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A novel improved class of ratio-product type exponential estimators of population variance

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Abstract. Several auxiliary information-based estimators of population variance are available in the existing literature on survey sampling. Mostly, these estimators are based on conventional dispersion measures of the auxiliary variable. In this study, a generalized class of ratio-product type exponential estimators of the population variance is proposed by integrating the nonconventional auxiliary information under Simple Random Sampling (SRS). The performance of the proposed estimators was compared, theoretically and numerically, with several existing estimators of the population variance. It was established that the proposed class of estimators outperformed the existing estimators in terms of Mean Squared Error (MSE) and Relative Root Mean Square Error (RRMSE). Moreover, Percentage Relative Efficiency (PRE) of the proposed estimators was much higher than that of their counterparts.

1. Introduction

In survey sampling, auxiliary information, if available or easily obtainable without involving much cost, can be advantageously used in choosing the appropriate sampling design, selection of sampling units for inquiry or measurement processes, and the estimation of the characteristics of interest. For example, to study sugar cane production, the auxiliary information about area under cultivation, the market price of sugar, the incentive in terms of support price given to the farmers, the production of sugar cane in the previous year, etc. can play a vital role in the efficient estimation of the expected sugar cane production. Ratio, product, regression, and exponential type estimators are popular choices, in practice, to efficiently estimate population mean and variance in the presence of auxiliary information correlated with the study variable. The use of these estimators is expanding to a variety of fields such as yield estimation in agriculture, demographic studies, environmental studies, statistical process monitoring in industry, medical and biological sciences, and many other related fields; see for example [1–7].

Along with population mean, the estimation of variance is of great interest to make certain policy decisions in many practical situations such as agricul-

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ture, business, stock investment, production planning in manufacturing industry, services industry, ecology, seismology, and medical sciences [8–10]. Therefore, efficient estimation of the mean and variance are equally important for effective decision making. The estimation of variance in the context of ratio-type methods of estimation, using auxiliary information, has been considered by various researchers. Usually, conventional auxiliary measures such as mean, median, quartiles, variance, coefficient of kurtosis, variation, skewness, and the correlation between the study and auxiliary variables are employed under ratio and regression type estimation structures to improve the efficiency of the estimators of variance. For example, see [10–27] as well as their cited references for details on this subject. The auxiliary measures used in most of the existing ratio-type estimators of variance are non-resistant to the presence of outliers. The use of such measures can undermine the efficiency of the ratio-type estimators of variance if some outliers are present in the data. Thus, there is need for incorporation of some outlier resistant auxiliary measures to develop more stable ratio-type estimators.

Recently, ratio-type estimators for the estimation of population mean have been developed, which incorporate auxiliary information by using nonconventional measures of location [28–32]. These non-conventional measures are somewhat robust and outlier-resistant, which aids in stabilizing the Mean Square Error (MSE) of estimators in the presence of outliers [8,33,34]. Use of non-conventional or robust measures for the estimation of population variance, in the context of ratio or regression methods of estimation, is still a neglected area. Efficient estimation of variance in the presence of outliers is of paramount interest in several practical situations, such as in agriculture, business, production processes, and so forth. Therefore, the problem of estimating the finite population variance is dealt with in this study by incorporating auxiliary information in some non-conventional and robust measures of dispersion, as detailed in Section 3, to develop more stable and outlier-resistant ratio-product type exponential estimators. It is assumed that the auxiliary information on the non-conventional measures is readily available or can be obtained economically. Suppose a finite population $\Omega = \{\Omega_1, \Omega_2, \cdots, \Omega_N\}$ consists of N different and identifiable units. Let (y, x) be the measurable study and auxiliary variables, respectively, with their values (y_i, x_i) being ascertained on $\Omega_i (i = 1, 2, \cdots, N)$. The purpose of the measurement process is to efficiently estimate the population variance of the variable of interest, $S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$, by drawing a random sample of size n from Ω using Simple Random Sampling Without Replacement (SRSWOR). Let $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$ and $s_x^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ be the sample variance of the study and the auxiliary variable, respectively. Furthermore, let ρ_{yx} be the population coefficient of correlation between the study and the auxiliary variables and $C_y = S_y/\bar{Y}$ and $C_x = S_x/\bar{X}$ be the population coefficients of variation for y and x, respectively.

To determine the bias and MSE of the existing and the proposed estimators, the following preliminaries regarding the relative error terms are considered: Let:

$$\xi_0 = (s_y^2 - S_y^2)/S_y^2,$$

and:

$$\xi_1 = \left(s_x^2 - S_x^2\right)/S_x^2,$$

so that:

$$E(\xi_1) = E(\xi_2) = 0,$$

$$E(\xi_0^2) = \eta(\beta_{2(y)} - 1) = \beta^{\star}_{2(y)},$$

$$E(\xi_1^2) = \eta(\beta_{2(x)} - 1) = \beta^{\star}_{2(x)},$$

$$E(\xi_0\xi_1) = \eta(\lambda_{22} - 1) = \lambda^{\star}_{22},$$

where:

$$\eta = \left(\frac{1}{n} - \frac{1}{N}\right).$$

$\beta_{2(y)}$ and $\beta_{2(x)}$ are the population coefficients of kurtosis of the study variable y and auxiliary variable x, respectively, and:

$$\lambda_{22} = \frac{\mu_{22}}{\mu_{20}\mu_{02}},$$

with:

$$\mu_{rs} = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^r\left(x_i - \bar{X}\right)^s.$$

2. Some existing estimators of variance under SRS

Numerous estimators of finite population variance are available in the literature. In this section, we briefly describe the structure of some of the existing estimators of finite population variance based on SRS.

The usual unbiased estimator of variance under SRS as defined in Cochran [35] is given as:

$$\widehat{S}_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2.$$

The variance of $\widehat{S}_y^2$ is given as:

$$Var\left(\widehat{S}_y^2\right) \cong S_y^4 \beta_{2(y)}^\star. \tag{1}$$

Isaki [13] proposed a ratio type estimator of S_y^2, which is given as:

$$\widehat{S}_R^2 = s_y^2 \left(\frac{S_x^2}{s_x^2}\right).$$

The MSE of $\widehat{S}_R^2$, to the first degree of approximation, is given as:

$$MSE\left(\widehat{S}_R^2\right) \cong S_y^4 \left[\beta_{2(y)}^\star + \beta_{2(x)}^\star - 2\lambda_{22}^\star\right]. \tag{2}$$

The conventional regression type estimator, according to Isaki [13], is given as:

$$\widehat{S}_{Reg}^2 = s_y^2 + b_{(s_y^2,s_x^2)}(S_x^2 - s_x^2),$$

where $b_{(s_y^2,s_x^2)}$ represents the regression coefficient to be estimated from the sample.

The MSE of $\widehat{S}_{Reg}^2$, up to the first degree of approximation, is given as:

$$MSE\left(\widehat{S}_{Reg}^2\right) \cong S_y^4 \beta_{2(y)}^\star \left[1 - \rho_{(S_y^2,S_x^2)}^2\right], \tag{3}$$

where:

$$\rho_{(S_y^2,S_x^2)} = \lambda_{22}^\star / \sqrt{\beta_{2(y)}^\star \beta_{2(x)}^\star},$$

denotes the population correlation coefficient between y and x.

The difference type estimator of Singh et al. [19] is given as:

$$\widehat{S}_d^2 = c_1 s_y^2 + c_2(S_x^2 - s_x^2),$$

where c_1 and c_2 are unknown constants and their optimal values are determined in such a manner that the MSE of $\widehat{S}_d^2$ is minimized.

The minimum MSE of $\widehat{S}_d^2$ at optimum values $c_{1(opt)} = \beta_{2(x)}^\star / (\beta_{2(x)}^\star + \beta_{2(y)}^\star \beta_{2(x)}^\star - \lambda_{22}^{\star 2})$ and $c_{2(opt)} = S_x^2 \lambda_{22}^\star / (S_y^2(\beta_{2(x)}^\star + \beta_{2(y)}^\star \beta_{2(x)}^\star - \lambda_{22}^{\star 2}))$, up to the first degree of approximation, is given by:

$$MSE\left(\widehat{S}_d^2\right)_{\min} \cong \frac{S_y^4 \beta_{2(y)}^\star \left[1 - \rho_{(S_y^2,S_x^2)}^2\right]}{1 + \beta_{2(y)}^\star \left[1 - \rho_{(S_y^2,S_x^2)}^2\right]}. \tag{4}$$

The ratio-type exponential estimator proposed by Bahl and Tuteja [11] is given as:

$$\widehat{S}_{BT}^2 = s_y^2 \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right).$$

The minimum MSE of $\widehat{S}_{BT}^2$, up to the first degree of approximation, is given as:

$$MSE\left(\widehat{S}_{BT}^2\right) \cong S_y^4 \left[\beta_{2(y)}^\star + \frac{1}{4}\beta_{2(x)}^\star - \lambda_{22}^\star\right]. \tag{5}$$

Upadhyaya and Singh [25] used the coefficient of kurtosis of the auxiliary variable to propose a modified ratio-type estimator of population variance, which is given as:

$$\widehat{S}_{US}^2 = s_y^2 \left(\frac{S_x^2 + \beta_{2(x)}}{s_x^2 + \beta_{2(x)}}\right).$$

The MSE of $\widehat{S}_{US}^2$, up to the first degree of approximation, is given as:

$$MSE\left(\widehat{S}_{US}^2\right) \cong S_y^4 \left[\beta_{2(y)}^\star + \gamma_{US}^2 \beta_{2(x)}^\star - 2\gamma_{US}\lambda_{22}^\star\right], \tag{6}$$

where:

$$\gamma_{US} = \frac{S_x^2}{S_x^2 + \beta_{2(x)}}.$$

Kadilar and Cingi [14] utilized population coefficient of variation and the population coefficient of kurtosis of the auxiliary variable to suggest some modified estimators of population variance as:

$$\widehat{S}_{KC1}^2 = s_y^2 \left(\frac{S_x^2 + C_x}{s_x^2 + C_x}\right),$$

$$\widehat{S}_{KC2}^2 = s_y^2 \left(\frac{C_x S_x^2 + \beta_{2(x)}}{C_x s_x^2 + \beta_{2(x)}}\right),$$

$$\widehat{S}_{KC3}^2 = s_y^2 \left(\frac{\beta_{2(x)} S_x^2 + C_x}{\beta_{2(x)} s_x^2 + C_x}\right).$$

