherical fuzzy set has been done by Kifayat et al. [7]. In addition, the limitations of the existing similarity measures for IFSs and PFSs have been provided in view of extended features of TSFS. Further, different similarity measures for TSFS have also been provided by them along with suitable applications. Garg et al. [8] presented a new improved interactive aggregation operators for TSFSs with application in decision-making. Next, Liu et al. [9] discussed the power Muirhead mean operator along with their properties over $T$-spherical fuzzy environment and provided a novel approach for decision-making problems. Recently, Jin et al. [10] introduced the notion of linguistic spherical fuzzy set (LSFS) and investigated its various aggregation operators to develop a new approach to solve the decision-making problems. Also, Guleria and Bajaj [11] presented eigen spherical fuzzy sets with an algorithm to find the greatest and the least eigen spherical fuzzy sets to solve some of the decision-making problems.

Engineering problems, socio-economic problems, decision-making issues encounters different types of incompleteness, vagueness and impreciseness. However, in ordinary circumstances, there are significant theories and literatures available to deal with, but they are not sufficiently capable due to the involvement of the parameterization tool. The ability to beat such impediments was shown by a new set called 'soft set' introduced by Molodtsov [12] who set forward some significant deliberations. Further, the new extensions (fuzzy soft set (FSS), intuitionistic fuzzy soft set (IFSS)) were given by Maji et al. [13] [14][15] with different binary operations and applications. The literature on Pythagorean fuzzy soft set (PyFSS) was laid down by Peng et al. [16] for solving soft computing problems. In the field of aggregation operators, Wang & Li [17] extended power Bonferroni mean operator and proposed Pythagorean fuzzy interaction power Bonferroni mean operator with its weighted form without loosing the main characteristic of power operator. Further, an illustrative example of multi-attribute decision-making problem has been successfully solved based on the proposed aggregation operators. Recently, Guleria et al. [18] introduced the concept of $T$-spherical fuzzy soft set with aggregation operators in the decision-making problem.

Naim and Serdar [19] proposed the notion of soft matrices for connecting and computing the information of the soft set with decision applications. Such representing form of the information in terms of matrix was extended by Yong et al. [20] first. Chetia et al. [21] utilized the fuzzy and intuitionistic fuzzy information to handle the decision-making problems. Yang et al. [22] presented an algorithm on the basis of adjustable soft discernibility matrix utilizing the level soft set of PFSS for decision-making problems. Kamaci [23] studied the symmetric difference operation the soft sets and matrices along with the similarity measure for the soft matrices. Further, with the help of soft matrices, the Scilab code for various computational processes has been developed. Recently, Guleria at al. [24] proposed the Pythagorean fuzzy soft matrices for dealing with the problems of medical diagnosis and decision-making. Next, Bajaj and Guleria [25] utilized the notion of Pythagorean fuzzy soft matrix to develop a new dimensionality reduction technique to solve the decision-making problem.

Also, in the field of pattern recognition, Wu et al. [26] applied new distance/divergence measures of $T$-spherical fuzzy sets and discussed its added advantage along with the limitations of the existing measures. The proposed divergence measure has been utilizing the concept of Jensen-Shannon divergence which has the capability to eliminate the counter-intuitive observations. Chen et al. [27] presented a kind of generalized parametric $T$-spherical fuzzy set and devised various geometric aggregation operators. Further, these have been extended in terms of group-generalized parameter and used for proposing an algorithm for the multi-attribute decision making (MADM) problem. Garg et al. [28] introduced the theory of power aggregation operators from the concept of weighted/order weighted/hybrid averaging/geometric operators and obtained the relationships between the various attributes in the form of introduced power operator. Finally, the proposed power aggregation operators have been used to solve the MADM problem and the comparative study has also been carried out in order to validate the proposed concept.

In order to have a better understanding of the sequential development of various extensions of fuzzy set, we present a road map given in Figure 1.

It may be There is no need to say that the concept of matrix plays a very important and vital role in various computational techniques and in the study of dimensionality feature of different engineering problems, which certainly motivates the research community to think over further extensions. In view of the current status of the extensions stated above and to fulfill the research gap, we present a novel kind of matrix termed as $T$-spherical fuzzy soft matrix in connection and association with the $T$-spherical fuzzy soft set. The novel extension and its format is capable to handle the uncertainty and impreciseness of the incomplete information in a more closer sense, i.e., spherical fuzzy information and its four parameters of fuzziness. With the introduction of the proposed notion, the decision-making problems and the selection process problems can be dealt in a better and broader sense of human opinion.

The propositions in the current communication have been structured as follows. The preliminary concepts in connection with the proposed work have been provided in Section 2. In Section 3, we introduce a novel kind of matrix termed as $T$-spherical fuzzy soft matrix with its different types and categories. Subsequently, different types of standard binary operations and their operational laws have also been discussed in detail. In Section 4, a new decision-making algorithm has been provided by incorporating the proposed revised choice matrix and its weighted form along with a numerical example for solving a general problem of decision-making. Next, in Section 5, another new algorithm for selection process problem has been outlined by incorporating the proposed score matrix and utility matrix along with a numerical example. Finally, the paper is concluded in Section 6 with possible scope for future work.

## 2 Preliminary Concepts

Some of the basic preliminaries and notions in connection with the $T$-spherical fuzzy soft sets are being outlined in this section.
Let $U = \{u_1, u_2, \ldots, u_m\}$ be the domain of discourse and $\mu : U \to [0,1]$, $\eta : U \to [0,1]$ and $\nu : U \to [0,1]$ are the characterizing function for membership, neutral-membership & non-membership grades respectively.

- *A* ***picture fuzzy set*** *[5] $A$ in $U$ is given by*

$$A = \{< u, \mu_A(u), \eta_A(u), \nu_A(u) >\mid u \in U\};$$

*and for every $u \in U$, the following condition is satisfied:*

$$\mu_A(u) + \eta_A(u) + \nu_A(u) \leq 1.$$

*The following residual equation gives the degree of refusal as*

$$r_A(u) = 1 - (\mu_A(u) + \eta_A(u) + \nu_A(u)).$$

- *A **spherical fuzzy set** [6] $S$ in $U$ is given by*

$$S = \{< u, \mu_S(u), \eta_S(u), \nu_S(u) >|\ u \in U\};$$

*and for every $u \in U$, the following condition is satisfied:*

$$\mu_S^2(u) + \eta_S^2(u) + \nu_S^2(u) \leq 1, \quad \forall u \in U.$$

*The following residual equation gives the degree of refusal as*

$$r_S(u) = \sqrt{1 - (\mu_S^2(u) + \eta_S^2(u) + \nu_S^2(u))}.$$

- *A $T$**-spherical fuzzy set** [6] $S$ in $U$ is given by*

$$S = \{< u, \mu_S(u), \eta_S(u), \nu_S(u) >|\ u \in U\};$$

*and for every $u \in U$, the following condition is satisfied:*

$$\mu_S^q(u) + \eta_S^q(u) + \nu_S^q(u) \leq 1, \quad \forall\ u \in U;\ q = 1, 2, 3, \ldots.$$

*Similarly, the following equation gives the degree of refusal as*

$$r_S(u) = \sqrt[q]{1 - (\mu_S^q(u) + \eta_S^q(u) + \nu_S^q(u))}; \quad q = 1, 2, 3, \ldots.$$

Also, various generalizations and extensions of soft sets are being listed below for ready reference:
Let $P = \{p_1, p_2, \ldots, p_n\}$ be the set of parameters under the same universe of discourse $U$. The pair $(\Phi, P)$ is called

- *"**soft set** [12] over $U$ iff $\Phi : P \to \mathcal{P}(U)$, where $\mathcal{P}(U)$ is the power set of $U$."*

- *"**Pythagorean fuzzy soft set** [16] over $U$ if $\Phi : P \to PYFS(U)$ and can be represented as*

$$(\Phi, P) = \{(p, \Phi(p)) : p \in P,\ \Phi(p) \in PYFS(U)\},$$

*where $PYFS(U)$ represents the set of all PyFSs of $U$."*

- *"**picture fuzzy soft set** [5] over $U$ if $\Phi : P \to PFS(U)$ and can be represented as*

$$(\Phi, P) = \{(p, \Phi(p)) : p \in P,\ \Phi(p) \in PFS(U)\},$$

*where $PFS(U)$ represents the set of all PFSs of $U$."*

- *"$T$-spherical fuzzy soft set [18] over $U$ if $\Phi : P \to TSFS(U)$ and can be represented as*

$$(\Phi, P) = \{(p, \Phi(p)) : p \in P,\ \Phi(p) \in TSFS(U)\},$$

*where $TSFS(U)$ represents the set of all TSFSs of $U$."*

- *"The subset $U \times P$ is uniquely defined with the help of $R_P = \{(u,p), p \in E, u \in \Phi(p)\}$ and the characteristic function of $R_P$ as $\chi_{R_P} : U \times P \to [0,1]$ given by*

$$\chi_{R_P}(u,p) = \begin{cases} 1 & if\ \ (u,p) \in E \\ 0 & if\ \ (u,p) \notin E \end{cases}.$$

*If $a_{ij} = \chi_{R_P}(u_i, p_j)$, then a matrix $[a_{ij}] = [\chi_{R_P}(u_i, p_j)]$ is called **soft matrix** of the soft set $(\Phi, P)$ over $U$."*

In the literature of soft matrix theory, Naim & Serdar [29] used the concept of soft set theory to define the product of two soft matrices as well as product of two fuzzy soft matrices (FSMs) with their different theoretical properties respectively. Finally, an application based on the soft max-min decision-making method has been presented for both types of matrices for the sake of better clarity and readability. Broumi et al. [30] defined the notion of a different kind of FSM based on reference function and also presented some new operations related to its complement and trace. Further, the concept of reference function has been utilized for addressing a decision-making problem. Next, Petchimuthu et al. [31] generalized the products of two FSMs and presented mean operators/normalized fuzzy weighted mean operators for the FSMs based on which two algorithms for multi-criteria group decision-making problems. Recently, Naim & Serdar [32] proposed the concept of fuzzy parameterized fuzzy soft matrices with their fundamental properties. With the help of the proposed study, they devised Pervalence Effect Method for noise removal filters in performance-based value assignment. For the sake of further deliberations, in the next sections, we propose the notion of $T$-spherical fuzzy soft matrices (TSFSM) with various operations and applications.

