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A combined approximation method for nonlinear foam drainage equation

M. Izadi*

Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.

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Abstract. The aim of this study is to develop a combined approximation technique to find a numerical solution to the foam drainage equation in various absorption and distillation processes. In this approach, first, discretization of time is performed with the aid of the Taylor expansion series. Hence, a collocation method based on novel Bessel polynomials is utilized for the space variable. Thus, the solution is found by solving a linear system of algebraic equations in each time step. Numerical simulations are provided to check the accuracy and efficiency of the presented algorithm. The numerical results are compared with exact solutions as well as the outcomes of other existing numerical methods.

1. Introduction

This research aims to develop an efficient approximation algorithm to solve the nonlinear foam drainage equation [1]:

$$\frac{\partial W}{\partial t}+\frac{\partial}{\partial x}\Big(W^2-\frac{\sqrt{W}}{2}\frac{\partial W}{\partial x}\Big)=0, \tag{1}$$

with initial condition:

$$W|_{t=0}=W_0(x). \tag{2}$$

Here, $W(x,t)$ as the scaled coordinates and t the time position) denotes the cross-section of a channel formed where three films meet, usually indicated as "Plateau border". Foams naturally appear in numerous applications and technological processes and have attracted the attention of many researchers given their significance, see cf. [2,3].

By substituting $W(x,t):=w^2(x,t)$ in Eqs. (1) and (2), we arrive at the following nonlinear initial-value problem:

$$\begin{cases} w_t+2w^2\, w_x-w_x^2, -\frac{1}{2}\, w\, w_{xx}=0, \\ \qquad\qquad\qquad 0\le x\le L, \quad 0\le t\le T \\ w|_{t=0}=w_0(x):=\sqrt{W_0(x)}, \qquad 0<x<L \end{cases} \tag{3}$$

where $T>0$ is a given final time and $L>0$ is a real constant. In addition, the following boundary conditions are supplemented with the initial-value problem (3):

$$w(0,t)=h_0(t), \quad w(L,t)=h_1(t), \quad 0\le t\le T, \tag{4}$$

where $h_0(t)$ and $h_1(t)$ are two prescribed functions. Over the last few decades, researchers have proposed several analytical techniques as well as approximative algorithms to solve the foam drainage equation. Among these methods, we mention the Tahn and Adomian decomposition methods [4], the Homotopy Perturbation Method (HPM) [5], the symmetry Lie group approaches [6,7], the series solution based on the homotopy analysis method [8], the Exp-function

*. *E-mail address: izadi@uk.ac.ir*

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approach [9], the Homotopy Perturbation Transform Method (HPTM) [10], a semi-analytical approach based upon the quasilinearization and the Haar wavelet bases, and a hybrid computational approach based on the generalized Chebyshev polynomials and quasilinearization technique [11].

The main objective of this research study is to derive a new combined approximation technique based on a combination of Taylor-series approach and spectral collocation scheme for the numerical treatment of the nonlinear foam-drainage equation. On the one hand, the considered equation is a time-dependent model problem and, thus, it is of interest to develop an accurate time-marching algorithm for our model. On the other hand, collocation-based methods have been applied successfully to many model problems in science and engineering due to their efficiency as well as simple applicability while giving high accuracy. Among many existing approaches based on the collocation strategy, we can mention the meshfree methods [12,13], the spectral collocation approach based on diverse polynomial bases such as Legendre, Chebyshev, Chelyshkov, etc. utilized in [14–17], over the past decades.

In approximate terms, the Taylor approach with second-order accuracy is first employed to discretize the time variable. Then, in each time step, it is supposed that the underlying model problem has a solution in terms of the novel Bessel series expansion of the unknown function. Afterwards, representing all involved unknowns in the Bessel matrix form together with the proper usage of a suitable set of collocation points helps determine the unknown series coefficients by solving a linear system of matrix equations. Indeed, the Bessel polynomial of order ℓ is defined explicitly as [18]:

$$\mathbb{B}_\ell(x) = \sum_{\kappa=0}^{\ell} \frac{(\ell+\kappa)!}{(\ell-\kappa)!}\frac{x^\kappa}{2^\kappa\,\kappa!}, \quad \ell = 0,1,\dots. \tag{5}$$

See also [19–22] for recent applications. Besides the fact that all coefficients of $\mathbb{B}_\ell(x)$ are positive, they also satisfy the second-order differential equation:

$$x^2\,\mathbb{B}_\ell''(x) + 2(x+1)\,\mathbb{B}_\ell'(x) - \ell\,(\ell+1)\,\mathbb{B}_\ell(x) = 0.$$

It should be noted that the considered Bessel functions $\mathbb{B}_\ell(x)$ differ from the traditional Bessel functions of the first kind, which have previously been utilized in various research papers, see cf. [17,23].

