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Efficient estimation of population median using supplementary variable

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Abstract. The present study aims to introduce an exponential ratio-cum-product type class of estimators for population median under simple random sampling scheme using the supplementary variable. Expressions for the bias and Mean Square Error (MSE) were obtained up to the first order of approximation. The proposed class of estimators was more efficient than all other considered estimators under certain conditions. Four real datasets and simulation studies were carried out to assess the performances of these estimators. Both numerical and simulation studies demonstrated that the proposed class of estimators outperformed the other considered estimators.

1. Introduction

In situations where the supplementary variable is correlated with the study variable, application of the appropriate supplementary variable may result in a considerable reduction in Mean Squared Error (MSE) of the regression, ratio, and product estimators. The main objective of a number of statistical analyses is to estimate the location parameter for the desired distribution, which describes the data in good manners. In case the entire data are homogeneous, the sample mean performs better. There are situations in survey sampling when the study variable indicates an extremely skewed distribution such as expenditure, taxes, income, production, and consumption. In addition, if some of the observations in the data greatly differ from the others, the sample mean performs very poorly, and the results are either under- or over-estimated. To handle such situations, median is used as an alternative measure of location since it performs better in case of skewed variable and gives an efficient estimation of the population parameter. A considerably rich literature is available on the estimation of the mean and total; however, comparatively less efforts have been made to propose efficient estimators of the population median (M_y). Gross [1] was the first to have proposed a weighted estimator without a supplementary variable for population median (M_y) and derived its asymptotic distribution. Kuk and Mak [2] suggested the classical ratio method for population median using supplementary information. Singh et al. [3] presented a class of generalized ratio type estimators for population median using the supplementary variable. Singh et al. [4,5] introduced certain estimators for population median using known parameters of the supplementary variables. Singh et al. [6] suggested chain ratio and regression estimators for population median. Gupta et al. [7] recommended a number of difference-type estimators for population median using the supplementary variables. Other important contributions in this area are the studies conducted by Aladag and Cingi [8], Singh and Solanki [9], Jhajj et al. [10], Sharma and Singh [11], Aladag and Cingi [12], Shabbir

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and Gupta [13,14], Solanki and Singh [15], Enang et al. [16], Jhajj et al. [17], Koyuncu [18], Irfan et al. [19], Yadav et al. [20], Bandyopadhyay et el. [21], Javed et al. [22], and Baig et al. [23]. These researchers proposed a large number of estimators to estimate the population parameters under different situations using the supplementary variables.

In the present study, a new class of ratio-cum-product type exponential estimators for population median (M_y) was proposed under Simple Random Sampling (SRS) scheme using the supplementary variable. This research primarily aims to propose an efficient class of estimators to increase the precision of the median estimators.

Consider a finite population $U = \{U_1, U_2, \cdots, U_N\}$ with N units. Let (y_i, x_i) be the ith values of the survey variable y and supplementary variable x, respectively. A sample of size n was selected from this population using Simple Random Sample Without Replacement (SRSWOR) scheme. Let $\widehat{M}_y$ and $\widehat{M}_x$ be the sample medians corresponding to the population medians M_y and M_x, respectively. Let $f_y(M_y)$ and $f_x(M_x)$ be the probability density functions of M_y and M_x, respectively. Let $\rho_{(\widehat{M}_y,\widehat{M}_x)} = \rho_{yx} = 4P_{11}(y,x) - 1$ be the coefficient of correlation between $\widehat{M}_y$ and $\widehat{M}_x$, where P_{11} indicates the fraction of values in the population with $Y \leq M_Y$ and $X \leq M_X$. It is assumed that (y,x) follows a continuous distribution with $f_y(M_y)$ and $f_x(M_x)$ as their marginal densities.

To get the bias and MSE of the estimators, the following relative error terms are used.

Let:

$$\zeta_0 = \left(\frac{\widehat{M}_y - M_y}{M_y}\right) \quad \text{and} \quad \zeta_1 = \left(\frac{\widehat{M}_x - M_x}{M_x}\right),$$

such that:

$$E(\zeta_j) = 0, \qquad (j = 0, 1), \qquad E(\zeta_0^2) = \lambda\psi_{M_y}^2,$$

$$E(\zeta_1^2) = \lambda\psi_{M_x}^2, \qquad E(\zeta_0\zeta_1) = \lambda\psi_{M_{yx}},$$

where:

$$\psi_{M_y} = \frac{1}{[M_y f_y(M_y)]}, \qquad \psi_{M_x} = \frac{1}{[M_x f_x(M_x)]},$$

$$\psi_{M_{yx}} = \rho_{yx}\psi_{M_y}\psi_{M_x},$$

and:

$$\lambda = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{N}\right).$$

2. Estimators in literature

This section discusses the following usual median per unit estimator and some existing estimators of the population median M_y:

(i) The most common estimator of the median M_y is sample median estimator $\widehat{M}_y$ whose variance is calculated by:

$$V\left(\widehat{M}_y\right) = \lambda M_y^2 \psi_{M_y}^2. \tag{1}$$

(ii) The usual ratio estimator of the median suggested by Kuk and Mak [2] is:

$$\widehat{M}_R = \widehat{M}_y \left(\frac{M_x}{\widehat{M}_x}\right). \tag{2}$$

The bias and MSE of $\widehat{M}_R$ are given by:

$$B\left(\widehat{M}_R\right) \cong \lambda M_y \left(\psi_{M_x}^2 - \psi_{M_{yx}}\right), \tag{3}$$

and:

$$MSE\left(\widehat{M}_R\right) \cong \lambda M_y^2 \left(\psi_{M_y}^2 + \psi_{M_x}^2 - 2\psi_{M_{yx}}\right). \tag{4}$$

(iii) The usual exponential estimator is:

$$\widehat{M}_{\text{Exp}} = \widehat{M}_y \exp\left(\frac{M_x - \widehat{M}_x}{M_x + \widehat{M}_x}\right). \tag{5}$$

