Scientia Iranica E (2022) 29(4), 2230–2240

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions E: Industrial Engineering
http://scientiairanica.sharif.edu

The business advantage of identifying and solving pseudo-continuous-integer periodical linear problems

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Received 28 November 2019; received in revised form 31 July 2020; accepted 26 October 2020

KEYWORDS
Business profit;
Integer programming;
Linear programming;
Operations management;
Operations planning.

Abstract. Many optimization applications require the final value of decision variables to be integer. In many cases, the relaxed optimal solution does not satisfy the integrality constraint; therefore, the problem must be solved using integer or mix-integer programming algorithms with significant computational effort and most likely a worsen objective function value. The contribution of this paper is two-fold: (a) identification of a type of problems in which the relaxed optimal objective function value can be kept at the implementation phase by modifying the planning horizon and (b) identification of a multi-period-based solution procedure. Three small instances are provided in order to illustrate the methodology as well as the economic impact involved. In addition, a fourth industrial-scale case is included for the benefit of practitioners. This work shows that business profit can be increased for pseudo-continuous-integer periodical linear problems by identifying optimal decision-making periods.

1. Introduction

In a competitive business environment, practitioners are interested in solutions that can be obtained and implemented in a simple way. They require that the generated solutions provide better profits (or lower costs) in order to promote business growth and competitiveness.

Linear, integer, and mixed integer programming models are a flourishing field of optimization. Nowadays, they are applied to an immense variety of real-life problems in a number of disciplines [1–10]. The development of these models has provided the ground for enhancement of efficient solutions over decades and they are still progressing quickly. For linear models, the well-known simplex method derived in [11], the interior point method developed in [12], and the one later improved in [13] are used to solve this class of problems. For integer programming models, algorithms and methods such as branch and bound [14], cutting planes [15], and branch and cut [16] exist. It is important to remark that the integer and mixed integer programming models are complex problems, many of which are NP-hard. Therefore, the computational complexity is high with often long computational time.

The process of obtaining the solution of integer and mix integer models is far more complex than that of simple linear programming ones. A particular network problem called transshipment enjoys unique characteristics such that if all demands are integer, all vertexes in the feasible region are integers; therefore, the solution of the network simplex [11,17] is integer without considering the problem as an integer model.

Often, there is a loss in the objective function

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doi: 10.24200/sci.2020.54580.3815

value when a problem is transformed from continuous to integer (or mix-integer). Hence, the search to reduce this loss is a continuous concern for practitioners and academics. There is scant treatment in the operational research academic literature in this regard; therefore, this concern has become the main motivation of this paper.

The contribution of this article is two-fold. The first is identifying a special class of problems whose solution must be integer while the continuous solution is not. However, an integer solution can be obtained by identifying special characteristics of the application. The second contribution is providing a method to convert the continuous solution into an integer solution (which can be implemented) without losing value in the objective function for these kinds of problems.

This paper is organized as follows: Section 2 presents a class of problems (to be named pseudo-continuous-integer periodical linear problems) in which its special characteristics are the focus of this work. Section 3 includes the methodology to transform a continuous solution from this class of problems into an integer one. Section 4 presents three small numerical instances to illustrate the functionality of the method. An industrial size case is represented for the benefit of practitioners. Finally, Section 5 provides conclusions and further research.

2. Special characteristics of pseudo-continuous integer periodical linear problems

Let us define a Pseudo-Continuous-Integer Periodical Linear Problem (PCIPLP) as the one that satisfies the following five characteristics:

1. Pure integer programming problems with no binary variables;
2. Single period planning horizon that repeats identically over a non-limited number of consecutive periods;
3. The time to make decisions can be transformed from every period to once every T periods, where T is an integer number to be determined;
4. Between two consecutive periods, fractional values (resources, demands, etc.) can be conveyed;
5. The objective function of the problem can change from a fixed periodical number to an average per period.

Many special applications in practice satisfy the above conditions including service management (public and private), transportation, production, order acceptance [18] and manufacturing, among others. Some illustrations are included in Section 4.