The respective MSEs of $\widehat{S}_{KC1}^2$, $\widehat{S}_{KC2}^2$, and $\widehat{S}_{KC3}^2$, up to the first degree of approximation, are given as:

$$MSE\left(\widehat{S}_{KC1}^2\right) \cong S_y^4 \left[\beta_{2(y)}^\star + \gamma_{KC1}^2 \beta_{2(x)}^\star - 2\gamma_{KC1}\lambda_{22}^\star\right], \tag{7}$$

$$MSE\left(\widehat{S}_{KC2}^2\right) \cong S_y^4 \left[\beta_{2(y)}^\star + \gamma_{KC2}^2 \beta_{2(x)}^\star - 2\gamma_{KC2}\lambda_{22}^\star\right], \tag{8}$$

$$MSE\left(\widehat{S}_{KC3}^2\right) \cong S_y^4 \left[\beta_{2(y)}^\star + \gamma_{KC3}^2 \beta_{2(x)}^\star - 2\gamma_{KC3}\lambda_{22}^\star\right], \tag{9}$$

where:

$$\gamma_{KC1} = \frac{S_x^2}{S_x^2 + C_x}, \qquad \gamma_{KC2} = \frac{C_x S_x^2}{C_x S_x^2 + \beta_{2(x)}},$$

$$\gamma_{KC3} = \frac{\beta_{2(x)} S_x^2}{\beta_{2(x)} S_x^2 + C_x}.$$

The estimator of population variance S_y^2, given by Shabbir and Gupta [17], is:

$$\widehat{S}_{SG}^2 = c_3 s_y^2 + c_4(S_x^2 - s_x^2) \exp\left\{\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right\},$$

where c_3 and c_4 are unknown quantities to be determined in a manner to minimize the MSE of $\widehat{S}_{SG}^2$.

$$c_{3(opt)} = \frac{\beta^{\star}_{2(x)}}{8}\left(\frac{8-\beta^{\star}_{2(x)}}{\beta^{\star}_{2(x)}+\beta^{\star}_{2(y)}\beta^{\star}_{2(x)}-\lambda^{\star 2}_{22}}\right),$$

and:

$$c_{4(opt)} = \frac{S_y^2}{8S_x^2}\left(\frac{-4\beta^{\star}_{2(x)}+\beta^{\star}_{2(x)}+8\lambda^{\star}_{22}-\lambda^{\star}_{22}\beta^{\star}_{2(x)}+4\beta^{\star}_{2(y)}\beta^{\star}_{2(x)}-4\lambda^{\star 2}_{22}}{\beta^{\star}_{2(x)}+\beta^{\star}_{2(y)}\beta^{\star}_{2(x)}-\lambda^{\star 2}_{22}}\right).$$

Box I

The optimum values of c_3 and c_4 that minimize the MSE of $\widehat{S}^2_{SG}$ are given by the equations shown in Box I, whereas the minimized MSE of $\widehat{S}^2_{SG}$ is:

$$MSE\left(\widehat{S}^2_{SG}\right)_{\min} \cong \frac{S_y^4}{64} \left(\frac{-\beta^{\star 2}_{2(x)}-16\beta^{\star}_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)\left(\beta^{\star}_{2(x)}-4\right)}{1+\beta^{\star}_{2(y)}(1-\rho^2_{(S_y^2,S_x^2)})}\right). \quad (10)$$

Subramani and Kumarapandiyan [36] used the median of the auxiliary variable to propose an estimator of the population variance, which is defined as:

$$\widehat{S}^2_{SK1} = s_y^2\left(\frac{S_x^2+M_x}{s_x^2+M_x}\right).$$

The MSE of $\widehat{S}^2_{SK1}$, up to the first degree of approximation, is given as:

$$MSE\left(\widehat{S}^2_{SK1}\right) \cong S_y^4\left[\beta^{\star}_{2(y)}+\gamma^2_{SK1}\beta^{\star}_{2(x)}-2\gamma_{SK1}\lambda^{\star}_{22}\right], \quad (11)$$

where:

$$\gamma_{SK1} = \frac{S_x^2}{S_x^2+M_x}.$$

Taking motivation from Kadilar and Cingi [14] and Subramani and Kumarapandiyan [36], a new ratio-type estimator of the population variance was introduced by Subramani and Kumarapandiyan [22] that utilized the population information for the coefficient of variation and the median of the auxiliary variable, as given below:

$$\widehat{S}^2_{SK2} = s_y^2\left(\frac{C_xS_x^2+M_x}{C_xs_x^2+M_x}\right).$$

The MSE of $\widehat{S}^2_{SK2}$, to the first order of approximation, is given as:

$$MSE\left(\widehat{S}^2_{SK2}\right) \cong S_y^4\left[\beta^{\star}_{2(y)}+\gamma^2_{SK2}\beta^{\star}_{2(x)}-2\gamma_{SK2}\lambda^{\star}_{22}\right], \quad (12)$$

where:

$$\gamma_{SK2} = \frac{C_xS_x^2}{C_xS_x^2+M_x}.$$

Khan and Shabbir [15] used upper quartile and the population correlation coefficient to suggest an improved ratio estimator of population variance as:

$$\widehat{S}^2_{KS} = s_y^2\left(\frac{\rho_{yx}S_x^2+Q_{3(x)}}{\rho_{yx}s_x^2+Q_{3(x)}}\right).$$

The MSE of $\widehat{S}^2_{KS}$, up to the first degree of approximation, is given as:

$$MSE\left(\widehat{S}^2_{KS}\right) \cong S_y^4\left[\beta^{\star}_{2(y)}+\gamma^2_{KS}\beta^{\star}_{2(x)}-2\gamma_{KS}\lambda^{\star}_{22}\right], \quad (13)$$

where:

$$\gamma_{KS} = \frac{\rho_{yx}S_x^2}{\rho_{yx}S_x^2+Q_{3(x)}}.$$

The generalized estimator of population variance proposed by Swain [24] is given below:

$$\widehat{S}^2_{SW} = s_y^2\left[k\left(\frac{S_x^2}{s_x^2}\right)^q+(1-k)\left(\frac{s_x^2}{S_x^2}\right)^h\right]^{\delta},$$

where k, q, and h are suitably chosen constant and $\delta=(1,-1)$. The minimum MSE of $\widehat{S}^2_{SW}$, up to the first degree of approximation, at the optimum value of $k=(\delta h+(\lambda^{\star}_{22}/\beta^{\star}_{2(y)}))/(\delta(g+h))$, is given by:

$$MSE\left(\widehat{S}^2_{SW}\right)_{\min} \cong S_y^4\beta^{\star}_{2(y)}(1-\rho^2_{(S_y^2,S_x^2)}). \quad (14)$$

It is to be noted that the $MSE(\widehat{S}^2_{SW})_{\min}$ is equal to $MSE(\widehat{S}^2_{Reg})$.

The general estimator class for population variance proposed by Yadav et al. [26] is given by:

$$\widehat{S}^2_{YG} = \left[c_5s_y^2+c_6(S_x^2-s_x^2)\right]\left\{\lambda\left(\frac{aS_x^2+b}{as_x^2+b}\right) +(1-\lambda)\exp\left(\frac{a(S_x^2-s_x^2)}{a(S_x^2+s_x^2)+2b}\right)\right\},$$

where c_5 and c_6 are suitably chosen constants that minimize the MSE of $\widehat{S}^2_{YG}$, while λ can take the values of 0 or 1 and a, b are the known values of the auxiliary variable parameters. The minimum MSE of $\widehat{S}^2_{YG}$, up to the first degree of approximation, at the optimum values is:

$$c_{5(opt)}=\left(\frac{1-\frac{1}{8}g^2(1+3\lambda+4\lambda^2)\beta^\star_{2(x)}}{1-\frac{1}{4}g^2\lambda(1+3\lambda)\beta^{\star 2}_{2(x)}+\beta^\star_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)}\right),$$

and:

$$c_{6(opt)}=\frac{S_y^2}{S_x^2}\left(\frac{1}{2}g(1+\lambda)+c_{5(opt)}\left(\frac{\lambda^\star_{22}}{\beta^\star_{2(x)}}-g(1+\lambda)\right)\right),$$

is given as:

$$MSE\left(\widehat{S}^2_{YG}\right)_{\min}\cong S_y^4\left\{\left(1-\frac{1}{4}g^2(1+\lambda)^2\beta^\star_{2(x)}\right)\right.$$
$$\left.-\frac{\left(1-\frac{1}{8}g^2(1+3\lambda+4\lambda^2)\beta^\star_{2(x)}\right)^2}{1-\frac{1}{4}g^2\lambda(1+3\lambda)\beta^{\star 2}_{2(x)}+\beta^\star_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)}\right\}, \quad (15)$$

where:

$$g=\frac{aS_x^2}{(aS_x^2+b)}.$$

The minimum MSE of $\widehat{S}^2_{YG}$, up to the first order of approximation at $(\lambda,a,b)=(1,1,0)$, is given below:

$$MSE\left(\widehat{S}^2_{YG}\right)_{\min}\cong S_y^4\left(\frac{S_y^{-4}MSE\left(\widehat{S}^2_{Reg}\right)(1-\beta^\star_{2(x)})}{1-\beta^\star_{2(x)}+S_y^{-4}MSE\left(\widehat{S}^2_{Reg}\right)}\right). \quad (16)$$

Yadav and Kadilar [37] proposed a ratio-product type estimator of population variance, which is given as:

$$\widehat{S}^2_{YK}=s_y^2\left[\alpha_1\left\{\frac{(1-\beta_1)s_x^2+\beta_1S_x^2}{\beta_1s_x^2+(1-\beta_1)S_x^2}\right\}\right.$$
$$\left.+(1-\alpha_1)\left\{\frac{\beta_1s_x^2+(1-\beta_1)S_x^2}{(1-\beta_1)s_x^2+\beta_1S_x^2}\right\}\right],$$

where α_1 and β_1 are constant.