The bias and MSE of $\widehat{M}_{\text{Exp}|}$, are given by:

$$B\left(\widehat{M}_{\text{Exp}}\right) \cong \lambda M_y \left(\frac{3}{8}\psi_{M_x}^2 - \frac{1}{2}\psi_{M_{yx}}\right), \tag{6}$$

and:

$$MSE\left(\widehat{M}_{\text{Exp}}\right) \cong \lambda M_y^2 \left(\psi_{M_y}^2 + \frac{1}{4}\psi_{M_x}^2 - \psi_{M_{yx}}\right). \tag{7}$$

(iv) Singh and Solanki [9] proposed the following estimators for M_y as:

$$\widehat{M}_{SS1} = k_1\widehat{M}_y + (1-k_1)\left(M_x - \widehat{M}_x\right), \tag{8}$$

$$\widehat{M}_{SS2} = k_2\widehat{M}_y + k_3\left(M_x - \widehat{M}_x\right), \tag{9}$$

where k_1, k_2, and k_3 are the constants.

To the first degree of approximation, the biases and minimum MSEs of $\widehat{M}_{SSi} (i = 1, 2)$ are given by:

$$B\left(\widehat{M}_{SS1}\right) \cong (k_1 - 1)M_y, \tag{10}$$

$$B\left(\widehat{M}_{SS2}\right) \cong (k_2 - 1)M_y, \tag{11}$$

$$MSE\left(\widehat{M}_{SS1}\right)_{\min} \cong M_y^2 \left[1 + \frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 - \frac{(1 + \frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 + \frac{M_x}{M_y}\lambda\psi_{M_{yx}})^2}{1 + \lambda\psi_{M_y}^2 + \frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 + 2\frac{M_x}{M_y}\lambda\psi_{M_{yx}}}\right], \tag{12}$$

$$MSE\left(\widehat{M}_{SS2}\right)_{\min} \cong \frac{\lambda M_y^2 \psi_{M_y}^2 (1-\rho_{yx}^2)}{1+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)}. \quad (13)$$

The optimal values of $k_j (j = 1,2,3)$ are:

$$k_{1(\text{opt})} = \frac{M_y^2 + M_x^2\lambda\psi_{M_x}^2 - M_y M_x \lambda\psi_{M_{yx}}}{M_y^2 + M_y^2\lambda\psi_{M_y}^2 + M_x^2\lambda\psi_{M_x}^2},$$

$$k_{2(\text{opt})} = \frac{1}{1+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)},$$

$$k_{3(\text{opt})} = \frac{M_y}{M_x}\left[\frac{\frac{\rho_{yx}\psi_{M_y}}{\psi_{M_x}}}{1+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)}\right].$$

(v) The difference estimator of the median is calculated in the following:

$$\widehat{M}_D = \widehat{M}_y - k_4(\widehat{M}_x - M_X), \quad (14)$$

where k_4 is the unknown constant whose value is to be found.

The minimum MSE of $\widehat{M}_D$ can be obtained through the following equation:

$$MSE\left(\widehat{M}_D\right)_{\min} \cong \lambda M_y^2\psi_{M_y}^2\left(1-\rho_{yx}^2\right). \quad (15)$$

The optimum value of k_4 is:

$$k_{4(\text{opt})} = \frac{f_x(M_x)}{f_y(M_y)}\rho_{yx}.$$

(vi) Solanki and Singh [15] proposed a generalized class of estimators for median as follows:

$$\widehat{M}_{SS3} = \widehat{M}_y\left[k_5\left(\frac{cM_X+d}{c\widehat{M}_x+d}\right)+k_6\left(\frac{c\widehat{M}_x+d}{cM_x+d}\right)\right], \quad (16)$$

where k_5 and k_6 are the unknown constants; c and d the known population parameters of the supplementary variable.

The bias and minimum MSE of $\widehat{M}_{SS3}$, at $(c,d) = (1,0)$, are given by Eqs. (17) and (18) as shown in Box I. The optimum values of k_5 and k_6 are obtained by the equations shown in Box II.

(vii) Shabbir and Gupta [14] suggested the following difference-type estimators of the median as follows:

$$\widehat{M}_{D1} = \left[k_7\widehat{M}_y + k_8\left(M_x - \widehat{M}_x\right)\right]\left(\frac{M_x}{\widehat{M}_x}\right), \quad (19)$$

$$\widehat{M}_{D2} = \left[k_9\widehat{M}_y + k_{10}\left(M_x - \widehat{M}_x\right)\right] \exp\left(\frac{M_x - \widehat{M}_x}{M_x + \widehat{M}_x}\right), \quad (20)$$

where $k_j (j = 7,8,9,10)$ are the constants.

The biases and minimum MSEs of $\widehat{M}_{Di} (i = 1,2)$ are given by:

$$B\left(\widehat{M}_{SS3}\right) \cong M_y[k_5(1-\lambda\psi_{M_{yx}}+\lambda\psi_{M_x}^2)+k_6(1+\lambda\psi_{M_{yx}})-1], \quad (17)$$

and:

$$MSE\left(\widehat{M}_{SS3}\right)_{\min} \cong \frac{\lambda M_y^2\begin{pmatrix}\lambda\psi_{M_x}^6+\lambda\psi_{M_y}^2\psi_{M_x}^4+2\lambda\psi_{M_x}^4\psi_{M_{yx}}-4\lambda\psi_{M_y}^2\psi_{M_x}^2\psi_{M_{yx}}-4\lambda\psi_{M_x}^2\psi_{M_{yx}}^2\\+4\lambda\psi_{M_y}^2\psi_{M_{yx}}^2-4\psi_{M_y}^2\psi_{M_x}^2+4\psi_{M_{yx}}^2\end{pmatrix}}{16\lambda\psi_{M_{yx}}^2-4\psi_{M_x}^2-8\lambda\psi_{M_x}^2\psi_{M_{yx}}-4\lambda\psi_{M_y}^2\psi_{M_x}^2-3\lambda\psi_{M_x}^4}. \quad (18)$$

