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Novel decision-making framework based on complex q-rung orthopair fuzzy information

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Abstract. Assessing uncertainty in decision-making is a major challenge for Decision-Makers (DMs), and the q-Rung Orthopair Fuzzy Set (q-ROFS) as the direct extension of Intuitionistic Fuzzy Set (IFS) and Pythagorean Fuzzy Set (PFS) play a crucial role in this aspect. The Complex q-Run Orthopair Fuzzy Set (Cq-ROFS) is a strong tool to deal with imprecision, vagueness, and fuzziness by expanding the scope of Membership Degree (MD) and Non-Membership Degree (NMD) of q-ROFS from real to complex unit disc. In this paper, we develop some new Cq-ROF Hamacher Aggregation Operators (AOs), i.e., the Cq-ROF Hamacher Weighted Averaging (Cq-ROFHWA) operator, the Cq-ROFH Weighted Geometric (Cq-ROFHWG) operator, the Cq-ROFH Ordered Weighted Averaging (Cq-ROFHOWA) operator, and the Cq-ROFH Ordered Weighted Geometric (Cq-ROFHOWG) operator. Subsequently, we establish a novel Cq-ROF graph framework based on the Hamacher operator called Cq-ROFH Graphs (Cq-ROFHGs) and evaluate its energy and Randić energy. In particular, we compute the energy of a splitting Cq-ROFHG and shadow Cq-ROFHG. Further, we describe the notions of Cq-ROFH digraphs (Cq-ROFHDGs). Moreover, an algorithm is given to solve Multiple Attribute Group Decision-Making (MAGDM) problems and the main steps are discussed clearly. Finally, a numerical instance related to the Facade Clothing Systems (FCS) selection is presented to show the effectiveness of the developed concepts in decision-making circumstances. In order to verify the effectiveness of our proposed scheme, a comparative analysis with previous approaches is provided.

1. Introduction

In the real world, being a complex cognitive computing method, Multiple Attribute Decision-Making (MADM) intends to make scientific decisions from a finite number of alternatives by utilizing the multiple attributes perspective and then, to rank a series of alternatives or pursue the appropriate one through effective information aggregation rules and decision analysis tools. A lot of soft computing methods have been used for MADM research over the last decades, and most of them are addressed in the form of generalized Fuzzy Sets (FSs) [1–5]. Multiple Attribute Group Decision-Making (MAGDM), an integrative research field that combines group decision-making with MADM, usually provides structures for acquiring group preference in-

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formation through individual preference information and specifically evaluating various alternatives through different theoretical decision-making templates. For some MAGDM problems, Decision-Makers (DMs) experience problems in depicting attribute values of alternatives by utilizing crisp numbers. To describe the uncertainties, a novel notion of FS was initiated by Zadeh [6] whose element has only Membership Degree (MD) in [0, 1]. Further, Intuitionistic Fuzzy Sets (IFSs) [7], Pythagorean Fuzzy Set (PFSs) [8], and Fermatean FSs (FFSs) [9], whose elements are pairs of fuzzy numbers, were proposed. Each of them demonstrates the MD and Non-Membership Degree (NMD). The restriction of MD and NMD is that the sum and square sum of both belongs to [0, 1]. Yager revealed that the current frameworks of IFSs and PFSs are not capable enough to represent human opinions in a more realistic setting and has invented the concept of q-Rung Orthopair Fuzzy Set (q-ROFS) [10], which effectively broadens the scope of information by developing a new subjective constraint, where the qth sum of MD and NMD belongs to [0, 1]. If $q = 3$, $q = 2$, and $q = 1$, then the q-ROFS is reduced to the FFS, PFS, and IFS, respectively.

If DMs change the codomain of FSs from [0, 1] to unit disc, then to tackle certain type of problems, Ramot et al. [11] designed the idea of Complex Fuzzy Sets (CFSs) expressed by complex-valued mapping with codomain as a unit circle in the complex plane. Moreover, to represent the complex-valued NMD, Alkouri and Salleh [12,13] generalized the idea of CFS to Complex IFSs (CIFSs) and proposed the concept of CIF relations and a distance measure in CIF circumstances. Further, Garg and Rani set forward the CIF robust averaging-geometric AOs [14], explored certain series of distance measures between the two CIFSs [15], Archimedean t-norm and t-conorm-based generalized CIF Bonferroni mean AOs [16], exponential, logarithmic, and compensative generalized AOs with CIF information [17], CIF power AOs [18], and presented their applications in the field of decision-making. The idea of q-ROFS deals with only one aspect at a time, which sometimes causes data loss. In real life, however, we notice complex natural phenomena in which measuring the second dimension of the expression of the MD and NMD becomes essential. Complete facts are projected into a collection by creating the second dimension, which prevents any loss of information. Liu et al. [19] put forward an efficient and powerful tool to express unclear anomalies, called Complex q-Rung Orthopair Fuzzy Set (Cq-ROFSs), and developed the weighted averaging operator and weighted geometric operator based on Cq-ROF circumstances. The amplitude term represents the degree to which an object belongs in a Cq-ROFS and the phase terms are usually related to periodicity. The Complex Set (CS) and traditional q-ROFS theories are differentiated by certain phase terms. Garg et al. [20] presented multiple forms of operators such as power averaging, power weighted averaging, power hybrid averaging, power geometric, power weighted geometric, and power hybrid geometric operator in the context of Cq-ROFSs. Liu et al. [21] introduced the concept of Cq-RO Linguistic (Cq-ROL) sets and developed operators like the Cq-ROL Heronian mean, Cq-ROL weighted Heronian mean, Cq-ROL geometric Heronian mean, and Cq-ROL weighted geometric Heronian mean operator. Cq-ROFSs are incredibly versatile and efficient as compared to many current FSs theories.

Graphs can be used to design numerous types of relations and methodologies of physical, biological, social, and information technology and have a massive variety of valuable applications. Graph theory is eventually the study of relationships and offers a useful resource to quantify and simplify several components of dynamic systems. Studying graphs in a system offers responses to a broad variety of configurations, networking, optimization, matching, and operational issues. Graph properties in relation to the characteristic polynomial and matrix values associated with the graph are studied in spectral graph theory, such as its adjacency matrix, Harmonic matrix, Zagreb matrix, and geometric-arithmetic matrix. The idea of the graph energy was set up by Gutman [22] and lower and upper limits were explored. Vaidya and Popat [23] proposed the energy of splitting and shadow graphs. The Randić Matrix (RM)$R(\mathcal{G}) = (\mathfrak{y}_{ij})$ of a graph of a graph $\mathcal{G}$ whose vertex $\mathfrak{y}_i$ has degree d_i is defined by $\mathfrak{y}_{ij} = \frac{1}{\sqrt{d_i d_j}}$ if the vertices $\mathfrak{y}_i$ and $\mathfrak{y}_j$ are adjacent and $\mathfrak{y}_{ij} = 0$, otherwise. The sum of absolute values of the eigenvalues of $R(\mathcal{G})$ is Randić Energy (RE). Graph vertices and edges uncertainties are typical in this context due to a number of factors, such as noise measurements and conflicting sources of information. In order to deal with such complexities in objects and connections, Rosenfeld [24] originated the concept of Fuzzy Graph (FG) and established its framework. Anjali and Mathew [25] set forward the energy of an FG. Akram et al. [26–28] developed several novel concepts of graphs in generalized fuzzy circumstances. Thirunavukarasu et al. [29] determined the energy of Complex FGs (CFGs). Luqman et al. developed the concepts of complex fuzzy hypergraphs [30] and complex neutrosophic hypergraphs [31]. Naz et al. put forward the concepts of Pythagorean Fuzzy Graphs (PFGs) [32] and complex PFGs [33] as well as their pertinent applications in decision-making. Habib et al. [34] designed a new definition of q-ROF Graphs (q-ROFGs) and presented its use in the soil ecosystem. Yin et al. [35] elaborated on some product operations on q-ROFGs. Further, Akram et al. [36] presented q-rung orthopair fuzzy graphs under Hamacher operators. Moreover, Guleria and Bajaj [37] designed the notion of T-spherical FGs along with the operations. Recently, Naz et al. [38,39] extended q-ROFGs to the dual hesitant q-ROF scenario and proposed several types of energy like geometric arithmetic energy, atom bond connectivity energy, Zegrab energy, and harmonic energy.

Information AO plays a significant role in the decision-making process, especially in MADM. In 1978, Hamacher [40] introduced the Hamacher operations like Hamacher product and Hamacher sum, which are more and more general and flexible than the algebraic and Einstein product and sum. Inspired by the theory of q-ROFGs, it is essential to expand q-ROFG to Complex q-ROFG (Cq-ROFG), since Cq-ROFG is a powerful concept for dealing with uncertain and unpredictable information and is also a general form of FGs, whose restriction is quite similar to q-ROFG. However, the MD and NMD range are bound to unit disc in a complex plane rather than [0, 1]. Moreover, Hamacher operators are more flexible and parameterized. Thus, we define the Cq-ROF relations and put forward the innovative concept of Cq-ROFGs utilizing Hamacher operator. The newly proposed Cq-ROF Hamacher Graphs (Cq-ROFHGs) are extremely versatile and efficient and can coordinate the expert decision-making opinions in a complex state compared to many existing FSs theories. We also establish the energy and RE of the developed Cq-ROFHGs and Cq-ROF Hamacher digraphs (Cq-ROFHDGs) and provide their pertinent application in MAGDM.

The format of the paper is as follows. Section 2 reviews some fundamental concepts of Cq-ROFSs. Section 3 puts forward some Cq-ROF Hamacher AOs. Section 4 proposes the innovative idea of Cq-ROFHGs and Cq-ROFHDGs, and examines their energy. The concept of splitting Cq-ROFHG and shadow Cq-ROFHG with their energies are also discussed in this section. Section 5 refers to RE of Cq-ROFHGs and CqROFHDGs. Further, in Section 6, a novel MAGDM approach is established based on energy and RE of Cq-ROFHDGs. In Section 7, a case study and an appropriate comparative analysis are discussed to illustrate the usefulness and effectiveness of the established ideas of Cq-ROFHGs in decision-making. Finally, Section 8 concludes the entire paper and points out several future research topics. The graphical interpretation of the paper is given in Figure 1.

2. Preliminaries

In this section, the basic notions like Cq-ROFSs along the operations and t-norms are reviewed for a better understanding in the next sections.

Definition 1 [41]. A Cq-ROFS $\pounds$ is defined as:

$$\pounds = \{(\mathfrak{s}, \breve{\tau}_{\pounds}(\mathfrak{s}), \breve{\upsilon}_{\pounds}(\mathfrak{s})) : \mathfrak{s} \in R\},$$

where $\breve{\tau}_{\pounds}$,$\breve{\upsilon}_{\pounds}$:$R \longrightarrow !\{c : c \in C, |c| \leq 1\}$ are the complex-valued membership and non-membership functions, respectively, and defined as:

$$\breve{\tau}_{\pounds}(\mathfrak{s}) = \breve{\wp}_{\pounds}(\mathfrak{s})e^{i2\pi\omega_{\breve{\wp}_{\pounds}}(\mathfrak{s})}; \breve{\upsilon}_{\pounds}(\mathfrak{s}) = \breve{\Re}_{\pounds}(\mathfrak{s})e^{i2\pi\omega_{\breve{\Re}_{\pounds}}(\mathfrak{s})},$$

where $0 \leq \breve{\wp}_{\pounds}(\mathfrak{s})$, $\breve{\Re}_{\pounds}(\mathfrak{s})$, $\breve{\wp}^q_{\pounds}(\mathfrak{s})+ \breve{\Re}^q_{\pounds}(\mathfrak{s}) \leq 1$ and $0 \leq \omega^q_{\breve{\wp}_{\pounds}}(\mathfrak{s}), \omega^q_{\breve{\Re}_{\pounds}}(\mathfrak{s})$, $\omega^q_{\breve{\wp}_{\pounds}}(\mathfrak{s})+ \omega^q_{\breve{\Re}_{\pounds}}(\mathfrak{s}) \leq 1$. Further, $\breve{\pi}_{\pounds}(\mathfrak{s})= (1 - (\breve{\wp}^q_{\pounds}(\mathfrak{s})+ \breve{\Re}^q_{\pounds}(\mathfrak{s})))^{\frac{1}{q}}$ and $\omega_{\breve{\pi}_{\pounds}}(\mathfrak{s})= (1 - (\omega^q_{\breve{\wp}_{\pounds}}(\mathfrak{s})+ \omega^q_{\breve{\Re}_{\pounds}}(\mathfrak{s})))^{\frac{1}{q}}$ are complex hesitancy degree of $\mathfrak{s}$. For simplicity, the pair $\breve{\mathfrak{I}} = ((\breve{\wp}, \omega_{\breve{\wp}}), (\breve{\Re}, \omega_{\breve{\Re}}))$ is called the Cq-ROF number (Cq-ROFN), where $0 \leq \breve{\wp}, \breve{\Re}, \breve{\wp}^q+ \breve{\Re}^q \leq 1$, and $0 \leq \omega_{\breve{\wp}}$,$\omega_{\breve{\Re}}$, $\omega^q_{\breve{\wp}}+\omega^q_{\breve{\Re}} \leq 1$.

Definition 2 [41]. Let $\pounds$={($\mathfrak{s}$, ($\breve{\wp}_{\pounds}(\mathfrak{s})$,$\omega_{\breve{\wp}_{\pounds}}(\mathfrak{s})$),($\breve{\Re}_{\pounds}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds}}(\mathfrak{s})$)): $\mathfrak{s}\in R$}, $\pounds_1$={($\mathfrak{s}$, ($\breve{\wp}_{\pounds_1}(\mathfrak{s})$, $\omega_{\breve{\wp}_{\pounds_1}}(\mathfrak{s})$), ($\breve{\Re}_{\pounds_1}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds_1}}(\mathfrak{s})$)): $\mathfrak{s} \in R$}, and $\pounds_2$={($\mathfrak{s}$, ($\breve{\wp}_{\pounds_2}(\mathfrak{s})$, $\omega_{\breve{\wp}_{\pounds_2}}(\mathfrak{s})$), ($\breve{\Re}_{\pounds_2}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds_2}}(\mathfrak{s})$)): $\mathfrak{s} \in R$} be the Cq-ROFSs in R, then:

(i) $\pounds_1 \subseteq \pounds_2$ if and only if $\breve{\wp}_{\pounds_1}(\mathfrak{s}) \leq \breve{\wp}_{\pounds_2}(\mathfrak{s}), \breve{\Re}_{\pounds_1}(\mathfrak{s}) \geq \breve{\Re}_{\pounds_2}(\mathfrak{s})$ for amplitude terms and $\omega_{\breve{\wp}_{\pounds_1}}(\mathfrak{s}) \leq \omega_{\breve{\wp}_{\pounds_2}}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds_1}}(\mathfrak{s}) \geq \omega_{\breve{\Re}_{\pounds_2}}(\mathfrak{s})$ for phase terms, for all $\mathfrak{s} \in R$;

(ii) $\pounds_1 = \pounds_2$ if and only if $\breve{\wp}_{\pounds_1}(\mathfrak{s})= \breve{\wp}_{\pounds_2}(\mathfrak{s})$, $\breve{\Re}_{\pounds_1}(\mathfrak{s})= \breve{\Re}_{\pounds_2}(\mathfrak{s})$ for amplitude terms and $\omega_{\breve{\wp}_{\pounds_1}}(\mathfrak{s})= \omega_{\breve{\wp}_{\pounds_2}}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds_1}}(\mathfrak{s})= \omega_{\breve{\Re}_{\pounds_2}}(\mathfrak{s})$ for phase terms, for all $\mathfrak{s}\in R$;

(iii) $\overline{\pounds}$= {($\mathfrak{s}$,($\breve{\Re}_{\pounds}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds}}(\mathfrak{s})$), ($\breve{\wp}_{\pounds}(\mathfrak{s})$, $\omega_{\breve{\wp}_{\pounds}}(\mathfrak{s})$)): $\mathfrak{s} \in R$}.

Definition 3 [42]. Let: $\pounds$={($\mathfrak{s}$, ($\breve{\wp}_{\pounds}(\mathfrak{s})$,$\omega_{\breve{\wp}_{\pounds}}(\mathfrak{s})$), ($\breve{\Re}_{\pounds}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds}}(\mathfrak{s})$)): $\mathfrak{s} \in R$}, $\pounds_1$={($\mathfrak{s}$,($\breve{\wp}_{\pounds_1}(\mathfrak{s})$, $\omega_{\breve{\wp}_{\pounds_1}}(\mathfrak{s})$), ($\breve{\Re}_{\pounds_1}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds_1}}(\mathfrak{s})$)): $\mathfrak{s}\in R$}, and $\pounds_2$={($\mathfrak{s}$,($\breve{\wp}_{\pounds_2}(\mathfrak{s})$,$\omega_{\breve{\wp}_{\pounds_2}}(\mathfrak{s})$), ($\breve{\Re}_{\pounds_2}(\mathfrak{s})$, $\omega_{\breve{\Re}_{\pounds_2}}(\mathfrak{s})$)): $\mathfrak{s}\in R$} be the Cq-ROFSs in R, then:

1. $\pounds_1 \oplus \pounds_2= \left(\left(\sqrt[q]{\breve{\wp}^q_{\pounds_1} + \breve{\wp}^q_{\pounds_2} - \breve{\wp}^q_{\pounds_1}\breve{\wp}^q_{\pounds_2}}, \sqrt[q]{\omega^q_{\breve{\wp}_{\pounds_1}} + \omega^q_{\breve{\wp}_{\pounds_2}} - \omega^q_{\breve{\wp}_{\pounds_1}}\omega^q_{\breve{\wp}_{\pounds_2}}}\right), (\breve{\Re}_{\pounds_1} \breve{\Re}_{\pounds_2}, \omega_{\breve{\Re}_{\pounds_1}} \omega_{\breve{\Re}_{\pounds_2}})\right)$;

2. $\pounds_1 \otimes \pounds_2= \left((\breve{\wp}_{\pounds_1}\breve{\wp}_{\pounds_2}, \omega_{\breve{\wp}_{\pounds_1}}\omega_{\breve{\wp}_{\pounds_2}}), \left(\sqrt[q]{\breve{\Re}^q_{\pounds_1} + \breve{\Re}^q_{\pounds_2} - \breve{\Re}^q_{\pounds_1}\breve{\Re}^q_{\pounds_2}}, \sqrt[q]{\omega^q_{\breve{\Re}_{\pounds_1}} + \omega^q_{\breve{\Re}_{\pounds_2}} - \omega^q_{\breve{\Re}_{\pounds_1}}\omega^q_{\breve{\Re}_{\pounds_2}}}\right)\right)$;

3. $\lambda\pounds= \left(\left(\sqrt[q]{1-(1-\breve{\wp}^q_{\pounds})^{\lambda}}, \sqrt[q]{1-(1-\omega^q_{\breve{\wp}_{\pounds}})^{\lambda}}\right), \left((\breve{\Re}_{\pounds})^{\lambda}, (\omega_{\breve{\Re}_{\pounds}})^{\lambda}\right)\right)$, $\lambda > 0$;