3. Methodology

Let us consider an integer problem that meets the characteristics of the latter section. Let the integer period problem be:

$$Opt \quad c^t x,$$

s.t.:

$$Ax = b, \qquad x \in \mathbb{Z}^{n+}, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{Z}^{n+}$, and $\mathbb{Z}^{n+}$ be the n dimensional setof non-negative integers.

The relaxed linear model associated to Eq. (1) is:

$$Opt \quad c^t y,$$

s.t.:

$$Ay = b, \qquad y \geq 0, \tag{2}$$

where $y \in \mathbb{R}^n$. Let $I = \{1, \cdots, n\}$ be the index set and y^* be the optimal solution of Eq. (2), with $y_i^* = d_i/e_i$, where $d_i, e_i \in \mathbb{Z}^+ \ \forall \ i \in I$. It is important to remark that all optimal linear programming solutions involve rational numbers because of the computational nature of the algorithms [19]. Let T be the minimum common multiple of all e_i's.

By multiplying both sides of the constraint $Ay = b$ by the scalar T in Eq. (2) and making $w_i = Ty_i$, $\forall \ i \in I$. Then, the one period model in Eq. (2) is transformed into a T-period model as follows:

$$Opt \quad c^t w, \tag{3}$$

s.t.

$$Aw = Tb, \tag{4}$$

$$w \geq 0, \tag{5}$$

where $w \in \mathbb{R}^n$. Relation (3) represents the objective function value taking into account T periods and Eq. (4) represents the technical constraints for a T-period problem.

Notice that the model in Eq. (2) is mathematically equivalent to Relations (3) to (5); however, the latter happens to have integer solutions for all its variables. Since both problems share the same A, c, and $T > 0$, they share the same optimal basis (basic columns of matrix A) and the value for the dual variables. Thus, the sensitivity linear programming tools apply to Relations (3) to (5). Hence, sensitivity analysis interpretations are valid on Relations (3) to (5) as long as the resulting solution remains with all integer values. The latter can be a challenging task.

Besides, Problems (3) to (5) have only a change in the time horizon in Eq. (1) as long as the latter satisfies all the characteristics listed in Section 2.

Table 1. Method for solving the single period problem in Eq. (1) through the multiple-period problem in Relations (3) through (5).

Step 1:	Solve to optimality the continuous linear programming model in Eq. (2) as: $Opt\ c^t y$ s.t. $Ay = b, \quad y \geq 0$
Step 2:	Let the optimal solution of Eq. (2) be $y_i^* = d_i/e_i$, $\forall\ i \in I$, where $d_i,\ e_i \in \mathbb{Z}^+$
Step 3:	Determine T as the minimum common multiple for all e_i, $\forall\ i \in I$, where T determines the planning horizon
Step 4:	Define the optimal solution of the problem in Relations (3) through (5) as: $Opt\ c^t w$ s.t.: $Aw = Tb, \quad w \geq 0$ as $w_i^* = \left(\frac{T}{d_i}\right) e_i$, where $w_i^* \in \mathbb{Z}^+$, $\forall\ i \in I$.
Step 5:	Sensitivity analysis, since the problem in Relations (3) through (5) is a continuous linear programming problem with optimal solution $w_i^* \in \mathbb{Z}^+$, all regular sensitivity analysis techniques are available for solving the problem in Relations (3) through (5) as long as the new solution remains integer

As a matter of fact, all the constraints in Eq. (4) from $Ay = b$ in Eq. (2) have just moved their limits parallel-wise by a factor T.

The optimal solution of Relations (3) to (5) is given by:

$$w_i^* = \left(\frac{T}{e_i}\right) d_i, \qquad \forall\ i \in I, \tag{6}$$

where $\frac{T}{e_i} \in \mathbb{Z}^+$; thus, $w_i^* \in \mathbb{Z}^+$ is an optimal solution to Relations (3) to (5).

Notice that the problem in Relations (3) to (5) is a regular linear programming problem that occurs to have integer solutions for all its variables. Hence, all sensitivity analysis techniques are applied as long as the new solution remains integer.

In summary, a problem that satisfies the conditions of a PCIPLP in Section 2 is re-formulated from a single period to a T period planning horizon with an optimal solution as in Eq. (6) by solving the linear programming continuous problem in Eq. (2).