2. Taylor scheme for time discretization

First, an attempt is made to discretize the foam drainage equation with respect to time variable. In this respect, the interval $[0,T]$ can be subdivided into $(M+1)$ grid points:

$$0 =: t_0 < t_1 = \Delta t < \ldots < t_M := M\Delta t = T,$$

being $\Delta t = t_n - t_{n-1}$ the uniform time step. To get a time-accurate discretization scheme, according to the Taylor series representation for $w^n = w(x,t_n)$ we obtain the following:

$$w_t^n = \frac{w^{n+1}-w^n}{\Delta t} - \frac{1}{2}\Delta t\, w_{tt}^n + \mathcal{O}(\Delta t^2). \tag{6}$$

To proceed, we differentiate Eq. (3) with respect to t to get:

$$\begin{aligned} w_{tt}^n &= \left[\frac{1}{2}w^n\,w_{xx}^n + (w_x^n)^2 - 2(w^n)^2\,w_x^n\right]_t \\ &= \frac{1}{2}w_t^n\,w_{xx}^n + \frac{1}{2}w^n\,(w_t^n)_{xx} + 2w_x^n\,(w_t^n)_x \\ &\quad -4w^n\,w_t^n\,w_x^n - 2(w^n)^2\,(w_t^n)_x. \end{aligned}$$

By replacing the first-order derivatives $w_t^n \approx (w^{n+1} - w^n)/\,\Delta t$ in all occurrences, we may write w_{tt}^n as follows:

$$\begin{aligned} \Delta t\, w_{tt}^n &= \left[\frac{1}{2}w_{xx}^n - 4w^n\,w_t^n\right](w^{n+1}-w^n) \\ &\quad +2\big[w_x^n - (w^n)^2\big](w_x^{n+1}-w_x^n) \\ &\quad +\frac{1}{2}w^n(w_{xx}^{n+1}-w_{xx}^n). \end{aligned} \tag{7}$$

Next, Eq. (7) is inserted into the right-hand side of Eq. (6) using the time discretized form of Eq. (3), i.e.:

$$w_t^n = -2(w^n)^2\,w_x^n + (w_x^n)^2 + \frac{1}{2}\,w^n\,w_{xx}^n,$$

for the left-hand side of Eq. (6). After some manipulations, the following time discretized equation for Eq. (3) with second-order accuracy in time is obtained:

$$\begin{aligned} &\Big[\Delta t(2w^n\,w_x^n - \frac{1}{4}w_{xx}^n) + 1\Big]w^{n+1} \\ &\quad -\Delta t\Big[w_x^n - (w^n)^2\Big]w_x^{n+1} - \frac{\Delta t}{4}\,w^n\,w_{xx}^{n+1} \\ &\quad = w^n\Big[1 + \Delta t\,w^n\,w_x^n\Big], \end{aligned} \tag{8}$$

for $n = 0,1,\ldots$. To start computations in Eq. (8), we need $w^0 = w_0(x)$, which is obtained from the initial condition in Eq. (3). Moreover, the boundary conditions obtained from Eq. (4) at $x = 0, L$ are:

$$\begin{aligned} &w^{n+1}(0) := h_0^{n+1} = h_0(t_{n+1}), \\ &w^{n+1}(L) := h_1^{n+1} = h_1(t_{n+1}), \quad n = 0,1,\ldots,M-1. \end{aligned} \tag{9}$$

3. Bessel functions: Basic matrix relations

Now, the first stage in discretizing the foam drainage equation in time is carried out through Eq. (8). In the second stage, it is required to approximate the solution of the original model (1) with respect to the space variable through solving Eq. (8). To do so, it is assumed that the solution w^{n+1} of Eq. (8) can be written as a combination of $\mathbb{B}_\ell(x)$. In the first time step, i.e., for $n=0$, we use the initial condition $w_0(x)$ to determine w^0 exactly. Let $\mathcal{W}_{n,N}(x)$ denotes the approximate solution of w^n at time level t_n. Then, the $\mathcal{W}_{n+1,N}(x)$ is sought after at the next time level t_{n+1} as follows:

$$\mathcal{W}_{n+1,N}(x) = \sum_{\ell=0}^{N} a_{\ell,n}\,\mathbb{B}_\ell(x), \quad x \in [0,L], \tag{10}$$

for $n = 0,1,\ldots,M-1$. Here, $a_{\ell,n}$, $\ell = 0,1,\ldots,N$ as the unknown Bessel coefficients must be found. Let us introduce the Bessel vector $\mathbf{B}_N(x)$ as well as the unknown vector $\mathbf{A}_{n,N}$ in the forms:

$$\mathbf{B}_N(x) = [\mathbb{B}_0(x) \quad \mathbb{B}_1(x) \quad \ldots \quad \mathbb{B}_N(x)],$$

$$\mathbf{A}_{n,N} = [a_{0,n}\ a_{1,n}\ \ldots\ a_{n,N}]^T.$$

With the help of these vectors, we are able to rewrite Eq. (10) in a compact representation as follows:

$$\mathcal{W}_{n+1,N}(x) = \mathbf{B}_N(x)\,\mathbf{A}_{n,N}, \tag{11}$$

additionally, by introducing the matrix $\mathbf{D}$ shown in Box I, and the monomial vector $\mathbf{X}_N(x) = \begin{bmatrix}1 & x & x^2 & \ldots & x^N\end{bmatrix}$, we shall express $\mathbf{B}_N(x)$ as follows:

$$\mathbf{B}_N(x) = \mathbf{X}_N(x)\,\mathbf{D}^T. \tag{12}$$

We are left with finding a relationship between $\mathbf{X}_N(x)$ and $\frac{d}{dx}\mathbf{X}_N(x)$. A straightforward calculation shows that:

$$\frac{d}{dx}\mathbf{X}_N(x) = \mathbf{X}_N(x)\,\mathbf{M}^T,$$

$$\mathbf{M}^T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)\times(N+1)}. \tag{13}$$

We consider the result of the convergence of the Bessel functions as $N \to \infty$. This property indicates that the Bessel function is exponentially convergent in the weighted L_2 norm with respect to weight function $r(x) = exp(-2L/x)$.