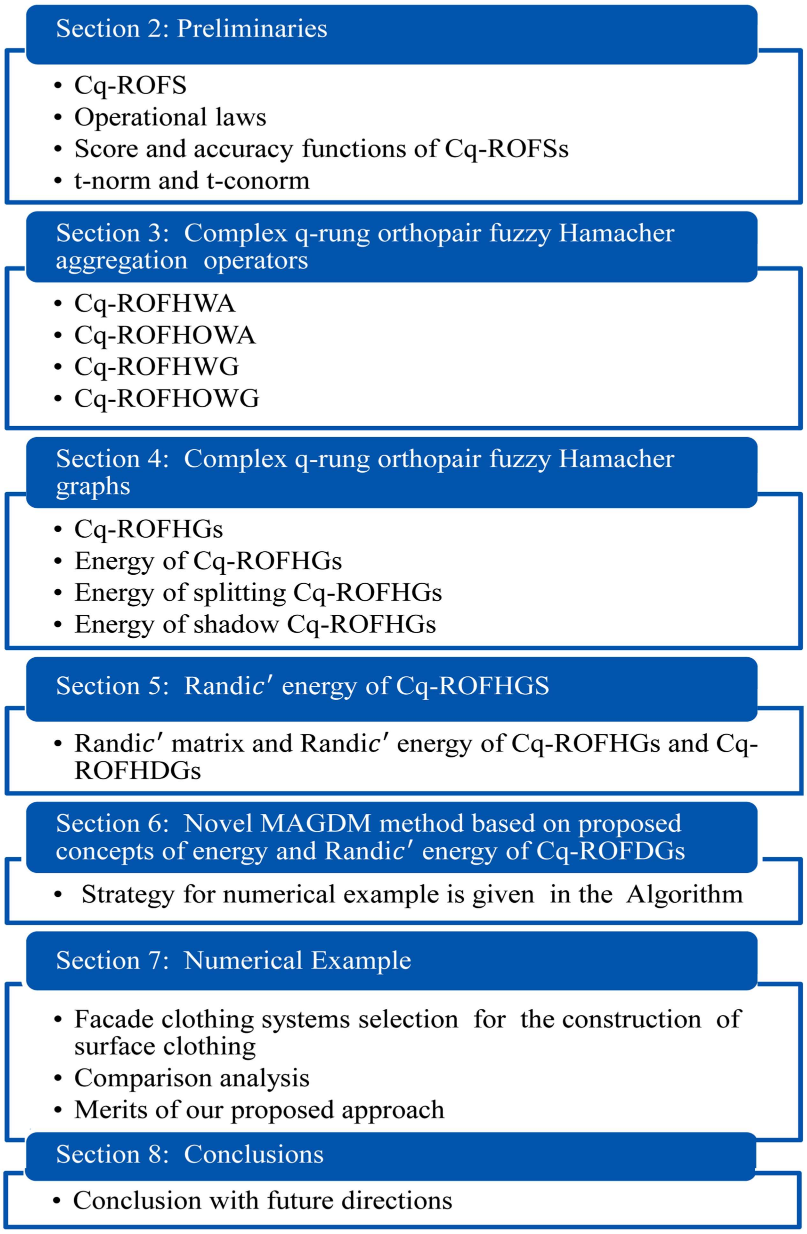

Figure 1. Graphical representation of the paper.

4. $\pounds^{\lambda}=\left(((\breve{\wp}_{\pounds})^{\lambda}, (\omega_{\breve{\wp}_{\pounds}})^{\lambda}), \left(\sqrt[q]{1-(1-\breve{\Re}^q_{\pounds})^{\lambda}}, \sqrt[q]{1-(1-\omega^q_{\breve{\Re}_{\pounds}})^{\lambda}}\right)\right), \lambda > 0.$

Example 1. Suppose that a fixed set R has only one element $\mathfrak{s}, \breve{\wp}_{\pounds}(\mathfrak{s}) = 0.5$, $\omega_{\breve{\wp}_{\pounds}}(\mathfrak{s}) = 0.8$, $\breve{\Re}_{\pounds}(\mathfrak{s}) = 0.8$, $\omega_{\breve{\Re}_{\pounds}}(\mathfrak{s}) = 0.9$. Then $\breve{\gimel}=\{(\mathfrak{s},(0.5, 0.8),(0.8, 0.9))\}$ is a C5-ROFN, represented as $\breve{\gimel}=((0.5, 0.8),(0.8, 0.9))$ for simplicity.

Definition 4. The score function $\digamma$ and the accuracy function $\mathfrak{H}$ of a Cq-ROFN $\breve{\gimel}=((\breve{\wp},\omega_{\breve{\wp}}), (\breve{\Re},\omega_{\breve{\Re}}))$, are defined as $\digamma(\breve{\gimel}) = \frac{1}{4}((1 + \breve{\wp}^q - \breve{\Re}^q) + (1 + \omega^q_{\breve{\wp}} - \omega^q_{\breve{\Re}}))$, $\digamma(\breve{\gimel}) \in [0, 1]$ and $\mathfrak{H}(\breve{\gimel})= (\breve{\wp}^q + \breve{\Re}^q)+ (\omega^q_{\breve{\wp}}+ \omega^q_{\breve{\Re}})$, $\mathfrak{H}(\breve{\gimel})\in[0, 1]$, respectively.

Definition 5. Let $\breve{\gimel}_1 = ((\breve{\wp}_1, \omega_{\breve{w}p_1}), (\breve{\Re}_1, \omega_{\breve{\Re}_1}))$ and $\breve{\gimel}_2 = ((\breve{\wp}_2, \omega_{\breve{\wp}_2}), (\breve{\Re}_2, \omega_{\breve{\Re}_2}))$ be two Cq-ROFNs. Then we have:

1. If $\digamma(\breve{\gimel}_1) > \digamma(\breve{\gimel}_2)$, then $\breve{\gimel}_1 \succ \breve{\gimel}_2$;
2. If $\digamma(\breve{\gimel}_1) = \digamma(\breve{\gimel}_2)$, then:
   - If $\mathfrak{H}(\breve{\gimel}_1) > \mathfrak{H}(\breve{\gimel}_2)$, then $\breve{\gimel}_1 \succ \breve{\gimel}_2$;
   - If $\mathfrak{H}(\breve{\gimel}_1) = \mathfrak{H}(\breve{\gimel}_2)$, then $\breve{\gimel}_1 = \breve{\gimel}_2$.

To extend the existing t-norm (TN) and t-conorm (TCN) operations,the Hamacher product and the Hamacher sum defined by Hamacher [40] are as follows:

$$\mathfrak{T}_{\aleph}^{H}(\mathfrak{s},\mathfrak{t})=\begin{cases}\frac{\mathfrak{st}}{\aleph+(1-\aleph)(\mathfrak{s}+\mathfrak{t}-\mathfrak{st})} & \text{if } \aleph>0,\\ \frac{\mathfrak{st}}{\mathfrak{s}+\mathfrak{t}-\mathfrak{st}} & \text{if } \aleph=0,\end{cases}$$

$$(\mathfrak{T}^{*})_{\aleph}^{H}(\mathfrak{s},\mathfrak{t})=\begin{cases}\frac{\mathfrak{s}+\mathfrak{t}-\mathfrak{st}-(1-\aleph)\mathfrak{st}}{1-(1-\aleph)\mathfrak{st}} & \text{if } \aleph>0,\\ \frac{\mathfrak{s}+\mathfrak{t}-2\mathfrak{st}}{1-\mathfrak{st}} & \text{if } \aleph=0.\end{cases}$$

Here $P(\mathfrak{s},\mathfrak{t}) \leq \frac{\mathfrak{st}}{\mathfrak{s}+\mathfrak{t}-\mathfrak{st}} \leq M(\mathfrak{s},\mathfrak{t})$ and $M^{*}(\mathfrak{s},\mathfrak{t}) \leq \frac{\mathfrak{s}+\mathfrak{t}-2\mathfrak{st}}{1-\mathfrak{st}} \leq P^{*}(\mathfrak{s},\mathfrak{t})$.

3. Complex q-rung orthopair fuzzy Hamacher aggregation operators

Some Hamacher operations, that is, the Hamacher product and the Hamacher sum of two Cq-ROFNs $\breve{\beth}_1$ and $\breve{\beth}_2$, $\aleph>0$, are defined as follows:

Definition 6. Let $\breve{\beth}$=$((\breve{\wp},\omega_{\breve{\wp}}),(\breve{\Re},\omega_{\breve{\Re}}))$, $\breve{\beth}_1$=$((\breve{\wp}_1,\omega_{\breve{\wp}_1}),(\breve{\Re}_1,\omega_{\breve{\Re}_1}))$, and $\breve{\beth}_2$=$((\breve{\wp}_2,\omega_{\breve{\wp}_2}),(\breve{\Re}_2,\omega_{\breve{\Re}_2}))$ be the Cq-ROFNs; then, their basic Hamacher operations can be defined by equations are shown in Box I. Utilizing the Hamacher operations among the Cq-ROFNs, in this section, we develop the weighted averaging and geometric Hamacher AOs with Cq-ROFS, such as Cq-ROFHWA operator, Cq-ROFH Ordered Weighted Averaging (Cq-ROFHOWA) operator, Cq-ROFH Weighted Geometric (Cq-ROFHWG) operator, and Cq-ROF Hamacher ordered weighted geometric (Cq-ROFHOWG) operator.

Definition 7. Consider $\breve{\beth}_j$=$((\breve{\wp}_j,\omega_{\breve{\wp}_j}),(\breve{\Re}_j,\omega_{\breve{\Re}_j}))(j=1,2,\ldots,n)$ is a collection of Cq-ROFNs; then, the Cq-ROFHWA operator is described as:

$$Cq\text{–}ROFHWA_w(\breve{\beth}_1,\breve{\beth}_2,\ldots,\breve{\beth}_n)=\oplus_{j=1}^{n}(\varpi_j\breve{\beth}_j),$$

where $w=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ can be the $\breve{\beth}_j$ $(j=1,2,\ldots,n)$, weight vector and $\varpi_j>0$, $\sum_{j=1}^{n}\varpi_j=1$.

Theorem 1. Let $\breve{\beth}_j$=$((\breve{\wp}_j,\omega_{\breve{\wp}_j}),(\breve{\Re}_j,\omega_{\breve{\Re}_j}))$ $(j=1,2,\ldots,n)$ be a collection of Cq-ROFNs, where $\aleph>0$. Then, its aggregated value by utilizing Cq-ROFHWA operator is also a Cq-ROFN, and Cq-ROFHWA$_w$ obtained as shown in Box II, when $\aleph=1$, Cq-ROFHWA operator reduces to the Cq-ROF weighted averaging (Cq-ROFWA) operator as follows:

$$Cq-ROFWA_w(\breve{\beth}_1,\breve{\beth}_2,\ldots,\breve{\beth}_n)$$

$$=\left(\left(\sqrt[q]{1-\prod_{j=1}^{n}(1-(\breve{\wp}_j)^q)^{\varpi_j}},\sqrt[q]{1-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\wp}_j})^q\right)^{\varpi_j}}\right),\left(\prod_{j=1}^{n}(\breve{\Re}_j)^{\varpi_j},\prod_{j=1}^{n}(\omega_{\breve{\Re}_j})^{\varpi_j}\right)\right).$$

Definition 8. Let $\breve{\beth}_j=((\breve{\wp}_j,\omega_{\breve{\wp}_j}),(\breve{\Re}_j,\omega_{\breve{\Re}_j}))(j=1,2,\ldots,n)$ be a collection of Cq-ROFNs, then the Cq-ROFHOWA operator is shown in Box III, where $(\daleth(1),\daleth(2),\ldots,\daleth(n))$ is a permutation of $(1,2,\ldots,n)$, such that $\breve{\beth}_{\daleth(j-1)}\geq\breve{\beth}_{\daleth(j)}$ for every $j=2,\ldots,n$, and $w=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ is the weight vector such that $\varpi_j>0$, $\sum_{j=1}^{n}\varpi_j=1,\aleph>0$.

When $\aleph=1$, Cq-ROFHOWA operator reduces to the Cq-ROF Ordered Weighted Averaging (Cq-ROFOWA) operator as follows:

$$Cq-ROFOWA_w(\breve{\beth}_1,\breve{\beth}_2,\ldots,\breve{\beth}_n)=$$

$$\left(\left(\sqrt[q]{1-\prod_{j=1}^{n}\left(1-(\breve{\wp}_{\daleth(j)})^q\right)^{\varpi_j}},\sqrt[q]{1-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\wp}_{\daleth(j)}})^q\right)^{\varpi_j}}\right)\left(\prod_{j=1}^{n}(\breve{\Re}_{\daleth(j)})^{\varpi_j},\prod_{j=1}^{n}(\omega_{\breve{\Re}_{\daleth(j)}})^{\varpi_j}\right)\right).$$

Definition 9. Let $\breve{\beth}_j=((\breve{\wp}_j,\omega_{\breve{\wp}_j}),(\breve{\Re}_j,\omega_{\breve{\Re}_j}))$ $(j=1,2,\ldots,n)$ be a collection of Cq-ROFNs, then the Cq-ROFHWG operator is formally defined as:

$$Cq-ROFHWG_w(\breve{\beth}_1,\breve{\beth}_2,\ldots,\breve{\beth}_n)=\otimes_{j=1}^{n}(\varpi_j\breve{\beth}_j).$$

Theorem 2. Let $\breve{\beth}_j$= $((\breve{\wp}_j,\omega_{\breve{\wp}_j}),(\breve{\Re}_j,\omega_{\breve{\Re}_j}))$ $(j=1,2,\ldots,n)$ be a collection of Cq-ROFNs, where $\aleph>0$. Then its aggregated value by utilizing Cq-ROFHWG operator is also a Cq-ROFN, and Cq-ROFWG obtained as shown in Box IV, when $\aleph=1$, the Cq-ROFHWG operator is converted into the Cq-ROF Weighted Geometric (Cq-ROFWG) operator as follows:

$$Cq-ROFWG_w(\breve{\beth}_1,\breve{\beth}_2,\ldots,\breve{\beth}_n)=$$

$$\left(\left(\prod_{j=1}^{n}(\breve{\Re}_j)^{\varpi_j},\prod_{j=1}^{n}(\omega_{\breve{\Re}_j})^{\varpi_j}\right),\sqrt[q]{1-\prod_{j=1}^{n}(1-(\breve{\wp}_j)^q)^{\varpi_j}},\right)\left(\sqrt[q]{1-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\wp}_j})^q\right)^{\varpi_j}}\right).$$

Definition 10. Let $\breve{\beth}_j=((\breve{\wp}_j,\omega_{\breve{\wp}_j}),(\breve{\Re}_j,\omega_{\breve{\Re}_j}))$ $(j=1,2,\ldots,n)$ be a collection of Cq-ROFNs, and the Cq-ROFHOWG operator be described as shown in Box V, when $\aleph=1$, Cq-ROFHOWG operator transforms

1.

$$\breve{\beth}_1 \oplus \breve{\beth}_2 = \Bigg(\Bigg(\sqrt[q]{\frac{(\breve{\wp}_1)^q + (\breve{\wp}_2)^q - (\breve{\wp}_1)^q(\breve{\wp}_2)^q - (1-\aleph)(\breve{\wp}_1)^q(\breve{\wp}_2)^q}{1-(1-\aleph)(\breve{\wp}_1)^q(\breve{\wp}_2)^q}},$$

$$\sqrt[q]{\frac{(\omega_{\breve{\wp}_1})^q + (\omega_{\breve{\wp}_2})^q - (\omega_{\breve{\wp}_1})^q(\omega_{\breve{\wp}_2})^q - (1-\aleph)(\omega_{\breve{\wp}_1})^q(\omega_{\breve{\wp}_2})^q}{1-(1-\aleph)(\omega_{\breve{\wp}_1})^q(\omega_{\breve{\wp}_2})^q}}\Bigg),$$

$$\Bigg(\frac{\breve{\Re}_1\breve{\Re}_2}{\sqrt[q]{\aleph+(1-\aleph)((\breve{\Re}_1)^q+(\breve{\Re}_2)^q-(\breve{\Re}_1)^q(\breve{\Re}_2)^q)}}, \frac{\omega_{\breve{\Re}_1}\omega_{\breve{\Re}_2}}{\sqrt[q]{\aleph+(1-\aleph)((\omega_{\breve{\Re}_1})^q+(\omega_{\breve{\Re}_2})^q-(\omega_{\breve{\Re}_1})^q(\omega_{\breve{\Re}_2})^q)}}\Bigg)\Bigg)$$

$\aleph > 0$;

2.

$$\breve{\beth}_1 \otimes \breve{\beth}_2 = \Bigg(\Bigg(\frac{\breve{\wp}_1\breve{\wp}_2}{\sqrt[q]{\aleph+(1-\aleph)((\breve{\wp}_1)^q+(\breve{\wp}_2)^q-(\breve{\wp}_1)^q(\breve{\wp}_2)^q)}}, \frac{\omega_{\breve{\wp}_1}\omega_{\breve{\wp}_2}}{\sqrt[q]{\aleph+(1-\aleph)((\omega_{\breve{\wp}_1})^q+(\omega_{\breve{\wp}_2})^q-(\omega_{\breve{\wp}_1})^q(\omega_{\breve{\wp}_2})^q)}}\Bigg),$$

$$\Bigg(\sqrt[q]{\frac{(\breve{\Re}_1)^q+(\breve{\Re}_2)^q-(\breve{\Re}_1)^q(\breve{\Re}_2)^q-(1-\aleph)(\breve{\Re}_1)^q(\breve{\Re}_2)^q}{1-(1-\aleph)(\breve{\Re}_1)^q(\breve{\Re}_2)^q}}, \sqrt[q]{\frac{(\omega_{\breve{\Re}_1})^q+(\omega_{\breve{\Re}_2})^q-(\omega_{\breve{\Re}_1})^q(\omega_{\breve{\Re}_2})^q-(1-\aleph)(\omega_{\breve{\Re}_1})^q(\omega_{\breve{\Re}_2})^q}{1-(1-\aleph)(\omega_{\breve{\Re}_1})^q(\omega_{\breve{\Re}_2})^q}}\Bigg)\Bigg)$$

$\aleph > 0$;

3.

$$\lambda\breve{\beth}_1 = \Bigg(\Bigg(\sqrt[q]{\frac{(1+(\aleph-1)(\breve{\wp}_1)^q)^\lambda-(1-(\breve{\wp}_1)^q)^\lambda}{(1-(\aleph-1)(\breve{\wp}_1)^q)^\lambda+(\aleph-1)(1-(\breve{\wp}_1)^q)^\lambda}}, \sqrt[q]{\frac{(1+(\aleph-1)(\omega_{\breve{\wp}_1})^q)^\lambda-(1-(\omega_{\breve{\wp}_1})^q)^\lambda}{(1-(\aleph-1)(\omega_{\breve{\wp}_1})^q)^\lambda+(\aleph-1)(1-(\omega_{\breve{\wp}_1})^q)^\lambda}}\Bigg),$$

$$\Bigg(\frac{\sqrt[q]{\aleph}(\breve{\Re}_1)^\lambda}{\sqrt[q]{(1-(\aleph-1)(1-(\breve{\Re}_1)^q))^\lambda+(\aleph-1)(\breve{\Re}_1)^{q\lambda}}}, \frac{\sqrt[q]{\aleph}(\omega_{\breve{\Re}_1})^\lambda}{\sqrt[q]{(1-(\aleph-1)(1-(\omega_{\breve{\Re}_1})^q))^\lambda+(\aleph-1)(\omega_{\breve{\Re}_1})^{q\lambda}}}\Bigg)\Bigg) \quad \aleph > 0;$$