# 3 $T$-spherical Fuzzy Soft Matrices & Operations

Here, we first present a new kind of soft matrix which is a generalized notion of Pythagorean fuzzy soft matrix and can also be viewed as an extension of $T$-spherical fuzzy soft set. Next, we introduce various types of binary operations over these matrices.

Let $(\Phi, P)$ be a $T$-spherical fuzzy soft set over $U$ (universe). As mentioned earlier, $R_P$ can be defined by its membership & non-membership function $\mu_{R_P} : U \times P \to [0,1]$ and $\nu_{R_P} : U \times P \to [0,1]$ respectively.

If $(\mu_{ij}, \eta_{ij}, \nu_{ij}) = (\mu_{R_P}(u_i, p_j), \eta_{R_P}(u_i, p_j), \nu_{R_P}(u_i, p_j))$, where $\mu_{R_P}(u_i, p_j)$ represents membership/belongingness of $u_i$ in the $T$-spherical fuzzy set $F(p_j)$, $\eta_{R_P}(u_i, p_j)$ depicts the neutral/abstain membership of $u_i$ in the $T$-spherical fuzzy set $F(p_j)$, and $\nu_{R_P}(u_i, p_j)$ represents the non-membership of $u_i$ in the TSFS $F(p_j)$ respectively, then we propose a matrix, termed as $T$-spherical fuzzy soft matrix (TSFSM) over $U$, which is given by

$$[M] = [m_{ij}]_{m\times n} = [(\mu_{ij}^M, \eta_{ij}^M, \nu_{ij}^M)]_{m\times n} = \begin{bmatrix} (\mu_{11}, \eta_{11}, \nu_{11}) & (\mu_{12}, \eta_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \eta_{13}, \nu_{1n}) \\ (\mu_{21}, \eta_{21}, \nu_{21}) & (\mu_{22}, \eta_{22}, \nu_{22}) & \cdots & (\mu_{2n}, \eta_{2n}, \nu_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m1}, \eta_{m1}, \nu_{m1}) & (\mu_{m2}, \eta_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \eta_{mn}, \nu_{mn}) \end{bmatrix}.$$

For a better understanding, let us consider a hypothetical example where $U = \{u_1, u_2, u_3\}$ and $P = \{p_1, p_2, p_3\}$ with

$$\Phi(p_1) = \{(u_1, 0.5, 0.5, 0.2), (u_2, 0.8, 0.3, 0.5), (u_3, 0.6, 0.7, 0.2)\},$$
$$\Phi(p_2) = \{(u_1, 0.7, 0.5, 0.3), (u_2, 0.3, 0.3, 0.9), (u_3, 0.6, 0.3, 0.4)\},$$
$$\Phi(p_3) = \{(u_1, 0.5, 0.2, 0.6), (u_2, 0.7, 0.6, 0.2), (u_3, 0.8, 0.4, 0.5)\},$$

then $(\Phi, P)$ is the parameterized family of $\Phi(p_1), \Phi(p_2), \Phi(p_3)$ over $U$.

Hence, the $T$-spherical fuzzy soft matrix $M$ can be written as

$$[M] = [(\mu_{ij}^M, \eta_{ij}^M, \nu_{ij}^M)]_{m\times n} = \begin{bmatrix} (0.5, 0.5, 0.2) & (0.7, 0.5, 0.3) & (0.5, 0.2, 0.6) \\ (0.8, 0.3, 0.5) & (0.3, 0.3, 0.9) & (0.7, 0.6, 0.2) \\ (0.6, 0.7, 0.2) & (0.6, 0.3, 0.4) & (0.8, 0.4, 0.5) \end{bmatrix}.$$

Let $TSFSM_{m\times n}$ be the set of all the $T$-spherical fuzzy soft matrices over $U$. Further, different types of $T$-spherical fuzzy soft matrices are being accordingly provided. A matrix $M = [(\mu_{ij}^M, \eta_{ij}^M, \nu_{ij}^M)] \in TSFSM_{m\times n}$ is called **$T$-spherical fuzzy soft**:

- "**zero matrix** if $\mu_{ij}^M = 0$, $\eta_{ij}^M = 0$ & $\nu_{ij}^M = 0; \forall i, j$ the matrix $0 = [0, 0, 0]$."
- "**square matrix** if $m = n$."
- "**row matrix** if $n = 1$."
- "**column matrix** if $m = 1$."
- "**diagonal matrix** if all its non-diagonal entries are zero $\forall\ i, j$."
- "**$\mu$-universal matrix** if $\mu_{ij}^M = 1$, $\eta_{ij}^M = 0$ & $\nu_{ij}^M = 0$; $\forall\ i$ & $j$, denoted by $P_\mu$."
- "**$\eta$-universal matrix** if $\mu_{ij}^M = 0$, $\eta_{ij}^M = 1$ & $\nu_{ij}^M = 0$; $\forall\ i$ & $j$, denoted by $P_\eta$."
- "**$\nu$-universal matrix** if $\mu_{ij}^M = 0$, $\eta_{ij}^M = 0$ & $\nu_{ij}^M = 1$; $\forall\ i$ & $j$, denoted by $P_\nu$."
- "**Scalar multiplication:** for any scalar $k$, we define $kM = [(k\mu_{ij}^M, k\eta_{ij}^M, k\nu_{ij}^M)]$, $\forall\ i$ & $j$."

Next, we present some set-theoretic relations for given $T$-spherical fuzzy soft matrices $M = [(\mu_{ij}^M, \eta_{ij}^M, \nu_{ij}^M)]$ and $N = [(\mu_{ij}^N, \eta_{ij}^N, \nu_{ij}^N)] \in TSFSM_{m\times n}$.

- "**Subsethood:** $M \subseteq N$ if $\mu_{ij}^M \leq \mu_{ij}^N$, $\eta_{ij}^M \geq \eta_{ij}^N$ & $\nu_{ij}^M \geq \nu_{ij}^N$; $\forall\ i$ & $j$."

- "**Containment:** $M \supseteq N$ if $\mu_{ij}^M \geq \mu_{ij}^N$, $\eta_{ij}^M \leq \eta_{ij}^N$ & $\nu_{ij}^M \leq \nu_{ij}^N$; $\forall$ $i$ & $j$."
- "**Equality:** $M = N$ if $\mu_{ij}^M = \mu_{ij}^N$, $\eta_{ij}^M = \eta_{ij}^N$ & $\nu_{ij}^M = \nu_{ij}^N$; $\forall$ $i$ & $j$."
- "**Max Min Product:**
Let $M = [a_{ij}] = [(\mu_{ij}^M, \eta_{ij}^M, \nu_{ij}^M)] \in TSFSM_{m\times n}$ & $N = [b_{jk}] = [(\mu_{jk}^N, \eta_{jk}^N, \nu_{jk}^N)] \in TSFSM_{n\times p}$ then
$M * N = [c_{ik}]_{m\times p} = \left[\left(\max(\min_j(\mu_{ij}^M, \mu_{jk}^N)), \min(\min_j(\eta_{ij}^M, \eta_{jk}^N)), \min(\max_j(\nu_{ij}^M, \nu_{jk}^N))\right)\right]$; $\forall$ $i$, $j$ & $k$."
- "**Average Max Min Product:**
Let $M = [a_{ij}] = [(\mu_{ij}^M, \eta_{ij}^M, \nu_{ij}^M)] \in TSFSM_{m\times n}$ & $N = [b_{jk}] = [(\mu_{jk}^N, \eta_{jk}^N, \nu_{jk}^N)] \in TSFSM_{n\times p}$ be two $T$-spherical fuzzy soft matrices then
$M *_A N = [c_{ik}]_{m\times p} = \left[\left(\max_j(\frac{\mu_{ij}^M + \mu_{jk}^N}{2}), \min_j(\frac{\eta_{ij}^M + \eta_{jk}^N}{2}), \min_j(\frac{\nu_{ij}^M, \nu_{jk}^N}{2})\right)\right]$; $\forall$ $i$, $j$ & $k$."

**Remark:** It may be noted that Klement et al. [33], [34] have provided different types of *triangular norm* (t-*norm*) and *triangular conorm* (t-*conorm*) in an elaborated way. Accordingly, different combinations of these norms can also be considered for the proposed $T$-spherical fuzzy soft matrices. Here, we have only taken the composition of maximum and minimum operator.