Box I

$$k_{5(\text{opt})} = \left(\frac{\lambda\psi_{M_y}^2\psi_{M_x}^2-3\lambda\psi_{M_x}^2\psi_{M_{yx}}-2\lambda\psi_{M_y}^2\psi_{M_{yx}}+4\lambda\psi_{M_{yx}}^2-2\psi_{M_x}^2+2\psi_{M_{yx}}}{16\lambda\psi_{M_{yx}}^2-4\psi_{M_x}^2-8\lambda\psi_{M_x}^2\psi_{M_{yx}}-4\lambda\psi_{M_y}^2\psi_{M_x}^2-3\lambda\psi_{M_x}^4}\right),$$

and:

$$k_{6(\text{opt})} = \left(\frac{\lambda\psi_{M_x}^4+\lambda\psi_{M_y}^2\psi_{M_x}^2+3\lambda\psi_{M_x}^2\psi_{M_{yx}}-2\lambda\psi_{M_y}^2\psi_{M_{yx}}-4\lambda\psi_{M_{yx}}^2+2\psi_{M_x}^2+2\psi_{M_{yx}}}{16\lambda\psi_{M_{yx}}^2-4\psi_{M_x}^2-8\lambda\psi_{M_x}^2\psi_{M_{yx}}-4\lambda\psi_{M_y}^2\psi_{M_x}^2-3\lambda\psi_{M_x}^4}\right).$$

Box II

$$B\left(\widehat{M}_{D1}\right) \cong (k_7-1)M_y + k_8\lambda M_y(\psi^2_{M_x}-\psi_{M_{yx}}) + k_8\lambda M_x\psi^2_{M_x}, \quad (21)$$

$$B\left(\widehat{M}_{D2}\right) \cong (k_9-1)M_y + k_9\lambda M_y\left(\frac{3}{8}\psi^2_{M_x}-\frac{1}{2}\psi_{M_{yx}}\right) + \frac{1}{2}k_{10}\lambda M_x\psi^2_{M_x}, \quad (22)$$

$$MSE\left(\widehat{M}_{D1}\right)_{\min} \cong M_y^2 \left[\frac{(1-\lambda\psi^2_{M_x})\lambda\psi^2_{M_y}(1-\rho^2_{yx})}{(1-\lambda\psi^2_{M_x})+\lambda\psi^2_{M_y}(1-\rho^2_{yx})}\right], \quad (23)$$

$$MSE\left(\widehat{M}_{D2}\right)_{\min} \cong M_y^2\left[1-\frac{1}{4}\lambda\psi^2_{M_x} - \frac{(1-\frac{1}{8}\lambda\psi^2_{M_x})^2}{1+\lambda\psi^2_{M_y}(1-\rho^2_{yx})}\right]. \quad (24)$$

The optimal values of $k_j(j=7,8,9,10)$ are:

$$k_{7(\text{opt})} = \frac{(1-\lambda\psi^2_{M_x})}{1-\lambda\psi^2_{M_x}+\lambda\psi^2_{M_y}(1-\rho^2_{yx})},$$

$$k_{8(\text{opt})} = \frac{M_y}{M_x}\left[1+k_{7(opt)}\left(\frac{\rho_{yx}\psi_{M_y}}{\psi_{M_x}}-2\right)\right],$$

$$k_{9(\text{opt})} = \frac{1-\frac{1}{8}\lambda\psi^2_{M_x}}{1+\lambda\psi^2_{M_y}(1-\rho^2_{yx})},$$

and:

$$k_{10(\text{opt})} = \frac{M_y}{M_x}\left[\frac{1}{2}+k_{9(\text{opt})}\left(\frac{\rho_{yx}\psi_{M_y}}{\psi_{M_x}}-1\right)\right].$$

3. The proposed estimator

Motivated by Solanki and Singh [15] and Muneer et al. [24], we proposed the following class of exponential ratio-cum-product type estimators of population median M_y as follows:

$$\widehat{M}_{\text{Prop}} = \widehat{M}_y\left[k_{11}\left(\frac{M_x}{\widehat{M}_x}\right)+k_{12}\left(\frac{\widehat{M}_x}{M_x}\right)\right] \left[\alpha\exp\left(\frac{M_x-\widehat{M}_x}{M_x+\widehat{M}_x}\right) + (1-\alpha)\exp\left(\frac{\widehat{M}_x-M_x}{M_x+\widehat{M}_x}\right)\right], \quad (25)$$

where $(k_j, j=11,12)$ are the unknown constants whose values are to be determined and α is the known constant that takes values of $(0,1)$. By substituting the values of α into Eq. (25), we get the following two estimators:

(i) Put $\alpha=0$; then, the proposed estimator is reduced to:

$$\widehat{M}_{\text{Prop}_{(\alpha=0)}} = \widehat{M}_y\left[k_{11}\left(\frac{M_x}{\widehat{M}_x}\right)+k_{12}\left(\frac{\widehat{M}_x}{M_x}\right)\right] \exp\left(\frac{\widehat{M}_x-M_x}{M_x+\widehat{M}_x}\right). \quad (26)$$

(ii) Put $\alpha=1$, and the proposed estimator is reduced to:

$$\widehat{M}_{\text{Prop}_{(\alpha=1)}} = \widehat{M}_y\left[k_{11}\left(\frac{M_x}{\widehat{M}_x}\right)+k_{12}\left(\frac{\widehat{M}_x}{M_x}\right)\right] \exp\left(\frac{M_x-\widehat{M}_x}{M_x+\widehat{M}_x}\right). \quad (27)$$