4.

$$\breve{\beth}_1^\lambda = \Bigg(\Bigg(\frac{\sqrt[q]{\aleph}(\breve{\wp}_1)^\lambda}{\sqrt[q]{(1-(\aleph-1)(1-(\breve{\wp}_1)^q))^\lambda+(\aleph-1)(\breve{\wp}_1)^{q\lambda}}}, \frac{\sqrt[q]{\aleph}(\omega_{\breve{\wp}_1})^\lambda}{\sqrt[q]{(1-(\aleph-1)(1-(\omega_{\breve{\wp}_1})^q))^\lambda+(\aleph-1)(\omega_{\breve{\wp}_1})^{q\lambda}}}\Bigg),$$

$$\Bigg(\sqrt[q]{\frac{(1+(\aleph-1)(\breve{\Re}_1)^q)^\lambda-(1-(\breve{\Re}_1)^q)^\lambda}{(1-(\aleph-1)(\breve{\Re}_1)^q)^\lambda+(\aleph-1)(1-(\breve{\Re}_1)^q)^\lambda}}, \sqrt[q]{\frac{(1+(\aleph-1)(\omega_{\breve{\Re}_1})^q)^\lambda-(1-(\omega_{\breve{\Re}_1})^q)^\lambda}{(1-(\aleph-1)(\omega_{\breve{\Re}_1})^q)^\lambda+(\aleph-1)(1-(\omega_{\breve{\Re}_1})^q)^\lambda}}\Bigg)\Bigg) \quad \aleph > 0.$$

Box I

into the Cq-ROF Ordered Weighted Geometric (Cq-ROFOWG) operator as:

$$Cq-ROFOWG_w(\breve{\beth}_1, \breve{\beth}_2, \ldots, \breve{\beth}_n) = \Bigg(\Bigg(\prod_{j=1}^{n}(\breve{\wp}_{\gimel(j)})^{\varpi_j},$$

$$\prod_{j=1}^{n}(\omega_{\breve{\wp}_{\gimel(j)}})^{\varpi_j}\Bigg), \Bigg(\sqrt[q]{1-\prod_{j=1}^{n}\left(1-(\breve{\Re}_{\gimel(j)})^q\right)^{\varpi_j}},$$

$$\sqrt[q]{1-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\Re}_{\gimel(j)}})^q\right)^{\varpi_j}}\Bigg)\Bigg).$$

4. Complex q-Rung Orthopair Fuzzy Hamacher Graphs (Cq-ROFHGs)

In this section, a new concept of Cq-ROFG based on Hamacher operator termed as Cq-ROFHG is formed first, and then its energy along with the relevant

$$
\begin{aligned}
&Cq\text{–}ROFHWA_w(\breve{\gimel}_1,\breve{\gimel}_2,\ldots,\breve{\gimel}_n)=\oplus_{j=1}^{n}(\varpi_j\breve{\gimel}_j)\\
&=\left(\left(\left(\sqrt[q]{\frac{\prod_{j=1}^{n}(1+(\aleph-1)(\breve{\wp}_j)^q)^{\varpi_j}-\prod_{j=1}^{n}(1-(\breve{\wp}_j)^q)^{\varpi_j}}{\prod_{j=1}^{n}(1+(\aleph-1)(\breve{\wp}_j)^q)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(1-(\breve{\wp}_j)^q)^{\varpi_j}}},\right.\right.\right.\\
&\left.\sqrt[q]{\frac{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\wp}_j})^q\right)^{\varpi_j}-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\wp}_j})^q\right)^{\varpi_j}}{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\wp}_j})^q\right)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}\left(1-(\omega_{\breve{\wp}_j})^q\right)^{\varpi_j}}}\right),\\
&\left.\left.\left(\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\breve{\Re}_j)^{\varpi_j}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\breve{\Re}_j)^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\breve{\Re}_j)^{q\varpi_j}}},\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\omega_{\breve{\Re}_j})^{\varpi_j}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\omega_{\breve{\Re}_j})^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\omega_{\breve{\Re}_j})^{q\varpi_j}}}\right)\right)\right).
\end{aligned}
$$

Box II

properties is determined. Subsequently, inspired by the theory of splitting energy and shadow energy of a graph, we determine the energy of splitting Cq-ROFHG and the energy of shadow Cq-ROFHG.

Definition 11. A Cq-ROFS Ξ in $R\times R$ is said to be a Cq-ROF relation (Cq-ROFR) in R, denoted by:

$$\Xi=\{\langle\mathfrak{st},(\breve{\wp}_{\Xi}(\mathfrak{st}),\omega_{\breve{\wp}_{\Xi}}(\mathfrak{st})),(\breve{\Re}_{\Xi}(\mathfrak{st}),\omega_{\breve{\Re}_{\Xi}}(\mathfrak{st}))\rangle|\mathfrak{st}\in R\times R\},$$

$$
\begin{aligned}
&Cq-ROFHOWA_w(\breve{\gimel}_1,\breve{\gimel}_2,\ldots,\breve{\gimel}_n)=\oplus_{j=1}^{n}(\varpi_j\breve{\gimel}_{\beth(j)})\\
&=\left(\left(\left(\sqrt[q]{\frac{\prod_{j=1}^{n}\left(1+(\aleph-1)(\breve{\wp}_{\beth(j)})^q\right)^{\varpi_j}-\prod_{j=1}^{n}\left(1-(\breve{\wp}_{\beth(j)})^q\right)^{\varpi_j}}{\prod_{j=1}^{n}\left(1+(\aleph-1)(\breve{\wp}_{\beth(j)})^q\right)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}\left(1-(\breve{\wp}_{\beth(j)})^q\right)^{\varpi_j}}},\right.\right.\right.\\
&\left.\sqrt[q]{\frac{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\wp}_{\beth(j)}})^q\right)^{\varpi_j}-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\wp}_{\beth(j)}})^q\right)^{\varpi_j}}{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\wp}_{\beth(j)}})^q\right)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}\left(1-(\omega_{\breve{\wp}_{\beth(j)}})^q\right)^{\varpi_j}}}\right),\\
&\left(\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\breve{\Re}_{\beth(j)})^{\varpi_j}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\breve{\Re}_{\beth(j)})^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\breve{\Re}_{\beth(j)})^{q\varpi_j}}},\right.\\
&\left.\left.\left.\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\omega_{\breve{\Re}_{\beth(j)}})^{\varpi_j}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\omega_{\breve{\Re}_{\beth(j)}})^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\omega_{\breve{\Re}_{\beth(j)}})^{q\varpi_j}}}\right)\right)\right).
\end{aligned}
$$

Box III

$$Cq-ROFHWG(\breve{\beth}_1,\breve{\beth}_2,\ldots,\breve{\beth}_n)$$

$$=\left(\left(\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\breve{\wp}_j)^{\varpi_j}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\breve{\wp}_j)^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\breve{\wp}_j)^{q\varpi_j}}},\right.\right.$$

$$\left.\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\omega_{\breve{\wp}_j})^{\varpi_j}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\omega_{\breve{\wp}_j})^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\omega_{\breve{\wp}_j})^{q\varpi_j}}}\right),$$

$$\left(\sqrt[q]{\frac{\prod_{j=1}^{n}\left(1+(\aleph-1)(\breve{\Re}_j)^q\right)^{\varpi_j}-\prod_{j=1}^{n}\left(1-(\breve{\Re}_j)^q\right)^{\varpi_j}}{\prod_{j=1}^{n}\left(1+(\aleph-1)(\breve{\Re}_j)^q\right)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}\left(1-(\breve{\Re}_j)^q\right)^{\varpi_j}}},\right.$$

$$\left.\left.\sqrt[q]{\frac{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\Re}_j})^q\right)^{\varpi_j}-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\Re}_j})^q\right)^{\varpi_j}}{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\Re}_j})^q\right)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}\left(1-(\omega_{\breve{\Re}_j})^q\right)^{\varpi_j}}}\right)\right).$$

Box IV

where $\breve{\wp}_\Xi$, $\breve{\Re}_\Xi$, $\omega_{\breve{\wp}_\Xi}$, $\omega_{\breve{\Re}_\Xi}$: $R\times R\to[0,1]$ indicate the membership and non-membership function of Ξ, respectively, such that $0\le\breve{\wp}^q_\Xi(\mathfrak{st})+\breve{\Re}^q_\Xi(\mathfrak{st})\le 1$ and $0\le\omega_{\breve{\wp}^q_\Xi}(\mathfrak{st})+\omega_{\breve{\Re}^q_\Xi}(\mathfrak{st})\le 1$ for all $\mathfrak{st}\in R\times R$.

We define the Cq-ROF preference relations (Cq-ROFPRs) as follows.

Definition 12. A Cq-ROFPR on the set R is given by a matrix:

$$\mathscr{R}=(\breve{\beth}_{ij})_{n\times n},\breve{\beth}_{ij}=((\breve{\wp}_{ij},\omega_{\breve{\wp}_{ij}}),(\breve{\Re}_{ij},\omega_{\breve{\Re}_{ij}}))$$

$$(i,j=1,2,\ldots,n),$$

where $(\breve{\wp}_{ij},\omega_{\breve{\wp}_{ij}})$ and $(\breve{\Re}_{ij},\omega_{\breve{\Re}_{ij}})$ represent the complex MD and complex NMD, and

$$\pi_{ij}=\sqrt[q]{(1-\breve{\wp}^q_{ij}-\breve{\Re}^q_{ij})+(1-\omega^q_{\breve{\wp}_{ij}}-\omega^q_{\breve{\Re}_{ij}})},$$

indicates the hesitation degree, subject to the following conditions:

$$0\le\breve{\wp}_{ij},\breve{\Re}_{ij},\breve{\wp}^q_{ij}+\breve{\Re}^q_{ij}\le 1,$$

$$\breve{\wp}_{ij}=\breve{\Re}_{ji},\breve{\wp}_{ii}=\breve{\Re}_{ii}=0.5,$$

$$0\le\omega_{\breve{\wp}_{ij}},\omega_{\breve{\Re}_{ij}},\omega^q_{\breve{\wp}_{ij}}+\omega^q_{\breve{\Re}_{ij}}\le 1,\omega_{\breve{\wp}_{ij}}=\omega_{\breve{\Re}_{ji}},$$

$$\omega_{\breve{\wp}_{ii}}=\omega_{\breve{\Re}_{ii}}=0.5,$$

for all $i,j=1,2,\ldots,n$.

Definition 13. A Cq-ROFHG on a non-empty set R is an ordered pair $\mathfrak{G}=(\pounds,\Xi)$, where $\pounds$ is a Cq-ROFS on R and Ξ is a Cq-ROFR on R such that:

$$\breve{\wp}_\Xi(\mathfrak{st})\le\frac{\breve{\wp}_\pounds(\mathfrak{s})\breve{\wp}_\pounds(\mathfrak{t})}{\breve{\wp}_\pounds(\mathfrak{s})+\breve{\wp}_\pounds(\mathfrak{t})-\breve{\wp}_\pounds(\mathfrak{s})\breve{\wp}_\pounds(\mathfrak{t})},$$

$$\breve{\Re}_\Xi(\mathfrak{st})\le\frac{\breve{\Re}_\pounds(\mathfrak{s})+\breve{\Re}_\pounds(\mathfrak{t})-2\breve{\Re}_\pounds(\mathfrak{s})\breve{\Re}_\pounds(\mathfrak{t})}{1-\breve{\Re}_\pounds(\mathfrak{s})\breve{\Re}_\pounds(\mathfrak{t})},$$

for amplitude terms,

$$\omega_{\breve{\wp}_\Xi}(\mathfrak{st})\le\frac{\omega_{\breve{\wp}_\pounds}(\mathfrak{s})\omega_{\breve{\wp}_\pounds}(\mathfrak{t})}{\omega_{\breve{\wp}_\pounds}(\mathfrak{s})+\omega_{\breve{\wp}_\pounds}(\mathfrak{t})-\omega_{\breve{\wp}_\pounds}(\mathfrak{s})\omega_{\breve{\wp}_\pounds}(\mathfrak{t})},$$

$$\omega_{\breve{\Re}_\Xi}(\mathfrak{st})\le\frac{\omega_{\breve{\Re}_\pounds}(\mathfrak{s})+\omega_{\breve{\Re}_\pounds}(\mathfrak{t})-2\omega_{\breve{\Re}_\pounds}(\mathfrak{s})\omega_{\breve{\Re}_\pounds}(\mathfrak{t})}{1-\omega_{\breve{\Re}_\pounds}(\mathfrak{s})\omega_{\breve{\Re}_\pounds}(\mathfrak{t})},$$

for phase terms, where $0\le\breve{\wp}^q_\Xi(\mathfrak{st})+\breve{\Re}^q_\Xi(\mathfrak{st})\le 1$ and $0\le\omega^q_{\breve{\wp}_\Xi}(\mathfrak{st})+\omega^q_{\breve{\Re}_\Xi}(\mathfrak{st})\le 1$ for all $\mathfrak{s},\mathfrak{t}\in R$. We call $\pounds$

$$Cq-ROFHWG(\breve{\gimel}_1,\breve{\gimel}_2,\ldots,\breve{\gimel}_n)=\oplus_{j=1}^{n}(\breve{\gimel}_{\beth\,(j)})^{\varpi_j}=\left(\left(\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\breve{\wp}_{\beth_{(j)}})^{\varpi_{(j)}}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\breve{\wp}_{\beth_{(j)}})^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\breve{\wp}_{\beth_{(j)}})^{q\varpi_j}}},\right.\right.$$

$$\left.\frac{\sqrt[q]{\aleph}\prod_{j=1}^{n}(\omega_{\breve{\wp}_{\beth_{(j)}}})^{\varpi_j}}{\sqrt[q]{\prod_{j=1}^{n}(1+(\aleph-1)(1-(\omega_{\breve{\wp}_{\beth_{(j)}}})^q))^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}(\omega_{\breve{\wp}_{\beth_{(j)}}})^{q\varpi_j}}}\right),$$

$$\left(\sqrt[q]{\frac{\prod_{j=1}^{n}\left(1+(\aleph-1)(\breve{\Re}_{\beth_{(j)}})^q\right)^{\varpi_j}-\prod_{j=1}^{n}\left(1-(\breve{\Re}_{\beth_{(j)}})^q\right)^{\varpi_j}}{\prod_{j=1}^{n}\left(1+(\aleph-1)(\breve{\Re}_{\beth_{(j)}})^q\right)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}\left(1-(\breve{\Re}_{\beth_{(j)}})^q\right)^{\varpi_j}}},\right.$$

$$\left.\left.\sqrt[q]{\frac{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\Re}_{\beth_{(j)}}})^q\right)^{\varpi_j}-\prod_{j=1}^{n}\left(1-(\omega_{\breve{\Re}_{\beth_{(j)}}})^q\right)^{\varpi_j}}{\prod_{j=1}^{n}\left(1+(\aleph-1)(\omega_{\breve{\Re}_{\beth_{(j)}}})^q\right)^{\varpi_j}+(\aleph-1)\prod_{j=1}^{n}\left(1-(\omega_{\breve{\Re}_{\beth_{(j)}}})^q\right)^{\varpi_j}}}\right)\right).$$

Box V

a Cq-ROFS of vertices and Ξ a Cq-ROFS of edges in $\mathfrak{G}$. Here, we consider Ξ a symmetric Cq-ROFR on $\pounds$. In case of no symmetryic on $\pounds$, $\mathfrak{D}=(\pounds,\overrightarrow{\Xi})$ is called Cq-ROF Hamacher Digraph (Cq-ROFHDG).

Example 2. Consider a graph $\mathcal{G}=(\breve{V},\breve{E})$, where $\breve{V}=\{\mathfrak{s}_1,\mathfrak{s}_2,\ \mathfrak{s}_3,\mathfrak{s}_4,\mathfrak{s}_5,\ \mathfrak{s}_6\}$ and $\breve{E}=\{\mathfrak{s}_1\ \mathfrak{s}_2,\mathfrak{s}_1\mathfrak{s}_3,\ \mathfrak{s}_1\mathfrak{s}_4,\mathfrak{s}_1\ \mathfrak{s}_5,\mathfrak{s}_1\mathfrak{s}_6\}$ are the vertex and edge set of $\mathcal{G}$, respectively. Let $\mathfrak{G}=(\pounds,\Xi)$ be a C3-ROFHG on $\breve{V}$, as presented in Figure 2, defined by:

$$\pounds=\left(\frac{\mathfrak{s}_1}{((0.9,0.7),(0.6,0.8))},\frac{\mathfrak{s}_2}{((0.7,0.9),(0.7,0.5))},\frac{\mathfrak{s}_3}{((0.8,0.7),(0.5,0.8))},\frac{\mathfrak{s}_4}{((0.8,0.5),(0.6,0.9))},\frac{\mathfrak{s}_5}{((0.9,0.6),(0.5,0.9))},\frac{\mathfrak{s}_6}{((0.8,0.6),(0.7,0.7))}\right),$$

$$\Xi=\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{((0.63,0.62),(0.75,0.82))},\frac{\mathfrak{s}_1\mathfrak{s}_3}{((0.72,0.52),(0.66,0.88))},\frac{\mathfrak{s}_1\mathfrak{s}_4}{((0.71,0.40),(0.70,0.90))},\frac{\mathfrak{s}_1\mathfrak{s}_5}{((0.80,0.45),(0.71,0.91))},\frac{\mathfrak{s}_1\mathfrak{s}_6}{((0.70,0.47),(0.78,0.85))}\right).$$

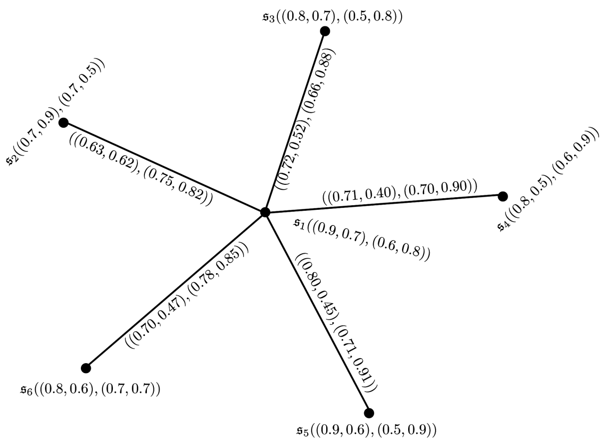

Figure 2. C3-ROFHG.

Definition 14. The Adjacency Matrix (AM) $A(\mathfrak{G})=(A(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j),\ \omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\ \mathfrak{s}_j)),\ A(\breve{\Re}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}_{\Xi}}(\mathfrak{s}_i\ \mathfrak{s}_j)))$ of a Cq-ROFHG $\mathfrak{G}=(\pounds,\Xi)$ is a square matrix $A(\mathfrak{G})=[a_{ij}]$,

a_{ij}=(($\breve{\wp}_\Xi(\mathfrak{s}_i\ \mathfrak{s}_j)$,$\omega_{\breve{\wp}_\Xi}$ $(\mathfrak{s}_i\mathfrak{s}_j)$), ($\breve{\Re}_\Xi(\mathfrak{s}_i\mathfrak{s}_j)$, $\omega_{\breve{\Re}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j)$)), where ($\breve{\wp}_\Xi(\mathfrak{s}_i\ \mathfrak{s}_j)$,$\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\ \mathfrak{s}_j)$) and ($\breve{\Re}_\Xi(\mathfrak{s}_i\ \mathfrak{s}_j)$, $\omega_{\breve{\Re}_\Xi}$ $(\mathfrak{s}_i\mathfrak{s}_j)$) indicate the strength of relationship and non-relationship between $\mathfrak{s}_i$ and $\mathfrak{s}_j$, respectively, in complex scenario.