The minimum MSE of $\widehat{S}^2_{YK}$, up to the first degree of approximation, is given below:

$$MSE\left(\widehat{S}^2_{YK}\right)_{\min}\cong S_y^4\left[\left(\beta^\star_{2(y)}+\beta^\star_{2(x)}-2\lambda^\star_{22}\right)\right.$$
$$+16\alpha_1\beta_1\beta^\star_{2(x)}(1-\alpha_1-\beta_1+\alpha_1\beta_1)$$
$$\left.+4\lambda^\star_{22}(\alpha_1-\beta_1)^2+4\beta^\star_{2(x)}(-\alpha_1-\beta_1+\alpha_1^2+\beta_1^2)\right]. \quad (17)$$

The minimum MSE of $\widehat{S}^2_{YK}$, up to the first degree of approximation at $(\alpha_{1(opt)},\beta_{1(opt)})=\left(\frac{1}{2},\frac{1}{2}\right)$, is:

$$MSE\left(\widehat{S}^2_{YK}\right)_{\min}\cong S_y^4\beta^\star_{2(y)}. \quad (18)$$

And when $(\alpha_1,\beta_1)=(((\beta^\star_{2(x)}-2\lambda^\star_{22})/2\beta^\star_{2(x)}),0)$, the minimum MSE of $\widehat{S}^2_{YK}$ is given as:

$$MSE\left(\widehat{S}^2_{YK}\right)_{\min}\cong S_y^4\beta^\star_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right). \quad (19)$$

Recently, Yaqub and Shabbir [27] proposed an improved class of estimators for population variance given as:

$$\widehat{S}^2_{YS}=s_y^2\left[c_7+c_8(S_x^2-s_x^2)\right]\left(\frac{aS_x^2+b}{as_x^2+b}\right)$$
$$\left\{\frac{1}{2}\exp\left(\frac{a(S_x^2-s_x^2)}{aS_x^2+s_x^2)+2b}\right)\right.$$
$$\left.+frac12\exp\left(\frac{a(s_x^2-S_x^2)}{a(s_x^2+S_x^2)+2b}\right)\right\},$$

where c_7 and c_8 are suitably chosen constants and a and b be the known population parameters of the auxiliary variable. Assuming $a=1$ and $b=0$, the minimum MSE of $\widehat{S}^2_{YS}$, up to the first degree of approximation, based on the optimum values is calculated by Eq. (20) as shown in Box II.

3. The proposed generalized estimator of variance

This section presents a generalized ratio product type exponential estimator of population variance, which incorporates information on some outlier resistant non-conventional measures of dispersion of the auxiliary variable. The non-conventional measures are used in a linear combination within the structure of the proposed estimator to make it robust against possible outliers in the data. The non-conventional somewhat robust measures of the auxiliary variable considered in this study include:

(i) **The interquartile range:** The interquartile range (IQR) is the difference between the upper quartile ($Q_{3(x)}$) and lower quartile ($Q_{1(x)}$). Symbolically, it is given as:

$$IQR_x=Q_{3(x)}-Q_{1(x)}.$$

It is the most known, somewhat, robust measure of dispersion with a breakdown point of 25%.

(ii) **The Gini's mean difference estimator:** Gini [38] suggested an estimator of dispersion, which is also known as the Gini's mean difference estimator. It is given as:

$$c_{7(opt)}=\frac{\beta^{\star}_{2(x)}}{2}\left(\frac{1+7\left(1-\beta^{\star}_{2(x)}\right)}{\beta^{\star 2}_{2(x)}+4\beta^{\star}_{2(x)}\left(1-\beta^{\star}_{2(x)}\right)+4\beta^{\star}_{2(y)}\beta^{\star}_{2(x)}-4\lambda^{\star 2}_{22}}\right),$$

and:

$$c_{8(opt)}=\frac{S_y^2}{2S_x^2}\left(\frac{\lambda^{\star}_{22}+7\lambda^{\star}_{22}\left(1-\beta^{\star}_{2(x)}\right)-8\beta^{\star}_{2(x)}\left(1-\beta^{\star}_{2(x)}\right)+8\beta^{\star}_{2(y)}\beta^{\star}_{2(x)}-8\lambda^{\star 2}_{22}}{\beta^{\star 2}_{2(x)}+4\beta^{\star}_{2(x)}\left(1-\beta^{\star}_{2(x)}\right)+4\beta^{\star}_{2(y)}\beta^{\star}_{2(x)}-4\lambda^{\star 2}_{22}}\right),$$

hence:

$$MSE\left(\widehat{S}^2_{YS}\right)_{\min}\cong\frac{S_y^4}{16}\left[\frac{64\left(1-\beta^{\star}_{2(x)}\right)S_y^{-4}MSE\left(\widehat{S}^2_{Reg}\right)-\beta^{\star 2}_{2(x)}}{\beta^{\star}_{2(x)}+4\left(1-\beta^{\star}_{2(x)}\right)+4S_y^{-4}MSE\left(\widehat{S}^2_{Reg}\right)}\right]. \tag{20}$$

Box II

$$GIN_x=\frac{4}{N-1}\sum_{i=1}^{N}\left(\frac{2i-N-1}{2N}\right)x_{(i)},$$

where $x_{(i)}$ denotes the ith order statistics. It is robust to outliers and more efficient than the estimators based on range and standard deviation (cf. [39]).

(iii) **The Downton's estimator:** Like GIN_x, Downton [40] suggested a robust and highly efficient estimator of dispersion. It is defined as:

$$DOW_x=\frac{2\sqrt{\pi}}{N(N-1)}\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)x_{(i)},$$

where $x_{(i)}$ denotes the ith order statistics. The asymptotic efficiency of DOW_x is 97.8% (cf. [39]).

(iv) **The probability weighted moment estimator:** Another similar estimator to GIN_x and DOW_x is the probability weighted moment estimator given in [41]. It is defined as:

$$SPW_x=\frac{\sqrt{\pi}}{N^2}\sum_{i=1}^{N}(2i-N-1)x_{(i)},$$

where $x_{(i)}$ denotes the ith order statistics. Its properties are similar to those of GIN_x and DOW_x, as all these three estimators are proportional to each other.

(v) **Median absolute deviation from median:** Hampel [42] suggested an estimator based on the median of the absolute deviations taken from the median, which is given as:

$$MADM_x=m\left[\text{median}\left|x_i-\widetilde{X}\right|\right]$$

$$\text{for}\quad i=1,2,\cdots,N,$$

where m is the consistency coefficient and $\widetilde{X}$ denotes the median of the observations. $MADM_x$ is robust against outliers with a breakdown point of 50%, but under normality its efficiency is relatively low, i.e., 37%. To make $MADM_x$ a consistent estimator of σ under normal distribution, the value of m is set equal to 1.4826.

(vi) **The median of pairwise distances:** Shamos [43] and Bickel and Lehmann [44] suggested an estimator of dispersion based on the median of pairwise distances as median$\{|x_i-x_l|; i<l\}$. Rousseeuw and Croux [45] suggested to pre-multiply it by 1.0483 to achieve consistency under the Gaussian distribution. The resultant estimator can be defined as:

$$Bn_x=1.0483[\text{median}\{|x_i-x_l|; i<l\}].$$

Bn_x is somewhat robust to outliers with a breakdown point of 29% and has a relatively high efficiency (about 86%) under normality.

(vii) **The ordered statistic of subranges:** Croux and Rousseeuw [46] proposed a class of location-free robust estimators of dispersion based on ordered statistics of subranges defined as:

$$Sr_x^{\alpha}=C_{\alpha}\left\{\left|x_{(i+[\alpha N]+1)}-x_{(i)}\right|_{\left(\left[\frac{N}{2}\right]-\alpha N\right)}\right\},$$

where $0<\alpha<0.5$ and $x_{(1)}\leq x_{(2)}\leq\cdots\leq x_{(N)}$ are ordered statistics (here, the symbol $[\cdot]$ represents the integer part). The value of Sr_x^{α} is determined by first, sorting the observations x_i and then, calculating the absolute differences $|x_{(i+[\alpha N]+1)}-x_{(i)}|$ for $i=1,2,\cdots,N-[\alpha N]-1$. From these calculated quantities, the $\left(\left[\frac{N}{2}\right]-\alpha N\right)$th order statistics yield the desired estimator. The constant C_α is chosen in such a way that Sr_x^α becomes a consistent estimator for a given value of α. In the present study, we use $\alpha = 0.25$, which corresponds to $C_\alpha = 1.4826$ under normality. Sr_x^α has a 50% breakdown point and it is more efficient than $MADM_x$ for small samples.

(viii) **The trimmed mean of median deviations:** Rousseeuw and Croux [47] proposed an estimator with a high breakdown point of 50% and efficiency of 52% under normality, which is relatively higher than $MADM_x$. It is defined as:

$$Tn_x = \frac{1.38}{h}\sum_{k=1}^{h}[\text{median}\{|x_i - x_l|, i \neq l\}]_{(k)},$$

where for each $i = 1, 2, \cdots, N$, we compute the median of $|x_i - x_l|$, $l = 1, 2, 3, \cdots, N$ that yields N values. The average of the first h order statistics gives the desired estimator (here, $h = \left[\frac{N}{2}\right] + 1$, which is roughly half the number of observations).

(ix) **The 0.25-quantile of pairwise distances:** Similar to Bn_x, Rousseeuw and Croux [45] suggested a robust estimator of dispersion based on the 0.25-quantile of pairwise distances between the observations. It is given as:

$$Qn_x = d\{\text{median}|x_i - x_l|; i < l\}_{(p)},$$

where:

$$p = \binom{h}{2} \approx \binom{N}{2}/4 \quad \text{and} \quad h = \left[\frac{N}{2}\right] + 1.$$

Hence, the pth order statistic of the $\binom{N}{2}$ inter-point distances yields the desired estimator. The value of d is set equal to 2.2219 for Qn_x to be a consistent estimator under normality. The estimator Qn_x has a 50% breakdown point and high Gaussian asymptotic efficiency of 82%.

(x) **The median of the median of distances:** Rousseeuw and Croux [45] suggested another robust estimator, which has a high breakdown point of 50%. it is defined as:

$$Sn_x = q[\text{median}_i\{\text{median}_j|x_i - x_j|; i \neq j\}],$$

where q is the consistency factor with a default value of 1.1926 under the normal population. To compute Sn_x, first, we determine the median of $\{|x_i - x_j|; j = 1, 2, \cdots, N\}$ for each i, which results in N values. Finally, the median of these N values yields Sn_x.

For more details on these non-conventional measures of dispersion, the readers may see [38–40,43–47] and the references cited therein.