**Standard Binary Operations for $T$-spherical Fuzzy Soft Matrices:**
Suppose that there are two $T$-spherical fuzzy soft matrices $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})]$ and $S_2 = [(\mu_{ij}^{S_2}, \eta_{ij}^{S_2}, \nu_{ij}^{S_2})] \in TSFSM_{m\times n}$. Then some of the binary operations may be given as follows:

- $S_1^c = \left[\left(\nu_{ij}^{S_1}, \eta_{ij}^{S_1}, \mu_{ij}^{S_1}\right)\right]$; $\forall$ $i$ and $j$.
- $S_1 \cup S_2 = \left[\left(\max(\mu_{ij}^{S_1}, \mu_{ij}^{S_2}), \min(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \min(\nu_{ij}^{S_1}, \nu_{ij}^{S_2})\right)\right]$; $\forall$ $i$ and $j$.
- $S_1 \cap S_2 = \left[\left(\min(\mu_{ij}^{S_1}, \mu_{ij}^{S_2}), \min(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \max(\nu_{ij}^{S_1}, \nu_{ij}^{S_2})\right)\right]$ $\forall$; $i$ and $j$.
- $S_1 \otimes S_2 = \left[\left(\mu_{ij}^{S_1} \cdot \mu_{ij}^{S_2}, \eta_{ij}^{S_1} \cdot \eta_{ij}^{S_2}, \sqrt[n]{(\nu_{ij}^{S_1})^2 + (\nu_{ij}^{S_2})^2 - (\nu_{ij}^{S_1})^2 \cdot (\nu_{ij}^{S_2})^2}\right)\right]$; $\forall$ $i$ and $j$.
- $S_1 \oplus S_2 = \left[\left(\sqrt[n]{(\mu_{ij}^{S_1})^2 + (\mu_{ij}^{S_2})^2 - (\mu_{ij}^{S_1})^2 \cdot (\mu_{ij}^{S_2})^2}, \eta_{ij}^{S_1} \cdot \eta_{ij}^{S_2}, \nu_{ij}^{S_1} \cdot \nu_{ij}^{S_2}\right)\right]$; $\forall$ $i$ and $j$.
- $S_1 @ S_2 = \left[\left(\frac{\mu_{ij}^{S_1} + \mu_{ij}^{S_2}}{2}, \frac{\eta_{ij}^{S_1} + \eta_{ij}^{S_2}}{2}, \frac{\nu_{ij}^{S_1} + \nu_{ij}^{S_2}}{2}\right)\right]$; $\forall$ $i$ and $j$.
- $S_1 @_w S_2 = \left[\left(\frac{w_1\mu_{ij}^{S_1} + w_2\mu_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\eta_{ij}^{S_1} + w_2\eta_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\nu_{ij}^{S_1} + w_2\nu_{ij}^{S_2}}{w_1 + w_2}\right)\right]$; $\forall$ $i$ and $j$ ; where $w_1, w_2 > 0$ are the weights.
- $S_1 \$ S_2 = \left[\left(\sqrt[n]{\mu_{ij}^{S_1} \cdot \mu_{ij}^{S_2}}, \sqrt[n]{\eta_{ij}^{S_1} \cdot \eta_{ij}^{S_2}}, \sqrt[n]{\nu_{ij}^{S_1} \cdot \nu_{ij}^{S_2}}\right)\right]$; $\forall$ $i$ and $j$.
- $S_1 \$_w S_2 = \left[\left(((\mu_{ij}^{S_1})^{w_1} \cdot (\mu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\eta_{ij}^{S_1})^{w_1} \cdot (\eta_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\nu_{ij}^{S_1})^{w_1} \cdot (\nu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}\right)\right]$; $\forall$ $i$ and $j$, where $w_1, w_2 > 0$ are the weights.
- $S_1 \bowtie S_2 = \left[\left(2 \cdot \frac{\mu_{ij}^{S_1} \cdot \mu_{ij}^{S_2}}{\mu_{ij}^{S_1} + \mu_{ij}^{S_2}}, 2 \cdot \frac{\eta_{ij}^{S_1} \cdot \eta_{ij}^{S_2}}{\eta_{ij}^{S_1} + \eta_{ij}^{S_2}}, 2 \cdot \frac{\nu_{ij}^{S_1} \cdot \nu_{ij}^{S_2}}{\nu_{ij}^{S_1} + \nu_{ij}^{S_2}}\right)\right]$; $\forall$ $i$ and $j$.
- $S_1 \bowtie_w S_2 = \left[\left(\frac{w_1 + w_2}{\frac{w_1}{\mu_{ij}^{S_1}} + \frac{w_2}{\mu_{ij}^{S_2}}}, \frac{w_1 + w_2}{\frac{w_1}{\eta_{ij}^{S_1}} + \frac{w_2}{\eta_{ij}^{S_2}}}, \frac{w_1 + w_2}{\frac{w_1}{\nu_{ij}^{S_1}} + \frac{w_2}{\nu_{ij}^{S_2}}}\right)\right]$; $\forall$ $i$ and $j$ ; where $w_1, w_2 > 0$ are the weights.

**Proposition 1** *Let $S_1$ and $S_2 \in TSFSM_{m\times n}$ then the following laws hold:*

*(i)* $S_1 \cup S_2 = S_2 \cup S_1$

*(ii)* $S_1 \cap S_2 = S_2 \cap S_1$

*(iii)* $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$

*(iv)* $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$

*(v)* $(S_1^c \cap S_2^c)^c = S_1 \cup S_2$

*(vi)* $(S_1^c \cup S_2^c)^c = S_1 \cap S_2$.

**Proof :** Let $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})], S_2 = [(\mu_{ij}^{S_2}, \eta_{ij}^{S_2}, \nu_{ij}^{S_2})] \in TSFSM_{m\times n}$. Then $\forall$ $i$ & $j$ we get,

(i)

$$\begin{aligned} S_1 \cup S_2 &= \Big[\big(\max(\mu_{ij}^{S_1}, \mu_{ij}^{S_2}), \min(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \min(\nu_{ij}^{S_1}, \nu_{ij}^{S_2})\big)\Big] \\ &= \Big[\big(\max(\mu_{ij}^{S_2}, \mu_{ij}^{S_1}), \min(\eta_{ij}^{S_2}, \eta_{ij}^{S_1}), \min(\nu_{ij}^{S_2}, \nu_{ij}^{S_1})\big)\Big] = S_2 \cup S_1. \end{aligned}$$

(ii)

$$\begin{aligned} S_1 \cup S_2 &= \Big[\big(\min(\mu_{ij}^{S_1}, \mu_{ij}^{S_2}), \min(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \max(\nu_{ij}^{S_1}, \nu_{ij}^{S_2})\big)\Big] \\ &= \Big[\big(\min(\mu_{ij}^{S_2}, \mu_{ij}^{S_1}), \min(\eta_{ij}^{S_2}, \eta_{ij}^{S_1}), \max(\nu_{ij}^{S_2}, \nu_{ij}^{S_1})\big)\Big] = S_2 \cup S_1. \end{aligned}$$

(iii)

$$\begin{aligned} (S_1 \cup S_2)^c &= \Big(\big([(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})] \cup [(\mu_{ij}^{S_2}, \eta_{ij}^{S_2}, \nu_{ij}^{S_2})]\big)\Big)^c \\ &= [\max(\mu_{ij}^{S_1}, \mu_{ij}^{S_2}), \min(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \min(\nu_{ij}^{S_1}, \nu_{ij}^{S_2})]^c \\ &= \big[\big(\min(\nu_{ij}^{S_1}, \nu_{ij}^{S_2}), \min(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \max(\mu_{ij}^{S_1}, \mu_{ij}^{S_2})\big)\big] \\ &= \big[\big([(\nu_{ij}^{S_1}, \eta_{ij}^{S_1}, \mu_{ij}^{S_1})] \cap [(\nu_{ij}^{S_2}, \eta_{ij}^{S_2}, \mu_{ij}^{S_2})]\big)\big] = S_1^c \cap S_2^c. \end{aligned}$$

Similarly, $(iv)$, $(v)$ and $(vi)$ can be proved easily.

**Proposition 2** *Let $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})] \in TSFSM_{m\times n}$. Then the following laws hold as per the proposed definitions:*

*(i)* $(S_1^c)^c = S_1$

*(ii)* $(P_\mu)^c = P_\nu$

*(iii)* $(P_\eta)^c = P_\eta$

*(iv)* $(P_\nu)^c = P_\mu$

*(v)* $S_1 \cup S_1 = S_1$

*(vi)* $S_1 \cup P_\mu = P_\mu$

*(vii)* $S_1 \cap P_\nu = S_1$

*(viii)* $S_1 \cap S_1 = S_1$

*(ix)* $S_1 \cap P_\mu = S_1$

*(x)* $S_1 \cap P_\nu = P_\nu$.