Definition 15. The spectrum of AM of a Cq-ROFHG $A(\mathfrak{G})$ is characterized as ($\mathfrak{Y}$,$\mathfrak{Z}$), where $\mathfrak{Y}$ and $\mathfrak{Z}$ represent the sets of eigenvalues of $A(\breve{\wp}_\Xi\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j))$ and $A(\breve{\Re}_\Xi\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\Re}_\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j))$, respectively.

Definition 16. The energy of a Cq-ROFHG $\mathfrak{G} = (\pounds, \Xi)$ is defined as:

$$E(\mathfrak{G}) =(E(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}),\ E(\breve{\Re}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}))$$

$$= \left(\left(\sum_{\substack{j=1\\ \breve{\psi}_j\in\mathfrak{Y}}}^{n} |\breve{\psi}_j|,\ \sum_{\substack{j=1\\ \omega_{\breve{\psi}_j}\in\mathfrak{Y}}}^{n} |\omega_{\breve{\psi}_j}|\right), \left(\sum_{\substack{j=1\\ \breve{\chi}_j\in\mathfrak{Z}}}^{n} |\breve{\chi}_j|,\ \sum_{\substack{j=1\\ \omega_{\breve{\chi}_j}\in\mathfrak{Z}}}^{n} |\omega_{\breve{\chi}_j}|\right)\right).$$

Theorem 3. Let $\mathfrak{G} = (\pounds, \Xi)$ be a Cq-ROFHG and let $A(\mathfrak{G})$ be its AM. If $\breve{\psi}_1 \geq \breve{\psi}_2 \geq \ldots \geq \breve{\psi}_n$ and $\breve{\chi}_1 \geq \breve{\chi}_2 \geq \ldots \geq \breve{\chi}_n$ are the eigenvalues of $A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\ \omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\ \mathfrak{s}_j))$ and $A(\breve{\Re}_\Xi(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))$, respectively, then:

(i)

$$\left(\sum_{\substack{j=1\\ \breve{\psi}_j\in\mathfrak{Y}}}^{n} \breve{\psi}_j,\ \sum_{\substack{j=1\\ \omega_{\breve{\psi}_j}\in\mathfrak{Y}}}^{n} \omega_{\breve{\psi}_j}\right) = (0,0),$$

$$\left(\sum_{\substack{j=1\\ \breve{\chi}_j\in\mathfrak{Z}}}^{n} \breve{\chi}_j,\ \sum_{\substack{j=1\\ \omega_{\breve{\chi}_j}\in\mathfrak{Z}}}^{n} \omega_{\breve{\chi}_j}\right) = (0,0).$$

(ii)

$$\left(\sum_{\substack{j=1\\ \breve{\psi}_j\in\mathfrak{Y}}}^{n} \breve{\psi}_j^2,\ \sum_{\substack{j=1\\ \omega_{\breve{\psi}_j}\in\mathfrak{Y}}}^{n} \omega_{\breve{\psi}_j}^2\right) = 2 \left(\sum_{1\leq i<j\leq n} \breve{\wp}_\Xi^2(\mathfrak{s}_i\mathfrak{s}_j),\ \sum_{1\leq i<j\leq n} \omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_i\mathfrak{s}_j)\right),$$

$$\left(\sum_{\substack{j=1\\ \breve{\chi}_j\in\mathfrak{Z}}}^{n} \breve{\chi}_j^2,\ \sum_{\substack{j=1\\ \omega_{\breve{\chi}_j}\in\mathfrak{Z}}}^{n} \omega_{\breve{\chi}_j}^2\right) = 2 \left(\sum_{1\leq i<j\leq n} \breve{\Re}_\Xi^2(\mathfrak{s}_i\mathfrak{s}_j),\ \sum_{1\leq i<j\leq n} \omega_{\breve{\Re}_\Xi}^2(\mathfrak{s}_i\mathfrak{s}_j)\right).$$

Proof.

(i) Obvious

(ii) Since

$$tr(A^2(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})) = \left(\sum_{\substack{j=1\\ \breve{\psi}_j\in\mathfrak{Y}}}^{n} \breve{\psi}_j^2,\ \sum_{\substack{j=1\\ \breve{\psi}_j\in\mathfrak{Y}}}^{n} \omega_{\breve{\psi}_j}^2\right),$$

where matrix obtained as shown in Box VI.

Hence:

$$\begin{aligned} tr\left(A^2(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})\right) =&(0 + (\breve{\wp}_\Xi^2(\mathfrak{s}_1\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_1\mathfrak{s}_2)) + \ldots + (\breve{\wp}_\Xi^2(\mathfrak{s}_1\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_1\mathfrak{s}_n)))\\ &+ ((\breve{\wp}_\Xi^2(\mathfrak{s}_2\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_2\mathfrak{s}_1)) + 0 + \ldots + (\breve{\wp}_\Xi^2(\mathfrak{s}_2\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_2\mathfrak{s}_n)))\\ &\vdots\\ &+ ((\breve{\wp}_\Xi^2(\mathfrak{s}_n\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_n\mathfrak{s}_1)) + (\breve{\wp}_\Xi^2(\mathfrak{s}_n\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_n\mathfrak{s}_2)) + \ldots + 0)\\ =&2\left(\sum_{1\leq i<j\leq n} \breve{\wp}_\Xi^2(\mathfrak{s}_i\mathfrak{s}_j),\ \sum_{1\leq i<j\leq n} \omega_{\breve{\wp}_\Xi}^2(\mathfrak{s}_i\mathfrak{s}_j)\right). \end{aligned}$$

Box VI

$$\left(\sum_{\substack{j=1\\ \breve{\psi}_j\in\mathfrak{Y}}}^{n} \breve{\psi}_j^2,\ \sum_{\substack{j=1\\ \omega_{\breve{\psi}_j}\in\mathfrak{Y}}}^{n} \omega_{\breve{\psi}_j}^2\right)$$

$$=2\left(\sum_{1\le i<j\le n} \breve{\wp}_{\Xi}^2(\mathfrak{s}_i\mathfrak{s}_j),\ \sum_{1\le i<j\le n} \omega_{\breve{\wp}_{\Xi}}^2(\mathfrak{s}_i\mathfrak{s}_j)\right).$$

Analogously,

$$\left(\sum_{\substack{j=1\\ \breve{\chi}_j\in\mathfrak{Z}}}^{n} \breve{\chi}_j^2,\ \sum_{\substack{j=1\\ \omega_{\breve{\chi}_j}\in\mathfrak{Z}}}^{n} \omega_{\breve{\chi}_j}^2\right)$$

$$=2\left(\sum_{1\le i<j\le n} \breve{\Re}_{\Xi}^2(\mathfrak{s}_i\mathfrak{s}_j),\ \sum_{1\le i<j\le n} \omega_{\breve{\Re}_{\Xi}}^2(\mathfrak{s}_i\mathfrak{s}_j)\right). \qquad \square$$

Example 3. Let $\mathfrak{G} = (\pounds, \Xi)$ be a C4-ROFHG on $\breve{V} = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6, \mathfrak{s}_7\}$ and $\breve{E} = \{\mathfrak{s}_1\mathfrak{s}_2, \mathfrak{s}_2\mathfrak{s}_3, \mathfrak{s}_3\mathfrak{s}_4, \mathfrak{s}_3\mathfrak{s}_5, \mathfrak{s}_3\mathfrak{s}_6, \mathfrak{s}_3\mathfrak{s}_7, \mathfrak{s}_2\mathfrak{s}_6, \mathfrak{s}_1\mathfrak{s}_3\}$, as given in Figure 3.

The $A(\mathfrak{G})$, $Spec(\mathfrak{G})$, and $E(\mathfrak{G})$ of a C4-ROFHG given in Figure 3 are shown in Box VII. Now, $E(\breve{\wp}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))=(4.3475, 2.9555)$ and $E(\breve{\Re}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\Re}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))=(5.5563, 6.0622)$. Therefore, $E(\mathfrak{G})=((4.3475, 2.9555), (5.5563, 6.0622))$.

Theorem 4. Let $\mathfrak{G}=(\pounds, \Xi)$ be a Cq-ROFHG on n vertices and $A(\mathfrak{G})=(A(\breve{\wp}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j)),\ A(\breve{\Re}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\Re}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j)))$ be the AM of $\mathfrak{G}$. Then, obtained inqualities are shown in Box VIII.

Now we determine the energy of a splitting Cq-ROFHG and a shadow Cq-ROFHG.

Definition 17. The splitting Cq-ROFHG $\mathcal{S}(\mathfrak{G})$ of a Cq-ROFHG $\mathfrak{G}$ is attained by adding to each vertex $\mathfrak{s}$ another vertex $\mathfrak{s}'$, such that $\mathfrak{s}'$ is adjacent to each vertex that is adjacent to $\mathfrak{s}$ in $\mathfrak{G}$, and MD and NMD remain unchanged.

Theorem 5. Let $\mathcal{S}\ (\mathfrak{G})$ be a splitting Cq-ROFHG of a Cq-ROFHG $\mathfrak{G}$. Then $E(\mathcal{S}(\mathfrak{G}))=\sqrt{5}E(\mathfrak{G})$.

Proof. Consider a Cq-ROFHG with a set of vertices $\{\mathfrak{s}_1,\ \mathfrak{s}_2, \ldots, \mathfrak{s}_n\}$. Then its AM is $A(\mathfrak{G})=(A(\breve{\wp}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))\ (\mathfrak{G}),\ A(\breve{\wp}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\Re}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))\ (\mathfrak{G}))$, where obtained as shown in Box IX. To obtain $\mathcal{S}\ (\breve{\wp}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))\ (\mathfrak{G})$, let $\mathfrak{s}_1', \mathfrak{s}_2', \ldots, \mathfrak{s}_n'$ be the vertices corresponding to $\mathfrak{s}_1,\ \mathfrak{s}_2, \ldots, \mathfrak{s}_n$, which are included in $\mathfrak{G}$, such that, $\mathcal{N}\ (\mathfrak{s}_i)=\mathcal{N}\ (\mathfrak{s}_i')$, $i = 1, 2, \ldots, n$. At that point we can represent $A(\breve{\wp}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))\ (\mathcal{S}\ (\mathfrak{G}))$ as a block matrix as shown in Box X, i.e.,

$$A(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathcal{S}(\mathfrak{G})) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\otimes A(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}).$$

Since $\frac{1\pm\sqrt{5}}{2}$ and $(\breve{\psi}_i, \omega_{\breve{\psi}_i})$, $(i = 1, 2, \ldots, n)$, are the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $A(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$, respectively. So,

$$E(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathcal{S}(\mathfrak{G}))=\sum_{i=1}^{n}\left|\left(\frac{1\pm\sqrt{5}}{2}\right)(\breve{\psi}_i, \omega_{\breve{\psi}_i})\right|$$

$$=\sum_{i=1}^{n}\left(\frac{1+\sqrt{5}}{2}+\frac{\sqrt{5}-1}{2}\right)(|\breve{\psi}_i|, |\omega_{\breve{\psi}_i}|)$$

$$=\sqrt{5}\sum_{i=1}^{n}(|\breve{\psi}_i|, |\omega_{\breve{\psi}_i}|).$$

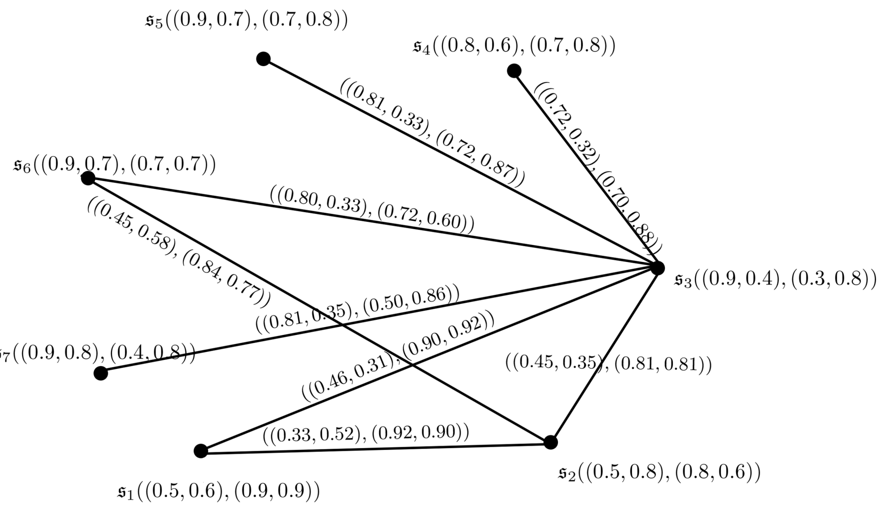

Figure 3. C4-ROFHG.

$$A(\mathfrak{G}) = \begin{bmatrix} (0,0) & ((0.33,0.52),(0.92,0.90)) & (0.46,0.31),(0.90,0.92)) \\ (0.33,0.52),(0.92,0.90)) & (0,0) & (0.45,0.35),(0.81,0.81)) \\ (0.46,0.31),(0.90,0.92)) & (0.45,0.35),(0.81,0.81)) & (0,0) \\ (0,0) & (0,0) & (0.72,0.32),(0.70,0.88)) \\ (0,0) & (0,0) & (0.81,0.33),(0.72,0.87)) \\ (0,0) & (0.45,0.58),(0.84,0.77)) & (0.80,0.33),(0.72,0.60)) \\ (0,0) & (0,0) & (0.81,0.35),(0.50,0.86)) \end{matrix}$$

$$\begin{matrix} (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.45,0.58),(0.84,0.77)) & (0,0) \\ (0.72,0.32),(0.70,0.88)) & (0.81,0.33),(0.72,0.87)) & (0.80,0.33),(0.72,0.60)) & (0.81,0.35),(0.50,0.86)) \\ (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) \end{bmatrix}.$$

$$Spec(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)) = \{(-1.6444, -0.8067), (-0.5294, -0.6710), (-0.0000, -0.0000), (0.0000, 0.0000), (0.0000, 0.0000), (0.3639, 0.3246), (1.8099, 1.1532)\}.$$

$$Spec(\breve{\Re}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)) = \{(-1.6131, -1.8712), (-1.1651, -1.1599), (-0.0000, -0.0000), (0.0000, -0.0000), (0.0000, -0.0000), (0.4480, 0.6122), (2.3302, 2.4189)\}.$$

Box VII

$$\sqrt{2\left(\sum_{1\le i<j\le n} \breve{\wp}^2_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \sum_{1\le i<j\le n} \omega^2_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)\right) + n(n-1)|\det(A(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)))|^{\frac{2}{n}}}$$

$$\le E(\breve{\wp}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)) \le \sqrt{2n\left(\sum_{1\le i<k\le n} \breve{\wp}^2_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \sum_{1\le i<j\le n} \omega^2_{\breve{\wp}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)\right)}.$$

$$\sqrt{2\left(\sum_{1\le i<j\le n} \breve{\Re}^2_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \sum_{1\le i<j\le n} \omega^2_{\breve{\Re}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)\right) + n(n-1)|\det(A(\breve{\Re}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)))|^{\frac{2}{n}}}$$

$$\le E(\breve{\Re}_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)) \le \sqrt{2n\left(\sum_{1\le i<j\le n} \breve{\Re}^2_{\Xi}(\mathfrak{s}_i\mathfrak{s}_j), \sum_{1\le i<j\le n} \omega^2_{\breve{\Re}_{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)\right)}.$$

Box VIII

Similarly, we can show that $E(\breve{\Re}_{\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\Re}_{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))$ $(\mathcal{S}(\mathfrak{G})) = \sqrt{5}\sum_{i=1}^{n}(|\breve{\chi}_i|,\ |\omega_{\breve{\chi}_i}|)$.

Hence, $E(\mathcal{S}(\mathfrak{G})) = \sqrt{5}E(\mathfrak{G})$ □.

Definition 18. The shadow Cq-ROFHG $\mathcal{SH}(\mathfrak{G})$ of a connected Cq-ROFHG $\mathfrak{G}$ is designed by taking two duplicates of $\mathfrak{G}$, say $\mathfrak{G}'$ and $\mathfrak{G}''$. Connect each vertex $\mathfrak{s}'$ in $\mathfrak{G}'$ to the neighbors of the corresponding vertex $\mathfrak{s}''$ in $\mathfrak{G}''$ with the same MD and NMD.

Theorem 6. Let $\mathcal{SH}(\mathfrak{G})$ be a shadow Cq-ROFHG of

$$A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$$
$$=\begin{bmatrix} 0 & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_1)) & (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_2)) & \dots & 0 \end{bmatrix}.$$

Box IX

$$A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathcal{S}(\mathfrak{G}))$$
$$=\left[\begin{array}{cccc} 0 & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_1)) & (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_2)) & \dots & 0 \\ \hline 0 & (\breve{\wp}_\Xi(\mathfrak{s}'_1\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_1\mathfrak{s}_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}'_1\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_1\mathfrak{s}_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}'_2\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_2\mathfrak{s}_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}'_2\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_2\mathfrak{s}_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}'_n\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_n\mathfrak{s}_1)) & (\breve{\wp}_\Xi(\mathfrak{s}'_n\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_n\mathfrak{s}_2)) & \dots & 0 \end{array}\right|$$

$$\left|\begin{array}{cccc} 0 & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}'_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}'_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}'_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}'_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}'_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}'_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}'_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}'_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}'_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}'_1)) & (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}'_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}'_1)) & \dots & 0 \\ \hline 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right]$$

Box X

a Cq-ROFHG $\mathfrak{G}$. Then $E(\mathcal{SH}(\mathfrak{G}))= 2E(\mathfrak{G})$.