Taking motivation from Shabbir and Gupta [17] and Naz et al. [34], we integrate the above-mentioned non-conventional robust measures of dispersion to design a stable ratio product type exponential estimator of population variance defined as:

$$\widehat{S}^2_{\Pr op} = s_y^2\left\{p_1\left(\frac{\varphi S_x^2 + \theta}{\varphi s_x^2 + \theta}\right)^{\frac{\varphi S_x^2}{\varphi S_x^2 + \theta}} + p_2\left(\frac{\varphi s_x^2 + \theta}{\varphi S_x^2 + \theta}\right)^{\frac{\varphi S_x^2}{\varphi S_x^2 + \theta}}\right\}\exp\left\{\frac{\varphi(S_x^2 - s_x^2)}{\varphi(S_x^2 + s_x^2) + 2\theta}\right\}, \tag{21}$$

where p_1 and p_2 are suitably chosen constants and their values are to be determined later in such a manner that MSE of $\widehat{S}^2_{\Pr op}$ is minimized. φ can either be some known real value or function of a known conventional population parameter–such as $\rho_{(s_y^2, s_x^2)}$, C_x, or any other value–of the auxiliary variable, whereas θ can be one of the above-mentioned non-conventional measures.

Setting $\omega = \frac{\varphi S_x^2}{\varphi S_x^2 + \theta}$ and expressing $\widehat{S}^2_{\Pr op}$ in terms of ξ's, we have:

$$\widehat{S}^2_{\Pr op} = S_y^2(1 + \xi_0)\left[p_1\{1 + \omega\xi_1\}^{-\omega} + p_2\{1 + \omega\xi_1\}^{\omega}\right] \exp\left[-\frac{\omega\xi_1}{2}\left(1 + \frac{\omega\xi_1}{2}\right)^{-1}\right]. \tag{22}$$

For simplification, expanding Eq. (22) and retaining terms only up to the 2nd order in ξ's, we have:

$$\widehat{S}^2_{\Pr op} - S_y^2 \cong S_y^2\Bigg[p_1\left\{1 + \xi_0 - \left(\omega^2 + \frac{\omega}{2}\right)\xi_1 + \left(\frac{\omega^4}{2} + \omega^3 + \frac{3}{8}\omega^2\right)\xi_1^2 - \left(\omega^2 + \frac{\omega}{2}\right)\xi_0\xi_1\right\} + p_2\left\{1 + \xi_0 + \left(\omega^2 - \frac{\omega}{2}\right)\xi_1 + \left(\frac{\omega^4}{2} - \omega^3 + \frac{3}{8}\omega^2\right)\xi_1^2 + \left(\omega^2 - \frac{\omega}{2}\right)\xi_0\xi_1\right\} - 1\Bigg]. \tag{23}$$

By applying expectation on both sides of Eq. (23), we get the bias of $\widehat{S}^2_{PR}$ as follows:

$$\text{Bias}\left(\widehat{S}^2_{PR}\right) \cong S_y^2\Bigg[(p_1 + p_2 - 1) + p_1\left\{\left(\frac{\omega^4}{2} + \omega^3 + \frac{3}{8}\omega^2\right)\beta^\star_{2(x)} - \left(\omega^2 + \frac{\omega}{2}\right)\lambda^\star_{22}\right\} + p_2\left\{\left(\frac{\omega^4}{2} - \omega^3 + \frac{3}{8}\omega^2\right)\beta^\star_{2(x)} + \left(\omega^2 - \frac{\omega}{2}\right)\lambda^\star_{22}\right\} - 1\Bigg]. \tag{24}$$

$$p_{1(opt)}=\frac{1}{8}\left[\frac{\left\{\begin{array}{l}\omega(-1+2\omega)\left(\omega^2(-7+4\omega(1+\omega))\beta^{\star 2}_{2(x)}-16\lambda^{\star 2}_{22}+8\beta^{\star}_{2(x)}(1+3\omega\lambda^{\star}_{22})\right)\\ \quad +16\left(\omega\beta^{\star}_{2(x)}-\lambda^{\star}_{22}\right)\beta^{\star}_{2(y)}+16\lambda^{\star}_{22}\end{array}\right\}}{\omega^2\left\{\omega^2(4\omega^2-5)\beta^{\star 2}_{2(x)}-16\lambda^{\star 2}_{22}+4\beta^{\star}_{2(x)}(1+4\omega\lambda^{\star}_{22}+\beta^{\star}_{2(y)})\right\}}\right],$$

and:

$$p_{2(opt)}=\frac{1}{8}\left[\frac{\left\{\begin{array}{l}\omega(1+2\omega)\left(\omega^2\left(-7+4\omega(-1+\omega)\right)\beta^{\star 2}_{2(x)}-16\lambda^{\star 2}_{22}+8\beta^{\star}_{2(x)}\left(1+3\omega\lambda^{\star}_{22}\right)\right)\\ \quad +16(-\omega\beta^{\star}_{2(x)}+\lambda^{\star}_{22})\beta^{\star}_{2(y)}-16\lambda^{\star}_{22}\end{array}\right\}}{\omega^2\left\{\omega^2(4\omega^2-5)\beta^{\star 2}_{2(x)}-16\lambda^{\star 2}_{22}+4\beta^{\star}_{2(x)}(1+4\omega\lambda^{\star}_{22}+\beta^{\star}_{2(y)})\right\}}\right].$$

Box III

$$MSE\left(\widehat{S}^2_{PR}\right)_{\min}\cong\frac{S^4_y}{16}\left[\frac{\left\{\begin{array}{l}16\omega^2\beta^{\star 2}_{2(x)}\left(\left(\omega-4\omega^3\right)\lambda^{\star}_{22}-4\beta^{\star}_{2(y)}\right)-\omega^4\left(1-4\omega^2\right)^2\beta^{\star 3}_{2(x)}-64\lambda^{\star 2}_{22}\left(1+\beta^{\star}_{2(y)}\right)\\ \quad +16\beta^{\star}_{2(x)}\left(\omega^2\left(4\omega^2-1\right)\lambda^{\star 2}_{22}+4(1+2\omega\lambda^{\star}_{22})\beta^{\star}_{2(y)}\right)\end{array}\right\}}{\omega^2(4\omega^2-5)\beta^{\star 2}_{2(x)}-16\lambda^{\star 2}_{22}+4\beta^{\star}_{2(x)}\left(1+4\omega\lambda^{\star}_{22}+\beta^{\star}_{2(y)}\right)}\right]. \tag{26}$$

Box IV

Squaring both sides of Eq. (23) and then, applying expectation, we get the MSE of $\widehat{S}^2_{PR}$ as follows:

$$\begin{aligned}MSE\left(\widehat{S}^2_{PR}\right)&=S^4_y+p_1^2S^4_y\Bigg\{1+\beta^{\star}_{2(y)}\\&+(2\omega^4+3\omega^3+\omega^2)\beta^{\star}_{2(x)}-4\left(\omega^2+\frac{\omega}{2}\right)\lambda^{\star}_{22}\Bigg\}\\&+p_2^2S^4_y\Bigg\{1+\beta^{\star}_{2(y)}+\left(2\omega^4-3\omega^3+\omega^2\right)\beta^{\star}_{2(x)}\\&+4\left(\omega^2-\frac{\omega}{2}\right)\lambda^{\star}_{22}\Bigg\}+2p_1p_2S^4_y\left\{1+\beta^{\star}_{2(y)}\right.\\&\left.+\omega^2\beta^{\star}_{2(x)}-2\omega\lambda^{\star}_{22}\right\}-2p_1S^4_y\Bigg\{1\\&+\left(\frac{\omega^4}{2}+\omega^3+\frac{3}{8}\omega^2\right)\beta^{\star}_{2(x)}-\left(\omega^2+\frac{\omega}{2}\right)\lambda^{\star}_{22}\Bigg\}\\&-2p_2S^4_y\Bigg\{1+\left(\frac{\omega^4}{2}-\omega^3+\frac{3}{8}\omega^2\right)\beta^{\star}_{2(x)}\\&+\left(\omega^2-\frac{\omega}{2}\right)\lambda^{\star}_{22}\Bigg\}.\end{aligned} \tag{25}$$

To get the optimal values of p_1 and p_2, we minimize Eq. (25) with respect to p_1 and p_2, which gives the equations shown in Box III. The optimal values of p_1 and p_2 are then substituted in Eq. (25), which gives the minimum MSE of $\widehat{S}^2_{PR}$ by Eq. (26) as shown in Box IV. Many estimators of population variance can be generated from the class of estimators given in Eq. (21) by setting different values of φ and θ. A number of selected estimators, which are members of the proposed class, are given in Table 1.

Table 1. Some new members of the proposed class I estimators.

Estimator	Value of constant	
	φ	θ
$\widehat{S}^2_{PR-1} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2 + IQR_x}{\rho_{(s_y^2,\,s_x^2)} s_x^2 + IQR_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + IQR_x}} + p_2 \left(\frac{\rho_{(s_y^2,\,s_x^2)} s_x^2 + IQR_x}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + IQR_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + IQR_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\,s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\,s_x^2)} (S_x^2 + s_x^2) + 2IQR_x} \right\}$	$\rho_{(s_y^2,\,s_x^2)}$	IQR_x
$\widehat{S}^2_{PR-2} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2 + GIN_x}{\rho_{(s_y^2,\,s_x^2)} s_x^2 + GIN_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + GIN_x}} + p_2 \left(\frac{\rho_{(s_y^2,\,s_x^2)} s_x^2 + GIN_x}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + GIN_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + GIN_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\,s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\,s_x^2)} (S_x^2 + s_x^2) + 2GIN_x} \right\}$	$\rho_{(s_y^2,\,s_x^2)}$	GIN_x
$\widehat{S}^2_{PR-3} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2 + DOW_x}{\rho_{(s_y^2,\,s_x^2)} s_x^2 + DOW_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + DOW_x}} + p_2 \left(\frac{\rho_{(s_y^2,\,s_x^2)} s_x^2 + DOW_x}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + DOW_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + DOW_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\,s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\,s_x^2)} (S_x^2 + s_x^2) + 2DOW_x} \right\}$	$\rho_{(s_y^2,\,s_x^2)}$	DOW_x
$\widehat{S}^2_{PR-4} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2 + SPW_x}{\rho_{(s_y^2,\,s_x^2)} s_x^2 + SPW_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + SPW_x}} + p_2 \left(\frac{\rho_{(s_y^2,\,s_x^2)} s_x^2 + SPW_x}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + SPW_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + SPW_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\,s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\,s_x^2)} (S_x^2 + s_x^2) + 2SPW_x} \right\}$	$\rho_{(s_y^2,\,s_x^2)}$	SPW_x
$\widehat{S}^2_{PR-5} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2 + MADM_x}{\rho_{(s_y^2,\,s_x^2)} s_x^2 + MADM_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + MADM_x}} + p_2 \left(\frac{\rho_{(s_y^2,\,s_x^2)} s_x^2 + MADM_x}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + MADM_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + MADM_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\,s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\,s_x^2)} (S_x^2 + s_x^2) + 2MADM_x} \right\}$	$\rho_{(s_y^2,\,s_x^2)}$	$MADM_x$
$\widehat{S}^2_{PR-6} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2 + Bn_x}{\rho_{(s_y^2,\,s_x^2)} s_x^2 + Bn_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + Bn_x}} + p_2 \left(\frac{\rho_{(s_y^2,\,s_x^2)} s_x + Bn_x}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + Bn_x} \right)^{\frac{\rho_{(s_y^2,\,s_x^2)} S_x^2}{\rho_{(s_y^2,\,s_x^2)} S_x^2 + Bn_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\,s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\,s_x^2)} (S_x^2 + s_x^2) + 2Bn_x} \right\}$	$\rho_{(s_y^2,\,s_x^2)}$	Bn_x

Table 1. Some new members of the proposed class I estimators (continued).