**Proposition 3** *Let $S_1$ and $S_2 \in TSFSM_{m\times n}$. Then the following laws w.r.t. the weighted form hold:*

*(i)* $(S_1^c @_w S_2^c)^c = S_1 @_w S_2$

*(ii)* $(S_1^c \$_w S_2^c)^c = S_1 \$_w S_2$

*(iii)* $(S_1^c \bowtie_w S_2^c)^c = S_1 \bowtie_w S_2$

*(iv)* $S_1 @_w S_2 = S_2 @_w S_1$

*(v)* $S_1 \$_w S_2 = S_2 \$_w S_1$

*(vi)* $S_1 \bowtie_w S_2 = S_2 \bowtie_w S_1$.

**Proof :** Let $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})], S_2 = [(\mu_{ij}^{S_2}, \eta_{ij}^{S_2}, \nu_{ij}^{S_2})] \in TSFSM_{m\times n}$. Then $\forall\ i, j$ & $w_1, w_2 > 0$, we get,

(i)

$$\begin{aligned}
(S_1^c @_w S_2^c)^c &= \left(\left[\left((\nu_{ij}^{S_1}, \eta_{ij}^{S_1}, \mu_{ij}^{S_1}) @_w (\nu_{ij}^{S_2}, \eta_{ij}^{S_2}, \mu_{ij}^{S_2})\right)\right]\right)^c \\
&= \left(\left[\left(\frac{w_1\nu_{ij}^{S_1} + w_2\nu_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\eta_{ij}^{S_1} + w_2\eta_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\mu_{ij}^{S_1} + w_2\mu_{ij}^{S_2}}{w_1 + w_2}\right)\right]\right)^c \\
&= \left[\left(\frac{w_1\mu_{ij}^{S_1} + w_2\mu_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\eta_{ij}^{S_1} + w_2\eta_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\nu_{ij}^{S_1} + w_2\nu_{ij}^{S_2}}{w_1 + w_2}\right)\right] = S_1 @_w S_2.
\end{aligned}$$

(ii)

$$\begin{aligned}
(S_1^c \$_w S_2^c)^c &= \left(\left[\left((\nu_{ij}^{S_1}, \eta_{ij}^{S_1}, \mu_{ij}^{S_1}) \$_w (\nu_{ij}^{S_2}, \eta_{ij}^{S_2}, \mu_{ij}^{S_2})\right)\right]\right)^c \\
&= \left(\left[\left(((\nu_{ij}^{S_1})^{w_1} \cdot (\nu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\eta_{ij}^{S_1})^{w_1} \cdot (\eta_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\mu_{ij}^{S_1})^{w_1} \cdot (\mu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}\right)\right]\right)^c \\
&= \left[\left(((\mu_{ij}^{S_1})^{w_1} \cdot (\mu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\eta_{ij}^{S_1})^{w_1} \cdot (\eta_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\nu_{ij}^{S_1})^{w_1} \cdot (\nu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}\right)\right] \\
&= S_1 \$_w S_2.
\end{aligned}$$

Similar proof for $(iii)$.

(iv)

$$\begin{aligned}
S_1 @_w S_2 &= \left[\left(\frac{w_1\mu_{ij}^{S_1} + w_2\mu_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\eta_{ij}^{S_1} + w_2\eta_{ij}^{S_2}}{w_1 + w_2}, \frac{w_1\nu_{ij}^{S_1} + w_2\nu_{ij}^{S_2}}{w_1 + w_2}\right)\right] \\
&= \left[\left(\frac{w_2\mu_{ij}^{S_2} + w_1\mu_{ij}^{S_1}}{w_2 + w_1}, \frac{w_2\eta_{ij}^{S_2} + w_1\eta_{ij}^{S_1}}{w_2 + w_1}, \frac{w_2\nu_{ij}^{S_2} + w_1\nu_{ij}^{S_1}}{w_2 + w_1}\right)\right] \\
&= S_2 @_w S_1.
\end{aligned}$$

(v)

$$\begin{aligned}
S_1 \$_w S_2 &= \left[\left(((\mu_{ij}^{S_1})^{w_1} \cdot (\mu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\eta_{ij}^{S_1})^{w_1} \cdot (\eta_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}, ((\nu_{ij}^{S_1})^{w_1} \cdot (\nu_{ij}^{S_2})^{w_2})^{\frac{1}{w_1+w_2}}\right)\right] \\
&= \left[\left(((\mu_{ij}^{S_2})^{w_2} \cdot (\mu_{ij}^{S_1})^{w_1})^{\frac{1}{w_2+w_1}}, ((\eta_{ij}^{S_2})^{w_2} \cdot (\eta_{ij}^{S_1})^{w_1})^{\frac{1}{w_2+w_1}}, ((\nu_{ij}^{S_2})^{w_2} \cdot (\nu_{ij}^{S_1})^{w_1})^{\frac{1}{w_2+w_1}}\right)\right] \\
&= S_2 \$_w S_1.
\end{aligned}$$

Similarly, $(vi)$ can easily be verified.

**Proposition 4** *Let* $S_1$, $S_2$ *and* $S_3 \in TSFSM_{m\times n}$ *be three matrices then the following laws in connection with the associativity hold:*

*(i)* $(S_1 \cup S_2) \cup S_3 = S_1 \cup (S_2 \cup S_3)$

*(ii)* $(S_1 \cap S_2) \cap S_3 = S_1 \cap (S_2 \cap S_3)$

*(iii)* $(S_1 @ S_2) @ S_3 = S_1 @ (S_2 @ S_3)$

*(iv)* $(S_1 \$ S_2) \$ S_3 = S_1 \$ (S_2 \$ S_3)$

*(v)* $(S_1 \bowtie S_2) \bowtie S_3 = S_1 \bowtie (S_2 \bowtie S_3)$.

**Proof:** For all $i$ & $j$ we write,

(i)

$$\begin{aligned}(S_1 \cup S_2) \cup S_3 &= \left[\left([(\max\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, \min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}), \min\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}] \cup [(\mu_{ij}^{S_3}, \nu_{ij}^{S_3})]\right)\right] \\ &= \left[\left(\max\{(\mu_{ij}^{S_1}, \mu_{ij}^{S_2}), \mu_{ij}^{S_3}\}, \min\{(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \eta_{ij}^{S_3}\}, \min\{(\nu_{ij}^{S_1}, \nu_{ij}^{S_2}), \nu_{ij}^{S_3}\}\right)\right] \\ &= \left[\left(\max\{(\mu_{ij}^{S_1}, (\mu_{ij}^{S_2}, \mu_{ij}^{S_3}))\}, \min\{\eta_{ij}^{S_1}, (\eta_{ij}^{S_2}, \eta_{ij}^{S_3})\}, \min\{\nu_{ij}^{S_1}, (\nu_{ij}^{S_2}, \nu_{ij}^{S_3})\}\right)\right] \\ &= S_1 \cup (S_2 \cup S_3).\end{aligned}$$

(ii)

$$\begin{aligned}(S_1 \cap S_2) \cap S_3 &= \left[\left((\min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, (\min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}, \max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}) \cup (\mu_{ij}^{S_3}, \nu_{ij}^{S_3})\right)\right] \\ &= \left[\left(\min\{(\mu_{ij}^{S_1}, \mu_{ij}^{S_2}), \mu_{ij}^{S_3}\}, (\min\{(\eta_{ij}^{S_1}, \eta_{ij}^{S_2}), \eta_{ij}^{S_3}\}, , \max\{(\nu_{ij}^{S_1}, \nu_{ij}^{S_2}), \nu_{ij}^{S_3}\})\right)\right] \\ &= \left[\left(\min\{(\mu_{ij}^{S_1}, (\mu_{ij}^{S_2}, \mu_{ij}^{S_3}))\}, \big(\min\{(\eta_{ij}^{S_1}, (\eta_{ij}^{S_2}, \mu_{ij}^{S_3}))\}, \max\{\nu_{ij}^{S_1}, (\nu_{ij}^{S_2}, \nu_{ij}^{S_3})\}\big)\right)\right] \\ &= S_1 \cap (S_2 \cap S_3).\end{aligned}$$

Similar proof for (iii), (iv) and (v).

**Proposition 5** *Let $S_1$, $S_2$ and $S_3 \in TSFSM_{m\times n}$ are soft matrices. The following laws in connection with the distributivity hold:*

*(i)* $S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$

*(ii)* $(S_1 \cap S_2) \cup S_3 = (S_1 \cup S_3) \cap (S_2 \cup S_3)$

*(iii)* $S_1 \cup (S_2 \cap S_3) = (S_1 \cup S_2) \cap (S_1 \cup S_3)$

*(iv)* $(S_1 \cup S_2) \cap S_3 = (S_1 \cap S_3) \cup (S_2 \cap S_3)$

*(v)* $(S_1 \cap S_2) @ S_3 = (S_1 @ S_3) \cap (S_2 @ S_3)$

*(vi)* $(S_1 \cap S_2) \bowtie S_3 = (S_1 \bowtie S_3) \cap (S_2 \bowtie S_3)$

*(vii)* $S_1 \cup (S_2 @ S_3) = (S_1 \cup S_2) @ (S_1 \cup S_3)$

*(viii)* $(S_1 \cup S_2) \bowtie S_3 = (S_1 \bowtie S_3) \cup (S_2 \bowtie S_3)$

*(ix)* $S_1 @ (S_2 \cup S_3) = (S_1 @ S_2) \cup (S_1 @ S_3)$

*(x)* $S_1 @ (S_2 \cap S_3) = (S_1 @ S_2) \cap (S_2 @ S_3)$

*(xi)* $S_1 \$ (S_2 \cup S_3) = (S_1 \$ S_2) \cup (S_1 \$ S_3)$

*(xii)* $(S_1 \cup S_2) \$ S_3 = (S_1 \$ S_3) \cup (S_2 \$ S_3)$

*(xiii)* $S_1 \cup (S_2 \bowtie S_3) = (S_1 \cup S_2) \bowtie (S_1 \cup S_3)$

*(xiv)* $S_1 \bowtie (S_2 \cup S_3) = (S_1 \bowtie S_2) \cup (S_1 \bowtie S_3)$

*(xv)* $S_1 \$ (S_2 \cap S_3) = (S_1 \$ S_2) \cap (S_2 \$ S_3)$

*(xvi)* $(S_1 \cap S_2) \$ S_3 = (S_1 \$ S_3) \cap (S_2 \$ S_3)$.