Proof. Consider a Cq-ROFHG with set of vertices $\{v_1, v_2, \dots, v_n\}$. Then its AM is shown in Box XI. To obtain $\mathcal{SH}(\breve{\wp}_\Xi\ (\mathfrak{s}_i\mathfrak{s}_j),\ \omega_{\breve{\wp}_\Xi}\ (\mathfrak{s}_i\mathfrak{s}_j))\ (\mathfrak{G})$, let $\mathfrak{s}'_1$, $\mathfrak{s}'_2$, $\dots, \mathfrak{s}'_n$ be the vertices corresponding to $\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_n$, which are added in $\mathfrak{G}$, such that, $\mathcal{N}\ (\mathfrak{s}_i)= \mathcal{N}\ (\mathfrak{s}'_i)$, $i = 1, 2, \dots, n$. Then we can represent $A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),$ $\omega_{\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j)})(\mathcal{SH}(\mathfrak{G}))$ as a block matrix obtained as shown in Box XII. Since $0, 2$ and $(\breve{\psi}_i,\ \omega_{\breve{\psi}_i})$, $i = 1, 2, \dots, n$, are the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $A(\breve{\wp}_\Xi\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_\Xi}\ (\mathfrak{s}_i\ \mathfrak{s}_j))\ (\mathfrak{G})$, respectively. So,

$$E(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathcal{SH}(\mathfrak{G}))$$

$$=\sum_{i=1}^{n}\left|2(\breve{\psi}_i,\omega_{\breve{\psi}_i})\right| = 2\left(\sum_{i=1}^{n}|\breve{\psi}_i|,\sum_{i=1}^{n}|\omega_{\breve{\psi}_i}|\right).$$

Analogously,

$$E(\breve{\Re}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathcal{SH}(\mathfrak{G})) =$$

$$2\left(\sum_{i=1}^{n}|\breve{\chi}_i|,\sum_{i=1}^{n}|\omega_{\breve{\chi}_i}|\right).$$

Hence, $E(\mathcal{SH}(\mathfrak{G})) = 2E(\mathfrak{G})$ □.

$$A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$$

$$=\begin{bmatrix} 0 & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_1)) & (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_2)) & \dots & 0 \end{bmatrix}.$$

Box XI

$$A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathcal{SH}(\mathfrak{G}))$$

$$=\left[\begin{array}{cccc} 0 & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_n),\omega\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_1)) & (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}_2)) & \dots & 0 \\ \hline 0 & (\breve{\wp}_\Xi(\mathfrak{s}'_1\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_1\mathfrak{s}_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}'_1\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_1\mathfrak{s}_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}'_2\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_2\mathfrak{s}_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}'_2\mathfrak{s}_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_2\mathfrak{s}_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}'_n\mathfrak{s}_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_n\mathfrak{s}_1)) & (\breve{\wp}_\Xi(\mathfrak{s}'_n\mathfrak{s}_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_n\mathfrak{s}_2)) & \dots & 0 \end{array}\right|$$

$$\left.\begin{array}{cccc} 0 & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}'_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_1\mathfrak{s}'_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}'_n),\omega\breve{\wp}_\Xi(\mathfrak{s}_1\mathfrak{s}'_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}'_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}'_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}_2\mathfrak{s}'_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_2\mathfrak{s}'_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}'_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}'_1)) & (\breve{\wp}_\Xi(\mathfrak{s}_n\mathfrak{s}'_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_n\mathfrak{s}'_2)) & \dots & 0 \\ \hline 0 & (\breve{\wp}_\Xi(\mathfrak{s}'_1\mathfrak{s}'_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_1\mathfrak{s}'_2)) & \dots & (\breve{\wp}_\Xi(\mathfrak{s}'_1\mathfrak{s}'_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_1\mathfrak{s}'_n)) \\ (\breve{\wp}_\Xi(\mathfrak{s}'_2\mathfrak{s}'_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_2\mathfrak{s}'_1)) & 0 & \dots & (\breve{\wp}_\Xi(\mathfrak{s}'_2\mathfrak{s}'_n),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_2\mathfrak{s}'_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (\breve{\wp}_\Xi(\mathfrak{s}'_n\mathfrak{s}'_1),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_n\mathfrak{s}'_1)) & (\breve{\wp}_\Xi(\mathfrak{s}'_n\mathfrak{s}'_2),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}'_n\mathfrak{s}'_2)) & \dots & 0 \end{array}\right].$$

i.e.,

$$A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathcal{SH}(\mathfrak{G}))=\begin{bmatrix} A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) & A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) \\ A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) & A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\otimes A(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}).$$

Box XII

Example 4. Let $\mathfrak{G}=(\pounds,\Xi)$ be a C2-ROFHG on $\breve{V}=\{\mathfrak{s}_1,\ \mathfrak{s}_2,\ \mathfrak{s}_3,\ \mathfrak{s}_4\}$ and $\breve{E}=\{\mathfrak{s}_1\ \mathfrak{s}_2,\ \mathfrak{s}_1\ \mathfrak{s}_3,\ \mathfrak{s}_1\ \mathfrak{s}_4\}$, as in Figure 4, defined by:

$$\pounds=\left(\frac{\mathfrak{s}_1}{((0.9,0.6),(0.3,0.7))},\frac{\mathfrak{s}_2}{((0.6,0.8),(0.7,0.5))},\right.$$

$$\left.\frac{\mathfrak{s}_3}{((0.5,0.6),(0.7,0.8))},\frac{\mathfrak{s}_4}{((0.4,0.9),(0.8,0.4))}\right)$$

$$\Xi=\left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{((0.54,0.50),(0.71,0.76))},\right.$$

$$\left.\frac{\mathfrak{s}_1\mathfrak{s}_3}{((0.45,0.43),(0.72,0.85))},\frac{\mathfrak{s}_1\mathfrak{s}_4}{((0.38,0.55),(0.81,0.73))}\right).$$

The $A(\mathfrak{G})$ and $E(\mathfrak{G})$ of a C2-ROFHG, given in Figure 4 are shown in Box XIII.

The splitting C2-ROFHG of the C2-ROFHG shown in Figure 4, is presented in Figure 5.

$A(\mathfrak{G}) =$

$$quad \begin{bmatrix} (0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\ (0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\ (0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\ (0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0) \end{bmatrix}$$

$E(\mathfrak{G}) = ((1.5981, 1.7174), (2.5912, 2.7078)).$

Box XIII

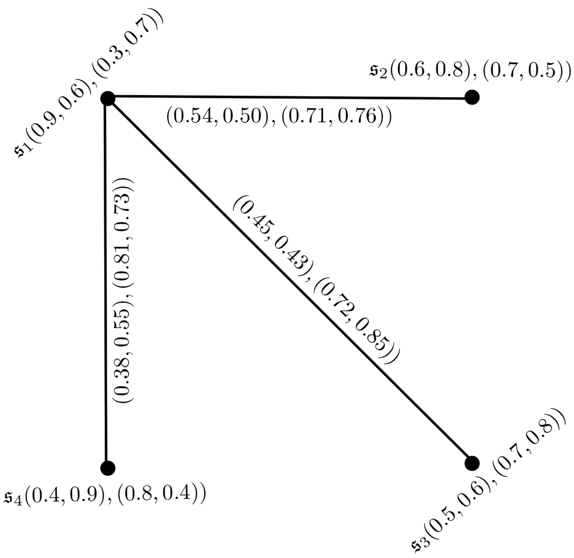

Figure 4. C2-ROFHG.

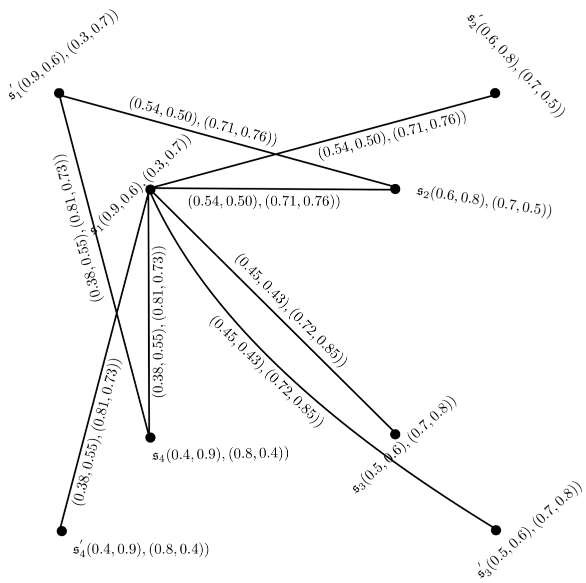

Figure 5. Splitting C2-ROFHG.

The $A(\mathcal{S}(\mathfrak{G}))$ and the $E(\mathcal{S}(\mathfrak{G}))$ of a splitting C2-ROFHG, given in Figure 5, are shown in Box XIV. The $A(\mathcal{SH}(\mathfrak{G}))$ and the $E(\mathcal{SH}(\mathfrak{G}))$ of a shadow C2-ROFHG, given in Figure 6, calculated in Box XV.

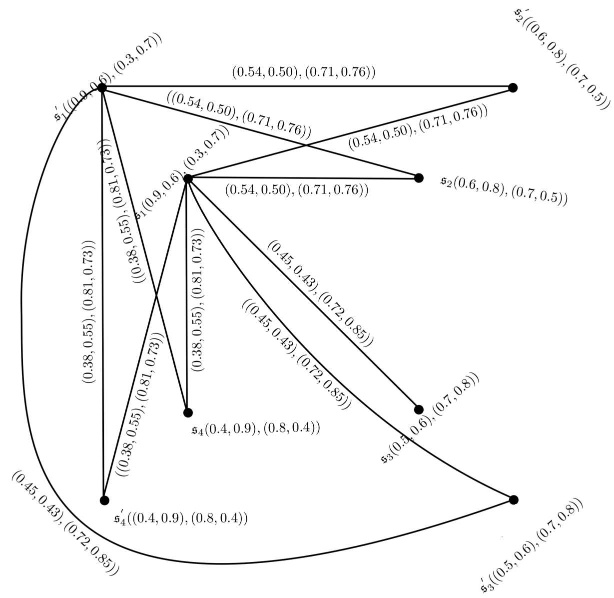

Figure 6. Shadow C2-ROFHG.

Definition 19 The energy of a Cq-ROFHDG $\mathfrak{D}$=($\pounds$, $\overrightarrow{\Xi}$) is defined as:

$$E(\mathfrak{D}) = \Big(E(\breve{\wp}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\overrightarrow{\Xi}}}(\mathfrak{s}_i\mathfrak{s}_j)),$$

$$E(\breve{Re}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{Re}_{\overrightarrow{\Xi}}}(\mathfrak{s}_i\mathfrak{s}_j))\Big)$$

$$= \left(\left(\sum_{\substack{i=1 \\ \breve{\psi}_i \in \mathcal{Y}}}^{n} |Re(\breve{\psi}_i)|, \sum_{\substack{i=1 \\ \breve{\psi}_i \in \mathcal{Y}}}^{n} |\omega_{Re}(\breve{\psi}_i)| \right), \left(\sum_{\substack{i=1 \\ \breve{\chi}_i \in \mathcal{Z}}}^{n} |Re(\breve{\chi}_i)|, \sum_{\substack{i=1 \\ \breve{\chi}_i \in \mathcal{Z}}}^{n} |\omega_{Re}(\breve{\chi}_i)| \right) \right),$$

where $\mathcal{Y}$ and $\mathcal{Z}$ represent the sets of eigenvalues of $A(\breve{\wp}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)})$ and $A(\breve{\Re}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j)})$, and Re($\breve{\psi}_i$) and Re($\breve{\chi}_i$) represent the real parts of the eigenvalues $\breve{\psi}_i$ and $\breve{\chi}_i$, respectively.

$A(\mathcal{S}(\mathfrak{G}))$

$$= \left[\begin{array}{cccc}
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\
(0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\
(0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0) \\
\hline
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\
(0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\
(0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0)
\end{array}\right|$$

$$\left.\begin{array}{cccc}
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\
(0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\
(0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0) \\
\hline
(0,0) & (0,0) & (0,0) & (0,0) \\
(0,0) & (0,0) & (0,0) & (0,0) \\
(0,0) & (0,0) & (0,0) & (0,0) \\
(0,0) & (0,0) & (0,0) & (0,0)
\end{array}\right].$$

$$E(\mathcal{S}(\mathfrak{G})) = ((3.5735, 3.8403), (5.7941, 6.0548))$$

$$= \sqrt{5}((1.5981, 1.7174), (2.5912, 2.7078)) = \sqrt{5}E(\mathfrak{G}).$$

Box XIV

$A(\mathcal{SH}(\mathfrak{G}))$

$$= \left[\begin{array}{cccc}
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\
(0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\
(0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0) \\
\hline
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\
(0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\
(0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0)
\end{array}\right|$$

$$\left.\begin{array}{cccc}
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\
(0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\
(0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0) \\
\hline
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0,0) & (0,0) & (0,0) \\
(0.45,0.43),(0.72,0.85) & (0,0) & (0,0) & (0,0) \\
(0.38,0.55),(0.81,0.73) & (0,0) & (0,0) & (0,0)
\end{array}\right].$$

$$E(\mathcal{SH}(\mathfrak{G})) = ((3.1962, 3.4349), (5.1824, 5.4155))$$

$$= 2((1.5981.1.7174), (2.5912, 2.7078)) = 2E(\mathfrak{G}).$$

Box XV

5. RE of Cq-ROFHGs

In this section, the novel concept of the RE of a Cq-ROFHG is introduced and its relevant properties are discussed in detail.

Definition 20. Let $\mathfrak{G}=(\pounds,\Xi)$ be a Cq-ROFHG on n vertices. The RM, $R(\mathfrak{G})$=($R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)), R(\breve{\Re}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}}(\mathfrak{s}_i\mathfrak{s}_j))$)=$[a'_{ij}]$, of $\mathfrak{G}$ is a

$$a'_{ij}=\begin{cases}0 & \text{if } i=j,\\ \frac{1}{\sqrt{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}} & \text{if the nodes } \mathfrak{s}_i \text{ and } \mathfrak{s}_j \text{ of the } Cq\text{–}ROFHG\ \mathfrak{G} \text{ are adjacent},\\ 0 & \text{if the nodes } \mathfrak{s}_i \text{ and } \mathfrak{s}_j \text{ of the } Cq\text{–}ROFHG\ \mathfrak{G} \text{ are non-adjacent}.\end{cases}$$

Definition 21. The RE of a Cq-ROFHG $\mathfrak{G}=(\pounds,\Xi)$ is defined as:

$$RE(\mathfrak{G})=\Big(RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)),RE(\breve{\Re}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}}(\mathfrak{s}_i\mathfrak{s}_j))\Big)$$

$$=\left(\left(\sum_{\substack{j=1\\ \breve{\delta}\in\mathfrak{Y}_R}}^{n}|\breve{\delta}_j|,\ \sum_{\substack{j=1\\ \omega_{\breve{\delta}_j}\in\mathfrak{Y}_R}}^{n}|\omega_{\breve{\delta}_j}|\right),\left(\sum_{\substack{j=1\\ j\in\mathfrak{Z}_R}}^{n}|\breve{\eta}_j|,\ \sum_{\substack{j=1\\ \omega_{\breve{\eta}_j}\in\mathfrak{Z}_R}}^{n}|\omega_{\breve{\eta}_j}|\right)\right),$$

where $\mathfrak{Y}_R$ and $\mathfrak{Z}_R$ are the sets of Randić eigenvalues of $R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$ and $R(\breve{\Re}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$, respectively.

First, we establish the trace of the RMs $R(\mathfrak{G})$, $R^2(\mathfrak{G})$, $R^3(\mathfrak{G})$, and $R^4(\mathfrak{G})$, i.e., $tr(R(\mathfrak{G}))$, $tr(R^2(\mathfrak{G}))$, $tr(R^3(\mathfrak{G}))$, and $tr(R^4(\mathfrak{G}))$. Further, utilizing these equalities, the upper and lower bounds for RE are derived.

Lemma 1. Let $\mathfrak{G}=(\pounds,\Xi)$ be a Cq-ROFHG on n vertices and $R(\mathfrak{G})$= ($R(\breve{\wp}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j)), R(\breve{\Re}_\Xi(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}_\Xi}(\mathfrak{s}_i\mathfrak{s}_j))$) be the RM of $\mathfrak{G}$. Then:

1. $tr(R(\mathfrak{G}))=0$,
2. $tr(R^2(\mathfrak{G}))=2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}$,
3. $tr(R^3(\mathfrak{G}))=2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\ k\sim j}}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_k)}\right)$,
4. $tr(R^4(\mathfrak{G}))=\sum_{i=1}^{n}\left(\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\right)^2$

$$+\sum_{i\neq j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\ k\sim j}}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_k)}\right)^2.$$

Proof.

1. Obvious.
2. For matrix $R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$. If $i=j$:

$$R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_i),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_i))(\mathfrak{G})=\sum_{j=1}^{n}R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})R(\breve{\wp}(\mathfrak{s}_j\mathfrak{s}_i),\omega_{\breve{\wp}}(\mathfrak{s}_j\mathfrak{s}_i))(\mathfrak{G})$$

$$=\sum_{j=1}^{n}(R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))^2(\mathfrak{G})$$

$$=\sum_{i\sim j}R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))^2(\mathfrak{G})$$

$$=\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}.$$

Whereas if $i\neq j$:

$$R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j)\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})=\sum_{k=1}^{n}R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_k),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_k))(\mathfrak{G})R(\breve{\wp}(\mathfrak{s}_k\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_k\mathfrak{s}_j))(\mathfrak{G})$$

$$R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_i),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_i))(\mathfrak{G})R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})+R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})R(\breve{\wp}(\mathfrak{s}_j\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_j\mathfrak{s}_j))(\mathfrak{G})+\sum_{k\sim i,k\sim j}R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_k),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_k))(\mathfrak{G})R(\breve{\wp}(\mathfrak{s}_k\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_k\mathfrak{s}_j))(\mathfrak{G})$$

$$=\frac{1}{\sqrt{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}}\sum_{\substack{k\sim i\\ k\sim j}}\frac{1}{(d_{\breve{\wp},\omega_{\breve{\wp}}})(\mathfrak{s}_k)}.$$

Therefore, we have:

$$tr(R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})=\sum_{i=1}^{n}\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}=2\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}.$$

Similarly, we can show that: $tr(R^2(\breve{\Re}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})=2\sum_{i\sim j}\frac{1}{d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_i)d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_j)}$. Hence $tr(R^2(\mathfrak{G}))=2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}$.

3. Now, we determine the matrix $R^3(\breve{\wp}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}}\ (\mathfrak{s}_i\ \mathfrak{s}_j)))\ (\mathfrak{G})$.

$$R^3(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_i),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_i))(\mathfrak{G})=\sum_{j=1}^{n}R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),$$

$$\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})R^2(\breve{\wp}(\mathfrak{s}_j\mathfrak{s}_k),\omega_{\breve{\wp}(\mathfrak{s}_j\mathfrak{s}_k)})(\mathfrak{G})$$

$$=\sum_{i\sim j}\frac{1}{\sqrt{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}}R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),$$

$$\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})=\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}$$

$$\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_k)}\right).$$

Therefore:

$$tr(R^3(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$$

$$=\sum_{i=1}^{n}\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_k)}\right)$$

$$=2\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_k)}\right).$$

Similarly, we can show that:

$$tr(R^3(\breve{\Re}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$$

$$=2\sum_{i\sim j}\frac{1}{d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_i)d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_k)}\right).$$

Hence,

$$tr(R^3(\mathfrak{G}))=2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_k)}\right).$$

4. We now calculate $tr(R^4(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$. Because:

$$tr(R^4(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$$

$$=||R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})||_F^2,$$

where $||R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})||_F$ denotes the Frobenius norm of $R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_i),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$, we obtain:

$$tr(R^4(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$$
$$=\sum_{i,j=1}^{n}|R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})|^2$$

$$=\sum_{i=j}|R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})|^2$$

$$+\sum_{i\neq j}|R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})|^2$$

$$=\sum_{i=1}^{n}\left(\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}\right)^2$$

$$+\sum_{i\neq j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}$$

$$\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_k)}\right)^2.$$

Similarly, we can show that:

$$tr(R^4(\breve{\Re}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\Re}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$$

$$=\sum_{i=1}^{n}\left(\sum_{i\sim j}\frac{1}{d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_i)d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_j)}\right)^2$$

$$+\sum_{i\neq j}\frac{1}{d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_i)d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_k)}\right)^2.$$

Hence,

$$tr(R^4(\mathfrak{G}))=\sum_{i=1}^{n}\left(\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\right)^2$$

$$+\sum_{i\neq j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_k)}\right)^2. \qquad \square$$

Theorem 7. Let $\mathfrak{G}=(\pounds,\Xi)$ be a Cq-ROFHG on n vertices. Then, we have:

$$RE(\mathfrak{G})\leq\sqrt{2n\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}}.$$

Furthermore, $RE(\mathfrak{G})=\sqrt{2n\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}}$ if and only if $\mathfrak{G}$ is a Cq-ROFHG with only end vertices, or isolated vertices.