Theorem 3.1 [22]. *Let $Z_N(x) = \mathbf{B}_N(x)\,\mathbf{A}_{n,N}$ be the best square approximation to $Z(x)$. Under the assumptions $Z(x) \in \mathcal{C}^{N+1}[0,L]$ and $M_\infty := \max_{x\in[0,L]} |Z^{(N+1)}(x)|$, we have the following error bound:*

$$||Z(x) - Z_N(x)||_r \leq \frac{M_\infty}{\sqrt{2N+3}} \frac{L^{M+\frac{3}{2}}}{(N+1)!} \frac{1}{exp(1)}.$$

4. Taylor-Bessel collocation method

Now, the solution to the discretized model problem (8) is to be approximated via Eq. (10). To do so, we first define the set of collocation points $\{x_q\}_{q=0}^N$ on $[0,L]$ with:

$$x_q = \frac{L}{N}q, \quad q = 0,1,\ldots,N. \tag{14}$$

Next, we express the unknown functions w^{n+1}, w_x^{n+1}, and w_{xx}^{n+1} in Eq. (8) in a matrix form. By placing the collocation points (Eq. (14)) into the resulting equation, we get a linear matrix equation.

Our next task is to combine two previously asserted Relations (Eqs. (11) and (12)). In this way, Eq. (10) is rewritten in the matrix expression as follows:

$$\mathcal{W}_{n+1,N}(x) = \mathbf{X}_N(x)\,\mathbf{D}^T\,\mathbf{A}_{n,N}. \tag{15}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 3 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \frac{N!}{(N-2)!\,1!\,2^1} & \frac{(N+1)!}{(N-3)!\,2!\,2^2} & \ldots & \frac{(2N-2)!}{0!\,(N-1)!\,2^{N-1}} & 0 \\ 1 & \frac{(N+1)!}{(N-1)!\,1!\,2^1} & \frac{(N+2)!}{(N-2)!\,2!\,2^2} & \ldots & \frac{(2N-1)!}{1!\,(N-1)!\,2^{N-1}} & \frac{(2N)!}{0!\,N!\,2^N} \end{bmatrix}_{(N+1)\times(N+1)}$$

Box I

After evaluating the preceding equation at the collocation points (Eq. (14)) we arrive at:

$$\mathbf{W}_{n+1} = \mathbf{Y}\,\mathbf{D}^T\,\mathbf{A}_{n,N}, \qquad \mathbf{W}_{n+1} = \begin{bmatrix} \mathcal{W}_{n+1,N}(x_0) \\ \mathcal{W}_{n+1,N}(x_1) \\ \vdots \\ \mathcal{W}_{n+1,N}(x_N) \end{bmatrix},$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{X}_N(x_0) \\ \mathbf{X}_N(x_1) \\ \vdots \\ \mathbf{X}_N(x_N) \end{bmatrix}. \tag{16}$$

We can represent the first and second-orders derivatives in Eq. (8) through Eqs. (13) and (15) in the matrix forms:

$$\begin{cases} w_x^{n+1} \approx \mathcal{W}^{(1)}_{n+1,N}(x) = \mathbf{X}_N(x)\,\mathbf{M}^T\,\mathbf{D}^T\,\mathbf{A}_{n,N}, \\ w_{xx}^{n+1} \approx \mathcal{W}^{(2)}_{n+1,N}(x) = \mathbf{X}_N(x)\,(\mathbf{M}^T)^2\,\mathbf{D}^T\,\mathbf{A}_{n,N}. \end{cases} \tag{17}$$

Similarly, by evaluating them at the collocation points, the first and second derivatives in Eqs. (17) can be written in the matrix forms:

$$\dot{\mathbf{W}}_{n+1} = \mathbf{Y}\,\mathbf{M}^T\,\mathbf{D}^T\,\mathbf{A}_{n,N},$$

$$\dot{\mathbf{W}}_{n+1} = \begin{bmatrix} \mathcal{W}^{(1)}_{n+1,N}(x_0) \\ \mathcal{W}^{(1)}_{n+1,N}(x_1) \\ \vdots \\ \mathcal{W}^{(1)}_{n+1,N}(x_N) \end{bmatrix}, \tag{18}$$

$$\ddot{\mathbf{W}}_{n+1} = \mathbf{Y}\,(\mathbf{M}^T)^2\,\mathbf{D}^T\,\mathbf{A}_{n,N},$$

$$\ddot{\mathbf{W}}_{n+1} = \begin{bmatrix} \mathcal{W}^{(2)}_{n+1,N}(x_0) \\ \mathcal{W}^{(2)}_{n+1,N}(x_1) \\ \vdots \\ \mathcal{W}^{(2)}_{n+1,N}(x_N) \end{bmatrix}. \tag{19}$$