The pseudo-code in Table 1 transforms the planning horizon for decision-making from one period to T periods for PCIPLPs.

4. Numerical illustrations

Four PCIPLP cases are presented in this section. Three of these cases are small instances to illustrate the methodology in simple terms, while the last one represents an industrial size order acceptance case to show that the methodology can be applied to larger problems.

4.1. A public service management instance

A small city of 15,000 inhabitants consumes an average of 1,200,000 liters of drinkable water per day. The city obtains water from the central purifying facility where water is treated by conventional filtration and chloro-hydration methods. In addition, two chemical compounds (softener and purifier) are included. The city is evaluating two potential suppliers of these chemical compounds. Supplier A offers packages with 4 kg of softener and 1.5 kg of purifier for \$80 a package. Supplier B offers packages with 2 kg of softener and 4.5 kg of purifier for \$100 a package. In order to keep water drinkable, the city facility requires 75 kg of softener and 50 kg of purifier per day. The objective is to provide the daily levels of softener and purifier at a minimum cost for the city [20].

Let x_A and x_B be the number of packages per day to buy from each supplier. The relaxed linear programming model for the daily decision is:

$$\begin{aligned} &\min && 80y_A + 100y_B, && \\ &\text{s.t.:} && && \\ &\text{Softener:} && 4y_A + 2y_B && \geq 75, \\ &\text{Purifier:} && 1.5y_A + 4.5y_B && \geq 50, \\ & && y_A, y_B && \geq 0, \end{aligned} \tag{7}$$

with optimal continuous solution $y_A = 95/6$ and $y_B = 35/6$ and optimal daily cost of \$1,850.00.

The city cannot buy fractional packages from the suppliers. If the integrality constraint is added for both variables, the solution changes to $x_A = 16$ and $x_B = 6$ with an optimal solution of \$1,880.00. These values represent a \$30.00 increment per day, which is approximately 1.62% of increase in the daily cost.

Since the daily requirements of softener and purifier are fixed and the minimum common multiple in the denominator of the decision variables is $T = 6$, the city can buy every six days 95 packages (w_A^*) from supplier A and 35 packages (w_B^*) from supplier B.

Because the packages have separated containers for every element (softener and purifier), the city must measure 75 kg of softener and 50 kg of purifier every day and apply them to the city water supply. The comprehensive purchase will cost \$11,100.00 every six days; meanwhile, the average daily cost remains at its minimum at \$1,850.00.

The problem in Relation (7) is a numerical version of this problem in Eq. (2), while Problem (8) is equivalent to Relations (3) to (5):

$$\begin{aligned}
\min \quad & 80w_A + 100w_B, \\
\text{s.t.:} \quad & \\
\text{Softener } T\text{:} \quad & 4w_A + 2w_B \geq 75 \times 6 = 450, \\
\text{Purifier } T\text{:} \quad & 1.5w_A + 4.5w_B \geq 50 \times 6 = 300, \\
& w_A, w_B \geq 0. \qquad (8)
\end{aligned}$$

Figure 1 shows the three solutions x^*, y^*, and w^*. One can notice that the feasible region of Relation (8) is limited by the constraints Softener T and Purifier T and has re-scaled (moved parallel-wise) the boundaries of both constraints by a factor of $T = 6$ from the initial problem in Relation (7), limited by the original constraints softener and purifier.

Regarding sensitivity analysis, the dual variables of Relation (8) include softener $T = -14$ and purifier $T = -16$ (both integers). Thus, if one right-hand side is moved, it must be such that the new solution remains integer. Or, one can multiply both right-hand sides by multiples of $T = 6$ and keep the same dual variables and an integer solution.

In addition, the dual variables of Relation (7) are Softener $= -14$ and Purifier $= -16$ with the same values than the ones in Relation (8).

4.2. A production mix problem

A small manufacturing facility produces two products. Three machines are used in the manufacturing process, each with 44 available hours per week. The single period is considered as a week.

Table 2 shows operational information where columns 2-4 represent the number of manufacturing hours to produce a unit of each product. The last column in the table represents the marginal contribution obtained per product.

Table 2. Manufacturing operational data for production mix problem.