**Proof :** For all $i$ & $j$ we write,

(i)

$$
\begin{aligned}
S_1 \cap (S_2 \cup S_3) &= \Bigg[\Bigg(\big[\big(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1}\big)\big] \cap \big[\big(\max\{\mu_{ij}^{S_2}, \mu_{ij}^{S_3}\}, \min\{\eta_{ij}^{S_2}, \eta_{ij}^{S_3}\}, \min\{\nu_{ij}^{S_2}, \nu_{ij}^{S_3}\}\big)\big]\Bigg)\Bigg] \\
&= \Bigg[\Bigg(\min\{\mu_{ij}^{S_1}, \max\{\mu_{ij}^{S_2}, \mu_{ij}^{S_3}\}\}, \min\{\eta_{ij}^{S_1}, \min\{\eta_{ij}^{S_2}, \mu_{ij}^{S_3}\}\}, \\
&\qquad \max\{\nu_{ij}^{S_1}, \min\{\nu_{ij}^{S_2}, \nu_{ij}^{S_3}\}\}\Bigg)\Bigg].
\end{aligned}
$$

Now,

$$
\begin{aligned}
(S_1 \cap S_2) \cup (S_1 \cap S_3) &= \Big[\big(\min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, \min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}, \max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}\big)\Big] \cup \Big[\big(\min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_3}\}, \\
&\qquad \min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_3}\}, \max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_3}\}\big)\Big] \\
&= \Big[\big(\max(\min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, \min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_3}\}), \min(\min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}, \min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_3}\}), \\
&\qquad \min(\max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}, \max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_3}\})\big)\Big] \\
&= \Big[\big(\max(\mu_{ij}^{S_1}, \min\{\mu_{ij}^{S_2}, \mu_{ij}^{S_3}\}), \min(\eta_{ij}^{S_1}, \min\{\eta_{ij}^{S_2}, \eta_{ij}^{S_3}\}), \\
&\qquad \min(\nu_{ij}^{S_1}, \max\{\nu_{ij}^{S_2}, \nu_{ij}^{S_3}\})\big)\Big] \\
&= \Big[\big(\min(\mu_{ij}^{S_1}, \max\{\mu_{ij}^{S_2}, \mu_{ij}^{S_3}\}), \min(\mu_{ij}^{S_1}, \min\{\mu_{ij}^{S_2}, \mu_{ij}^{S_3}\}), \max(\nu_{ij}^{S_1}, \\
&\qquad \min\{\nu_{ij}^{S_2}, \nu_{ij}^{S_3}\})\big)\Big] = S_1 \cap (S_2 \cup S_3).
\end{aligned}
$$

Hence, $S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$ holds.

(ii)

$$
\begin{aligned}
(S_1 \cap S_2) \cup S_3 &= \big[\big(\min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, \min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}, \max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}\big)\big] \cup \big[\big(\mu_{ij}^{S_3}, \nu_{ij}^{S_3}\big)\big] \\
&= \Big[\big(\max(\min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, \mu_{ij}^{S_3}), \min(\min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}, \mu_{ij}^{S_3}), \\
&\qquad \min(\max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}, \nu_{ij}^{S_3})\big)\Big].
\end{aligned}
$$

Now,

$$
\begin{aligned}
(S_1 \cup S_3) \cap (S_2 \cup S_3) &= \big[\big(\max\{\mu_{ij}^{S_1}, \mu_{ij}^{S_3}\}, \min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_3}\}, \min\{\nu_{ij}^{S_1}, \nu_{ij}^{S_3}\}\big)\big] \cap \big[\big(\max\{\mu_{ij}^{S_2}, \mu_{ij}^{S_3}\}, \\
&\qquad \min\{\eta_{ij}^{S_2}, \eta_{ij}^{S_3}\}, \min\{\nu_{ij}^{S_2}, \nu_{ij}^{S_3}\}\big)\big] \\
&= \Big[\big(\min(\max\{\mu_{ij}^{S_1}, \mu_{ij}^{S_3}\}, \max\{\mu_{ij}^{S_2}, \mu_{ij}^{S_3}\}), \min(\min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_3}\}, \min\{\eta_{ij}^{S_2}, \eta_{ij}^{S_3}\}), \\
&\qquad \max(\min\{\nu_{ij}^{S_1}, \nu_{ij}^{S_3}\}, \min\{\nu_{ij}^{S_2}, \nu_{ij}^{S_3}\})\big)\Big] \\
&= \Big[\big(\min(\max\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, \mu_{ij}^{S_3}), \min(\min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}, \eta_{ij}^{S_3}), \\
&\qquad \max(\min\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}, \nu_{ij}^{S_3})\big)\Big] \\
&= \Big[\big(\max(\min\{\mu_{ij}^{S_1}, \mu_{ij}^{S_2}\}, \mu_{ij}^{S_3}), \min(\min\{\eta_{ij}^{S_1}, \eta_{ij}^{S_2}\}, \eta_{ij}^{S_3}), \\
&\qquad \min(\max\{\nu_{ij}^{S_1}, \nu_{ij}^{S_2}\}, \nu_{ij}^{S_3})\big)\Big] = (S_1 \cap S_2) \cup S_3
\end{aligned}
$$

Hence, $(S_1 \cap S_2) \cup S_3 = (S_1 \cup S_3) \cap (S_2 \cup S_3)$.

The rest of the laws may easily be obtained on similar lines.

# 4 Application of TSFSM in Decision-Making

In this section, we consider a general decision-making problem where the structure of information being considered in the format of $T$-spherical fuzzy soft matrix & propose some revised definitions which are utmost essential for solving the problem under consideration.

**Definition 1** *If* $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})] \in TSFSM_{m\times n}$, *then the* ***choice matrix*** *of TSFSM* $S_1$ *is defined as*

$$C(S_1) = \left[\left(\frac{\sum_{j=1}^{n}(\mu_{ij}^{S_1})^q}{n}, \frac{\sum_{j=1}^{n}(\eta_{ij}^{S_1})^q}{n}, \frac{\sum_{j=1}^{n}(\nu_{ij}^{S_1})^q}{n}\right)\right]_{m\times 1} \quad ;\ \forall\ i \textit{ if weights are same.}$$

**Definition 2** *If* $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})] \in TSFSM_{m\times n}$, *then the* ***weighted choice matrix*** *of TSFSM* $S_1$ *is defined by*

$$C_w(S_1) = \left[\left(\frac{\sum_{j=1}^{n} w_j(\mu_{ij}^{S_1})^q}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j(\eta_{ij}^{S_1})^q}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j(\nu_{ij}^{S_1})^q}{\sum w_j}\right)\right]_{m\times 1} \quad \forall\ i \textit{ where } w_j > 0 \textit{ are weights.}$$

On the basis of the above proposed definitions, we present a new algorithm to deal with the problem of decision-making which is being outlined in the flow chart given in Figure 2.

The proposed methodology for solving the multi-criteria decision-making problem is being illustrated with a numerical example as follows:

**Example 1.** Assume that an Indian multi-national company is planning some financial strategy for the upcoming year as per the group strategy objective. Four well defined investment alternatives have been taken into consideration and labeled as $\mathcal{A}^1$: investment in "*South Indian Markets*"; $\mathcal{A}^2$: investment in "*East Indian Markets*"; $\mathcal{A}^3$: investment in "*North Indian Markets*"; and $\mathcal{A}^4$: investment in "*West Indian markets*". After a preliminary screening for evaluation purpose, it has been decided to proceed by taking four criteria, namely as $\mathcal{C}^1$: "*growth*"; $\mathcal{C}^2$: "*risk analysis*"; $\mathcal{C}^3$: "*the socio-political impact*" and $\mathcal{C}^4$: "*the environmental and other factors*". Suppose that based on the financial strategies adopted for the welfare of the company, the weight vector is $\omega = (0.2, 0.3, 0.1, 0.4)^T$.

Here, for the simplicity of the computation for the example under consideration, we take the value of $q$ as 2 in the definitions. The computational steps for the the above stated problem using the proposed algorithm are below.