By introducing the following functions:

$$p_{n,0}(x) = 2\Delta t\, w^n\, w_x^n - \frac{1}{4}\Delta t\, w_{xx}^n + 1,$$

$$p_{n,1}(x) = -\Delta t\, w_x^n + \Delta t\,(w^n)^2,$$

$$p_{n,2}(x) = -\frac{1}{4}\Delta t\, w^n,$$

$$g_n(x) = w^n + \Delta t\,(w^n)^2\, w_x^n,$$

and using the approximations $\mathcal{W}_{n+1,N}(x), \mathcal{W}^{(1)}_{n+1,N}(x)$, $\mathcal{W}^{(2)}_{n+1,N}(x)$, we may rewrite Eq. (8) as:

$$p_{n,2}(x)\,\mathcal{W}^{(2)}_{n+1,N}(x) + p_{n,1}(x)\,\mathcal{W}^{(1)}_{n+1,N}(x) + p_{n,0}(x)\,\mathcal{W}_{n+1,N}(x) = g_n(x), \quad 0 \le x \le L. \tag{20}$$

By inserting the collocation points into Eq. (20), the following system is obtained:

$$\mathbf{P}_{n,2}\,\ddot{\mathbf{W}}_{n+1} + \mathbf{P}_{n,1}\,\dot{\mathbf{W}}_{n+1} + \mathbf{P}_{n,0}\,\mathbf{W}_{n+1} = \mathbf{G}_n. \tag{21}$$

In Eq. (21), the matrices $\mathbf{P}_{n,l}$, and the vector $\mathbf{G}_n$ take the forms:

$$\mathbf{P}_{n,l} = \begin{bmatrix} p_{n,l}(x_0) & 0 & \dots & 0 \\ 0 & p_{n,l}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{n,l}(x_N) \end{bmatrix}_{(N+1)\times(N+1)}$$

$$\mathbf{G}_n = \begin{bmatrix} g_n(x_0) \\ g_n(x_1) \\ \vdots \\ g_n(x_N) \end{bmatrix}_{(N+1)\times 1}.$$

$\ell = 0, 1, 2$. Let's place Eqs. (16), (18), and (19) into Eq. (21). This yields the fundamental matrix equation:

$$\mathbf{V}_n\,\mathbf{A}_{n,N} = \mathbf{G}_n, \tag{22}$$

where:

$$\mathbf{V}_n := \left\{\mathbf{P}_{n,2}\,\mathbf{Y}\,(\mathbf{M}^T)^2 + \mathbf{P}_{n,1}\,\mathbf{Y}\,\mathbf{M}^T + \mathbf{P}_{n,0}\,\mathbf{Y}\right\}\mathbf{D}^T.$$

Clearly, the fundamental matrix equation (Eq. (22)) is a set of $(N+1)$ linear equations in terms of $(N+1)$ unknown coefficients $a_{0,n}, a_{1,n}, \dots, a_{N,n}$ to be found.

To consider the boundary conditions (Eq. (9)), we must also convert them into matrix form. Based on the representation (Eq. (15)), these conditions, i.e. $\mathcal{W}_{n+1.N}(0) = h_0^{n+1}$ and $\mathcal{W}_{n+1,N}(1) = h_1^{n+1}$, can be expressed in the matrix notation:

$$\widehat{\mathbf{V}}_{n,0}\,\mathbf{A}_{n,N} = h_0^{n+1},$$

$$\widehat{\mathbf{V}}_{n,0} := \mathbf{X}_N(0)\,\mathbf{D}^T = [\hat{v}_{0,0} \quad \hat{v}_{0,1} \quad \dots \quad \hat{v}_{0,N}],$$

$$\widehat{\mathbf{V}}_{n,1}\,\mathbf{A}_{n,N} = h_1^{n+1},$$

$$\widehat{\mathbf{V}}_{n,1} := \mathbf{X}_N(1)\,\mathbf{D}^T = [\hat{v}_{1,0} \quad \hat{v}_{1,1} \quad \dots \quad \hat{v}_{1,N}].$$

Next, we substitute the first two rows of the augmented matrix $[\mathbf{V}_n; \mathbf{G}_n]$ by the vectors $[\widehat{\mathbf{V}}_{n,0}; h_0^{n+1}]$ and $[\widehat{\mathbf{V}}_{n,1}; h_1^{n+1}]$, for convenience. Thus, the modified linear system of equations is obtained by Eq. (23) as shown in Box II. Now, by solving the above linear system, we may to obtain the unknown Bessel coefficients in Eq. (15).

5. Numerical simulations

To testify the performance of the combined Taylor and Bessel-collocation approach, numerical simulations

$$\left[\widehat{\mathbf{V}}_n;\widehat{\mathbf{G}}_n\right]=\begin{bmatrix} \hat{v}_{0,0} & \hat{v}_{0,1} & \hat{v}_{0,2} & \hat{v}_{0,3} & \dots & \hat{v}_{0,N} & ; & h_0^{n+1} \\ \hat{v}_{1,0} & \hat{v}_{1,1} & \hat{v}_{1,2} & \hat{v}_{1,3} & \dots & \hat{v}_{1,N} & ; & h_1^{n+1} \\ v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} & \dots & v_{2,N} & ; & g_n(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & ; & \vdots \\ v_{N-1,0} & v_{N-1,1} & v_{N-1,2} & v_{N-1,3} & \dots & v_{N-1,N} & ; & g_n(x_{N-1}) \\ v_{N,0} & v_{N,1} & v_{N,2} & v_{N,3} & \dots & v_{N,N} & ; & g_n(x_N) \end{bmatrix}. \quad (23)$$

Box II

based on two test cases are given for the nonlinear initial and boundary value Problems (Eqs. (3) and (4)). Furthermore, comparisons of numerical results and the outcomes of diverse existing schemes are also made for validation. For implementations, MATLAB software (version 2017a) is employed.