Product	Compression (hr)	Cut (hr)	Polish (hr)	Marginal contribution ($)
1	0.5	0.49	0.21	50
2	0.9	0.7	0.39	105

The problem consists of finding the optimal production mix to maximize the sum of marginal contributions obtained by the production plan. In practice, it is not possible to manufacture and deliver a fraction of a unit.

Let x_i be the number of units of product i to be produced per week, where $i = 1, 2$. The relaxed linear programming model follows:

$$\begin{aligned}
\max \quad & 50y_1 + 105y_2, \\
\text{s.t.:} \quad & \\
\text{Compression:} \quad & 0.5y_1 + 0.9y_2 \leq 44, \\
\text{Cut:} \quad & 0.49y_1 + 0.7y_2 \leq 44, \\
\text{Polish:} \quad & 0.21y_1 + 0.39y_2 \leq 44, \\
& y_1, y_2 \geq 0. \qquad (9)
\end{aligned}$$

The optimal continuous solution is $y_1 = a_1/b_1 = 0$, $y_2 = a_2/b_2 = 440/9 = 48 + 8/9 \approx 48.8888\cdots$, with an objective function value of $\$5,133 + 1/3$. Thus, $T = 9$.

If the integrality constrain is added to Relation (9), the solution transforms to $x_1 = 1$, $x_2 = 48$, with objective function value $5,090.00. This integer solution translates to $43.33 less per period.

The original period of the problem is a week and

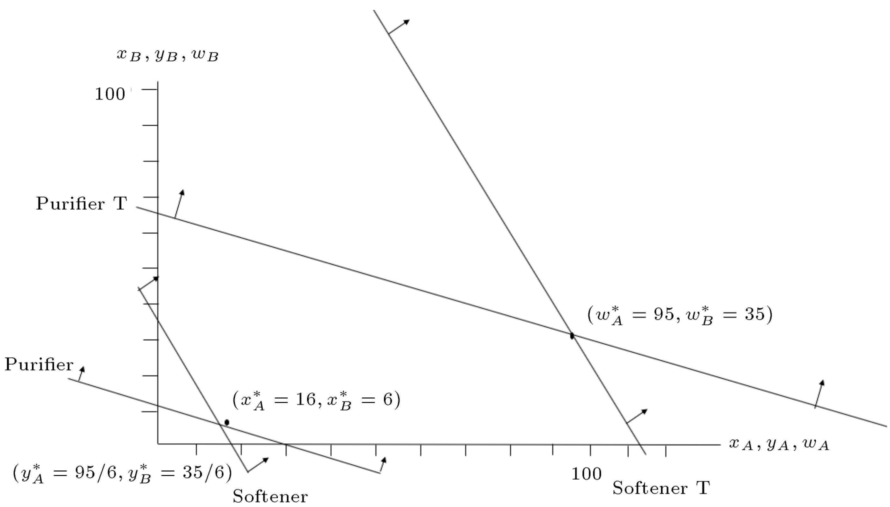

Figure 1. Geometrical representation of the three solution spaces for the illustration in Section 4.1.

Table 3. Airline fleet capacity and cost per route for the airline case.

Fleet	Seat capacity	Availability	Route 1	Route 2	Route 3	Route 4
1	200	5	5	4	3	2
2	150	3	7	5	5	4
3	120	2	9	7	6	6

that the manufacturing facility works for an undetermined number of weeks. A practical solution is to manufacture 48 units of product two. At the end of the first 44 hours per week, the 49-th unit will be finished at $8/9 \approx 0.8888(88.88\%)$. Assuming that the process can stop at no loss of any kind and resume in the next consecutive period, the 49-th unit (needing only 1/9 of the work) at the beginning of the second period is considered to continue the process until finishing it and the manufacturing process continues. In doing so, in eight out of nine weeks, the manufacturing process delivers 49 units of product 2 and only 48 units of product 2 will be delivered in one out of nine weeks. Hence, in 9 weeks, 440 units of product two are delivered, which is an integer number.

In doing so, the average production per week is $440/9 \approx 48.88$ units of product two and the average objective function value is $5,133.33 per week with all the constraints met. This is possible because an inventory of partially-finished units could be conveyed from one period to the next. Notice that the five characteristics of PCIPLPs are met.