By expressing $\widehat{M}_{\text{Prop}}$ in Eq. (25) in terms of $(\zeta_j, j=0,1)$ up to the first order of approximation, we get:

$$\widehat{M}_{\text{Prop}} \cong M_y(1+\zeta_0)\left[k_{11}(1-\zeta_1+\zeta_1^2) + k_{12}(1+\zeta_1)\right]\left[1+(1-2\alpha)\frac{\zeta_1}{2}-(1!-4\alpha)\frac{\zeta_1^2}{8}\right]. \quad (28)$$

Keeping $\zeta's$ up to power two and expanding Eq. (28), we have:

$$\widehat{M}_{\text{Prop}} - M_y \cong M_y\left[k_{11}\left\{1+\zeta_0-\left(\alpha+\frac{1}{2}\right)\zeta_1 - \left(\alpha+\frac{1}{2}\right)\zeta_0\zeta_1+\left(\frac{3}{8}+\frac{3\alpha}{2}\right)\zeta_1^2\right\} + k_{12}\left\{1+\zeta_0+\left(\frac{3}{2}-\alpha\right)\zeta_1+\left(\frac{3}{2}-\alpha\right)\zeta_0\zeta_1 + \left(\frac{3}{8}-\frac{\alpha}{2}\right)\zeta_1^2\right\}-1\right]. \quad (29)$$

Solving Eq. (29), the bias and MSE of $\widehat{M}_{\text{Prop}}$ are:

$$\text{Bias}\left(\widehat{M}_{\text{Prop}}\right) \cong M_y\Bigg[(k_{11}+k_{12}-1) + k_{11}\lambda\left\{\left(\frac{3}{8}+\frac{3\alpha}{2}\right)\psi^2_{M_x}-\left(\alpha+\frac{1}{2}\right)\psi_{M_{yx}}\right\} + k_{12}\lambda\left\{\left(\frac{3}{2}-\alpha\right)\psi_{M_{yx}}+\left(\frac{3}{8}-\frac{\alpha}{2}\right)\psi^2_{M_x}\right\}\Bigg], \quad (30)$$

and:

$$MSE\left(\widehat{M}_{\text{Prop}}\right) \cong M_y^2\Big[1 + k_{11}^2\Big\{1 + \lambda\psi_{M_y}^2 + \left(\alpha^2 + 4\alpha + 1\right)\lambda\psi_{M_x}^2 - (4\alpha + 2)\lambda\psi_{M_{yx}}\Big\} + k_{12}^2\Big\{1 + \lambda\psi_{M_y}^2 + \left(\alpha^2 - 4\alpha + 3\right)\lambda\psi_{M_x}^2 + (6 - 4\alpha)\lambda\psi_{M_{yx}}\Big\} + 2k_{11}k_{12}\Big\{1 + \lambda\psi_{M_y}^2 + (2 - 4\alpha)\lambda\psi_{M_{yx}} + \alpha^2\lambda\psi_{M_x}^2\Big\} - 2k_{11}\Big\{1 + \left(\frac{3}{8} + \frac{3\alpha}{2}\right)\lambda\psi_{M_x}^2 - \left(\alpha + \frac{1}{2}\right)\lambda\psi_{M_{yx}}\Big\} - 2k_{12}\Big\{1 + \left(\frac{3}{8} - \frac{\alpha}{2}\right)\lambda\psi_{M_x}^2 + \left(\frac{3}{2} - \alpha\right)\lambda\psi_{M_{yx}}\Big\}\Big]$$

or:

$$MSE\left(\widehat{M}_{Prop}\right) \cong M_y^2\left[1 + k_{11}^2 A_1 + k_{12}^2 A_2 + 2k_{11}k_{12}A_3 - 2k_{11}A_4 - 2k_{12}A_5\right], \qquad (31)$$

where:

$$A_1 = 1 + \lambda\psi_{M_y}^2 + \left(\alpha^2 + 4\alpha + 1\right)\lambda\psi_{M_x}^2 - (4\alpha + 2)\lambda\psi_{M_{yx}},$$

$$A_2 = 1 + \lambda\psi_{M_y}^2 + \left(\alpha^2 - 4\alpha + 3\right)\lambda\psi_{M_x}^2 + (6 - 4\alpha)\lambda\psi_{M_{yx}},$$

$$A_3 = 1 + \lambda\psi_{M_y}^2 + (2 - 4\alpha)\lambda\psi_{M_{yx}} + \alpha^2\lambda\psi_{M_x}^2,$$

$$A_4 = 1 + \left(\frac{3}{8} + \frac{3\alpha}{2}\right)\lambda\psi_{M_x}^2 - \left(\alpha + \frac{1}{2}\right)\lambda\psi_{M_{yx}},$$

$$A_5 = 1 + \left(\frac{3}{8} - \frac{\alpha}{2}\right)\lambda\psi_{M_x}^2 + \left(\frac{3}{2} - \alpha\right)\lambda\psi_{M_{yx}}.$$

Differentiate Eq. (31) with respect to k_{11} and k_{12} so that the optimal values of k_{11} and k_{12} can be obtained as follows:

$$k_{11(\text{opt})} = \left(\frac{A_2A_4 - A_3A_5}{A_1A_2 - A_3^2}\right),$$

and:

$$k_{12(\text{opt})} = \left(\frac{A_1A_5 - A_3A_4}{A_1A_2 - A_3^2}\right).$$

Substitute the optimal values of k_{11} and k_{12} into Eq. (31); then, the minimum MSE of $\widehat{M}_{\text{Prop}}$ can be obtained by:

$$MSE\left(\widehat{M}_{Prop}\right)_{\min} \cong M_y^2\left(1 - \frac{A_1A_5^2 - 2A_3A_4A_5 + A_2A_4^2}{A_1A_2 - A_3^2}\right). \qquad (32)$$