Test problem 5.1. *We first consider the foam drainage equation (Eq. (3)) with the following initial condition [4,24,11]:*

$$w_0(x)=-\tanh(x).$$

The exact solution is given by $w(x,t)=-\tanh(x-t)$.

We first employ Δt equal to $T=0.01$. Considering Eq. (10) with $N=5$, the following approximation for $0\le x\le L=1$ is obtained:

$$\mathcal{W}_{1,5}(x)=0.01807890048\,x^5-0.1934987886\,x^4$$
$$+0.4424756341\,x^3-0.03689672509\,x^2$$
$$-0.9975210119\,x+0.00999966668.$$

We plot the above obtained solution as an approximation to $w(x,T)$ in Figure 1. We also show the corresponding Absolute Errors (AE) $|w(x,T)-\mathcal{W}_{1,5}(x)|$ at $x\in[0,1]$ in Figure 2. Besides $\Delta t=0.01$, we use $\Delta t=0.1, 0.01$ to demonstrate the impact of different values of time step size on the computations.

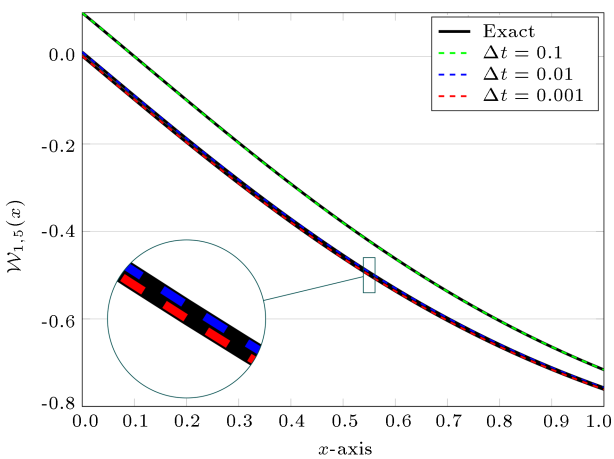

Figure 1. Graphs of exact and solutions at different time instants $t=\Delta t$ for $\Delta t=0.1, 0.01, 0.001$, $N=5$ in test problem 5.1.

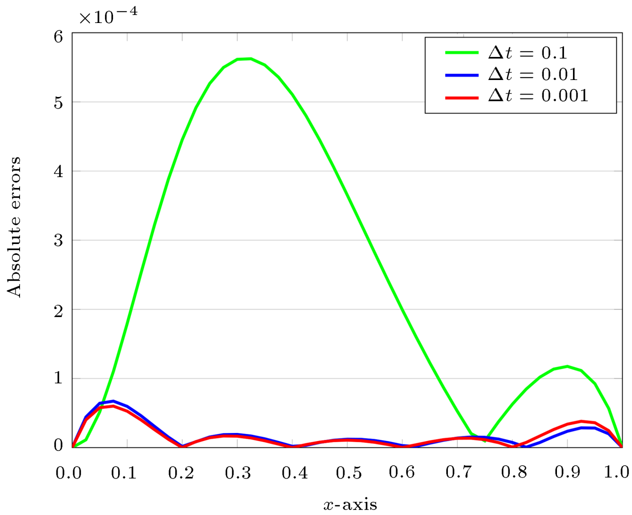

Figure 2. Graphs of absolute errors at different time instants $t=\Delta t$ for $\Delta t=0.1, 0.01, 0.001$, $N=5$ in test problem 5.1.

To validate our results, some comparisons are made in Tables 1 and 2, which show the numerical solutions obtained by the presented scheme evaluated at $t=0.01, 0.001$ and various $x\in[0,1]$. The corresponding AE are also reported in the second column of these tables. Furthermore, analogue results of the previously well-established methods are displayed in Table 1. These include the collocation method based on Bivariate Chebyshev Functions (BCF) [11], the Adomian Decomposition Method (ADM) [4], the Homotopy Perturbation Method (HPM) [5], the Haar Wavelet Quasilinearization Approach (HWQA) [24], and the Homotopy Perturbation Transform Method (HPTM) [10]. It can be observed that our numerical results are in good agreement with the corresponding exact solutions. However, our approach is more straightforward than other existing methods.

Test problem 5.2. *As the second example, we consider the following initial condition [4,24,11]:*

Table 1. The comparison of numerical results in Test Problem 5.1 for $N = 8$ and various $x \in [0, 1]$ at time $t = 0.01$.