- **Step 1.** First we write the following spherical fuzzy soft decision matrix $R = [(r_{ij})] = [(\mu_{ij}, \eta_{ij}, \nu_{ij})], (i, j = 1, 2, 3, 4)$ for the four alternatives $\mathcal{A}^i$ $(i = 1, 2, 3, 4)$ & the four criteria $\mathcal{C}^j$ $(j = 1, 2, 3, 4)$ based on the information provided by the experts:

$$R = \begin{array}{c} \\ \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \\ \mathcal{A}^4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{C}^1 \quad\quad & \quad\quad\mathcal{C}^2 \quad\quad & \quad\quad\mathcal{C}^3 \quad\quad & \quad\quad\mathcal{C}^4 \quad \end{array} \\ \begin{pmatrix} (0.2, 0.2, 0.6) & (0.5, 0.3, 0.2) & (0.5, 0.2, 0.3) & (0.4, 0.3, 0.2) \\ (0.3, 0.4, 0.4) & (0.6, 0.3, 0.1) & (0.5, 0.3, 0.2) & (0.2, 0.1, 0.7) \\ (0.4, 0.5, 0.2) & (0.6, 0.3, 0.2) & (0.7, 0.2, 0.2) & (0.3, 0.3, 0.5) \\ (0.3, 0.2, 0.6) & (0.2, 0.2, 0.6) & (0.2, 0.3, 0.6) & (0.4, 0.2, 0.4) \end{pmatrix} \end{array}.$$

- **Step 2.** Since $\mathcal{C}^2$ and $\mathcal{C}^3$ are the cost criterions whereas $\mathcal{C}^1$ and $\mathcal{C}^4$ are the benefit criterions, therefore, we have to normalize the decision matrix. Hence, we obtain the normalized decision matrix is as follows:

$$R = \begin{array}{c} \\ \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \\ \mathcal{A}^4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{C}^1 \quad\quad & \quad\quad\mathcal{C}^2 \quad\quad & \quad\quad\mathcal{C}^3 \quad\quad & \quad\quad\mathcal{C}^4 \quad \end{array} \\ \begin{pmatrix} (0.6, 0.2, 0.2) & (0.5, 0.3, 0.2) & (0.5, 0.2, 0.3) & (0.2, 0.3, 0.4) \\ (0.4, 0.4, 0.3) & (0.6, 0.3, 0.1) & (0.5, 0.3, 0.2) & (0.7, 0.1, 0.2) \\ (0.2, 0.5, 0.4) & (0.6, 0.3, 0.2) & (0.7, 0.2, 0.2) & (0.5, 0.3, 0.3) \\ (0.6, 0.2, 0.3) & (0.2, 0.2, 0.6) & (0.2, 0.3, 0.6) & (0.4, 0.2, 0.4) \end{pmatrix} \end{array}.$$

  We observe that the element (0.6, 0.2, 0.2) in the matrix $R$ represents the degree to which the alternative $A_1$ matches the criterion $C_1$ is 0.6, the degree to which $A_1$ is neutral to the criteria $C_1$ is 0.2 and the degree to which $A_1$ does not satisfy $C_1$ is 0.2. On similar pattern, rest elements of the matrix can be interpreted.

- **Step 3:**
  - **Case 1: Weights are Equal**
    We find the choice matrix for $R$ as:

$$C(R) = \begin{bmatrix} (0.225, 0.065, 0.0825) \\ (0.315, 0.0875, 0.045) \\ (0.285, 0.1175, 0.0825) \\ (0.15, 0.0525, 0.215) \end{bmatrix}.$$

- **Case 2: Weights are Unequal**
  If the weights $\omega = (0.2, 0.3, 0.1, 0.4)^T$ are given for the criteria $\mathcal{C}^1$: "*growth*"; $\mathcal{C}^2$: "*risk analysis*"; $\mathcal{C}^3$: "*the socio-political impact*" and $\mathcal{C}^4$: "*the environmental and other factors*", respectively. Then the weighted choice matrix for $R$ is as

$$C_w(R) = \begin{bmatrix} (0.188, 0.075, 0.093) \\ (0.361, 0.072, 0.041) \\ (0.265, 0.117, 0.084) \\ (0.152, 0.045, 0.215) \end{bmatrix}.$$

- **Step 4:**
  - **When Weights are Equal:** In view of the Step 3 that if equal preference is assigned to each and every criteria, then we get 0.315 as the maximum value of the membership, i.e., investment in "*East Indian Markets*". Therefore, in this case the most suitable market for investment is "*East Indian Markets*".
  - **When Weights are Unequal:** Suppose that if a company assumes the importance of the criteria "*the environmental and other factors*" over the other criteria, then 0.361 being the maximum value of the membership for market $A_2$. Therefore, in this case the most suitable market for investment is "*East Indian Markets*".

On the other hand, the solution based on the methodology outlined in [18] for the above same problem is:
The score value for each alternative has been calculated as

$$\text{``}S(T_{u_1}) = 0.051918,\ S(T_{u_2}) = 0.300067,\ S(T_{u_3}) = 0.102274,\ S(T_{u_4}) = -0.07275.''$$

On the basis of obtained values the ranking of the alternatives is done as $S(T_{u_2}) > S(T_{u_3}) > S(T_{u_1}) > S(T_{u_4})$, where $T_{u_i}$ is the aggregated/integrated representative identity in correspondence with each $\mathcal{A}^i$. Thus it has been found that the alternative $\mathcal{A}^2$ is the best one. Therefore, the best alternative strategy for the company is to invest in the East Indian Market.

**Comparative Remarks:**
On the basis of the computations carried out above and in view of the comparative analysis, some remarkable observations are being pointed out as below:

- Guleria et al. [18] discussed the decision-making process not using the notion of matrices and concluded that the *'East Indian Market'* is highly preferable for the company to invest.
- Also, as per the proposed methodology where we have put the available information in the matrix form, we equally concluded that the *'East Indian Market'* $\mathcal{A}^2$ is the most suitable investment.
- Hence, the proposed methodology is equally consistent. However, the advantage which we find is that for solving the application problem, it would be easier to work with the matrices which certainly gives the enhanced dimensionality feature and wider span of information.

## 5 Application of $T$-spherical Fuzzy Soft Matrix in Selection Processes

Here, we consider a general selection process problem where the format of information being considered as $T$-spherical fuzzy soft matrix and propose some revised definitions which are utmost essential for solving the problem under consideration.

**Definition 3** *Suppose* $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})] \in TSFSM_{m\times n}$. *Then its score matrix is defined by* $S(S_1) = [s_{ij}] = [((\mu_{ij}^{S_1})^q - (\eta_{ij}^{S_1})^q - (\nu_{ij}^{S_1})^q)]$ $\forall$ *i and j. The* $(i,j)^{th}$ *element of* $S(S_1)$ *depicts like an index for computing the optimized value of the membership (or non-membership) of the* $i^{th}$ *student getting* $j^{th}$ *opportunity.*

**Definition 4** *Suppose* $S_1 = [(\mu_{ij}^{S_1}, \eta_{ij}^{S_1}, \nu_{ij}^{S_1})], S_2 = [(\mu_{ij}^{S_2}, \eta_{ij}^{S_2}, \nu_{ij}^{S_2})] \in TSFSM_{m\times n}$. *Then the utility matrix is defined by* $U(S_1, S_2) = [u_{ij}]_{m\times n} = [S(S_1) - S(S_2)]$; $\forall$ *i & j. Observe that* $(i,j)^{th}$ *element in* $U(S_1, S_2)$ *gives an index for computing the value of belongingness with respect to its non-belongingness of* $i^{th}$ *student getting* $j^{th}$ *opportunity.*

Based on the above proposed definitions, we present a new algorithm to deal with the problem of selection processes which is being outlined in the flow chart given by Figure 3.

It may be noted that in the above considered selection problem, if there is any situation where there is a tie in the values of max $U_i$, then to resolve that issue we have to reassess the information regarding the student's skill. Further, we present the methodology involved in the proposed algorithm using a numerical example for better understanding.

**Methodology:**
Suppose we have a set of $m$ students $A = \{a_1, a_2, \ldots, a_m\}$ which are having some skills out of $n$ particular skills $S = \{s_1, s_2, \ldots, s_n\}$ in connection with a set of $k$ opportunities $Q = \{q_1, q_2, \ldots, q_k\}$. We use $T$-spherical fuzzy soft matrices to select the suitable profession based upon individual's own skill. A $T$-spherical fuzzy soft set $(F, S)$ over $A$, with $F : S \to \mathcal{P}(A)$ (power set of $A$) is framed which provides a set of tentative idea/detail of student's skills. $T$-spherical fuzzy soft set so obtained represents a soft matrix $M$ which may be termed as student-skills matrix. Next, we make out a different $T$-spherical fuzzy soft set $(G, Q)$ over $S$, where $G : Q \to \mathcal{P}(S)$, (power set of $S$), which provide a tentative details of the available opportunities based upon their own skills. $T$-spherical fuzzy soft set so obtained represents another matrix $N$ termed as skills-opportunities matrix. Next, the complements, i.e., $M^c$ & $N^c$ have been evaluated which may be $(F, S)^c$ & $(G, Q)^c$ respectively. Subsequently, the average max min product matrix $M *_A N$, given by $R_1$, is obtained providing the highest value of the membership of having the skill in the student. Similarly, the matrix $M^c *_A N^c$, given by $R_2$, determines the highest value of the membership of the non-suitability of the student with respect to the skill desired.

Further, we obtain the score matrices $S(R_1)$ and $S(R_2)$ in view of the definition (3), which gives the corresponding optimized value w.r.t. the sense of suitability, abstain and non-suitability of a student for a specific opportunity. Also, the utility matrix is being obtained in view of the definition (4) which is based on the obtained score matrices. Entries in the utility matrix accordingly gives that to what extent the alternative fulfills the opinion of a decision maker. As the entries of the utility matrix are real numbers, therefore, the values which will be maximum would represent as preferred alternatives.