Estimator	Value of constant φ	Value of constant θ
$\widehat{S}^2_{PR-7} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Sr_x^{\alpha}}{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Sr_x^{\alpha}} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Sr_x^{\alpha}}} + p_2 \left(\frac{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Sr_x^{\alpha}}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Sr_x^{\alpha}} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)}} S_x^2 + Sr_x^{\alpha}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\, s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\, s_x^2)} (S_x^2 + s_x^2) + 2Sr_x^{\alpha}} \right\}$	$\rho_{(s_y^2,\, s_x^2)}$	Sr_x^{α}
$\widehat{S}^2_{PR-8} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Tn_x}{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Tn_x} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Tn_x}} + p_2 \left(\frac{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Tn_x}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Tn_x} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Tn_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\, s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\, s_x^2)} (S_x^2 + s_x^2) + 2\,Tn_x} \right\}$	$\rho_{(s_y^2,\, s_x^2)}$	Tn_x
$\widehat{S}^2_{PR-9} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Qn_x}{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Qn_x} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Qn_x}} + p_2 \left(\frac{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Qn_x}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Qn_x} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Qn_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\, s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\, s_x^2)} (S_x^2 + s_x^2) + 2Qn_x} \right\}$	$\rho_{(s_y^2,\, s_x^2)}$	Qn_x
$\widehat{S}^2_{PR-10} = s_y^2 \left\{ p_1 \left(\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Sn_x}{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Sn_x} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Sn_x}} + p_2 \left(\frac{\rho_{(s_y^2,\, s_x^2)} s_x^2 + Sn_x}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Sn_x} \right)^{\frac{\rho_{(s_y^2,\, s_x^2)} S_x^2}{\rho_{(s_y^2,\, s_x^2)} S_x^2 + Sn_x}} \right\} \exp\left\{ \frac{\rho_{(s_y^2,\, s_x^2)} (S_x^2 - s_x^2)}{\rho_{(s_y^2,\, s_x^2)} (S_x^2 + s_x^2) + 2Sn_x} \right\}$	$\rho_{(s_y^2,\, s_x^2)}$	Sn_x
$\widehat{S}^2_{PR-11} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + IQR_x}{C_x s_x^2 + IQR_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + IQR_x}} + p_2 \left(\frac{C_x s_x^2 + IQR_x}{C_x S_x^2 + IQR_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + IQR_x}} \right\} \exp\left\{ \frac{C_x (S_x^2 - s_x^2)}{C_x (S_x^2 + s_x^2) + 2IQR_x} \right\}$	C_x	IQR_x
$\widehat{S}^2_{PR-12} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + GIN_x}{C_x s_x^2 + GIN_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + GIN_x}} + p_2 \left(\frac{C_x s_x^2 + GIN_x}{C_x S_x^2 + GIN_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + GIN_x}} \right\} \exp\left\{ \frac{C_x (S_x^2 - s_x^2)}{C_x (S_x^2 + s_x^2) + 2GIN_x} \right\}$	C_x	GIN_x
$\widehat{S}^2_{PR-13} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + DOW_x}{C_x s_x^2 + DOW_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + DOW_x}} + p_2 \left(\frac{C_x s_x^2 + DOW_x}{C_x S_x^2 + DOW_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + DOW_x}} \right\} \exp\left\{ \frac{C_x (S_x^2 - s_x^2)}{C_x (S_x^2 + s_x^2) + 2DOW_x} \right\}$	C_x	DOW_x

Table 1. Some new members of the proposed class I estimators (continued).

Estimator	**Value of constant**	
	$\boldsymbol{\varphi}$	$\boldsymbol{\theta}$
$\widehat{S}^2_{PR-14} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + SPW_x}{C_x s_x^2 + SPW_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + SPW_x}} + p_2 \left(\frac{C_x s_x^2 + SPW_x}{C_x S_x^2 + SPW_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + SPW_x}} \right\} \exp\left\{ \frac{C_x(S_x^2 - s_x^2)}{C_x(S_x^2 + s_x^2) + 2SPW_x} \right\}$	C_x	SPW_x
$\widehat{S}^2_{PR-15} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + MADM_x}{C_x s_x^2 + MADM_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + MADM_x}} + p_2 \left(\frac{C_x s_x^2 + MADM_x}{C_x S_x^2 + MADM_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + MADM_x}} \right\} \exp\left\{ \frac{C_x(S_x^2 - s_x^2)}{C_x(S_x^2 + s_x^2) + 2MADM_x} \right\}$	C_x	$MADM_x$
$\widehat{S}^2_{PR-16} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + Bn_x}{C_x s_x^2 + Bn_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Bn_x}} + p_2 \left(\frac{C_x s_x^2 + Bn_x}{C_x S_x^2 + Bn_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Bn_x}} \right\} \exp\left\{ \frac{C_x(S_x^2 - s_x^2)}{C_x(S_x^2 + s_x^2) + 2Bn_x} \right\}$	C_x	Bn_x
$\widehat{S}^2_{PR-17} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + Sr_x^\alpha}{C_x s_x^2 + Sr_x^\alpha} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Sr_x^\alpha}} + p_2 \left(\frac{C_x s_x^2 + Sr_x^\alpha}{C_x S_x^2 + Sr_x^\alpha} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Sr_x^\alpha}} \right\} \exp\left\{ \frac{C_x(S_x^2 - s_x^2)}{C_x(S_x^2 + s_x^2) + 2Sr_x^\alpha} \right\}$	C_x	Sr_x^α
$\widehat{S}^2_{PR-18} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + Tn_x}{C_x s_x^2 + Tn_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Tn_x}} + p_2 \left(\frac{C_x s_x^2 + Tn_x}{C_x S_x^2 + Tn_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Tn_x}} \right\} \exp\left\{ \frac{C_x(S_x^2 - s_x^2)}{C_x(S_x^2 + s_x^2) + 2Tn_x} \right\}$	C_x	Tn_x
$\widehat{S}^2_{PR-19} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + Qn_X}{C_x s_x^2 + Qn_X} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Qn_x}} + p_2 \left(\frac{C_x s_x^2 + Qn_x}{C_x S_x^2 + Qn_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Qn_x}} \right\} \exp\left\{ \frac{C_x(S_x^2 - s_x^2)}{C_x(S_x^2 + s_x^2) + 2Qn_x} \right\}$	C_x	Qn_x
$\widehat{S}^2_{PR-20} = s_y^2 \left\{ p_1 \left(\frac{C_x S_x^2 + Sn_x}{C_x s_x^2 + Sn_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Sn_x}} + p_2 \left(\frac{C_x s_x^2 + Sn_x}{C_x S_x^2 + Sn_x} \right)^{\frac{C_x S_x^2}{C_x S_x^2 + Sn_x}} \right\} \exp\left\{ \frac{C_x(S_x^2 - s_x^2)}{C_x(S_x^2 + s_x^2) + 2Sn_x} \right\}$	C_x	Sn_x

4. Theoretical and numerical efficiency comparisons

In this section, theoretical and numerical efficiency of the proposed generalized class of ratio-product type exponential estimators of population variance are compared with the existing estimators discussed in Section 2.

4.1. Theoretical comparison

For theoretical efficiency comparison, let the MSE of the proposed class of estimators $\widehat{S}^2_{PR}$ be written as:

$$MSE\left(\widehat{S}^2_{PR}\right)_{\min} = \frac{S_y^4}{16}\left(\frac{A}{B}\right), \tag{27}$$

where:

$$A=16\omega^2\beta^{\star 2}_{2(x)}\left(\left(\omega-4\omega^3\right)\lambda^\star_{22}-4\beta^\star_{2(y)}\right) - \omega^4\left(1-4\omega^2\right)^2\beta^{\star 3}_{2(x)}-64\lambda^{\star 2}_{22}\left(1+\beta^\star_{2(y)}\right) +16\beta^\star_{2(x)}\left(\omega^2\left(4\omega^2-1\right)\lambda^{\star 2}_{22}+4(1+2\omega\lambda^\star_{22})\beta^\star_{2(y)}\right),$$

and:

$$B=\omega^2\left(4\omega^2-5\right)\beta^{\star 2}_{2(x)}-16\lambda^{\star 2}_{22} + 4\beta^\star_{2(x)}\left(1+4\omega\lambda^\star_{22}+\beta^\star_{2(y)}\right).$$

then:

(i) The proposed class of estimators has superior efficiency to $\widehat{S}^2_y$ if $Var(\widehat{S}^2_y) - MSE(\widehat{S}^2_{PR})_{\min} > 0$. Thus, by Eqs. (1) and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left\{\frac{16B\beta^\star_{2(y)}-A}{B}\right\}>0.$$

(ii) Similarly, $\widehat{S}^2_{PR}$ is more efficient than $\widehat{S}^2_R$ if $MSE(\widehat{S}^2_R) - MSE(\widehat{S}^2_{PR})_{\min} > 0$. Thus, by Eqs. (2) and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left\{\frac{16B\left(\beta^\star_{2(y)}+\beta^\star_{2(x)}-2\lambda^\star_{22}\right)-A}{B}\right\}>0.$$

(iii) The estimators envisaged in the class $\widehat{S}^2_{PR}$ attain higher efficiency than $\widehat{S}^2_{Reg}$, $\widehat{S}^2_{SW}$, and $\widehat{S}^2_{YK}$ if:

$$MSE(\widehat{S}^2_{Reg}) - MSE(\widehat{S}^2_{PR})_{\min}>0,$$

$$MSE(\widehat{S}^2_{SW})_{\min} - MSE(\widehat{S}^2_{PR})_{\min}>0, \quad \text{and}$$

$$MSE(\widehat{S}^2_{YK})_{\min} - MSE(\widehat{S}^2_{PR})_{\min}>0.$$

Thus, by Eqs. (3), (14), (19), and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left\{\frac{16B\beta^\star_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)-A}{B}\right\}>0.$$

(iv) The efficiency of the proposed class $\widehat{S}^2_{PR}$ is higher than $\widehat{S}^2_d$ if $MSE(\widehat{S}^2_d)_{\min} - MSE(\widehat{S}^2_{PR})_{\min} > 0$. Thus, by Eqs. (4) and (27), the efficiency condition is given by the equation shown in Box V.