$\frac{x}{64}$	**Present**	**AE**	**BCF [11]**	**HWQA [24]**	**ADM [4]**	**HPM [5]**	**HPTM [10]**
1	−0.0056255222	5.82_{-7}	−0.005624	−0.005626	−0.004253	−0.004358	−0.048341
3	−0.0368592824	9.87_{-7}	−0.036858	−0.036874	−0.009002	−0.027011	−0.017108
5	−0.0680207213	9.16_{-7}	−0.068019	−0.068085	−0.015399	−0.058439	−0.014053
7	−0.0990498931	7.28_{-7}	−0.099049	−0.099199	−0.023360	−0.089857	−0.045083
9	−0.1298876683	5.79_{-7}	−0.129887	−0.130158	−0.032796	−0.121194	−0.075921
27	−0.3900638141	2.53_{-7}	−0.390063	−0.392716	−0.168432	−0.387936	−0.336218
29	−0.4162316016	2.05_{-7}	−0.416231	−0.419230	−0.187449	−0.414778	−0.362463
31	−0.4417276182	1.71_{-7}	−0.441727	−0.445063	−0.206944	−0.440899	−0.388074
33	−0.4665295588	1.49_{-7}	−0.466529	−0.470208	−0.226840	−0.466274	−0.413043
35	−0.4906189456	1.30_{-7}	−0.490618	−0.494646	−0.247065	−0.490881	−0.437369
55	−0.6907427887	2.57_{-8}	−0.690742	−0.697478	−0.453650	−0.693486	−0.651564
57	−0.7067322567	3.28_{-8}	−0.706732	−0.713630	−0.473606	−0.709521	−0.671453
59	−0.7220308937	8.46_{-8}	−0.722030	−0.729056	−0.493271	−0.724843	−0.691471
61	−0.7366545782	1.55_{-7}	−0.736654	−0.743775	−0.512612	−0.739469	−0.711761
63	−0.7506204287	1.27_{-7}	−0.750620	−0.757808	−0.531603	−0.753420	−0.732485

Table 2. The comparison of numerical results in Test problem 5.1 for $N = 8$ and various $x \in [0, 1]$ at time $t = 0.001$.

$\frac{x}{64}$	**Present**	**AE**	**BCF [11]**	**HWQA [24]**	**ADM [4]**	**HPM [5]**	**HPTM [10]**
1	−0.0146242559	2.99_{-7}	−0.014624	−0.014624	−0.000433	−0.013626	−0.039344
3	−0.0458432514	4.06_{-7}	−0.045843	−0.045847	−0.002700	−0.044858	−0.008125
5	−0.0769726894	2.46_{-7}	−0.076972	−0.076984	−0.006885	−0.076014	−0.023004
7	−0.1079527574	6.67_{-8}	−0.107953	−0.107973	−0.012948	−0.107033	−0.053984
9	−0.1387246409	4.28_{-8}	−0.138725	−0.138769	−0.020837	−0.137855	−0.084756
27	−0.3976673000	1.64_{-8}	−0.397667	−0.398034	−0.160042	−0.397455	−0.343819
29	−0.4236441749	1.58_{-8}	−0.423644	−0.423939	−0.181144	−0.423499	−0.369872
31	−0.4489424537	6.57_{-9}	−0.448942	−0.449301	−0.202947	−0.448860	−0.395283
33	−0.4735409398	6.06_{-9}	−0.473541	−0.473964	−0.225333	−0.473515	−0.420046
35	−0.4974222452	1.56_{-8}	−0.497422	−0.497826	−0.248187	−0.497448	−0.444159
55	−0.6954195363	5.22_{-8}	−0.695419	−0.696107	−0.480881	−0.695694	−0.655872
57	−0.7112084249	3.79_{-8}	−0.711208	−0.711883	−0.502829	−0.711487	−0.675449
59	−0.7263108975	1.93_{-7}	−0.726311	−0.726999	−0.524303	−0.726592	−0.695130
61	−0.7407432348	3.37_{-7}	−0.740744	−0.741464	−0.545268	−0.741025	−0.715055
63	−0.7545229508	2.54_{-7}	−0.754523	−0.755247	−0.565692	−0.754803	−0.735378

$$w_0(x) = (1 + e^x)^{-1} - \frac{1}{2}.$$

It is shown that the exact solution is given by $w(x,t) = (1 + e^{x-\frac{t}{4}})^{-1} - \frac{1}{2}$.

For this test problem, we consider $T = 0.1$, $\Delta t = 0.001$, and $N = 5$. The snapshots of numerical solutions at different time instants $t = s\Delta t$, $s = 1, 2, \ldots, 100$ are shown in Figure 3. In addition, the corresponding AE are also plotted in Figure 4. In this case, the approximate solutions at $t = \Delta t$ and $t = T$ are obtained as follows:

$$\mathcal{W}_{1,5}(x) = -0.001109924646\, x^5 - 0.001404371504\, x^4$$

$$+0.02164565497\, x^3 - 0.0002217269879\, x^2$$

$$-0.2499815546\, x + 0.00006249999967,$$

Table 3. The comparison of L_2 and L_∞ error norms in Test problem 5.2 for diverse $N = 4, 5, \dots, 8$, $\Delta t = 0.001, 0.01, 0.1$ evaluated at the final times $t = T$ with $T = 0.1, 0.5, 1$.