4. Theoretical comparison

Here, the MSE of the proposed estimator $\widehat{M}_{\text{Prop}}$ is compared with the MSEs of the considered estimators to derive the optimal conditions:

Condition (i): By Eqs. (1) and (32):

$$V\left(\widehat{M}_y\right) - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\left(\lambda\psi_{M_y}^2 - 1\right)\theta_2 + \theta_1\right) > 0,$$

where:

$$\theta_1 = A_1A_5^2 - 2A_3A_4A_5 + A_2A_4^2,$$

and:

$$\theta_2 = A_1A_2 - A_3^2.$$

Condition (ii): By Eqs. (4) and (32),

$$MSE\left(\widehat{M}_R\right) - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\left(\lambda\psi_{M_y}^2 + \lambda\psi_{M_x}^2 - 2\lambda\psi_{M_{yx}} - 1\right)\theta_2 + \theta_1\right) > 0.$$

Condition (iii): By Eqs. (7) and (32), we have:

$$MSE\left(\widehat{M}_{\text{Exp}}\right) - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\left(\lambda\psi_{M_y}^2 + \frac{1}{4}\lambda\psi_{M_x}^2 - \lambda\psi_{M_{yx}} - 1\right)\theta_2 + \theta_1\right) > 0.$$

Condition (iv): By Eqs. (12) and (32), we have:

$$MSE\left(\widehat{M}_{SS1}\right)_{\min} - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 - \frac{\left(1 + \frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 + \frac{M_x}{M_y}\lambda\psi_{M_{yx}}\right)^2}{1 + \lambda\psi_{M_y}^2 + \frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 + 2\frac{M_x}{M_y}\lambda\psi_{M_{yx}}} + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (v): By Eqs. (13) and (32), we have:

$$MSE\left(\widehat{M}_{SS2}\right)_{\min} - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\frac{\lambda\psi_{M_y}^2(1-\rho_{yx}^2)}{1+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)} - 1 + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (vi): By Eqs. (15) and (32), we have:

$$MSE\left(\widehat{M}_{D}\right)_{\min} - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\lambda\psi_{M_y}^2(1-\rho_{yx}^2) - 1 + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (vii): By Eqs. (18) and (32), we will have the equation shown in Box III.

Condition (viii): By Eqs. (23) and (32), we have:

$$MSE\left(\widehat{M}_{D1}\right)_{\min} - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\frac{(1-\lambda\psi_{M_x}^2)\lambda\psi_{M_y}^2(1-\rho_{yx}^2)}{(1-\lambda\psi_{M_x}^2)+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)} - 1 + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (ix): By Eqs. (24) and (32), we have:

$$MSE\left(\widehat{M}_{D2}\right)_{\min} - MSE\left(\widehat{M}_{Prop}\right)_{\min} > 0,$$

if:

$$M_y^2\left(\frac{\theta_1}{\theta_2} - \frac{1}{4}\lambda\psi_{M_x}^2 - \frac{(1-\frac{1}{8}\lambda\psi_{M_x}^2)^2}{1+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)}\right) > 0.$$

Note: The proposed estimator $\hat{M}_{\text{Prop}}$ performs better when Conditions (i)–(ix) are satisfied.

5. Numerical comparisons

To examine the performances of the estimators, the following data sets are used for comparison:

- **Population 1 [25]:**
  Y: Number of teaching staff,
  X: Number of students in four different types of schools under 36 districts in Punjab province of Pakistan.
- **Population 2 [12]:**
  Y: Number of teachers,
  X: Number of students in 340 districts with elementary schools in 2007.
- **Population 3 [21]:**
  Y: Graduate or higher degrees in the United States in 2007,
  X: Graduate or higher degrees in the United States in 2006.
- **Population 4 [26]:**
  Y: Oil prices in the current week from 1996 to 2017,
  X: Oil prices in previous week from 1996 to 2017.

The descriptive statistics of all populations are given in Table 1.

The following expression is employed to obtain the Percent Relative Efficiencies (PREs) with respect to Gross [1] unbiased per unit estimator:

$$PRE = \frac{V\left(\widehat{M}_y\right)}{MSE\left(\widehat{M}_j\right) \text{ or } MSE\left(\widehat{M}_j\right)_{\min}} \times 100,$$

where:

$$j = y, R, \text{Exp}, SS1, SS2, D, D1, D2, SS3,$$

$$\text{Prop}_{(\alpha=0)}, \text{Prop}_{(\alpha=1)}.$$

MSEs and PREs values based on four populations are given in Table 2.

According to Table 2, the MSE of the proposed class of estimators in all four populations is minimum compared to the considered estimators. It is also observed that when using different correlations, the proposed estimators are more efficient and stable.

$$MSE\left(\widehat{M}_{SS3}\right)_{\min} - MSE\left(\widehat{M}_{\text{Prop}}\right)_{\min} > 0, \quad \text{if:}$$

$$M_y^2\left[\frac{\lambda\left(\lambda\psi_{M_x}^6+\lambda\psi_{M_y}^2\psi_{M_x}^4+2\lambda\psi_{M_x}^4\psi_{M_{yx}}-4\lambda\psi_{M_y}^2\psi_{M_x}^2\psi_{M_{yx}}-4\lambda\psi_{M_x}^2\psi_{M_{yx}}^2+4\lambda\psi_{M_y}^2\psi_{M_{yx}}^2-4\psi_{M_y}^2\psi_{M_x}^2+4\psi_{M_{yx}}^2\right)}{16\lambda\psi_{M_{yx}}^2-4\psi_{M_x}^2-8\lambda\psi_{M_x}^2\psi_{M_{yx}}-4\lambda\psi_{M_y}^2\psi_{M_x}^2-3\lambda\psi_{M_x}^4}\right.$$

$$\left.-1+\frac{\theta_1}{\theta_2}\right] > 0.$$

Box III

Table 1. Descriptive statistics of all populations.