(v) $\widehat{S}^2_{PR}$ shows better efficiency than $\widehat{S}^2_{BT}$ if $MSE(\widehat{S}^2_{BT}) - MSE(\widehat{S}^2_{PR})_{\min} > 0$. Thus, by Eqs. (5) and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left\{\frac{4B\left(4\beta^\star_{2(y)}+\beta^\star_{2(x)}-4\lambda^\star_{22}\right)-A}{B}\right\}>0.$$

(vi) The proposed class of estimators $\widehat{S}^2_{PR}$ display superior efficiency to $\widehat{S}^2_{US}$, $\widehat{S}^2_{KC1}$, $\widehat{S}^2_{KC2}$, $\widehat{S}^2_{KC3}$, $\widehat{S}^2_{SK1}$, $\widehat{S}^2_{SK2}$, and $\widehat{S}^2_{KS}$ if their MSEs are greater than $MSE(\widehat{S}^2_{PR})_{\min}$. Thus, by Eqs. (6)–(9), (11)–(13), and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left\{\frac{16B\left(\beta^\star_{2(y)}+\gamma_i^2\beta^\star_{2(x)}-2\gamma_i\lambda^\star_{22}\right)-A}{B}\right\}>0,$$

where $\gamma_i = \gamma_{US}, \gamma_{KC1}, \gamma_{KC2}, \gamma_{KC3}, \gamma_{SK1}, \gamma_{SK2}$, and γ_{KS}, respectively.

(vii) The efficiency of $\widehat{S}^2_{PR}$ is higher than $\widehat{S}^2_{SG}$ if $MSE(\widehat{S}^2_{SG})_{\min} - MSE(\widehat{S}^2_{PR})_{\min} > 0$. Thus, by Eqs. (10) and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left[\frac{16B\left\{\beta^\star_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)\right\}-A\left\{1+\beta^\star_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)\right\}}{B\left\{1+\beta^\star_{2(y)}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)\right\}}\right]>0.$$

Box V

$$\frac{S_y^4}{64}\left[\frac{\left\{-\beta_{2(x)}^{\star 2}-16\beta_{2(y)}^{\star}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)\left(\beta_{2(x)}^{\star}-4\right)\right\}}{1+\beta_{2(y)}^{\star}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)}\right.$$

$$\left.-\frac{4A}{B}\right]>0.$$

(viii) The estimators envisaged in $\widehat{S}^2_{PR}$ achieve higher efficiency than $\widehat{S}^2_{YG}$ if $MSE(\widehat{S}^2_{YG})_{\min}-MSE(\widehat{S}^2_{PR})_{\min}>0$. Thus, by Eqs. (16) and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left[\frac{\left\{16\beta_{2(y)}^{\star}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)\left(1-\beta_{2(x)}^{\star}\right)\right\}}{1-\beta_{2(x)}^{\star}+\beta_{2(y)}^{\star}\left(1-\rho^2_{(S_y^2,S_x^2t)}\right)}\right.$$

$$\left.-\frac{A}{B}\right]>0.$$

(ix) The proposed estimators of class $\widehat{S}^2_{PR}$ are superior to $\widehat{S}^2_{YS}$ in terms of efficiency if $MSE(\widehat{S}^2_{YS})_{\min}-\mathrm{MSE}(\widehat{S}^2_{PR})_{\min}>0$. Thus, by Eqs. (20) and (27), the efficiency condition is given as:

$$\frac{S_y^4}{16}\left[\frac{\left\{64\beta_{2(y)}^{\star}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)-\beta_{2(x)}^{\star 2}\right\}}{4-3\beta_{2(x)}^{\star}+\beta_{2(y)}^{\star}\left(1-\rho^2_{(S_y^2,S_x^2)}\right)}-\frac{A}{B}\right]>0.$$

4.2. Numerical comparison in the presence of outliers

As pointed out in Section 3, the proposed class of estimators incorporate the non-conventional measures, which are somewhat robust to outliers. Therefore, numerical comparison of the proposed class of estimators of population variance with the existing estimators is made by using three population datasets, which contain some outliers. The boxplots given in Figures 1–3 clearly show that both the study and auxiliary variables in Populations 1 and 3 are crippled with outliers, while population 2 has only one outlier in its auxiliary variable. Thus, these datasets are a good realization of both cases with and without outlier observation. These population datasets are frequently used in many studies to compare the performance of various estimators of population mean and variance (see for example [28,29,33,48,49]). The description and various population characteristics are detailed as follows:

Population 1: This dataset is taken from Cochran

Figure 1. Boxplots for Population 1. (a) Study variable and (b) auxiliary variable.

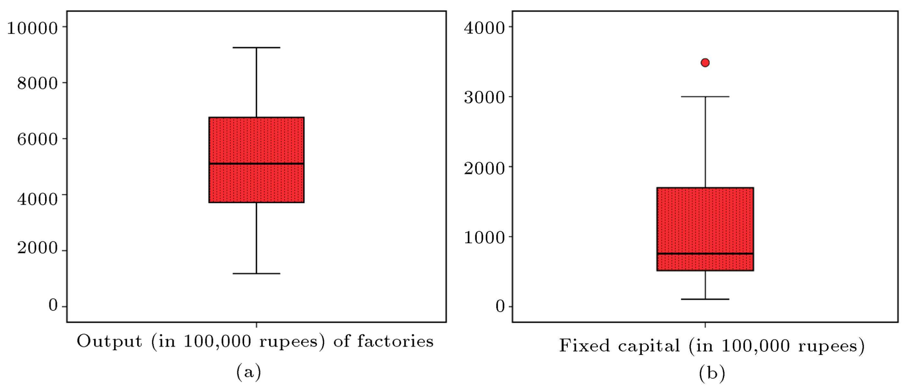

Figure 2. Boxplots for Population 2. (a) Study variable and (b) auxiliary variable.

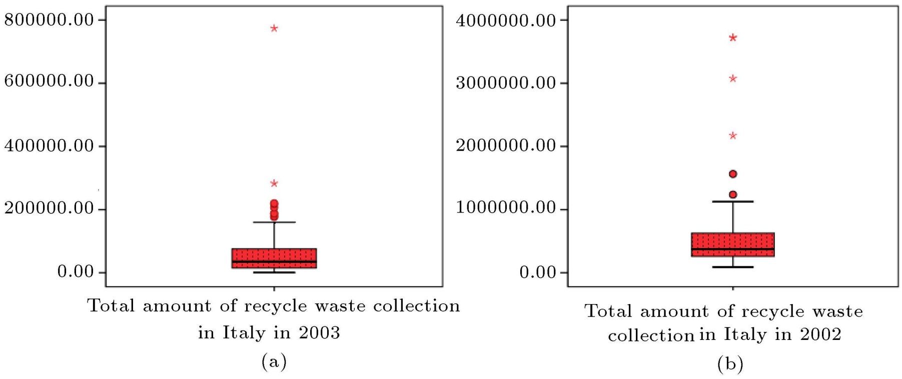

Figure 3. Boxplots for Population 3: (a) Study variable and (b) auxiliary variable.

[35], where y represents the number of inhabitants (in 1000's) in United States cities in 1930 and x is the number of inhabitants (in 1000's) in 1920.

$N = 49$, $n = 20$, $\overline{Y} = 127.7959$,

$\overline{X} = 103.1429$, $S_y^2 = 15158.8299$,

$S_x^2 = 10900.4249$, $\beta_{2(y)} = 4.9245$,

$\beta_{2(x)} = 5.9878$, $C_x = 1.0435$,

$\lambda_{22} = 4.6977$, $\eta = 0.02959$,

$\rho_{(S_y^2,S_x^2)} = 0.83577$, $Q_{1(x)} = 43.0$,

$Q_{3(x)} = 120.0$, $IQR_x = 77$,

$GIN_x = 97.7553$, $DOW_x = 86.6508$,

$SPW_x = 84.8456$, $MADM_x = 39.2889$,

$Bn_x = 52.415$, $Sr_x^\alpha = 34.0998$,

$Tn_x = 35.5488$, $Qn_x = 46.6599$,

$Sn_x = 40.5484$, $\rho_{yx} = 0.9817$,

$M_x = 64.0$.

Population 2: This dataset is obtained from Murthy [50], where y denotes the output (in 100,000 rupees) of factories in a region and x is fixed capital (in 100,000 rupees).

$N = 80$, $n = 20$, $\overline{Y} = 51.8264$,

$\overline{X} = 11.2646$, $S_y^2 = 336.9757$,

$S_x^2 = 70.6634$, $\beta_{2(y)} = 2.2667$,

$\beta_{2(x)} = 2.8664$, $C_x = 0.751$,

$\lambda_{22} = 2.2209$, $\eta = 0.0375$,

$\rho_{(S_y^2,S_x^2)} = 0.79311$, $Q_{1(x)} = 5.1500$,

$Q_{3(x)} = 16.975$, $IQR_x = 11.825$,

$GIN_x = 10.3613$, $DOW_x = 9.1844$,

$SPW_x = 9.0681$, $MADM_x = 4.8925$,

$Bn_x = 7.7060$, $Sr_x^\alpha = 4.0032$,

$Tn_x = 4.3265$, $Qn_x = 5.1770$,

$Sn_x = 4.6869$, $\rho_{yx} = 0.941$,

$M_x = 7.575$.

Population 3: This dataset is obtained from the Italian Bureau of Environment Protection-IBEP (2004) [source: http://www.osservatorionazionalerifuti.it (20-04)], where y denotes the amount of recyclable waste (in tons) collected in different cities of Italy in 2003 and x is the number of inhabitants living in those cities.