	$\Delta t = 0.001$				$\Delta t = 0.01$		$\Delta t = 0.1$	
	$T = 0.1$		$T = 0.5$		$T = 1$		$T = 1$	
N	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2
4	6.137_{-6}	1.533_{-6}	1.534_{-4}	2.034_{-5}	6.512_{-4}	3.349_{-4}	5.859_{-4}	2.966_{-4}
5	6.264_{-6}	2.606_{-7}	1.607_{-4}	4.955_{-5}	1.133_{-3}	5.212_{-4}	9.678_{-4}	4.496_{-4}
6	5.918_{-6}	2.316_{-7}	1.391_{-4}	5.211_{-5}	1.577_{-3}	7.531_{-4}	9.771_{-4}	5.598_{-4}
7	6.033_{-6}	6.316_{-8}	2.164_{-4}	7.232_{-5}	3.833_{-3}	1.668_{-3}	2.514_{-3}	9.270_{-4}
8	6.076_{-6}	2.276_{-7}	3.382_{-4}	9.884_{-5}	1.249_{-3}	4.766_{-3}	1.919_{-3}	1.041_{-3}

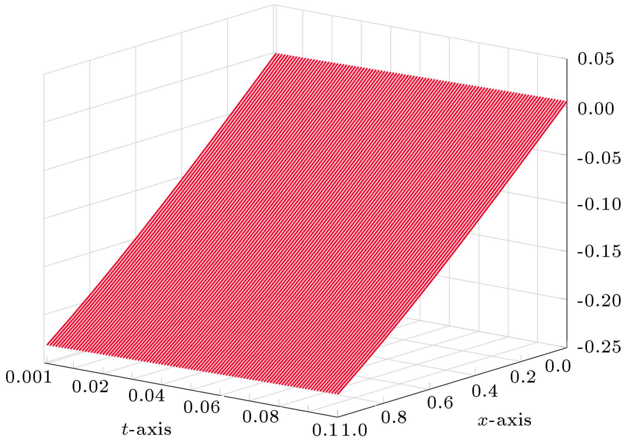

Figure 3. Graphs of numerical solutions in Test problem 5.2 at different time instants $t = s\Delta t$, $s = 1, 2, \dots, 100$ for $\Delta t = 0.001$, $T = 0.1$, and $N = 5$.

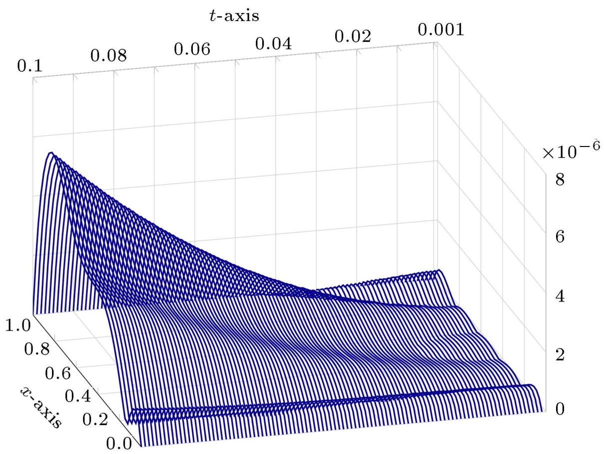

Figure 4. Graphs of absolute errors in Test problem 5.2 at different time instants $t = s\Delta t$, $s = 1, 2, \dots, 100$ for $\Delta t = 0.001$, $T = 0.1$, and $N = 5$.

$$\mathcal{W}_{100,5}(x) = -0.000731548639\,x^5 - 0.002111324555\,x^4 + 0.0224345299\,x^3 - 0.002035416473\,x^2 - 0.249920896\,x + 0.0062496745.$$

We next compute the maximum AE which are denoted by L_∞ and L_2 error norms evaluated at the final time $t = T$ via:

$$L_\infty := \max_{0\le x\le 1} |w(x,T) - \mathcal{W}_{M+1,N}(x)|,$$

$$L_2 := \left(\frac{1}{N+1}\int_0^1 [w(x,T) - \mathcal{W}_{M+1,N}(x)]^2 dx\right)^{\frac{1}{2}}.$$

We utilize various $N = 4, 5, \dots, 8$ and report the results of errors in Table 3. Also, different final times $T = 0.1, 0.5$, and $T = 1$ are used with the step sizes $\Delta t = 0.001, 0.001$, and $T = 0.1$.

Finally, our numerical results and computations are verified through a comparison with well-established numerical models and simulations. Tables 4 and 5 show these comparisons with the methods used in Tables 1 and 2 in Test problem 5.1. However, in Tables 4 and 5, the Laplace Decomposition Method (LDM) [25] is employed rather than HPM.

6. Conclusions

In this work, a space and time-accurate approximation technique was presented to solve the foam drainage equation. For the temporal discretization, the Taylor series expansion approach with order $O(\Delta t^2)$ was employed. Afterwards, at each time step, the novel Bessel based collocation approach with exponential accuracy was utilized to approximate the space variable. By using the matrix representations of these polynomials in conjunction with the collocation points, the scheme converts the underlying model problem into an algebraic linear system of equations. The utility and accuracy of the presented technique were examined by using numerical experiments. Comparisons with earlier computational and experimental studies were also made. The presented results demonstrated the reliability and the applicability of the presented combined algorithm for the nonlinear time-dependent foam drainage equation. The combined technique with

Table 4. The comparison of numerical results in Test problem 5.2 for $N = 8$ and various $x \in [0, 1]$ at time $t = 0.1$.