Notation	Population 1	Population 2	Population 3	Population 4
N	144	340	51	1134
n	10	49	11	210
λ	0.02327	0.00436	0.01782	0.00097
M_y	2023	178	25.80	48.55
M_x	64659	3526	25.60	48.49
$f_y(M_y)$	0.00024	0.00182	0.0728	0.0075349
$f_x(M_x)$	0.00001	0.00800	0.1080	0.0075437
ρ_{yx}	0.86110	0.92000	0.9956	0.9953
ψ_{M_y}	2.05965	3.08680	0.53242	2.73359
ψ_{M_x}	1.54658	0.03545	0.36168	2.73378
$\psi_{M_{yx}}$	2.74295	0.10067	0.19172	7.43791

Table 2. Mean Square Error (MSE) and Percent Relative Efficiencies (PRE) values of all considered and proposed estimators.

	Population 1		Population 2		Population 3		Population 4	
Estimator	MSE	*PRE*	MSE	*PRE*	MSE	*PRE*	MSE	*PRE*
$\widehat{M}_y$	403886.959	100.000	1318.303	100.000	3.36337	100.000	17.08538	100.000
$\widehat{M}_R$	109312.950	369.478	1290.618	102.145	0.36594	919.111	0.1606138	10637.548
$\widehat{M}_{\text{Exp}}$	199667.997	202.279	1304.417	101.065	1.47661	227.777	4.35105	392.672
$\widehat{M}_{SS1}$	102203.340	395.179	1242.712	106.083	3.31657	101.107	16.60673	102.883
$\widehat{M}_{SS2}$	174090.410	231.999	201.205	655.203	0.02953	11389.197	0.16022	10664.082
$\widehat{M}_D$	181824.980	222.129	202.492	651.042	0.02953	11388.691	0.16023	10663.357
$\widehat{M}_{D1}$	173655.141	232.579	201.205	655.203	0.02953	11389.198	0.16021	10664.087
$\widehat{M}_{D2}$	171479.031	235.532	201.205	655.203	0.02946	11417.774	0.15798	10814.411
$\widehat{M}_{SS3}$	103796.950	389.113	173.7406	758.776	0.03060	11440.852	0.16019	10665.607
$\widehat{M}_{\text{Prop}(\alpha=0)}$	**100940.420**	**400.124**	**173.066**	**761.732**	**0.01815**	**18528.773**	**0.11400**	**14986.947**
$\widehat{M}_{\text{Prop}(\alpha=1)}$	**99054.183**	**407.744**	**172.717**	**763.273**	**0.02924**	**11501.591**	**0.141623**	**12064.049**

6. Simulation study

The simulation study was carried out to evaluate the performance of the estimators both empirically and theoretically. To this end, three finite populations of size 1500 generated from a skewed multivariate normal distribution with the same theoretical means of $[Y, X]$ as $\mu = [5, 5]$ were taken into account and different variance covariance matrices were used, as shown in the following:

Population 1:

$$\Sigma = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}.$$

Population 2:

$$\Sigma = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}.$$

Population 3:

$$\Sigma = \begin{bmatrix} 9 & 1.9 \\ 1.9 & 4 \end{bmatrix}.$$

The sample sizes $n = 50, 100, 150, 300$ were attributed to each population, and the probability density functions $f_y(M_y)$ and $f_x(M_x)$ were calculated using normal distribution. Table 3 computes the empirical and theoretical MSEs and PREs. Then, the empirical MSE was estimated using 5000 samples of different sizes selected from each population.

Based on the analysis of the simulation results of all the three artificial populations in Table 3, it can be concluded that the proposed class of estimators

Table 3. Empirical and theoretical Mean Square Error (MSE) and Percent Relative Efficiencies (PRE) values of different estimators with respect to $\hat{M}_y$.