Proof. The variance of the numbers $|\breve{\delta}_i,\ \omega_{\breve{\delta}_i}|=\frac{1}{n}\left(\sum_{i=1}^{n}|\breve{\delta}_i|^2,\sum_{i=1}^{n}|\omega_{\breve{\delta}_i}|^2\right)-\left(\frac{1}{n}\left(\sum_{i=1}^{n}|\breve{\delta}_i|,\sum_{i=1}^{n}|\omega_{\breve{\delta}_i}|\right)\right)^2\geq 0$, $i=1,2,\ldots,n$. Now,

$$\left(\sum_{i=1}^{n}|\breve{\delta}_i|^2, \sum_{i=1}^{n}|\omega_{\breve{\delta}_i}|^2\right) = \left(\sum_{i=1}^{n}\breve{\delta}_i^2, \sum_{i=1}^{n}\omega_{\breve{\delta}_i}^2\right)$$

$$= tr(R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G}).$$

Therefore:

$$\frac{1}{n}tr(R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$$

$$-\left(\frac{1}{n}RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})\right)^2 \geq 0$$

$$\Longleftrightarrow \frac{1}{n}tr(R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j)))(\mathfrak{G})$$

$$\geq \left(\frac{1}{n}RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})\right)^2$$

$$\Longleftrightarrow RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$$

$$\leq \sqrt{n\ tr(R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}))}$$

$$\Longleftrightarrow RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) \leq$$

$$\sqrt{2n\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}}.$$

If $\mathfrak{G}$ is a Cq-ROFHG with only isolated vertices, i.e., without edges, then $(\breve{\delta}_i, \omega_{\breve{\delta}_i}) = (0,0)$ for all $i = 1, 2, \ldots, n$, and therefore $RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) = 0$. Since no vertices are adjacent, $\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)} = 0$. If $\mathfrak{G}$ is a Cq-ROFHG with only end vertices, i.e., incident with one edge, then $(\breve{\delta}_i, \omega_{\breve{\delta}_i}) = \pm d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)$; therefore, the variance of $(|\breve{\delta}_i|, |\omega_{\breve{\delta}_i}|)$=0, $i = 1, 2, \ldots, n$. Thus:

$$RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$$

$$= \sqrt{2n\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}}.$$

Analogously, we can show that:

$$\Longleftrightarrow RE(\breve{\Re}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$$

$$\leq \sqrt{2n\sum_{i\sim j}\frac{1}{d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_i)d_{(\breve{\Re},\omega_{\breve{\Re}})}(\mathfrak{s}_j)}}.$$

Hence:

$$RE(\mathfrak{G}) \leq \sqrt{2n\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}}.$$

Theorem 8. Let $\mathfrak{G} = (\pounds, \Xi)$ be a Cq-ROFHG on n vertices and at least one edge. Then, we have:

$$RE(\mathfrak{G}) \geq 2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}$$

$$\sqrt{\frac{2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}}{\sum_{i=1}^{n}\left(\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\right)^2 + \sum_{i\neq j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_k)}\right)^2}}.$$

Proof. According to the Hölder inequality, we have:

$$\sum_{i=1}^{n}l_im_i \leq \left(\sum_{i=1}^{n}l_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}m_i^q\right)^{\frac{1}{q}},$$

which holds for any real numbers $l_i,m_i \geq 0$, $i = 1, 2, \ldots, n$. Setting $l_i=(|\breve{\delta}_i|^{\frac{2}{3}}, |\omega_{\breve{\delta}_i}|^{\frac{2}{3}})$, $m_i=(|\breve{\delta}_i|^{\frac{4}{3}}, |\omega_{\breve{\delta}_i}|^{\frac{4}{3}})$, $p = \frac{3}{2}$, and $q = 3$, we obtain:

$$\left(\sum_{i=1}^{n}|\breve{\delta}_i|^2, \sum_{i=1}^{n}|\omega_{\breve{\delta}_i}|^2\right)$$

$$= \sum_{i=1}^{n}(|\breve{\delta}_i|^{\frac{2}{3}}, |\omega_{\breve{\delta}_i}|^{\frac{2}{3}})\left(|\breve{\delta}_i|^4, |\omega_{\breve{\delta}_i}|^4\right)^{\frac{1}{3}}$$

$$\leq \left(\sum_{i=1}^{n}|\breve{\delta}_i|, \sum_{i=1}^{n}|\omega_{\breve{\delta}_i}|\right)^{\frac{2}{3}}\left(\sum_{i=1}^{n}|\breve{\delta}_i|^4, \sum_{i=1}^{n}|\omega_{\breve{\delta}_i}|^4\right)^{\frac{1}{3}}.$$

If the Cq-ROFHG $\mathfrak{G}$ has at least one edge, then all $(\breve{\delta}_i, \omega_{\breve{\delta}_i})$'s are not equal to zero. Then $\left(\sum_{i=1}^{n}|\breve{\delta}_i|^4, \sum_{i=1}^{n}|\omega_{\breve{\delta}_i}|^4\right) \neq 0$ and we obtained inequalities are shown in Box XVI.

Theorem 9. Let $\mathfrak{G} = (\pounds, \Xi)$ be a Cq-ROFHG on n vertices. If $\mathfrak{G}$ is regular of degree $((p, \omega_p), (r, \omega_r))$ where $p, r > 0$, then:

$$RE(\mathfrak{G}) = \frac{1}{((p, \omega_p), (r, \omega_r))}E(\mathfrak{G}).$$

Proof. Assume that $\mathfrak{G}$ is a regular Cq-ROFHG of degree $((p, \omega_p), (r, \omega_r))$ and $p, r > 0$, i.e., $d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_1) = d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_2) = \ldots = d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_n) = (p, \omega_p)$. Then all non zero entries of $R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_i), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_i))(\mathfrak{G})$ are equal to $\frac{1}{(p,\omega_p)}$, implying that $R(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_i), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_i))(\mathfrak{G}) = \frac{1}{(p,\omega_p)}A(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_i), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_i))(\mathfrak{G})$. Therefore, for all $i = 1, 2, \ldots, n$.

$$(\breve{\delta}_i, \omega_{\breve{\delta}_i}) = \frac{1}{(p_i, \omega_{p_i})}(\breve{\psi}_i, \omega_{\breve{\psi}_i})$$

$$\left(\sum_{i=1}^{n}\breve{\delta}_i, \sum_{i=1}^{n}\omega_{\breve{\delta}_i}\right) = \frac{1}{(p_i, \omega_{p_i})}\left(\sum_{i=1}^{n}\breve{\psi}_i, \sum_{i=1}^{n}\omega_{\breve{\psi}_i}\right)$$

$$RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) = \frac{1}{(p_i,\omega_{p_i})}E(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}).$$

Similarly, we can show that:

$$RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) = \left(\sum_{i=1}^n |\breve{\delta}_i|, \sum_{i=1}^n |\omega_{\breve{\delta}_i}|\right) \geq \sqrt{\frac{\left(\sum_{i=1}^n |\breve{\delta}_i|^2, \sum_{i=1}^n |\omega_{\breve{\delta}_i}|^2\right)^3}{\left(\sum_{i=1}^n |\breve{\delta}_i|^4, \sum_{i=1}^n |\omega_{\breve{\delta}_i}|^4\right)}} = \sqrt{\frac{\left(\sum_{i=1}^n \breve{\delta}_i^2, \sum_{i=1}^n \omega_{\breve{\delta}_i^2}\right)^3}{\left(\sum_{i=1}^n \breve{\delta}_i^4, \sum_{i=1}^n \omega_{\breve{\delta}_i^4}\right)}}$$

$$= \sqrt{\frac{tr(R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}))^3}{tr(R^4(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}))}}$$

$$RE(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) \geq \sqrt{\frac{tr(R^2(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}))^3}{tr(R^4(\breve{\wp}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\wp}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}))}}$$

$$= 2\sum_{i\sim j} \frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)} \sqrt{\frac{2\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}}{\sum_{i=1}^n\left(\sum_{i\sim j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}\right)^2 + \sum_{i\neq j}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_i)d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\wp},\omega_{\breve{\wp}})}(\mathfrak{s}_k)}\right)^2}},$$

similarly,

$$RE(\breve{\mathfrak{R}}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\mathfrak{R}}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}) \geq 2\sum_{i\sim j} \frac{1}{d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_i)d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_j)}$$

$$\sqrt{\frac{2\sum_{i\sim j}\frac{1}{d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_i)d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_j)}}{\sum_{i=1}^n\left(\sum_{i\sim j}\frac{1}{d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_i)d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_j)}\right)^2 + \sum_{i\neq j}\frac{1}{d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_i)d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{(\breve{\mathfrak{R}},\omega_{\breve{\mathfrak{R}}})}(\mathfrak{s}_k)}\right)^2}}.$$

Hence, wehave :

$$RE(\mathfrak{G}) \geq 2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\sqrt{\frac{2\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}}{\sum_{i=1}^n\left(\sum_{i\sim j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\right)^2 + \sum_{i\neq j}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_i)d_{\mathfrak{G}}(\mathfrak{s}_j)}\left(\sum_{\substack{k\sim i\\k\sim j}}\frac{1}{d_{\mathfrak{G}}(\mathfrak{s}_k)}\right)^2}}. \qquad \square$$

Box XVI

$$RE(\breve{\mathfrak{R}}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\mathfrak{R}}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G})$$

$$= \frac{1}{(r_i,\omega_{r_i})}E(\breve{\mathfrak{R}}(\mathfrak{s}_i\mathfrak{s}_j),\omega_{\breve{\mathfrak{R}}}(\mathfrak{s}_i\mathfrak{s}_j))(\mathfrak{G}).$$

Hence, we have:

$$RE(\mathfrak{G}) = \frac{1}{((p_i,\omega_{p_i}),(r_i,\omega_{r_i}))}E(\mathfrak{G}).$$

Example 5. Let $\mathfrak{G} = (\pounds, \Xi)$ be a C5-ROFHG on $\breve{V} = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6\}$ and $\breve{E} = \{\mathfrak{s}_1\ \mathfrak{s}_2, \mathfrak{s}_2\ \mathfrak{s}_3, \mathfrak{s}_3\ \mathfrak{s}_4, \mathfrak{s}_4\ \mathfrak{s}_5, \mathfrak{s}_5\ \mathfrak{s}_6, \mathfrak{s}_6\ \mathfrak{s}_1, \mathfrak{s}_2\mathfrak{s}_6, \mathfrak{s}_3\mathfrak{s}_5\}$, as in Figure 7, calculated as shown

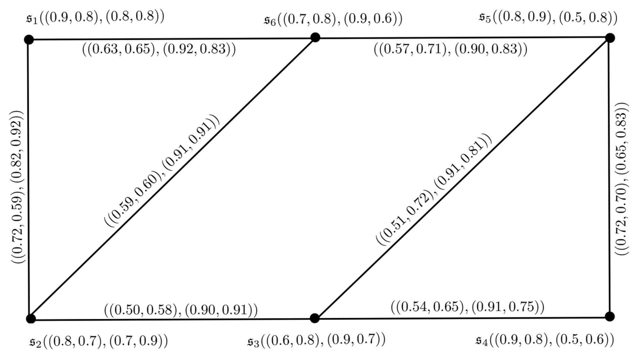

Figure 7. C5-ROFHG.

in Box XVII. The $A(\mathfrak{G})$, $R(\mathfrak{G})$, spectrum, and $RE(\mathfrak{G})$ of the C5-ROFHG, shown in Figure 7 are shown in Box XVIII. Now we will define the RM and RE of Cq-ROFHDG.

Definition 22. Let $\mathfrak{D} = (\pounds, \overrightarrow{\Xi})$ be a Cq-ROFHDG on n vertices. The RM, $R(\mathfrak{D}) = (R(\breve{\wp}_{\overrightarrow{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_{\overrightarrow{\Xi}}}\ (\mathfrak{s}_i\ \mathfrak{s}_j)),\ R(\breve{\Re}_{\overrightarrow{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\Re}_{\overrightarrow{\Xi}}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))) = [\mathfrak{s}_{ij}]$, of $\mathfrak{D}$ is a $n \times n$ matrix defined as:

$$\mathfrak{s}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \frac{1}{\sqrt{d_{\overrightarrow{\Xi}}(\mathfrak{s}_i) d_{\overrightarrow{\Xi}}(\mathfrak{s}_j)}} & \text{if the nodes } \mathfrak{s}_i \text{ and } \mathfrak{s}_j \\ & \text{of the C}q\text{-ROFHDG} \\ & \Xi \text{ are adjacent,} \\ 0 & \text{if the nodes } \mathfrak{s}_i \text{ and } \mathfrak{s}_j \\ & \text{of the C}q\text{-ROFHDG} \\ & \Xi \text{ are non adjacent.} \end{cases}$$

Definition 23. The RE of a Cq-ROFHDG $\mathfrak{D} = (\pounds, \overrightarrow{\Xi})$ is defined as:

$$RE(\mathfrak{D}) = \Big(RE(\breve{\wp}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\wp}_{\overrightarrow{\Xi}}}(\mathfrak{s}_i\mathfrak{s}_j)),$$

$$RE(\breve{\Re}_{\overrightarrow{\Xi}}(\mathfrak{s}_i\mathfrak{s}_j), \omega_{\breve{\Re}_{\overrightarrow{\Xi}}}(\mathfrak{s}_i\mathfrak{s}_j)) \Big)$$

$$= \left(\left(\sum_{\substack{i=1 \\ \breve{\delta}_i \in \mathcal{Y}_{\mathcal{R}}}}^{n} |Re(\breve{\delta}_i)|, \sum_{\substack{i=1 \\ \breve{\delta}_i \in \mathcal{Y}_{\mathcal{R}}}}^{n} |\omega_{Re}(\breve{\delta}_i)| \right), \right.$$

$$\left. \left(\sum_{\substack{i=1 \\ \breve{\eta}_i \in \mathcal{Z}_{\mathcal{R}}}}^{n} |Re(\breve{\eta}_i)|, \sum_{\substack{i=1 \\ \breve{\eta}_i \in \mathcal{Z}_{\mathcal{R}}}}^{n} |\omega_{Re}(\breve{\eta}_i)| \right) \right),$$

where $\mathcal{Y}_R$, $\mathcal{Z}_R$ are the sets of Randić eigenvalues of $R(\breve{\wp}_{\overrightarrow{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\wp}_{\overrightarrow{\Xi}}}\ (\mathfrak{s}_i\mathfrak{s}_j))\ (\mathfrak{D})$ and $R(\breve{\Re}_{\overrightarrow{\Xi}}\ (\mathfrak{s}_i\ \mathfrak{s}_j),\ \omega_{\breve{\Re}_{\overrightarrow{\Xi}}}\ (\mathfrak{s}_i\ \mathfrak{s}_j))\ (\mathfrak{D})$, whereas Re$(\breve{\delta}_i)$ and Re$(\breve{\eta}_i)$ express the real parts of the eigenvalues $\breve{\delta}_i$ and $\breve{\eta}_i$, respectively.

6. Novel MAGDM method based on proposed concepts of energy and RE of Cq-ROFGs

This section proposes a new MAGDM approach based on Cq-ROFDG to solve MAGDM problems where there are relationships between attributes. First of all, we present this kind of problem. We will then develop the procedure of the proposed method in depth.

The strategy is outlined in the given Algorithm 1. Figure 8 shows the flowchart of MAGDM based on Cq-ROFHWA operator ($\aleph = 1$).

7. Numerical example

The prior section presents a new MAGDM process. In order to further explain the methodology of the suggested decision-making approach, we are applying it to a real decision-making problem.

Suppose a group of DMs compare alternatives FCSs for the surface clothing of building compliant with their practical properties. The gathering interacts with four experts; $\mathfrak{e}_1$ = architect, $\mathfrak{e}_2$ =structural designer, $\mathfrak{e}_3$ = constructor, and $\mathfrak{e}_4$ = adviser, independently. The experts analyze four choices frameworks which are:

- $\daleth_1$: "Natural stone clothing";
- $\daleth_2$: "Plastic painting";
- $\daleth_3$: "Compact laminate clothing";
- $\daleth_4$: "Wood clothing".