$N = 103$, $n = 40$, $\overline{Y} = 62.6212$,

$\overline{X} = 556.5541$, $S_y^2 = 8345.7177$,

$S_y^2 = 8345.7177$, $S_y^2 = 8345.7177$,

$\beta_{2(x)} = 17.8738$, $C_x = 1.0963$,

$\lambda_{22} = 17.2220$, $\eta = 0.01529$,

$\rho_{(S_y^2,S_x^2)} = 0.6570$, $Q_{1(x)} = 259.3830$,

$Q_{3(x)} = 628.0235$, $IQR_x = 368.6405$,

$GIN_x = 457.666$, $DOW_x = 405.678$,

$SPW_x = 401.701$, $MADM_x = 223.169$,

$Bn_x = 241.697$, $Sr_x^\alpha = 191.317$,

Table 2. Estimated numerical results for the MSEs, RRMSEs, and PREs with respect to $\widehat{S}_y^2$ of the existing estimators.

Estimator	Measure	Pop-1	Pop-2	Pop-3	Estimator	Measure	Pop-1	Pop-2	Pop-3
$\widehat{S}_y^2$	MSE	26686254	5393.75	38482180	$\widehat{S}_{SG}^2$	MSE	7411158	1923.464	15515489
	RRMSE	0.3407832	0.2179449	0.7433034		RRMSE	0.179588	0.1301498	0.471975
	PRE	100	100	100		PRE	360.0821	280.4185	248.0243
$\widehat{S}_R^2$	MSE	10314786	2952.368	21898276	$\widehat{S}_{SK-1}^2$	MSE	10213530	2494.658	21896896
	RRMSE	0.2118675	0.1612452	0.5607138		RRMSE	0.2108251	0.14822	0.5606962
	PRE	258.7184	182.6923	175.7316		PRE	261.2834	216.212	175.7426
$\widehat{S}_{Reg}^2$	MSE	8045752	2000.995	21871290	$\widehat{S}_{SK-2}^2$	MSE	10217681	2389.708	21897016
	RRMSE	0.1871189	0.1327469	0.5603682		RRMSE	0.2108679	0.1450688	0.5606977
	PRE	331.6813	269.5533	175.9484		PRE	261.1772	225.7074	175.7417
$\widehat{S}_d^2$	MSE	7773572	1966.345	16644658	$\widehat{S}_{KS}^2$	MSE	10124389	2162.944	21895159
	RRMSE	0.1839266	0.1315925	0.4888479		RRMSE	0.209903	0.1380143	0.5606739
	PRE	343.2946	274.3033	231.1984		PRE	263.5839	249.3707	175.7566
$\widehat{S}_{BT}^2$	MSE	10021370	2188.727	25695997	$\widehat{S}_{SW}^2$	MSE	8045752	2000.995	21871290
	RRMSE	0.2088324	0.1388344	0.607392		RRMSE	0.1871189	0.1327469	0.5603682
	PRE	266.2935	246.4332	149.7594		PRE	331.6813	269.5533	175.9484
$\widehat{S}_{US}^2$	MSE	10305164	2750.183	21898209	$\widehat{S}_{YG}^2$	MSE	7728302	1963.789	15367096
	RRMSE	0.2117687	0.155626	0.560713		RRMSE	0.1833903	0.131507	0.4697126
	PRE	258.96	196.1233	175.7321		PRE	345.3055	274.6603	250.4193
$\widehat{S}_{KC-1}^2$	MSE	10313107	2895.45	21898272	$\widehat{S}_{YK}^2$	MSE	8045752	2000.995	21871290
	RRMSE	0.2118503	0.1596833	0.5607138		RRMSE	0.1871189	0.1327469	0.5603682
	PRE	258.7606	186.2836	175.7316		PRE	331.6813	269.5533	175.9484
$\widehat{S}_{KC-2}^2$	MSE	10305564	2691.57	21898215	$\widehat{S}_{YS}^2$	MSE	7335157	1919.253	14417414
	RRMSE	0.2117728	0.1539587	0.5607131		RRMSE	0.1786648	0.1300072	0.4549671
	PRE	258.95	200.3942	175.732		PRE	363.813	281.0338	266.9146
$\widehat{S}_{KC-3}^2$	MSE	10314506	2932.174	21898275					
	RRMSE	0.2118647	0.1606928	0.5607138					
	PRE	258.7255	183.9505	175.7316					

$Tn_x = 201.547,\qquad Qn_x = 223.029,$

$Sn_x = 221.654,\qquad \rho_{yx} = 0.7298,$

$M_x = 373.82.$

For numerical comparison, we computed the MSEs, Percentage Relative Efficiencies (PREs), and Relative Root Mean Square Errors (RRMSEs) based on the above-mentioned datasets. The PREs of the proposed estimators and the existing estimators relative to the usual SRS estimator of population variance ($\widehat{S}_y^2$) were obtained by using the following expression:

$$PRE\left(\bullet, \widehat{S}_y^2\right) = \frac{Var\left(\widehat{S}_y^2\right)}{MSE(\bullet) \text{ or } MSE(\bullet)_{\min}} \times 100, \quad (28)$$

where $MSE(\bullet)$ or $MSE(\bullet)_{\min}$ denotes the MSEs of the existing and proposed estimators of population variance considered in this study. An estimator with a higher value of PRE is considered superior to its counterparts. The RRMSE is obtained by using the following expression:

$$RRMSE = \frac{\sqrt{MSE\left(\widehat{\theta}_i\right)}}{\theta},$$

$$\widehat{\theta}_i = \widehat{S}_y^2, \widehat{S}_R^2, \widehat{S}_{Reg}^2, \cdots, \widehat{S}_{YS}^2, \widehat{S}_{PR-j}^2, \quad (29)$$

where θ is the true population variance, i.e., S_y^2, and $\widehat{S}_{PR-j}^2 (j = 1, 2, \cdots, 20)$ denotes the proposed estimators given in Table 1. An estimator with the lowest RRMSE is usually declared as the most efficient among the competing estimators.

The numerical results for MSEs, RRMSEs, and PREs of the existing estimators and proposed estimators are given in Tables 2 and 3, respectively. A

Table 3. Estimated numerical results for the MSEs, RRMSEs, and PREs with respect to $\widehat{S}_y^2$ of the proposed estimators.

Estimator	Measure	Pop-1	Pop-2	Pop-3	Estimator	Measure	Pop-1	Pop-2	Pop-3
	MSE	5458119	1862.839	13835683		MSE	5433607	1869.284	13823937
$\widehat{S}_{PR-1}^2$	RRMSE	0.1541189	0.1280823	0.4456938	$\widehat{S}_{PR-11}^2$	RRMSE	0.1537724	0.1283037	0.4455046
	PRE	488.9277	289.5445	278.1372		PRE	491.1333	288.5463	278.3735
	MSE	5490879	1847.107	13842745		MSE	5460177	1853.613	13828183
$\widehat{S}_{PR-2}^2$	RRMSE	0.1545807	0.1275403	0.4458075	$\widehat{S}_{PR-12}^2$	RRMSE	0.1541479	0.1277647	0.445573
	PRE	486.0106	292.0107	277.9953		PRE	488.7434	290.9857	278.288
	MSE	5473413	1832.747	13838622		MSE	5446001	1839.234	13825704
$\widehat{S}_{PR-3}^2$	RRMSE	0.1543346	0.1270436	0.4457412	$\widehat{S}_{PR-13}^2$	RRMSE	0.1539477	0.1272682	0.4455331
	PRE	487.5615	294.2986	278.0781		PRE	490.0156	293.2607	278.3379
	MSE	5470560	1831.236	13838307		MSE	5443688	1837.716	13825515
$\widehat{S}_{PR-4}^2$	RRMSE	0.1542944	0.1269912	0.4457361	$\widehat{S}_{PR-14}^2$	RRMSE	0.153915	0.1272157	0.44553
	PRE	487.8158	294.5414	278.0845		PRE	490.2238	293.5029	278.3418
	MSE	5397326	1763.154	13824116		MSE	5384508	1768.603	13816990
$\widehat{S}_{PR-5}^2$	RRMSE	0.1532582	0.1246082	0.4455075	$\widehat{S}_{PR-15}^2$	RRMSE	0.1530761	0.1248006	0.4453926
	PRE	494.4348	305.9149	278.3699		PRE	495.6117	304.9723	278.5135
	MSE	5418674	1812.166	13825591		MSE	5401720	1818.501	13817876
$\widehat{S}_{PR-6}^2$	RRMSE	0.153561	0.1263282	0.4455312	$\widehat{S}_{PR-16}^2$	RRMSE	0.1533205	0.1265488	0.4454069
	PRE	492.4868	297.6411	278.3402		PRE	494.0325	296.6042	278.4956
	MSE	5388830	1744.32	13821579		MSE	5377668	1749.249	13815467
$\widehat{S}_{PR-7}^2$	RRMSE	0.1531375	0.1239409	0.4454666	$\widehat{S}_{PR-17}^2$	RRMSE	0.1529788	0.1241158	0.4453681
	PRE	495.2143	309.2179	278.421		PRE	496.2422	308.3467	278.5442
	MSE	5391205	1751.381	13822394		MSE	5379580	1756.516	13815956
$\widehat{S}_{PR-8}^2$	RRMSE	0.1531713	0.1241915	0.4454797	$\widehat{S}_{PR-18}^2$	RRMSE	0.153006	0.1243734	0.445376
	PRE	494.996	307.9712	278.4046		PRE	496.0658	307.0709	278.5343
	MSE	5409339	1768.802	13824105		MSE	5394190	1774.389	13816983
$\widehat{S}_{PR-9}^2$	RRMSE	0.1534286	0.1248076	0.4455073	$\widehat{S}_{PR-19}^2$	RRMSE	0.1532136	0.1250046	0.4453925
	PRE	493.3367	304.938	278.3701		PRE	494.7222	303.9779	278.5136
	MSE	5399383	1758.961	13823996		MSE	5386165	1764.303	13816918
$\widehat{S}_{PR-10}^2$	RRMSE	0.1532874	0.1244599	0.4455055	$\widehat{S}_{PR-20}^2$	RRMSE	0.1530996	0.1246488	0.4453915
	PRE	494.2464	306.644	278.3723		PRE	495.4592	305.7156	278.5149

comparison of these results clearly establishes that all the proposed estimators as members of the class $\widehat{S}_{PR}^2$ have smaller MSEs and RRMSEs than the existing estimators of population variance in all the populations considered in this study. Moreover, the PREs of the proposed estimators are much higher than those of their existing counterparts. It is also observed that in most cases, the estimators that employ auxiliary information on C_x and the non-conventional measures in tandem, that is, $\widehat{S}_{PR-1}^2$ to $\widehat{S}_{PR-10}^2$, perform slightly better than other estimators of the proposed class $\widehat{S}_{PR}^2$.