This process can be extended to any number of products and resources in the same modeling.

Notice that the proposed solution is neither the solution to Relation (9) nor the integer version of Relation (9). It satisfies the practitioner's requirements in nine weeks.

4.3. An airline opening route decision

An airline company is analyzing the opening of four new routes to be assigned to their newly acquired fleet. Tables 3, 4, and 5 show the related data [20]. The relaxed model is defined as follows:

$$\min \quad 250,000y_{1,1} + 280,000y_{1,2} + 120,000y_{1,3}$$
$$+ 80,000y_{1,4} + 245,000y_{2,1} + 400,000y_{2,2}$$
$$+ 125,000y_{2,3} + 200,000y_{2,4} + 252,000y_{3,1}$$
$$+ 315,000y_{3,2} + 162,000y_{3,3} + 156,000y_{3,4}$$
$$+ 2,500nfp_1 + 3,000nfp_2 + 2,800nfp_3$$
$$+ 2,950nfp_4,$$

s.t.:

$$\text{Fleet 1} \qquad y_{1,1} + y_{1,2} + y_{1,3} + y_{1,4} \leq 5,$$
$$\text{Fleet 2} \qquad y_{2,1} + y_{2,2} + y_{2,3} + y_{2,4} \leq 3,$$
$$\text{Fleet 3} \qquad y_{3,1} + y_{3,2} + y_{3,3} + y_{3,4} \leq 2,$$
$$\text{Seats route 1} \qquad 1,000y_{1,1} + 1,050_{y2,1} + 1,080y_{3,1} + nfp_1 - es_1 = 2,500,$$
$$\text{Seats route 2} \qquad 800y_{1,2} + 750y_{2,2} + 840y_{3,2} + nfp_2 - es_2 = 2,000,$$
$$\text{Seats route 3} \qquad 600y_{1,3} + 750y_{2,3} + 720y_{3,3} + nfp_3 - es_3 = 2,200,$$
$$\text{Seats route 4} \qquad 400y_{1,4} + 600y_{2,4} + 720y_{3,4} + nfp_4 - es_4 = 1,800,$$
$$y_{i,j}, nfp_j, es_i \geq 0, \qquad \forall\ i=1,2,3, \quad \text{and} \quad j=1,2,3,4,$$

Table 4. Demand and opportunity cost per route for the airline case.

Route	Passenger demand per day	Opportunity cost per empty seat
1	2,500	2,500
2	2,000	3,000
3	2,200	2,800
4	1,800	2,950

Table 5. Operational cost per route for the airline case.

Fleet	Route 1	Route 2	Route 3	Route 4
1	50,000	70,000	40,000	40,000
2	35,000	80,000	25,000	50,000
3	28,000	45,000	27,000	26,000

where $y_{i,j}$ represents the number of airplanes from fleet i to be assigned to fly route j per a day; nfp_j defines the number of passengers (per day) under demand for route j (passenger not flown); and es_i is equal to the number of empty seats flown in a day for the jth route.

The optimal continuous solution is: $y_{1,1} = 5/2$, $y_{1,2} = 5/2$, $y_{2,3} = 44/15$, $y_{2,4} = 1/15$, $y_{3,4} = 2$, and $nfp_4 = 320$, as seen in Table 6, which shows the nonzero elements of the solution. The minimum daily cost is $2,961,000.00. If the integrality constraint is

Table 6. Optimal continuous daily solution for the airline case, with cost of $2,961,000.00.

Fleet	Route 1	Route 2	Route 3	Route 4
1	5/2	5/2		
2			44/15	1/15
3				2
nfp es				320

Table 7. Optimal continuous solution for the airline case, with $T = 30$, and average daily cost of $2,961,000.00.

Fleet	Route 1	Route 2	Route 3	Route 4
1	75	75		
2			88	2
3				60
nfp es				9,600

included, the objective function rises to $4,259,000.00, i.e., an approximate increase in the daily cost of 43.84%.

From Table 6, the minimum common multiple of $\{2, 15\}$ is $T = 30$. Considering that the flight plan contemplates 30 days now, Table 7 shows a solution for that period, maintaining a 30-day cost of $88,830,000.00, i.e., an average daily cost at $2,961,000.00.