The experts compare each pair of attributes $\daleth_i$ and $\daleth_j$ $(i, j = 1, 2, 3, 4)$, and provide Cq-ROFNs $\breve{\gimel}_{ij}^k = ((\breve{\wp}_{ij}^k, \omega_{\breve{\wp}_{ij}}^k), (\breve{\Re}_{ij}^k, \omega_{\breve{\Re}_{ij}}^k))$ $(k = 1, 2, 3, 4)$, composed of the complex MD $(\breve{\wp}_{ij}^k, \omega_{\breve{\wp}_{ij}}^k)$ to which $\daleth_i$ is preferable to $\daleth_j$ and the complex NMD $(\breve{\Re}_{ij}^k, \omega_{\breve{\Re}_{ij}}^k)$ t o which $\daleth_i$ is not preferable to $\daleth_j$, and then develop the Cq-ROFPRs $\mathscr{R}_k = (\breve{\gimel}_{ij}^k)_{4\times 4}$ $(k = 1, 2, 3, 4)$ as follows:

$$\pounds = \left(\frac{\mathfrak{s}_1}{((0.9, 0.8), (0.8, 0.8))}, \frac{\mathfrak{s}_2}{((0.8, 0.7), (0.7, 0.9))}, \frac{\mathfrak{s}_3}{((0.6, 0.8), (0.9, 0.7))}, \frac{\mathfrak{s}_4}{((0.9, 0.8), (0.5, 0.6))}, \right.$$

$$\left. \frac{\mathfrak{s}_5}{((0.8, 0.9), (0.5, 0.8))}, \frac{\mathfrak{s}_6}{((0.7, 0.8), (0.9, 0.6))} \right),$$

$$\Xi = \left(\frac{\mathfrak{s}_1\mathfrak{s}_2}{((0.72, 0.59), (0.82, 0.92))}, \frac{\mathfrak{s}_2\mathfrak{s}_3}{((0.50, 0.58), (0.90, 0.91))}, \frac{\mathfrak{s}_3\mathfrak{s}_4}{((0.54, 0.65), (0.91, 0.75))}, \frac{\mathfrak{s}_4\mathfrak{s}_5}{((0.72, 0.70), (0.65, 0.83))}, \right.$$

$$\left. \frac{\mathfrak{s}_5\mathfrak{s}_6}{((0.57, 0.71), (0.90, 0.83))}, \frac{\mathfrak{s}_6\mathfrak{s}_1}{((0.63, 0.65), (0.92, 0.83))}, \frac{\mathfrak{s}_2\mathfrak{s}_6}{((0.59, 0.60), (0.91, 0.91))}, \frac{\mathfrak{s}_3\mathfrak{s}_5}{((0.51, 0.72), (0.91, 0.81))} \right).$$

Box XVII

$$A(\mathfrak{G}) = \left[\begin{array}{ccc} (0,0) & ((0.72,0.59),(0.82,0.92)) & (0,0) \\ ((0.72,0.59),(0.82,0.92)) & (0,0) & ((0.50,0.58),(0.90,0.91)) \\ (0,0) & ((0.50,0.58),(0.90,0.91)) & (0,0) \\ (0,0) & (0,0) & ((0.54,0.65),(0.91,0.75)) \\ (0,0) & (0,0) & ((0.51,0.72),(0.91,0.81)) \\ ((0.63,0.65),(0.92,0.83)) & ((0.59,0.60),(0.91,0.91)) & (0,0) \end{array}\right.$$

$$\left.\begin{array}{ccc} (0,0) & (0,0) & ((0.63,0.65),(0.92,0.83)) \\ (0,0) & (0,0) & ((0.59,0.60),(0.91,0.91)) \\ ((0.54,0.65),(0.91,0.75)) & ((0.51,0.72),(0.91,0.81)) & (0,0) \\ (0,0) & ((0.72,0.70),(0.65,0.83)) & (0,0) \\ ((0.72,0.70),(0.65,0.83)) & (0,0) & ((0.57,0.71),(0.90,0.83)) \\ (0,0) & ((0.57,0.71),(0.90,0.83)) & (0,0) \end{array}\right],$$

$$R(\mathfrak{G}) = \left[\begin{array}{ccc} (0,0) & ((0.64,0.67),(0.47,0.46)) & (0,0) \\ ((0.64,0.67),(0.47,0.46)) & (0,0) & ((0.60,0.54),(0.37,0.38)) \\ (0,0) & ((0.60,0.54),(0.37,0.38)) & (0,0) \\ (0,0) & (0,0) & ((0.72,0.62),(0.49,0.51)) \\ (0,0) & (0,0) & ((0.60,0.49),(0.39,0.40)) \\ ((0.64,0.64),(0.46,0.47)) & ((0.56,0.54),(0.37,0.38)) & (0,0) \end{array}\right.$$

$$\left.\begin{array}{ccc} (0,0) & (0,0) & ((0.64,0.64),(0.46,0.47)) \\ (0,0) & (0,0) & ((0.56,0.54),(0.37,0.38)) \\ ((0.72,0.62),(0.49,0.51)) & ((0.60,0.49),(0.39,0.40)) & (0,0) \\ (0,0) & ((0.66,0.59),(0.51,0.51)) & (0,0) \\ ((0.66,0.59),(0.51,0.51)) & (0,0) & ((0.56,0.49),(0.39,0.40)) \\ (0,0) & ((0.56,0.49),(0.39,0.40)) & (0,0) \end{array}\right],$$

$$\begin{aligned} Spec(R(\mathfrak{G})) =& \{((-1.1650,-1.0379),(-0.7617,-0.7811)),((-0.9383,-0.8853),(-0.6825,-0.6910)),\\ & ((-0.5228,-0.5125),(-0.3994,-0.3995)),((0.0000,0.0000),(0.0001,0.0000)),\\ & ((0.9378,0.8872),(0.6806,0.6867)),((1.6883,1.5485),(1.1630,1.1849))\}. \end{aligned}$$

Therefore,

$$RE(\mathfrak{G}) = ((5.2522,4.8715),(3.6873,3.7434)).$$

Box XVIII

Step 1. The Cq-ROFDGs $\mathfrak{D}_k$ according to Cq-ROFPRs in Tables 1–4, are shown in Figure 9;

Step 2. The energy of each C3-ROFDG is determined in Table 5;

Step 3. Utilizing Eq. (6.1), each expert's weight (see, Table 6) can be calculated as;

Step 4. Compute the averaged Cq-ROFE $\breve{\gimel}_i^{(k)}$ of the FCS $\daleth_i$ overall the other FCSs for the experts $\mathfrak{e}_k (k = 1,2,3,4)$ by the Cq-ROF Hamacher averaging (Cq-ROFHA) operator with $\aleph = 1$ is shown in Box XIX. The final results are shown in Table 7.

Step 5. Compute a collective Cq-ROFE $\breve{\gimel}_i$ $(i = 1,2,3,4)$ of the FCS $\daleth_i$ over all the other FCSs based on the Cq-ROFHWA operator with $\aleph = 1$ (Table 8);

Step 6. Determine the score functions $\digamma(\breve{\gimel}_i)$ of $\breve{\gimel}_i (i = 1,2,3,4)$, utilizing Defenition 4.

$$\digamma(\breve{\gimel}_1) = 0.5549, \quad \digamma(\breve{\gimel}_2) = 0.5393,$$

$$\digamma(\breve{\gimel}_3) = 0.5285, \quad \digamma(\breve{\gimel}_4) = 0.5649.$$

Step 7. Rank all the FCSs $\daleth_i (i = 1,2,3,4)$ according to the values of $\digamma(\breve{\gimel}_i)(i = 1,2,3,4)$. Then, $\daleth_4 \succ \daleth_1 \succ \daleth_2 \succ \daleth_3$.

INPUT: A discrete set of FCSs (alternatives) $\mathfrak{X} = \{\daleth_1, \daleth_2, \ldots, \daleth_n\}$, a set of specialists $\mathfrak{e} = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_s\}$ and construction of Cq-ROFPR $\mathscr{R}_k = (\breve{\gimel}_{ij}^{(k)})_{n\times n}$ $(i, j = 1, 2, \ldots, n)$ for each specialist.

OUTPUT: The most advantageous FCS selection.

Step 1. Determine the $E(\mathfrak{D}_k)$ and $RE(\mathfrak{D}_k)$ $(k = 1, 2, \ldots, s)$ of each Cq-ROFDG.

Step 2. On the basis of $E(\mathfrak{D}_k)$ and $RE(\mathfrak{D}_k)$, calculate the wight vector of experts.

$$\varpi_k = \left(\left(\frac{E((\mathfrak{D}_{\wp})_k)}{\sum_{l=1}^{s} E((\mathfrak{D}_{\wp})_l)}, \frac{E((\mathfrak{D}_{\omega_{\wp}})_k)}{\sum_{l=1}^{s} E((\mathfrak{D}_{\omega_{\wp}})_l)} \right), \left(\frac{E((\mathfrak{D}_{\mathfrak{R}})_k)}{\sum_{l=1}^{s} E((\mathfrak{D}_{\mathfrak{R}})_l)}, \frac{E((\mathfrak{D}_{\omega_{\mathfrak{R}}})_k)}{\sum_{l=1}^{s} E((\mathfrak{D}_{\omega_{\mathfrak{R}}})_l)} \right) \right), \tag{6.1}$$

and

$$\varpi_k = \left(\left(\frac{RE((\mathfrak{D}_{\wp})_k)}{\sum_{l=1}^{s} RE((\mathfrak{D}_{\wp})_l)}, \frac{RE((\mathfrak{D}_{\omega_{\wp}})_k)}{\sum_{l=1}^{s} RE((\mathfrak{D}_{\omega_{\wp}})_l)} \right), \left(\frac{RE((\mathfrak{D}_{\mathfrak{R}})_k)}{\sum_{l=1}^{s} RE((\mathfrak{D}_{\mathfrak{R}})_l)}, \frac{RE((\mathfrak{D}_{\omega_{\mathfrak{R}}})_k)}{\sum_{l=1}^{s} RE((\mathfrak{D}_{\omega_{\mathfrak{R}}})_l)} \right) \right), \tag{6.2}$$

Step 3. Aggregate all $\breve{\gimel}_{ij}^{(k)}(i, j = 1, 2, \ldots, n)$ corresponding to the FCS $\daleth_i$, and get the Cq-ROF element (Cq-ROFE) $\breve{\gimel}_i^{(k)}$ of the FCS $\daleth_i$ over all the other FCSs for the specialist $\mathfrak{e}_k$ by using the Cq-ROFA operator.

Step 4. Aggregate all $\breve{\gimel}_i^{(k)}(k = 1, 2, \ldots, s)$ into a fused Cq-ROFN $\breve{\gimel}_i$ for the FCS $\daleth_i$ using the Cq-ROFWA operator.

Step 5. Determine the score functions $\digamma(\breve{\gimel}_i)$ of $\breve{\gimel}_i(i = 1, 2, \ldots, n)$, utilizing Def. 2.4.

Step 6. Rank all the FCSs $\daleth_i$ according to $\digamma(\breve{\gimel}_i)(i = 1, 2, \ldots, n)$.

Step 7. Output the best FCS.

Algorithm 1. The algorithm for the optimal Facade Clothing System (FCS) selection.

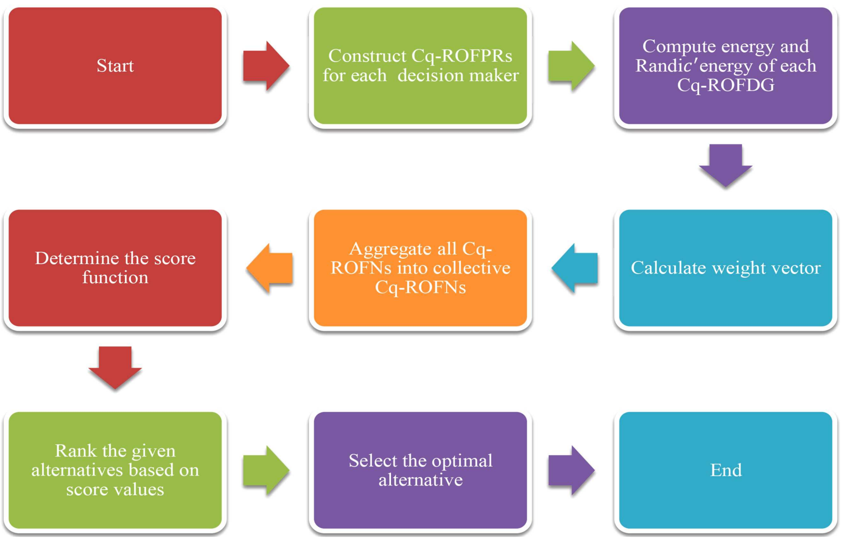

Figure 8. Flow chart of MAGDM based on Cq-ROFHWA operator $\aleph = 1$).

Table 1. Cq-ROFPRs of the architect.

$\mathscr{R}_1$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.5, 0.5), (0.5, 0.5))$	$((0.6, 0.8), (0.8, 0.7))$	$((0.8, 0.6), (0.4, 0.9))$	$((0.7, 0.9), (0.7, 0.5))$
$\daleth_2$	$((0.8, 0.7), (0.6, 0.8))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.7, 0.7), (0.8, 0.6))$	$((0.6, 0.7), (0.6, 0.8))$
$\daleth_3$	$((0.4, 0.9), (0.8, 0.6))$	$((0.8, 0.6), (0.7, 0.7))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.9, 0.6), (0.6, 0.8))$
$\daleth_4$	$((0.7, 0.5), (0.7, 0.9))$	$((0.6, 0.8), (0.6, 0.7))$	$((0.6, 0.8), (0.9, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$

$$\breve{\gimel}_i^{(k)} = \text{Cq-ROFHA}(\breve{\gimel}_{i1}^{(k)}, \breve{\gimel}_{i2}^{(k)}, \ldots, \breve{\gimel}_{in}^{(k)})$$

$$= \left(\left(\sqrt[q]{1 - \left(\prod_{j=1}^{n} \left(1 - (\breve{\wp}_{ij})^q\right) \right)^{1/n}}, \sqrt[q]{1 - \left(\prod_{j=1}^{n} \left(1 - (\omega_{\breve{\wp}_{ij}})^q\right) \right)^{1/n}} \right), \left(\left(\prod_{j=1}^{n} \breve{\Re}_{ij} \right)^{1/n}, \left(\prod_{j=1}^{n} \omega_{\breve{\Re}_{ij}} \right)^{1/n} \right) \right).$$

Box XIX

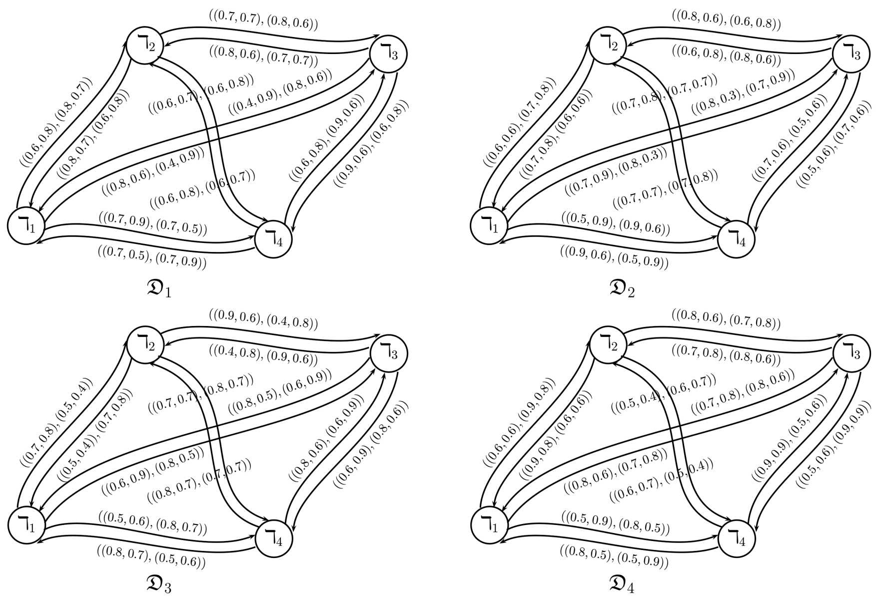

Figure 9. C3-ROFDG.

Table 2. Cq-ROFPRs of the structural designer.

$\mathscr{R}_2$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.5, 0.5), (0.5, 0.5))$	$((0.6, 0.6), (0.7, 0.8))$	$((0.7, 0.9), (0.8, 0.3))$	$((0.5, 0.9), (0.9, 0.6))$
$\daleth_2$	$((0.7, 0.8), (0.6, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.8, 0.6), (0.6, 0.8))$	$((0.7, 0.8), (0.7, 0.7))$
$\daleth_3$	$((0.8, 0.3), (0.7, 0.9))$	$((0.6, 0.8), (0.8, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.5, 0.6), (0.7, 0.6))$
$\daleth_4$	$((0.9, 0.6), (0.5, 0.9))$	$((0.7, 0.7), (0.7, 0.8))$	$((0.7, 0.6), (0.5, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$

Table 3. Cq-ROFPRs of the constructor.

$\mathscr{R}_3$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.5, 0.5), (0.5, 0.5))$	$((0.7, 0.8), (0.5, 0.4))$	$((0.8, 0.5), (0.6, 0.9))$	$((0.5, 0.6), (0.8, 0.7))$
$\daleth_2$	$((0.5, 0.4), (0.7, 0.8))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.9, 0.6), (0.4, 0.8))$	$((0.7, 0.7), (0.8, 0.7))$
$\daleth_3$	$((0.6, 0.9), (0.8, 0.5))$	$((0.4, 0.8), (0.9, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.6, 0.9), (0.8, 0.6))$
$\daleth_4$	$((0.8, 0.7), (0.5, 0.6))$	$((0.8, 0.7), (0.7, 0.7))$	$((0.8, 0.6), (0.6, 0.9))$	$((0.5, 0.5), (0.5, 0.5))$

Table 4. Cq-ROFPRs of the adviser.

$\mathscr{R}_4$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.5, 0.5), (0.5, 0.5))$	$((0.6, 0.6), (0.9, 0.8))$	$((0.7, 0.8), (0.8, 0.6))$	$((0.5, 0.9), (0.8, 0.5))$
$\daleth_2$	$((0.9, 0.8), (0.6, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.8, 0.6), (0.7, 0.8))$	$((0.5, 0.4), (0.6, 0.7))$
$\daleth_3$	$((0.8, 0.6), (0.7, 0.8))$	$((0.7, 0.8), (0.8, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$	$((0.5, 0.6), (0.9, 0.9))$
$\daleth_4$	$((0.8, 0.5), (0.5, 0.9))$	$((0.6, 0.7), (0.5, 0.4))$	$((0.9, 0.9), (0.5, 0.6))$	$((0.5, 0.5), (0.5, 0.5))$

Table 5. Energy of each C3-ROFDG.

Energy	$\breve{\wp}$	$\omega_{\breve{\wp}}$	$\breve{\mathfrak{R}}$	$\omega_{\breve{\mathfrak{R}}}$
$E(\mathfrak{D}_1)$	4.0943	4.3907	4.0943	4.3907
$E(\mathfrak{D}_2)$	4.0586	4.1770	4.0586	4.1770
$E(\mathfrak{D}_3)$	3.9962	4.1055	3.9962	4.1055
$E(\mathfrak{D}_4)$	4.0969	4.2162	4.0969	4.2162

Table 6. Weight of each C3-ROFDG.

Weights	$\breve{\wp}$	$\omega_{\breve{\wp}}$	$\breve{\mathfrak{R}}$	$\omega_{\breve{\mathfrak{R}}}$
ϖ_1	0.2520	0.2600	0.2520	0.2600
ϖ_2	0.2498	0.2473	0.2498	0.2473
ϖ_3	0.2460	0.2431	0.2460	0.2431
ϖ_4	0.2522	0.2496	0.2522	0.2496.

Step 8. Thus, the optimal FCS is $\daleth_4$.

Now, the RMs of the Cq-ROFDGs $R(\mathfrak{D}_k) = \mathscr{R}_k^R$ $(k = 1, 2, 3, 4)$, (Figure 9) are shown in Tables 9–12. We describe our method in the following algorithm.

Step 1. The Randić energy of each Cq-ROFDG is calculated in Table 13:

Step 2. Utilizing Eq. (6.2.), each expert's weight can be calculated in Table 14.

Step 3. Now, we will utilize above calculated weights and determine a collective Cq-ROFE $\breve{\gimel}_i$ $(i = 1, 2, 3, 4)$ of the FCS $\daleth_i$ over all the other FCSs based on the Cq-ROFHWA operator with $\aleph = 1$ (Table 15):

$$\breve{\gimel}_i = \text{C}q\text{-ROFHWA}(\breve{\gimel}_i^{(1)}, \breve{\gimel}_i^{(2)}, \ldots, \breve{\gimel}_i^{(n)}).$$

Step 4. Determine the score functions $\digamma(\breve{\gimel}_i)$ of the Cq-ROFE $\breve{\gimel}_i (i = 1, 2, 3, 4)$, utilizing Defenition 4.

$$\digamma(\breve{\gimel}_1) = 0.5556, \ \digamma(\breve{\gimel}_2) = 0.5395, \ \digamma(\breve{\gimel}_3) = 0.5292, \ \digamma(\breve{\gimel}_4) = 0.5651.$$

Step 5. Rank all the FCSs $\daleth_i (i = 1, 2, 3, 4)$ according to the values of $\digamma(\breve{\gimel}_i)(i = 1, 2, 3, 4)$. Then, $\daleth_4 \succ \daleth_1 \succ \daleth_2 \succ \daleth_3$.

Step 6. Thus, the optimal FCS is $\daleth_4$ among the four given FCSs.

Table 7. The fused results of the experts $\mathfrak{e}_k$ $(k = 1, 2, 3, 4)$.