n	Estimator	Population 1			Population 2			Population 3		
		MSE			MSE			MSE		
		Empirical	Theoretical	PRE	Empirical	Theoretical	PRE	Empirical	Theoretical	PRE
50	$\hat{M}_y$	0.306148	0.323727	100.00	0.183689	0.194236	100.00	0.275533	0.291354	100.00
	$\hat{M}_R$	0.236473	0.244817	132.23	0.097730	0.107067	181.41	0.327556	0.341346	85.35
	$\hat{M}_{\text{Exp}}$	0.255370	0.267997	120.79	0.123801	0.134467	144.45	0.269551	0.283504	102.77
	$\hat{M}_{SS1}$	0.318258	0.318130	101.76	0.188586	0.191566	101.39	0.285143	0.287387	101.38
	$\hat{M}_{SS2}$	0.242741	0.241913	133.82	0.099727	0.104712	185.50	0.274293	0.275908	105.60
	$\hat{M}_D$	0.238789	0.244085	132.63	0.098889	0.105124	184.77	0.267575	0.278749	104.52
	$\hat{M}_{D1}$	0.302483	0.241908	133.82	0.161117	0.104711	185.50	0.390574	0.275894	105.60
	$\hat{M}_{D2}$	0.258115	0.241766	133.90	0.114109	0.104646	185.61	0.302462	0.275563	105.73
	$\hat{M}_{SS3}$	0.296073	0.242102	133.71	0.131809	0.104845	185.26	0.411307	0.275573	105.73
	$\hat{M}_{\text{Prop}(\alpha=0)}$	0.330339	0.241733	133.92	0.194626	0.104634	185.63	0.413497	0.275120	105.90
	$\hat{M}_{\text{Prop}(\alpha=1)}$	0.252095	0.241607	133.99	0.108878	0.104620	185.66	0.277523	0.274128	106.28
100	$\hat{M}_y$	0.136593	0.156282	100.00	0.081956	0.093769	100.00	0.122933	0.140654	100.00
	$\hat{M}_R$	0.105485	0.118188	132.23	0.043630	0.051688	181.41	0.145648	0.164788	85.35
	$\hat{M}_{\text{Exp}}$	0.113646	0.129378	120.79	0.054098	0.064915	144.45	0.119849	0.136864	102.77
	$\hat{M}_{SS1}$	0.138803	0.154965	100.85	0.082791	0.093142	100.67	0.124701	0.139721	100.67
	$\hat{M}_{SS2}$	0.106430	0.117325	133.20	0.045065	0.050953	184.03	0.119171	0.133903	105.04
	$\hat{M}_D$	0.106030	0.117834	132.63	0.044935	0.050750	184.77	0.118115	0.134568	104.52
	$\hat{M}_{D1}$	0.132564	0.117325	133.20	0.070592	0.050853	184.39	0.173543	0.133901	105.04
	$\hat{M}_{D2}$	0.112452	0.117291	133.24	0.049209	0.050838	184.45	0.132336	0.133822	105.10
	$\hat{M}_{SS3}$	0.122492	0.117370	133.15	0.099777	0.050795	184.60	0.134908	0.133784	105.13
	$\hat{M}_{\text{Prop}(\alpha=0)}$	0.143380	0.117368	133.16	0.092437	0.050735	184.82	0.179402	0.133717	105.19
	$\hat{M}_{\text{Prop}(\alpha=1)}$	0.108088	0.117254	133.29	0.046540	0.050632	185.20	0.119465	0.133484	105.37
150	$\hat{M}_y$	0.087658	0.100467	100.00	0.052595	0.060280	100.00	0.078892	0.090420	100.00
	$\hat{M}_R$	0.068742	0.075978	132.23	0.028247	0.033228	181.41	0.093848	0.105935	85.35
	$\hat{M}_{\text{Exp}}$	0.073758	0.083172	120.79	0.034606	0.041731	144.45	0.077276	0.087984	102.77
	$\hat{M}_{SS1}$	0.088259	0.099921	100.55	0.052769	0.060020	100.43	0.079355	0.090033	100.43
	$\hat{M}_{SS2}$	0.068951	0.075540	133.00	0.029302	0.032585	184.99	0.076545	0.086233	104.86
	$\hat{M}_D$	0.069058	0.075750	132.63	0.029298	0.032625	184.77	0.076309	0.086508	104.52
	$\hat{M}_{D1}$	0.085043	0.075540	133.00	0.045133	0.032585	184.99	0.110684	0.086232	104.86
	$\hat{M}_{D2}$	0.072697	0.075526	133.02	0.031426	0.032579	185.00	0.084669	0.086199	104.90
	$\hat{M}_{SS3}$	0.107018	0.075559	132.97	0.048483	0.032598	184.92	0.089474	0.086899	104.06
	$\hat{M}_{\text{Prop}(\alpha=0)}$	0.089938	0.075510	133.05	0.061190	0.032576	185.04	0.114622	0.086156	104.95
	$\hat{M}_{\text{Prop}(\alpha=1)}$	0.069556	0.075001	133.95	0.029932	0.032147	187.51	0.076447	0.086059	105.07
300	$\hat{M}_y$	0.037044	0.044652	100.00	0.022226	0.026791	100.00	0.033340	0.040187	100.00
	$\hat{M}_R$	0.029300	0.033768	132.23	0.011986	0.014768	181.41	0.040547	0.047082	85.35
	$\hat{M}_{\text{Exp}}$	0.031229	0.036965	120.79	0.014392	0.018547	144.45	0.032560	0.039104	102.77
	$\hat{M}_{SS1}$	0.037163	0.044544	100.24	0.022263	0.026740	100.19	0.033432	0.040110	100.19
	$\hat{M}_{SS2}$	0.029297	0.033625	132.79	0.012613	0.014492	184.87	0.032058	0.038394	104.67
	$\hat{M}_D$	0.029387	0.033667	132.63	0.012624	0.014500	184.77	0.032029	0.038448	104.52
	$\hat{M}_{D1}$	0.035808	0.033625	132.79	0.018890	0.014492	184.87	0.048751	0.038394	104.67
	$\hat{M}_{D2}$	0.030652	0.033622	132.80	0.013044	0.014491	184.88	0.036065	0.038387	104.69
	$\hat{M}_{SS3}$	0.039510	0.033629	132.78	0.027763	0.014495	184.84	0.041918	0.038381	104.70
	$\hat{M}_{\text{Prop}(\alpha=0)}$	0.038825	0.033642	132.73	0.028340	0.014490	184.89	0.050344	0.038378	104.71
	$\hat{M}_{\text{Prop}(\alpha=1)}$	0.029498	0.033619	132.82	0.012789	0.014423	185.75	0.032083	0.038359	104.76

performs more efficiently, thus including less MSE than all the other considered estimators in all populations. Of note, the empirical MSE of $\widehat{M}_{SS3}$ is very high in all populations, hence performing quite poorly.

7. Conclusion

The current study proposed a new class of ratio-cum-product type exponential estimators for the population median (M_y) under simple random sampling using the supplementary variable. The bias and Mean Square Error (MSE) expressions were obtained up to first order of approximation. Theoretical conditions were derived under which the proposed class of estimators performed better. Tables 2 and 3 compared the proposed classes of estimators $\widehat{M}_{\text{Prop}_{(\alpha=0)}}$ and $\widehat{M}_{\text{Prop}_{(\alpha=1)}}$ with the unbiased estimator of median and other considered estimators both theoretically and empirically in terms of MSEs and Percent Relative Efficiencies (PREs) using different sample sizes. It was observed that the proposed class of estimators had minimum MSE; hence, it performed better in all the four populations. A simulation study reflected the same behavior in all populations. Therefore, the practical application of the proposed estimators $\widehat{M}_{\text{Prop}_{(\alpha=0)}}$ and $\widehat{M}_{\text{Prop}_{(\alpha=1)}}$ in different fields of engineering and data analysis is highly recommended.