**Example 2:** Suppose $A = \{a_1, a_2, a_3, a_4\}$ be a universal set, where $a_1, a_2, a_3, a_4$ represents students. Consider a set of skills $S = \{s_1, s_2, s_3, s_4\}$ which represents communication skill, presentation skill, analytical skill and technical skill respectively. Also, consider a set of opportunities, denoted by $Q = \{q_1, q_2, q_3\}$ which represents hardware, software and managerial job respectively. We formulate this scenario by taking $T$-spherical fuzzy soft set $(F, S)$ over $A$ representing the description of student's skill.

- **Step 1:**

$$(F,S) = \left\{ \begin{array}{l} F(s_1) = \{(a_1, 0.7, 0.5, 0.1), (a_2, 0.3, 0.2, 0.6), (a_3, 0.4, 0.6, 0.3), (a_4, 0.5, 0.7, 0.2)\} \\ F(s_2) = \{(a_1, 0.9, 0.1, 0.6), (a_2, 0.2, 0.6, 0.5), (a_3, 0.3, 0.4, 0.6), (a_4, 0.8, 0.2, 0.2)\} \\ F(s_3) = \{(a_1, 0.4, 0.5, 0.3), (a_2, 0.9, 0.1, 0.2), (a_3, 0.8, 0.2, 0.3), (a_4, 0.4, 0.5, 0.4)\} \\ F(s_4) = \{(a_1, 0.2, 0.3, 0.7), (a_2, 0.8, 0.2, 0.1), (a_3, 0.7, 0.4, 0.3), (a_4, 0.6, 0.2, 0.6)\} \end{array} \right\}.$$

By considering the $T$-spherical fuzzy soft set, the following $T$-spherical fuzzy soft matrix may be obtained after conversion as follows:

$$M = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ (0.7, 0.5, 0.1) & (0.9, 0.1, 0.6) & (0.4, 0.5, 0.3) & (0.2, 0.3, 0.7) \\ (0.3, 0.2, 0.6) & (0.2, 0.6, 0.5) & (0.9, 0.1, 0.2) & (0.8, 0.2, 0.1) \\ (0.4, 0.6, 0.3) & (0.3, 0.4, 0.6) & (0.8, 0.2, 0.3) & (0.7, 0.4, 0.3) \\ (0.5, 0.7, 0.2) & (0.8, 0.2, 0.2) & (0.4, 0.5, 0.4) & (0.6, 0.2, 0.6) \end{array}$$

Next, consider $T$-spherical fuzzy soft set $(G, Q)$ over $S$ representing the information about the opportunities based on skills.

$$(G,Q) = \left\{ \begin{array}{l} G(q_1) = \{(s_1, 0.7, 0.2, 0.2), (s_2, 0.8, 0.2, 0.2), (s_3, 0.4, 0.3, 0.7), (s_4, 0.5, 0.3, 0.3)\} \\ G(q_2) = \{(s_1, 0.5, 0.4, 0.4), (s_2, 0.3, 0.4, 0.6), (s_3, 0.8, 0.2, 0.2), (s_4, 0.6, 0.2, 0.4)\} \\ G(q_3) = \{(s_1, 0.4, 0.5, 0.3), (s_2, 0.3, 0.5, 0.5), (s_3, 0.5, 0.4, 0.3), (s_4, 0.3, 0.7, 0.2)\} \end{array} \right\}.$$

Again, the $T$-spherical fuzzy soft matrix may be formulated as follows:

$$N = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{ccc} q_1 & q_2 & q_3 \\ (0.7, 0.2, 0.2) & (0.5, 0.4, 0.4) & (0.4, 0.5, 0.3) \\ (0.8, 0.2, 0.2) & (0.3, 0.4, 0.6) & (0.3, 0.5, 0.5) \\ (0.4, 0.3, 0.7) & (0.8, 0.2, 0.2) & (0.5, 0.4, 0.3) \\ (0.5, 0.3, 0.3) & (0.6, 0.2, 0.4) & (0.3, 0.7, 0.2) \end{array}$$

- **Step 2:** Compute the complement matrices of the above TSFSMs which was obtained in Step 1 as follows:

$$M^c = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ (0.1, 0.5, 0.7) & (0.6, 0.1, 0.9) & (0.3, 0.5, 0.4) & (0.7, 0.3, 0.2) \\ (0.6, 0.2, 0.3) & (0.5, 0.6, 0.2) & (0.2, 0.1, 0.9) & (0.1, 0.2, 0.8) \\ (0.3, 0.6, 0.4) & (0.6, 0.4, 0.3) & (0.3, 0.2, 0.8) & (0.3, 0.4, 0.7) \\ (0.2, 0.7, 0.5) & (0.2, 0.2, 0.8) & (0.4, 0.5, 0.4) & (0.6, 0.2, 0.6) \end{array}$$

$$N^c = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{ccc} q_1 & q_2 & q_3 \\ (0.2, 0.2, 0.7) & (0.4, 0.4, 0.5) & (0.3, 0.5, 0.4) \\ (0.2, 0.2, 0.8) & (0.6, 0.4, 0.3) & (0.5, 0.5, 0.3) \\ (0.7, 0.3, 0.4) & (0.2, 0.2, 0.8) & (0.3, 0.4, 0.5) \\ (0.3, 0.3, 0.5) & (0.4, 0.2, 0.6) & (0.2, 0.7, 0.3) \end{array}$$

- **Step 3:** Evaluate the average max min products corresponding to the TSFSMs computed in Step 1 and Step 2 as follows:

$$R_1 = M *_A N = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{ccc} q_1 & q_2 & q_3 \\ (0.85, 0.15, 0.15) & (0.6, 0.25, 0.25) & (0.6, 0.3, 0.2) \\ (0.65, 0.2, 0.2) & (0.85, 0.15, 0.2) & (0.7, 0.25, 0.15) \\ (0.6, 0.25, 0.25) & (0.8, 0.2, 0.25) & (0.65, 0.3, 0.25) \\ (0.8, 0.2, 0.2) & (0.6, 0.2, 0.3) & (0.55, 0.35, 0.25) \end{array}$$

$$R_2 = M^c *_A N^c = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{ccc} q_1 & q_2 & q_3 \\ (0.5, 0.15, 0.35) & (0.6, 0.25, 0.4) & (0.55, 0.3, 0.25) \\ (0.45, 0.2, 0.5) & (0.55, 0.15, 0.25) & (0.45, 0.25, 0.25) \\ (0.5, 0.25, 0.55) & (0.6, 0.2, 0.3) & (0.55, 0.3, 0.3) \\ (0.8, 0.2, 0.4) & (0.5, 0.2, 0.5) & (0.4, 0.35, 0.45) \end{array}$$

- **Step 4:** Next, we compute the score matrices of TSFSMs $R_1$ and $R_2$ evaluated in Step 3 as follows:

$$S(R_1) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{ccc} q_1 & q_2 & q_3 \\ 0.6775 & 0.235 & 0.23 \\ 0.3425 & 0.66 & 0.405 \\ 0.235 & 0.5375 & 0.27 \\ 0.56 & 0.23 & 0.1175 \end{array}$$

$$S(R_2) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{ccc} q_1 & q_2 & q_3 \\ 0.105 & 0.1375 & 0.15 \\ -0.0875 & 0.2175 & 0.0775 \\ -0.115 & 0.23 & 0.1225 \\ 0.44 & -0.04 & -0.165 \end{array}$$

- **Step 5:** Corresponding to the matrices $S(R_1)$ and $S(R_2)$ evaluated in Step 4, we compute its utility matrix as follows:

$$U = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{ccc} q_1 & q_2 & q_3 \\ \mathbf{0.5725} & 0.0975 & 0.08 \\ 0.43 & \mathbf{0.4425} & 0.3275 \\ \mathbf{0.35} & 0.3075 & 0.1475 \\ 0.12 & 0.27 & \mathbf{0.2825} \end{array}$$

- **Step 6:** Thus, based on the obtained values in the utility matrix so computed in Step 5, it may be observed that students $\{a_1, a_3\}$ are best suitable for the hardware jobs ($q_1$), the student $a_2$ is best suitable for software job ($q_2$) and the student $a_4$ is best suitable for the managerial job ($q_3$). The computational values obtained above for the optimum allocation of the student's job based on their skills are represented by Figure 4.

**Advantages & Comparative Remarks:**
On the basis of the computations carried above, some comparative remarks are being pointed out as below:

- The example of job suitability for different students having different skills stated above demonstrates that there are no restrictions on the allocation due to the four determining parameters of $T$-spherical fuzzy information.
- Also, as per the proposed methodology where we have put the available information in the matrix form, we equally concluded that $\{a_1, a_3\}$ are best suitable for the hardware

jobs ($q_1$), the student $a_2$ is best suitable for software job ($q_2$) and the student $a_4$ is best suitable for the managerial job ($q_3$).

- However, the advantage which we find is that for solving the application problem, it would be easier to work with the matrices which certainly gives the enhanced dimensionality feature and wider span of information. This also indicates that TSFSM is more powerful in dealing with uncertain & imprecise infirmation than with Pythagorean or Picture fuzzy information.