$\frac{x}{64}$	Present	AE	BCF [11]	HWQA [24]	ADM [4]	HPM [5]	HPTM [10]
1	+0.0023435127	2.20_{-7}	+0.002344	+0.002344	+0.002098	+0.002083	+0.002084
3	−0.0054690211	4.89_{-7}	−0.005469	−0.005468	−0.005793	−0.005785	−0.005787
5	−0.0132787383	6.11_{-7}	−0.013278	−0.013278	−0.013664	−0.013650	−0.013651
7	−0.0210818974	6.53_{-7}	−0.021081	−0.021081	−0.021512	−0.021508	−0.021509
9	−0.0288747443	6.56_{-7}	−0.028874	−0.028874	−0.029357	−0.029354	−0.029353
27	−0.0979371596	5.45_{-7}	−0.097937	−0.097935	−0.098832	−0.098830	−0.098831
29	−0.1054263741	5.27_{-7}	−0.105426	−0.105424	−0.106358	−0.106357	−0.106358
31	−0.1128664061	5.08_{-7}	−0.112866	−0.112863	−0.113834	−0.113833	−0.113833
33	−0.1202541435	4.88_{-7}	−0.120254	−0.120251	−0.121255	−0.121254	−0.121254
35	−0.1275865681	4.69_{-7}	−0.127586	−0.127583	−0.128620	−0.128619	−0.128619
55	−0.1972794716	2.66_{-7}	−0.197279	−0.197274	−0.198532	−0.198532	−0.198532
57	−0.2038347870	2.50_{-7}	−0.203834	−0.203830	−0.205099	−0.205099	−0.205099
59	−0.2103071180	2.27_{-7}	−0.210307	−0.210304	−0.211582	−0.211582	−0.211582
61	−0.2166949149	1.81_{-7}	−0.216695	−0.216691	−0.217979	−0.217979	−0.217979
63	−0.2229967352	8.29_{-8}	−0.222997	−0.222994	−0.224288	−0.224288	−0.224288

Table 5. The comparison of numerical results in Test problem 5.2 for $N = 8$ and various $x \in [0, 1]$ at time $t = 0.01$.

$\frac{x}{64}$	Present	AE	BCF [11]	HWQA [24]	ADM [4]	HPM [5]	HPTM [10]
1	−0.0032812030	1.18_{-10}	−0.003281	−0.003281	−0.003309	−0.003307	−0.003307
3	−0.0110919303	3.50_{-10}	−0.011092	−0.011091	−0.011126	−0.011123	−0.011123
5	−0.0188972450	5.21_{-10}	−0.018897	−0.018897	−0.018939	−0.018935	−0.018934
7	−0.0266933472	6.16_{-10}	−0.026693	−0.026693	−0.026741	−0.026737	−0.026736
9	−0.0344764548	6.52_{-10}	−0.034476	−0.034476	−0.034529	−0.034525	−0.034524
27	−0.1033336837	5.28_{-10}	−0.103334	−0.103333	−0.103430	−0.103424	−0.103423
29	−0.1107878235	5.10_{-10}	−0.110788	−0.110787	−0.110888	−0.110883	−0.110880
31	−0.1181905313	4.91_{-10}	−0.118191	−0.118190	−0.118297	−0.118289	−0.118287
33	−0.1255387617	4.71_{-10}	−0.125539	−0.125538	−0.125649	−0.125641	−0.125638
35	−0.1328295665	4.52_{-10}	−0.132830	−0.132829	−0.132943	−0.132932	−0.132932
55	−0.2020073393	2.59_{-10}	−0.202007	−0.202007	−0.202149	−0.202136	−0.202132
57	−0.2085030871	2.27_{-10}	−0.208503	−0.208502	−0.208650	−0.208636	−0.208629
59	−0.2149147407	1.71_{-10}	−0.214915	−0.214914	−0.215065	−0.215047	−0.215041
61	−0.2212408533	9.18_{-11}	−0.221241	−0.221240	−0.221389	−0.221376	−0.221368
63	−0.2274800989	1.51_{-11}	−0.227480	−0.227480	−0.227639	−0.227618	−0.227608

inherited simplicity and ease of implementation can be easily extended to other nonlinear model problems in diverse disciplines of engineering and sciences.

Nomenclature

AE	Absolute Errors
BCF	Bivariate Chebyshev Functions
ADM	Adomian Decomposition Method
HPM	Homotopy Perturbation Method
HWQA	Haar Wavelet Quasilinearization Approach
HPTM	Homotopy Perturbation Transform Method
LDM	Laplace Decomposition Method

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Biography

Mohammad Izadi received his PhD degree from Leipzig University (2008–2012) in the group of "Scientific Computing" at Max-Planck Institute for Mathematics in the Sciences, Leipzig, Germany. Since 2013, he has served as an Assistant Professor at the Department of Applied Mathematics at Shahid Bahonar University of Kerman and since March 2021 is an Associated Professor. His interest areas include numerical analysis, numerical methods for (fractional) ordinary and partial differential equations, and spectral methods.