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References

1. Gross, S.T., *Median Estimation in Sample Surveys*, Survey Research Method Section, American Statistical Association, pp. 181–184 (1980).
2. Kuk, Y.C.A. and Mak, T.K. "Median estimation in the presence of auxiliary information", *Journal of Royal Statistical Society, B*, **51**, pp. 261–269 (1989).
3. Singh, S., Joarder, A.H., and Tracy, D.S. "Median estimation using double sampling", *Australian and Newzeland Journal of Statistics*, **43**(1), pp. 33–46 (2001).
4. Singh, H.P., Singh, S., and Joarder, A.H. "Estimation of population median when mode of an auxiliary variable is known", *J. Statist. Res.*, **37**(1), pp. 57–63 (2003).
5. Singh, H.P., Sidhu, S.S., and Singh, S. "Median estimation with known interquartile range of auxiliary variable", *Int. J. Appl. Math. Statist.*, **4**, pp. 68–80 (2006).
6. Singh, S., Singh, H.P., Upadhyaya, L.N. "Chain ratio and regression type estimators for median estimation in survey sampling", *Statistical Papers*, **48**(1), pp. 23–46 (2007).
7. Gupta, S., Shabbir, J., and Ahmad, S. "Estimation of median in two phase sampling using two auxiliary variables", *Communications in Statistics-Theory and Methods*, **37**(11), pp. 1815–1822 (2008).
8. Aladag, S. and Cingi, H. "New estimators for the population median in simple random sampling", *Proceedings of the Tenth Islamic Countries Conference on Statistical Sciences (ICCS-X)*, Held in New Cairo, Egypt, pp. 375–383 (2009).
9. Singh, H.P. and Solanki R.S. "Some classes of estimators for the population median using auxiliary information", *Communications in Statistics-Theory and Methods*, **42**, pp. 4222–4238 (2013).
10. Jhajj, H.S., Kaur, H., and Walia, G. "Efficient family of ratio-product type estimators of median", *Model Assisted Statistics and Applications*, **9**, pp. 277–282 (2014).
11. Sharma, P. and Singh, R. "Generalized class of estimators for population median using auxiliary information", *Hacettepe Journal of Mathematics and Statistics*, **44**(2), pp. 443–453 (2015).
12. Aladag, S. and Cingi, H. "Improvement in estimating the population median in simple random sampling and stratified random sampling using auxiliary information", *Communications in Statistics-Theory and Methods*, **44**, pp. 1013–1032 (2015).
13. Shabbir, J. and Gupta, S. "Improved estimation of finite population median under two-phase sampling when using two auxiliary variables", *Scientia Iranica*, **22**(3), pp. 1271–1277 (2015).
14. Shabbir, J. and Gupta, S. "A generalized class of difference-type estimator for population median in survey sampling", *Hacettepe Journal of Mathematics and Statistics*, **46**(5), pp. 1015–1028 (2017).
15. Solanki, R.S. and Singh, H.P. "Some classes of estimators for median estimation in survey sampling", *Communications in Statistics-Theory and Methods*, **44**(7), pp. 1450–1465 (2015).
16. Enang, E.I., Etuk, S.I., Ekpenyong, E.J., and Akpan, V.M. "An alternative exponential estimator of population median", *International Journal of Statistics and Economics*, **17**(3), pp. 85–97 (2016).
17. Jhajj, H.S., Kaur, H., and Jhajj, P. "Efficient family of estimators of median using twophase sampling design", *Communications in Statistics-Theory and Methods*, **45**(15), pp. 4325–31 (2016).
18. Koyuncu, N. "Improved ratio estimation of population mean under median and neoteric ranked set sampling", *AIP Conference Proceedings*, AIP Publishing, **1863**(1), pp. 1200–1204 (2017).

19. Irfan, M., Javed, M., and Lin, Z. "Efficient ratio-type estimators of finite population mean based on correlation coefficient", *Scientia Iranica*, **25**(4), pp. 2361–2372 (2018).

20. Yadav, D.K., Shukla, A.K., Tomer, S., and Kumar, B. "Predictive estimation of finite population mean using coefficient of kurtosis and median of an auxiliary variable under simple random sampling scheme", *International Journal of Mathematical Archive EISSN 2229-5046*, **9**(5), pp. 137–43 (2018).

21. Bandyopadhyay, A., Singh, G.N., and Das, S. "Estimation of population median in presence of non-response under two-phase sampling", *Sri Lankan Journal of Applied Statistics*, **17**(2), pp. 112–133 (2018).

22. Javed, M., Irfan, M., and Pang, T. "Hartley-Ross type unbiased estimators of population mean Using two auxiliary variables", *Scientia Iranica, E*, **26**(6), pp. 3835–3845 (2019). DOI: 10.24200/SCI.2018.5648.1397

23. Baig, A., Masood, S., and Tarray, T.A. "Improved class of difference-type estimators for population median in survey sampling", *Communications in Statistics-Theory and Methods*, **49**(23), pp. 5778–5793 (2019). DOI: 10.1080/03610926.2019.1622017

24. Muneer, S., Khalil, A., Shabbir, J., and Narjis, G. "A new improved ratio-product type exponential estimator of finite population variance using auxiliary information", *Journal of Statistical Computation and Simulation*, **88**(16), pp. 3179–3192 (2018).

25. "PDS Punjab Development Statistics", Bureau of Statistics, Government of the Punjab, Lahore, Pakistan, pp. 114-116 (2012).

26. Amir, M., Shabri, A., and Ishaq, M. "Improving forecasting accuracy of crude oil price using decomposition ensemble model with reconstruction of IMFs based on ARIMA model", *Malaysian Journal of Fundamental and Applied Sciences*, **14**(4), pp. 471–483 (2018).

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