4.4. Industrial size order acceptance case

An automotive Original Equipment Manufacturer (OEM) is asked to quote orders (products) for a new automotive platform. The potential contract includes a non-determined large number of periods. The OEM has manufacturing technical capabilities to quote ninety-three orders ($m = 93$). Each order to be quoted is expected to be manufactured on a highly specialized manufacturing cell. For the purpose of this case, the manufacturing cell can be considered a single machine. This manufacturing cell works three shifts of eight hours, each. The manufacturing cell utilization factor is 85 percent, making 36,720 working minutes available per month (available time, $AT = 36,720$).

Table 8 shows the monthly demand for each order, setup time (in minutes), marginal contribution of each unit in USD, the setup cost in USD, and the manufacturing standard time per unit, respectively. Five main raw materials ($m = 5$) are needed for every unit in each order. The current monthly availability of these raw materials is $RM = \{9,000; 8,000; 7,000; 6,000; 3,000\}$. The unitary requirement of each raw material is also shown in Table 8.

The relaxed mathematical formulation of the problem in a general form as follows:

$$\max \quad \sum_{i\in I} MC_i y_i - \sum_{i\in I} SUC_i a_i,$$

s.t.:

$$\text{Demand:} \qquad y_i \leq a_i d_i, \quad \forall\, i \in I,$$

$$\text{Time availability:} \qquad \sum_{i\in I}(SUT_i a_i + ST_i y_i) \leq AT,$$

$$\text{Row material availability:} \qquad \sum_{i\in I} c_{i,j} y_i \leq RM_j, \quad \forall\, j \in J,$$

$$a_i \in \mathbb{B}, \quad x_i \in \mathbb{R}, \quad \forall\, i \in I, \tag{10}$$

where $I = \{1, \cdots, n\}$ is the set of n orders and $J = \{1, \cdots, m\}$ is the set of m raw materials, d_i, MC_i, SUC_i, SUT_i, ST_i represent demand, marginal contribution, setup cost, and setup time for the ith order, respectively; $c_{i,j}$ defines the number of units from raw material j where each product in order i requires, AT states the available manufacturing time per period, and RM_j contains the availability of the jth raw material per period. The decision variable y_i defines the number of units to accept from the ith order, while a_i represents auxiliary variable to model the setups.

If all orders on full demand are accepted, an operational profit (marginal contribution minus setup costs) of 328,436.00 USD is achieved, but that solution requires 120,449 minutes per month (while only 36,700 are available) from the manufacturing cell, while the raw material requirements for this solution are 29,330 units of raw material one (while 9,000 are available); 28,392 units of raw material two (only 8,000 available); 23,791 units of raw material three (only 7,000 available); 20,021 units of raw material four (only 6,000 available), and finally 15,529 units of raw material five (only 3,000 available).

Since accepting all orders is not feasible, the order acceptance problem consists of maximizing the average monthly operational profit by deciding what orders to accept and what manufacturing level to run (units per month) in the case of accepted orders.

Table 9 shows the optimal relaxed solution (up to two digits after the decimal point), which makes 142,850.34 USD of operational profit, while the integer solution makes 137,314 USD. Notice that the relaxed solution increases the monthly operational profit by 5,536.34 USD (approximately 4.03% increase).

In the relaxed solution, orders 20 (14.32 units), 38 (113.26 units), and 43 (44.11 units) have no integer

Table 8. Industrial size order acceptance problem data.