# 6 Conclusions & Scope for Future Work

The concept of $T$-spherical fuzzy soft matrix has been successfully proposed with various binary operations, properties and propositions. In order to exhibit the computational applications of the proposed TSFSM with the idea of choice matrix and its weighted form, we have presented a new methodology for solving a decision-making problem with the help of an illustrative example. In addition to this, for solving a general selection process problem, another new methodology has been provided by well utilizing the concept of score and utility matrices. The inclusion of numerical examples for each application problem clearly illustrates the implementation part of the proposed methodologies. The comparative remarks which have been appended in the application sections gives a better understanding of the proposed technique with respect to the existing techniques. The application of the $T$-spherical fuzzy soft matrices may further be extended in future in various fields. Some of them are listed below.

The dimensionality reduction technique has been widely applied for solving the decision-making problems having the involvement of a large number of inter-related factors. For better coverage of the incomplete information, the impreciseness in the factors may be taken in the form of spherical fuzzy information and can be well addressed by utilizing $T$-spherical fuzzy set and $T$-spherical fuzzy soft matrix which may further help in developing a technique to handle/reduce the dimensionality of large sized data [25].

In literature, the divergence measure plays an important role in the process of pattern recognition. Wu et al. [26] presented for $T$-spherical fuzzy sets which may further be extended for $T$-spherical fuzzy soft matrices to enable various computational applications.

In our future work, we will extend the proposed notion of $T$-spherical fuzzy soft matrix to complex $T$-spherical fuzzy soft matrix/aggregation operators on the basis of deliberations given by Ali et al. [35] [36] which may be applied in identifying reference signal out of several transmitted signals. Such extension will also be helpful in tracking the cycle of pattern followed by various problems related to pattern recognition and decision-making.

# Declarations & Compliance with ethical standards

**Ethical approval:** This article does not contain any studies with human participants or animals performed by any of the authors.

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**Conflict of interest:** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Informed Consent:** This article does not contain any studies with human participants.

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**Figures & Tables**

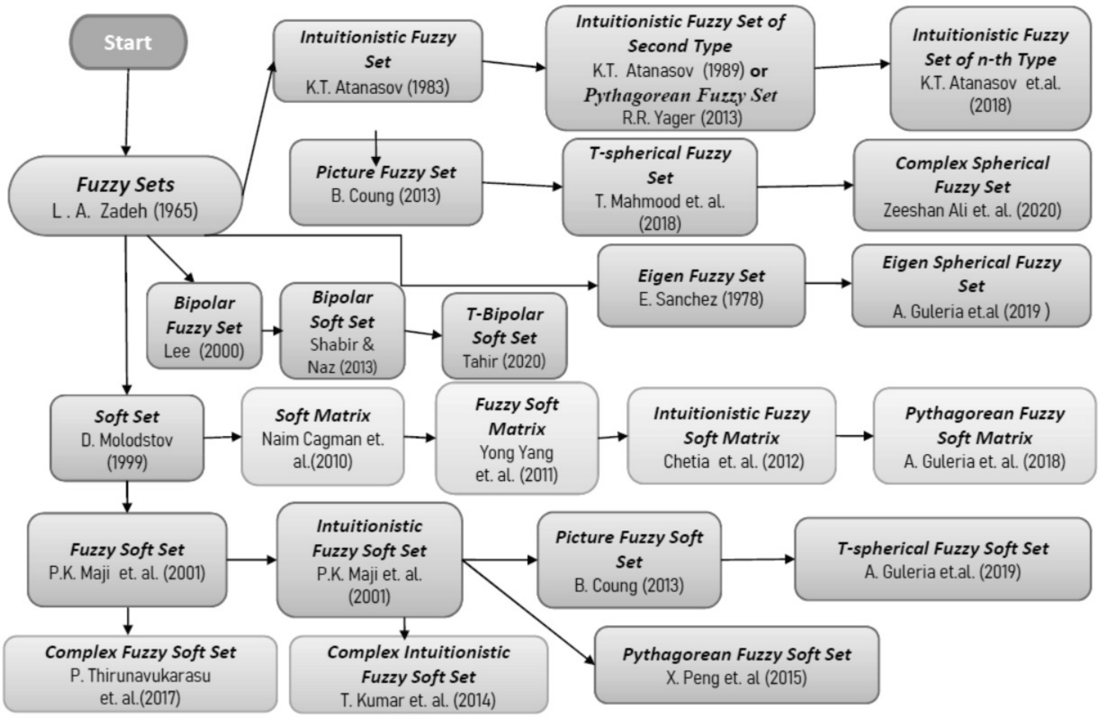


Figure 1: Extensions and Generalizations of Fuzzy Set

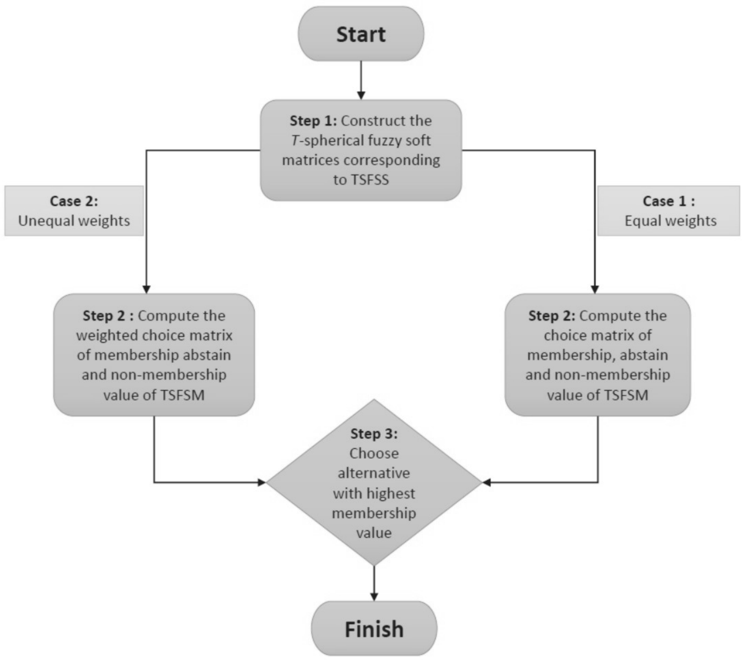


Figure 2: Proposed Methodology for Decision-making

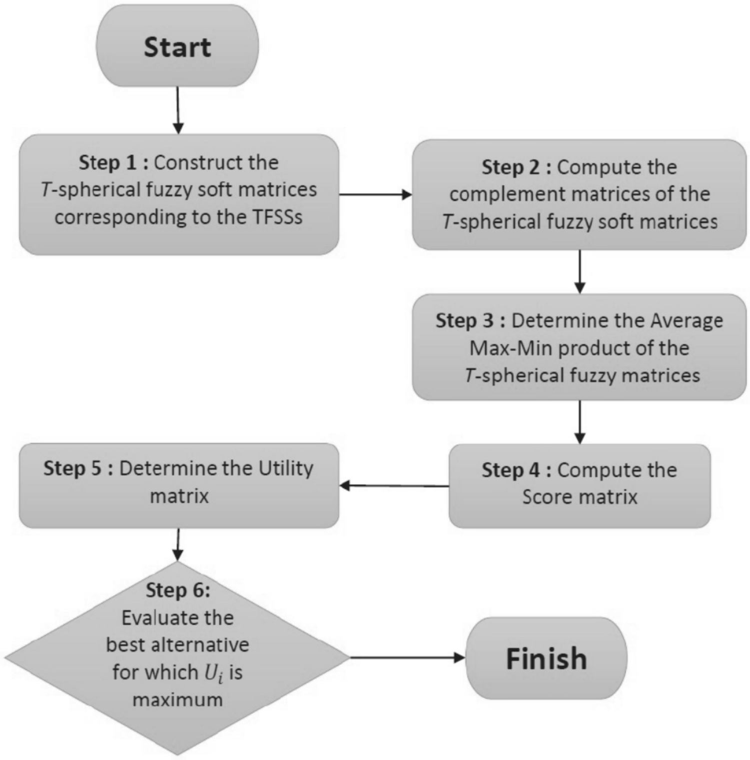


Figure 3: Flow Chart of the Algorithm for Selection Processes

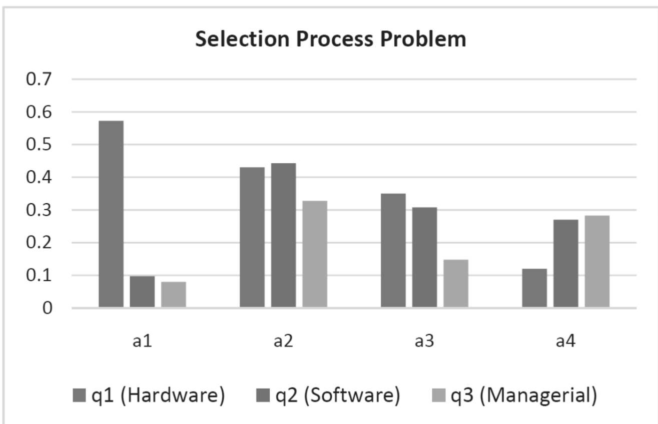


Figure 4: Representation of Optimum Allocation in Selection Process

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