5. Conclusion

In this study, we propose a new generalized class of ratio-product type exponential estimators of population variance under Simple Random Sampling (SRS), which incorporates both the conventional and somewhat robust non-conventional auxiliary information. The results for the bias, Mean Square Error (MSE), and efficiency conditions indicated that the proposed estimators were better than the existing estimators in the literature. Using three different datasets, which contained outlier observations, numerical efficiency comparison with the existing estimators was made based on MSEs, Relative Root Mean Square Errors (RRMSEs), and Percentage Relative Efficiency (PREs). It was established that the proposed estimators had superior efficiency to their counterparts.

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References

1. Cochran, W.G. "The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce", *The Journal of Agricultural Science*, **30**(2), pp. 262–275 (1940).
2. Stehman, S.V. "Use of auxiliary data to improve the precision of estimators of thematic map accuracy", *Remote Sensing of Environment*, **58**(2), pp. 169–176 (1996).
3. Ogliore, R.C., Huss, G.R., and Nagashima, K. "Ratio estimation in SIMS analysis", *Nuclear Instruments and Methods in Physics Research Section B: Beam Interactions with Materials and Atoms*, **269**(17), pp. 1910–1918 (2011).
4. Temesgen, H., Monleon, V., Weiskittel, A., et al. "Sampling strategies for efficient estimation of tree foliage biomass", *Forest Science*, **57**(2), pp. 153–163 (2011).
5. Sanusi, R.A., Abujiya, M.R., Riaz, M., et al. "Combined Shewhart CUSUM charts using auxiliary variable", *Computers and Industrial Engineering*, **105**, pp. 329–337 (2017).
6. Raza, M.A., Nawaz, T., and Aslam, M. "On designing CUSUM charts using ratio-type estimators for monitoring the location of normal processes", *Scientia Iranica*, **27**(3), pp. 1593–1605 (2020).
7. Sanusi, R.A., Abbas, N., and Riaz, M. "On efficient CUSUM-type location control charts using auxiliary information", *Quality Technology & Quantitative Management*, **15**(1), pp. 87–105 (2018).
8. Naz, F., Nawaz, T., Pang, T., et al. "Use of nonconventional dispersion measures to improve the efficiency of ratio-type estimators of variance in the presence of outliers", *Symmetry*, **12**(1), 16 (2020).
9. Özel, G., Çingi, H., and Oğuz, M. "Separate ratio estimators for the population variance in stratified random sampling", *Communications in Statistics-Theory and Methods*, **43**(22), pp. 4766–4779 (2014).
10. Solanki, R.S., Singh, H.P., and Pal, S.K. "Improved ratio-type estimators of finite population variance using quartiles", *Hacettepe Journal of Mathematics and Statistics*, **44**(3), pp. 747–754 (2015).
11. Bahl, S. and Tuteja, R.K. "Ratio and product type exponential estimators", *Journal of Information and Optimization Sciences*, **12**(1), pp. 159–164 (1991).
12. Das, A.K. and Tripathi, T.P. "Use of auxiliary information in estimating the finite population variance", *Sankhya C*, **40**, pp. 139–148 (1978).
13. Isaki, C.T. "Variance estimation using auxiliary information", *Journal of the American Statistical Association*, **78**, pp. 117–123 (1983).
14. Kadilar, C. and Cingi, H. "Ratio estimators for population variance in simple and stratified sampling", *Applied Mathematics and Computation*, **173**, pp. 1047–1058 (2006).
15. Khan, M. and Shabbir, J. "A ratio type estimator for the estimation of population variance using quartiles of an auxiliary variable", *Journal of Statistics Application and Probability*, **2**(3), pp. 319–325 (2013).
16. Maqbool, S. and Javaid, S. "Variance estimation using linear combination of tri-mean and quartile average", *American Journal of Biological and Environmental Statistics*, **3**(1), pp. 5–5 (2017).
17. Shabbir, J. and Gupta, S. "On improvement in variance estimation using auxiliary information", *Communications in Statistics-Theory and Methods*, **36**(12), pp. 2177–2185 (2007).
18. Singh, H.P. and Solanki, R.S. "Improved estimation of finite population variance using auxiliary information", *Communications in Statistics-Theory and Methods*, **42**(15), pp. 2718–2730 (2013).
19. Singh, H.P., Upadhyaya, L.N., and Namjoshi, U.D. "Estimation of finite population variance", *Current Science*, **57**(24), pp. 1331–1334 (1988).
20. Singh, R. and Malik, S. "Improved estimation of population variance using information on auxiliary attribute in simple random sampling", *Applied Mathematics and Computation*, **235**, pp. 43–49 (2014).
21. Subramani, J. and Kumarapandiyan, G. "Variance estimation using quartiles and their functions of an auxiliary variable", *International Journal of Statistics and Applications*, **2**(5), pp. 67–72 (2012).
22. Subramani, J. and Kumarapandiyan, G. "Estimation of variance using known coefficient of variation and median of an auxiliary variable", *Journal of Modern Applied Statistical Methods*, **12**(1), pp. 58–64 (2013).
23. Subramani, J. and Kumarapandiyan, G. "A class of modified ratio estimators for estimation of population variance", *Journal of Applied Mathematics, Statistics and Informatics*, **11**(1), pp. 91–114 (2015).

24. Swain, A.K.P.C. "Generalized estimator of finite population variance", *Journal of Statistical Theory and Applications*, **14**(1), pp. 45–51 (2015).

25. Upadhyaya, L.N. and Singh, H.P. "An estimator for population variance that utilizes the kurtosis of an auxiliary variable in sample surveys", *Vikram Mathematical Journal*, **19**, pp. 14–17 (1999).

26. Yadav, S.K., Kadilar, C., Shabbir, J., et al. "Improved family of estimators of population variance in simple random sampling", *Journal of Statistical Theory and Practice*, **9**(2), pp. 219–226 (2015).

27. Yaqub, M. and Shabbir, J. "An improved class of estimators for finite population variance", *Hacettepe Journal of Mathematics and Statistics*, **45**(5), pp. 1641–1660 (2016).

28. Abid, M., Abbas, N., Nazir, H.Z., et al. "Enhancing the mean ratio estimators for estimating population mean using non-conventional location parameters", *Revista Colombiana de Estadística*, **39**(1), pp. 63–79 (2016).

29. Abid, M., Nazir, H.Z., Riaz, M., et al. "Improved ratio estimators using some robust measures", *Hacettepe Journal of Mathematics and Statistics*, **47**(5), pp. 1375–1393 (2018).

30. Singh, H.P. and Yadav, A. "A new exponential approach for reducing the mean squared errors of the estimators of population mean using conventional and non-conventional location parameters", *Journal of Modern Applied Statistical Methods*, **18**(1), 26 (2020).

31. Subzar, M., Maqbool, S., Raja, T.A., et al. "Improved family of ratio type estimators for estimating population mean using conventional and non conventional location parameters", *Revista Investigacion Operacional*, **38**(5), pp. 499–513 (2017).

32. Abid, M., Sherwani, R.A.K., Abbas, N., et al. "Some improved modified ratio estimators based on decile mean of an auxiliary variable", *Pakistan Journal of Statistics and Operation Research*, **12**(4), pp. 787–797 (2016).

33. Abid, M., Ahmed, S., Tahir, M., et al. "Improved ratio estimators of variance based on robust measures", *Scientia Iranica*, **26**(4), pp. 2484–2494 (2019).

34. Naz, F., Abid, M., Nawaz, T., et al. "Enhancing efficiency of ratio-type estimators of population variance by a combination of information on robust location measures", *Scientia Iranica*, **27**(4), pp. 2040–2056 (2020).

35. Cochran, W.G., *Sampling Techniques*, 3rd Ed., John Wiley and Sons, New York (1977).

36. Subramani, J. and Kumarapandiyan, G. "Variance estimation using median of the auxiliary variable", *International Journal of probability and Statistics*, **1**(3), pp. 36–40 (2012).

37. Yadav, S.K. and Kadilar, C. "A two parameter variance estimator using auxiliary information", *Applied Mathematics and Computation*, **226**, pp. 117–122 (2014).

38. Gini, C. "Variabilità e mutabilità", in *Memorie di Metodologica Statistica Ed.*, Libreria Eredi Virgilio Veschi, Rome, (1912).

39. David, H.A. "Miscellanea: Gini's mean difference rediscovered", *Biometrika*, **55**(3), pp. 573–575 (1968).

40. Downton, F. "Linear estimates with polynomial coefficients", *Biometrika*, **53**(1–2), pp. 129–141 (1966).

41. Muhammad, F., Ahmad, S., and Abiodullah, M. "Use of probability weighted moments in the analysis of means", *Biometrical Journal*, **35**(3), pp. 371–378 (1993).

42. Hampel, F.R. "The influence curve and its role in robust estimation", *Journal of the American Statistical Association*, **69**(346), pp. 383–393 (1974).

43. Shamos, M.I. "Geometry and Statistics: Problems at the Interface", In *New Directions and Recent Results in Algorithms and Complexity*, Traub, J.F., Ed., Academic Press, New York, pp. 251–280 (1976).

44. Bickel, P.J. and Lehmann, E.L. "Descriptive statistics for nonparametric models IV. Spread", In *Contributions to Statistics*, Jureckova, J., Ed., Hajek Memorial Volume, Academia, Prague, pp. 33–40 (1979).

45. Rousseeuw, P.J. and Croux, C. "Alternatives to the median absolute deviation", *Journal of the American Statistical Association*, **88**(424), pp. 1273–1283 (1993).

46. Croux, C. and Rousseeuw, P.J. "A class of high-breakdown scale estimators based on subranges", *Communications in Statistics-Theory and Methods*, **21**(7), pp. 1935–1951 (1992).

47. Rousseeuw, P.J. and Croux, C. "Explicit scale estimators with high breakdown point", In *L1-Statistical Analysis and Related Methods*, Dodge, Y., Ed., North-Holland, Amsterdam, pp. 77–92 (1992).

48. Muneer, S., Khalil, A., Shabbir, J., et al. "A new improved ratio-product type exponential estimator of finite population variance using auxiliary information", *Journal of Statistical Computation and Simulation*, **88**(16), pp. 3179–3192 (2018).

49. Singh, H.P., Singh, A.K., and Solanki, R.S. "Estimation of finite population variance using auxiliary information in sample surveys", *Statistica*, **74**(1), pp. 99–116 (2014).

50. Murthy, M.N., *Sampling Theory and Methods*, Statistical Publishing Society, Calcutta (1967).

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