Order (i)	**Demand** (d_i)	**Set up** (SUT_i)	**Marg. cont.** (MC_i)	**Set up** (SUC_i)	**Std. time** (ST_i)	**Raw materials per unit**				
						$c_{i,1}$	$c_{i,2}$	$c_{i,3}$	$c_{i,4}$	$c_{i,5}$
1	62	86	68	66	14	1	6	3	3	1
2	129	63	26	100	14	7	1	3	4	3
3	114	82	23	84	18	3	4	2	3	2
4	94	89	15	77	17	4	3	5	1	1
5	20	52	73	70	11	2	2	1	4	1
6	27	57	25	50	23	2	4	1	4	2
7	125	67	70	98	18	1	4	1	4	1
8	95	82	7	92	20	1	4	3	2	1
9	74	80	68	95	11	1	4	5	2	3
10	143	26	29	34	20	2	6	2	4	1
11	113	61	70	10	7	3	6	2	4	3
12	20	8	79	96	21	1	1	4	1	3
13	21	20	59	95	9	4	1	5	3	3
14	130	45	28	64	23	5	2	1	1	2
15	114	64	34	91	23	3	6	4	2	1
16	100	96	25	38	12	6	5	1	2	1
17	124	87	62	23	9	5	5	2	3	1
18	136	89	77	10	7	1	2	4	4	3
19	34	61	50	46	8	4	4	1	2	2
20	70	99	56	12	8	6	4	3	2	2
21	148	24	47	82	16	7	4	5	1	2
22	88	32	82	92	13	6	5	5	3	3
23	103	56	76	31	7	6	6	2	2	2
24	132	7	65	69	10	3	6	1	2	3
25	95	73	6	61	18	3	3	4	4	3
26	42	45	19	7	19	2	6	2	2	1
27	130	35	31	51	8	2	2	4	4	3
28	143	73	29	22	19	2	3	2	4	2
29	127	25	8	40	8	6	4	5	1	2
30	33	41	16	7	13	1	5	2	4	3
31	55	34	21	90	7	3	5	1	3	1
32	52	94	25	62	9	1	4	1	1	3
33	19	35	29	19	22	7	6	1	4	3
34	138	53	27	91	22	4	3	4	4	3
35	73	49	68	38	19	3	4	1	1	1
36	76	32	59	89	12	3	1	1	3	2
37	93	58	42	66	6	3	5	3	2	1
38	116	57	48	85	22	2	6	2	4	1
39	127	35	21	44	6	4	1	3	1	2
40	63	20	67	66	14	5	1	3	4	1
41	142	50	22	59	12	4	6	2	1	3
42	96	58	13	78	18	5	6	4	1	1
43	128	53	54	62	21	7	4	5	3	1

Table 8. Industrial size order acceptance problem data (continued).

Order (i)	Demand (d_i)	Set up (SUT_i)	Marg. cont. (MC_i)	Set up (SUC_i)	Std. time (ST_i)	Raw materials per unit				
						$c_{i,1}$	$c_{i,2}$	$c_{i,3}$	$c_{i,4}$	$c_{i,5}$
44	78	69	66	65	15	4	1	5	1	3
45	50	73	13	11	25	7	1	2	1	1
46	69	91	78	71	18	3	1	4	1	3
47	123	24	27	29	6	6	5	2	4	3
48	39	44	59	10	6	3	6	3	2	3
49	45	11	62	32	16	7	3	5	2	1
50	67	31	41	91	8	4	2	2	4	2
51	50	29	10	68	18	7	3	5	1	2
52	23	74	30	38	11	3	3	5	1	1
53	128	11	37	58	18	3	5	3	1	2
54	113	68	70	82	16	4	4	5	3	3
55	57	49	43	26	13	7	2	4	3	2
56	125	67	64	57	6	3	1	5	4	1
57	103	86	60	42	8	2	1	5	1	3
58	76	50	80	16	13	4	5	2	3	2
59	121	78	74	35	11	4	6	4	4	2
60	137	33	52	26	16	6	2	4	1	1
61	118	71	45	77	7	3	6	3	4	1
62	74	19	16	83	23	2	4	3	2	1
63	143	56	68	59	14	2	6	4	2	2
64	22	24	63	94	24	4	5	3	3	2
65	54	18	37	67	10	3	3	3	4	3
66	47	99	28	12	15	1	2	1	3	3
67	148	98	15	92	24	7	6	1	1	3
68	20	33	55	19	14	1	4	3	1	1
69	54	27	8	76	7	1	3	5	2	2
70	114	77	81	26	24	4	1	5	1	1
71	62	42	11	60	9	4	1	3	2	2
72	43	64	20	47	21	1	6	5	1	2
73	67	14	19	41	16	3	4	2	1	1
74	53	82	34	52	19	4	5	3	3	3
75	141	33	71	44	22	1	3	2	3	3
76	81	60	14	83	23	4	6	2	3	2
77	19	70	23	69	8	2	2	4	3	3
78	141	31	73	61	14	6	1	4	4	2
79	18	69	59	50	22	6	1	4	1	3
80	119	77	15	76	18	3	2	1	1	1
81	19	23	77	94	16	5	3	1	1	1
82	23	59	53	51	21	4	1	5	3	2
83	42	80	49	26	24	5	4	4	2	3
84	146	45	83	81	23	5	2	5	4	1
85	47	82	38	65	19	4	1	3	3	3
86	112	60	8	94	7	1	2	2	4	1