Experts		The fused results of the experts
$\mathfrak{e}_1$	$\breve{\gimel}_1^{(1)}$	((0.6805, 0.7670), (0.5785, 0.6300))
	$\breve{\gimel}_2^{(1)}$	((0.6805, 0.6651), (0.6160, 0.6620))
	$\breve{\gimel}_3^{(1)}$	((0.7526, 0.7257), (0.6402, 0.6402))
	$\breve{\gimel}_4^{(1)}$	((0.6118, 0.7024), (0.6593, 0.6593))
$\mathfrak{e}_2$	$\breve{\gimel}_1^{(2)}$	((0.5921, 0.8073), (0.7085, 0.5180))
	$\breve{\gimel}_2^{(2)}$	((0.7012, 0.7142), (0.5958, 0.6402))
	$\breve{\gimel}_3^{(2)}$	((0.6418, 0.6253), (0.6654, 0.6344))
	$\breve{\gimel}_4^{(2)}$	((0.7573, 0.6118),(0.5439, 0.6817))
$\mathfrak{e}_3$	$\breve{\gimel}_1^{(3)}$	((0.6665, 0.6418), (0.5886, 0.5958))
	$\breve{\gimel}_2^{(3)}$	((0.7321, 0.5790), (0.5785, 0.6880))
	$\breve{\gimel}_3^{(3)}$	((0.5402, 0.8336), (0.7326, 0.5477))
	$\breve{\gimel}_4^{(3)}$	((0.7579, 0.6401), (0.5692, 0.6593))
$\mathfrak{e}_4$	$\breve{\gimel}_1^{(4)}$	((0.5921, 0.7670), (0.7326, 0.5886))
	$\breve{\gimel}_2^{(4)}$	((0.7582, 0.6315), (0.5958, 0.6402))
	$\breve{\gimel}_3^{(4)}$	((0.6665, 0.6575), (0.7085, 0.6817))
	$\breve{\gimel}_4^{(4)}$	((0.7670, 0.7321), (0.5000, 0.5733))

Table 8. Aggregated C3-ROFN.

Aggregated values	$\breve{\wp}$	$\omega_{\breve{\wp}}$	$\breve{\mathfrak{R}}$	$\omega_{\breve{\mathfrak{R}}}$
$\breve{\gimel}_1$	0.6367	0.7558	0.6487	0.5822
$\breve{\gimel}_2$	0.7203	0.6537	0.5965	0.6572
$\breve{\gimel}_3$	0.6646	0.7284	0.6855	0.6247
$\breve{\gimel}_4$	0.7327	0.6779	0.5652	0.6420

Table 9. RM of the Cq-ROFDG $\mathfrak{D}_1$.

$\mathscr{R}_1^R$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.50, 0.50), (0.50, 0.50))$	$((0.48, 0.46), (0.51, 0.47))$	$((0.48, 0.46), (0.50, 0.48))$	$((0.50, 0.46), (0.49, 0.47))$
$\daleth_2$	$((0.51, 0.47), (0.48, 0.46))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.48, 0.48), (0.49, 0.47))$	$((0.49, 0.48), (0.48, 0.45))$
$\daleth_3$	$((0.50, 0.48), (0.48, 0.46))$	$((0.49, 0.47), (0.48, 0.48))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.50, 0.48), (0.47, 0.47))$
$\daleth_4$	$((0.49, 0.47), (0.50, 0.46))$	$((0.48, 0.45), (0.49, 0.48))$	$((0.47, 0.47), (0.50, 0.48))$	$((0.50, 0.50), (0.50, 0.50))$

Table 10. RM of the Cq-ROFDG $\mathfrak{D}_2$.

$\mathscr{R}_2^R$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.50, 0.50), (0.50, 0.50))$	$((0.50, 0.44), (0.47, 0.53))$	$((0.54, 0.50), (0.44, 0.53))$	$((0.49, 0.47), (0.50, 0.51))$
$\daleth_2$	$((0.47, 0.53), (0.50, 0.44))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.49, 0.52), (0.49, 0.48))$	$((0.44, 0.49), (0.56, 0.46))$
$\daleth_3$	$((0.44, 0.53), (0.54, 0.50))$	$((0.49, 0.48), (0.49, 0.52))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.48, 0.56), (0.52, 0.46))$
$\daleth_4$	$((0.50, 0.51), (0.49, 0.47))$	$((0.56, 0.46), (0.44, 0.49))$	$((0.52, 0.46), (0.48, 0.56))$	$((0.50, 0.50), (0.50, 0.50))$

Table 11. RM of the Cq-ROFDG $\mathfrak{D}_3$.

$\mathscr{R}_3^R$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.50, 0.50), (0.50, 0.50))$	$((0.49, 0.56), (0.53, 0.47))$	$((0.56, 0.45), (0.46, 0.54))$	$((0.46, 0.51), (0.54, 0.48))$
$\daleth_2$	$((0.53, 0.47), (0.49, 0.56))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.55, 0.48), (0.46, 0.51))$	$((0.45, 0.54), (0.54, 0.44))$
$\daleth_3$	$((0.46, 0.54), (0.56, 0.45))$	$((0.46, 0.51), (0.55, 0.48))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.51, 0.44), (0.47, 0.52))$
$\daleth_4$	$((0.54, 0.48), (0.46, 0.51))$	$((0.54, 0.44), (0.45, 0.54))$	$((0.47, 0.52), (0.51, 0.44))$	$((0.50, 0.50), (0.50, 0.50)))$

Table 12. RM of the Cq-ROFDG $\mathfrak{D}_4$.

$\mathscr{R}_4^R$	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\daleth_1$	$((0.50, 0.50), (0.50, 0.50))$	$((0.50, 0.49), (0.46, 0.50))$	$((0.53, 0.47), (0.41, 0.48))$	$((0.49, 0.46), (0.52, 0.53))$
$\daleth_2$	$((0.46, 0.50), (0.50, 0.49))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.48, 0.53), (0.47, 0.46))$	$((0.44, 0.51), (0.59, 0.50))$
$\daleth_3$	$((0.41, 0.48), (0.53, 0.47))$	$((0.47, 0.46), (0.48, 0.53))$	$((0.50, 0.50), (0.50, 0.50))$	$((0.47, 0.49), (0.53, 0.48))$
$\daleth_4$	$((0.52, 0.53), (0.49, 0.46))$	$((0.59, 0.50), (0.44, 0.51))$	$((0.53, 0.48), (0.47, 0.49))$	$((0.50, 0.50), (0.50, 0.50))$

Table 13. Randić energy of each C3-ROFDG.

$RE(\mathfrak{D}_i)$	$\breve{\wp}$	$\omega_{\breve{\wp}}$	$\breve{\Re}$	$\omega_{\breve{\Re}}$
$RE(\mathfrak{D}_1)$	2.9345	2.8147	2.9345	2.8147
$RE(\mathfrak{D}_2)$	2.9547	2.9713	2.9547	2.9713
$RE(\mathfrak{D}_3)$	3.0075	2.9699	3.0075	2.9699
$RE(\mathfrak{D}_4)$	2.9368	2.9489	2.9368	2.9489

Table 14. Weight of each expert.

Weights	$\breve{\wp}$	$\omega_{\breve{\wp}}$	$\breve{\Re}$	$\omega_{\breve{\Re}}$
ϖ_1	0.2507	0.2405	0.2507	0.2405
ϖ_2	0.2497	0.2539	0.2497	0.2539
ϖ_3	0.2542	0.2537	0.2542	0.2537
ϖ_4	0.2482	0.2519	0.2482	0.2519

7.1. Comparison analysis

To examine the consequences of the technique utilized, we will compare our results with previous findings in the literature:

Table 15. Aggregated C3-ROFN.

Aggregated values	$\breve{\wp}$	$\omega_{\breve{\wp}}$	$\breve{\Re}$	$\omega_{\breve{\Re}}$
$\breve{\gimel}_1$	0.6376	0.7551	0.6471	0.5810
$\breve{\gimel}_2$	0.7208	0.6532	0.5955	0.6573
$\breve{\gimel}_3$	0.6642	0.7293	0.6851	0.6237
$\breve{\gimel}_4$	0.7334	0.6768	0.5645	0.6419

- We compare our approach with the Cq-ROF weighted averaging (Cq-ROFWA) operator, by taking parameters $\aleph = 1$ and $q = 3$. The ranking results which obtained from Cq-ROFWA operator are listed as: $\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$. We get the same ranking outcomes as in literature. Nevertheless, the methodology of graph adopted based on the Hamacher TN and TCN, and by taking parameter $\aleph = 1$, Hamacher TNs and TCNs are converted into the algebraic TNs and TCNs. As the algebraic TNs and TCNs are special instance of the Hamacher TNs and TCNs. So, the scheme described in this article is more comprehensive than the methodology proposed by Liu et al. [41];

- Compare with the Cq-ROF Einstein weighted averaging (Cq-ROFEWA) operator by taking parameters $\aleph = 2$, and $q = 3$. The ranking outcomes by utilizing Cq-ROFEWA operator are obtained as: $\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3 >$. But, the approach proposed in this article based on the Hamacher TN and TCN, and the Einstein TN and TCN are just a special case of the Hamacher TN and TCN when we take the parameter $\aleph = 2$. Hence, the proposed approach is more effective than the Cq-ROFEWA operator;
- We compare our proposed approach with the CIF Hamacher Weighted Averaging (CIFHWA) operator by taking $q = 1$. Utilizing CIFHWA operator, the ranking outcomes are acquired as: $\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$. The CIFHWA operator just aggregates the CIFNs, and the CIFN must meet the conditions that $0 \leq \breve{\wp} + \breve{\Re} \leq 1$ and $0 \leq \omega_{\breve{\wp}} + \omega_{\breve{\Re}} \leq 1$. Clearly, most of the assessment values do not follow the limit of condition described as: $0 \leq \breve{\wp} + \breve{\Re} \leq 1$ and $0 \leq \omega_{\breve{\wp}} + \omega_{\breve{\Re}} \leq 1$, so, this example, shows that the CIFHWA operator is not reasonable in described instance;
- Further, we compare our developed strategy with the CPF Hamacher Weighted Averaging (CPFHWA) operator (i.e., taking $q = 2$). The CPFHWA operator gives the ranking results as: $\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$. The CPFHWA operator just aggregates the CPFNs and has wider range than CIFHWA operator, and the CPFN should satisfy the limit of condition described as: $0 \leq \breve{\wp}^2 + \breve{\Re}^2 \leq 1$ and $0 \leq \omega_{\breve{\wp}}^2 + \omega_{\breve{\Re}}^2 \leq 1$. Clearly, most of the assessment values do not follow the limit of condition described as: $0 \leq \breve{\wp}^2 + \breve{\Re}^2 \leq 1$ and $0 \leq \omega_{\breve{\wp}}^2 + \omega_{\breve{\Re}}^2 \leq 1$ in this example. Hence, the CPFHWA operator is not appropriate for that described example;
- Now developed approach will be compared with the Complex Fermatean Fuzzy Hamacher Weighted Averaging (CFFHWA) operator by taking $q = 3$. According to CFFHWA operator, we get the ranking results as: $\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$. The CFFHWA operator satisfies the conditional limit of $0 \leq \breve{\wp}^3 + \breve{\Re}^3 \leq 1$ and $0 \leq \omega_{\breve{\wp}}^3 + \omega_{\breve{\Re}}^3 \leq 1$. However, the space of application of the CFFHWA operator is broader than the CPFHWA operator but limited than the Cq-ROF Hamacher weighted averaging operator discussed in our proposed scheme. Clearly, the assessment values in this decision-making problem meet the limit of conditions $0 \leq \breve{\wp}^q + \breve{\Re}^q \leq 1$ and $0 \leq \omega_{\breve{\wp}}^q + \omega_{\breve{\Re}}^q \leq 1$. So, in this example, the CFFHWA operator cannot completely deal the decision making problem.

Detailed evaluation results gained by using different MAGDM approaches are given in Tables 16, 17, and Figures 10, 11.

The q-ROFG deals with one-dimensional information at a time, which often results data loss. But the Cq-ROFG is a strong way of dealing with ambiguous information compared to the q-ROFG, since it incorporates two-dimensional information in a single element. Thus, the loss of data can be avoided by adding the second dimension of MD and NMD. If we consider the phase term of MD and NMD to be zero, then the Cq-ROFG is transformed to q-ROFG, and if we take $q = 1$ and $q = 2$ then the q-ROFG is converted to IFG and PFG, respectively. The comparison of Cq-ROFG with existing FG theories are shown in Table 18.

The merits of our approach are summarized in the following points:

Table 16. Comparison of decision results by utilizing different approaches (Energy).

Approaches	**Parameter**	$F(\breve{\gimel}_1)$	$F(\breve{\gimel}_2)$	$F(\breve{\gimel}_3)$	$F(\breve{\gimel}_4)$	**Order relation**
Cq-ROFWA operator	$\aleph = 1, q = 3$	0.5549	0.5393	0.5285	0.5649	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
Cq-ROFWE operator	$\aleph = 2, q = 3$	0.5532	0.5386	0.5257	0.5633	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
CIFHWA operator	$q = 1, \aleph = 3$	0.5379	0.5288	0.5161	0.5482	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
CPFHWA operator	$q = 2, \aleph = 3$	0.5511	0.5381	0.5223	0.5630	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
CFFHWA operator	$q = 3,\ \aleph = 3$	0.5524	0.5382	0.5240	0.5625	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$

Table 17. Comparison of decision results by utilizing different approaches (Randić energy).

Approaches	**Parameter**	$F(\breve{\gimel}_1)$	$F(\breve{\gimel}_2)$	$F(\breve{\gimel}_3)$	$F(\breve{\gimel}_4)$	**Order relation**
Cq-ROFWA operator	$\aleph = 1, q = 3$	0.5558	0.5398	0.5299	0.5654	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
Cq-ROFWE operator	$\aleph = 2, q = 3$	0.5543	0.5394	0.5271	0.5641	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
CIFHWA operator	$q = 1, \aleph = 3$	0.5392	0.5299	0.5175	0.5492	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
CPFHWA operator	$q = 2, \aleph = 3$	0.5524	0.5392	0.5239	0.5640	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$
CFFHWA operator	$q = 3,\ \aleph = 3$	0.5535	0.5391	0.5256	0.5633	$\daleth_4 > \daleth_1 > \daleth_2 > \daleth_3$

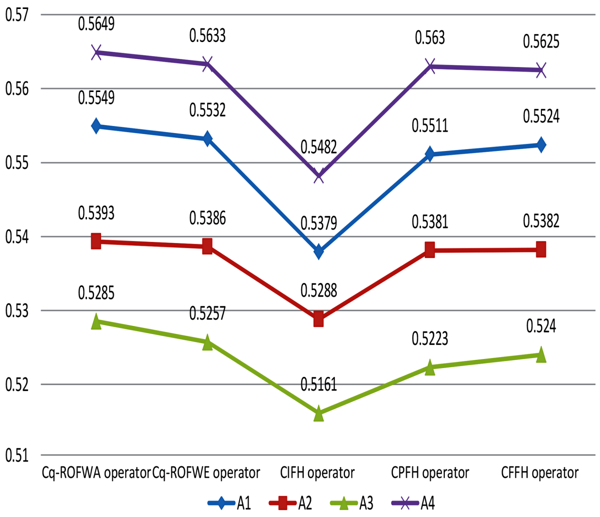

Figure 10. Comparison with some existing approaches (energy).

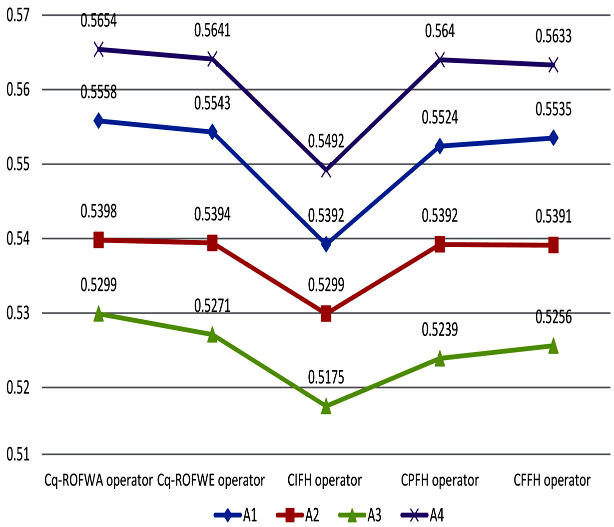

Figure 11. Comparison with some existing approaches (RE).

1. Our proposed scheme estimate that the sum of qth power of MD and NMD closed in complex plane of unit disc. The CIFS and CPFS loses their ability, when they tend to deal with such kind of information $((0.8, 0.7), (0.9, 0.8))$, provided by decision-making experts. Here, the Cq-ROFS proves their ability, due to its flexibility of qth power of MD and NMD in complex plane of unit disc;
2. Our proposed scheme is more broad than CIFS and CPFS. The notions of CIFS and CPFS can be originated from Cq-ROFS with the specific qth powers, such as $q = 1$ and $q = 2$. So, the proposed scheme is more preferable than CIFS and CPFS;
3. The new framework is evidently apparent and in the MAGDM environment, the Cq-ROF approach can be utilized effectively with the key role of minor data loss;
4. Usage of graph theory is one of the crucial aspect of proposed scheme, which shows its superiority on other existing methods;
5. For depicting information in realistic decision-making problem, the Cq-ROFSs approach can be implemented effectively;
6. Under Cq-ROF domain, to address the MAGDM problems, the Hamacher operator is a more powerful tool.

Table 18. Comparison of Cq-ROFG model with extant models in literature.

Model	$\breve{\wp}$	$\breve{\Re}$	$\breve{\pi}$	Periodicity	Represents two dimensional information
FG	✓	×	×	×	×
IFG	✓	✓	✓	×	×
PFG	✓	✓	✓	×	×
FFG	✓	✓	✓	×	×
q-ROFG	✓	✓	✓	×	×
CFG	✓	×	×	✓	✓
CIFG	✓	✓	✓	✓	✓
CPFG	✓	✓	✓	✓	✓
Cq-ROFG	✓	✓	✓	✓	✓

8. Conclusions

The Complex q-Rung Orthopair Fuzzy Set (Cq-ROFS) is an effective way to portray ambiguous data and is better than the Complex IFSs (CIFSs) and the CPFSs. Its prominent feature is that the total of the qth power of the amplitude term (similar to the phase term) of the complex-valued Membership Degree (MD) and the qth power of the amplitude term (similar to the phase term) of the complex-valued Non-Membership Degree (NMD) is equal to or less than 1. In this article, some new Cq-ROF Hamacher operations and Cq-ROF Hamacher aggregation operators, such as Cq-ROFHWA operator, Cq-ROFHOWA operator, Cq-ROFH Weighted Geometric (Cq-ROFHWG) operator and Cq-ROFHOWG operator have been developed for aggregating Cq-ROFNs. Subsequently, the novel idea of Cq-ROFGs utilizing Hamacher operator called Cq-ROFHGs is set forward and its energy and Randić energy is computed. In particular, the energy of a splitting Cq-ROFHG and shadow Cq-ROFHG has been developed. Finally, a quantitative example relating to the selection of Facade Clothing Systems (FCSs) has been provided to show the credibility of the concepts set out in the decision-making process. A Cq-ROFHG can well depict the network fuzziness. In future, our research work will be extended to:

1. Linguistic Cq-ROFGs;
2. 2-Tuple linguistic Cq-ROFGs;
3. Complex spherical fuzzy graphs;
4. 2-Tuple linguistic complex spherical fuzzy graphs.

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