Table 8. Industrial size order acceptance problem data (continued).

Order (i)	Demand (d_i)	Set up (SUT_i)	Marg. cont. (MC_i)	Set up (SUC_i)	Std. time (ST_i)	Raw materials per unit				
						$c_{i,1}$	$c_{i,2}$	$c_{i,3}$	$c_{i,4}$	$c_{i,5}$
87	46	7	76	32	23	7	3	3	2	2
88	149	84	14	28	6	7	2	2	3	3
89	29	18	80	50	10	1	1	5	4	1
90	67	41	8	20	14	1	2	1	1	3
91	36	24	49	43	19	3	3	3	4	2
92	114	60	9	80	24	2	4	4	4	2
93	83	58	28	43	10	2	6	2	1	2

Table 9. Relaxed versus integer solutions.

Op.[a] profit		$142,850.34	$137,314.00
Order	**Demand (u)**	**Relaxed**	**Integer**
1	62	62	62
5	20	20	20
7	125	125	100
15	114	0	63
17	124	124	100
18	136	0	10
20	70	**14.32**	70
23	103	103	100
35	73	73	73
36	76	76	75
37	93	93	93
38	116	**113.26**	100
40	63	63	63
43	128	**44.11**	100
46	69	69	69
49	45	45	45
56	125	125	100
58	76	76	76
59	121	121	100
60	137	137	100
61	118	0	98
63	143	143	100
64	22	0	22
68	20	20	20
70	114	114	100
78	141	141	100
81	19	19	19
84	146	146	100
87	46	46	46
89	29	29	29

[a]: Op.: Operational

production units. Therefore, order 20 will deliver 15 units in 32% of the months while 14 units in the rest; order 38 will deliver 114 units in 26% of the months and 113 in the rest; and finally, order 43 will deliver 45 units in 11% of the months and 44 in the rest.

For a literature review of order acceptance, please refer to [21].

5. Conclusions and further research

Many real-world problems require formulating and solving an integer programming model. The integer solution generally produces a significant worsened objective function value compared with the objective function value of the corresponding relaxed linear programming model. Thus, it is important to explore the possibility of generating a feasible integer solution that maintains the objective value of the relaxed problem at least on average per decision period.

This study shows a class of integer programming problems named Pseudo-Continuous-Integer Periodical Linear Problem (PCIPLP) in which the relaxed solution is used as the basis to construct a feasible integer solution maintaining the value of the relaxed solution by proposing a change in the problem planning horizon. One significant fact of our approach is that the provided integer solution is not an optimal solution for neither the original integer model proposed nor its relaxed version given the changes in the problem planning horizon made. The provided integer solution can be implemented in practice and maintains the objective function value of the relaxed problem on average (per period).

The set of PCIPLPs consists of integer single period problems that repeats indefinitely, but elements from one period can be conveyed to the next, thus a change in the problem planning horizon from one period to T periods is feasible, where T is computed as the minimum common multiple of the optimum continuous decision variable denominator values.

Further research pends ahead:

(a) Identification of some other kinds of integer problems that can be treated in a similar way;

(b) Identification of a set of mix-integer problems with similar characteristics;

(c) Search for classes of non-linear integer (or mix-integer) programming problems that can be solved in the same manner;

(d) Exploration of the effectiveness of this methodology for solving large-scale problems;

(e) Since sensitivity analysis can be applied to Relations (3) to (5) and its results are practical as long as the resultant solution remains fully integer, new techniques and new characteristics must be developed and found, respectively, to perform sensitivity analysis on a more practical matter.

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