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Nonlinear oscillations of viscoelastic microcantilever beam based on modified strain gradient theory

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Abstract. A viscoelastic microcantilever beam is analytically analyzed based on the modified strain gradient theory. Kelvin-Voigt scheme is used to model beam viscoelasticity. By applying Euler-Bernoulli inextensibility of the centerline condition based on Hamilton's principle, the nonlinear equation of motion and the related boundary conditions are derived from shortening effect theory and discretized by Galerkin method. Inner damping, nonlinear curvature effect, and nonlinear inertia terms are also taken into account. In the present study, the generalized derived formulation allows modeling any nonlinear combination such as nonlinear terms that arise due to inertia, damping, and stiffness, as well as modeling the size effect using modified coupled stress or modified strain gradient theories. First-mode nonlinear frequency and time response of the viscoelastic microcantilever beam are analytically evaluated using multiple time scale method and then, validated through numerical findings. The obtained results indicate that nonlinear terms have an appreciable effect on natural frequency and time response of a viscoelastic microcantilever. Moreover, further investigations suggest that due to the size effects, natural frequency would drastically increase, especially when the thickness of the beam and the length scale parameter are comparable. The findings elaborate the significance of size effects in analyzing the mechanical behavior of small-scale structures.

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1. Introduction

Microstructures have considerably drawn researchers' attention in recent years due to their wide variety of applications. Among different types of microstructures, microbeams are highlighted due to their numerous industrial applications such as Microelectromechanical Systems (MEMS), capacitors, resonators, micro switches, atomic force microscopes, and biosensors. Nevertheless, it is not surprising that the investigator focuses on the dynamical analysis of microbeams [1]. For instance, Ghommem and Abdelkefi [2] developed a nonlinear reduced-order model of an electrically actuated microcantilever beam with a tip mass using perturbation methods. They employed the resultant model as a resonant sensor for bio-mass detection and sensing, analyzed the nonlinear dynamics and effectiveness of the bio-mass sensor, and quantified the mass of biological entities.

On the contrary, numerous experimental results have proved the incapability of the classical theory to accurately predict the mechanical behavior of microscale beams. Studies on microscale structures showed that the classical theory could only estimate

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lower stiffness values, unlike experimental findings. The observations also indicated that the normalized deformation of microstructures depended on the size of the structure [3]. Due to the failure of classical theory in justifying and explaining the mechanical response of microscale structure, researchers proposed several non-classical methods including Modified Strain Gradient (MSG) and Modified Coupled Stress (MCS) theories [4,5]. Mindlin [4] considered strain besides the first and second strain derivatives to formulate strain gradient theory. Fleck and Hutchinson [6] ignored the effect of the second derivative of the strain and reformulated and simplified the strain gradient theory. Lam et al. [7] modified the formulation derived from Fleck and Hutchinson's study [6] by employing a non-classical equilibrium equation (i.e., the equilibrium equation of the moment of couples) as well as the classical equilibrium equations (i.e., equilibrium equations of forces and moment of forces) [8].

A number of researchers utilized the strain gradient or the MCS theories to study the mechanical behavior of the microscale structures [9–13]. Chen and Li [14] studied buckling and post-buckling of the imperfect microbeam, analytically considering the mid-plane stretching for different boundary conditions. They examined the size effects through the modified coupled stress theory on the critical buckling, static response, and dynamic properties of imperfect microbeams. Akgöz and Civalek [15] derived formulations to bend the linear elastic microbeam based on MSG theory for different boundary conditions and analytically solved them. In addition, they studied buckling of an axially loaded microbeam and compared the calculated critical buckling load, which was based on the modified strain gradient theory, via the MCS theory [16]. In another research, Miandoab et al. [17] calculated the length scale parameters along with Young's modulus of polysilicone based on the MSG theory by fitting the experimental reported results of static pull-in voltages to the predicted ones.

Recently, the size effects on nonlinear vibration of viscoelastic microstructures have gained significance [18–22] since viscoelastic materials such as Polydimethylsiloxane (PDMS) [23] have been developed and used in manufactured microbeam biosensors due to their compliance and biocompatibility [24]. Li et al. [25,26] formulated the governing equation of wave motion of fluid-conveying viscoelastic single-walled carbon nanotubes based on the nonlocal strain gradient theory. They discussed the importance of nonlocal as well as small-scale material parameters in the dispersion relation between the phase velocity and the wave number under magnetic field. Attia and Mohamed [27] considered the size effects based on the MCS theory and derived a non-classical nonlinear continuum model of an electrically actuated viscoelastic microbeams for different boundary conditions. Their model accounts for the nonlinearities due to axial residual stress, geometric nonlinearities, and electrical forcing with fringing effect. They modeled the viscoelasticity of the microbeam using Maxwell scheme and solved the resulting equations by the differential/integral quadrature model. In another research, they examined the effects of viscoelastic relaxation time, material length scale parameter, and size on the bending response of the Euler-Bernoulli nanobeams using nonlocal couple stress model [28]. They also studied free vibration of Functionally Graded (FG) viscoelastic nanobeams and investigated the effects of different parameters, including Poisson effect, surface elasticity, material length-scale, and material damping gradient index. They assessed the size effects based on MCS theory and formulated the viscoelasticity effect via the Kelvin Voigt model [29].

The oscillation of viscoelastic micro- and nanoscale Timoshenko beams based on fractional Kelvin-Voigt model was studied by Ansari et al. [30]. They employed Galerkin and predictor-corrector methods to analyze the impacts of length scale parameters, viscoelasticity coefficient, and fractional-order. The nonlinear dynamic stability of a viscoelastic microbeam was analyzed by Fu et al. [31]. They used numerical methods to solve the formulations and investigate the effects of creep quantity, geometric nonlinearity, environmental, and inner damping with symmetric electrostatic load on the principal region of instability. However, they did not take the length scale parameters into account. Fu and Zhang [32] examined the electromechanical dynamic buckling behavior of a viscoelastic microbeam. They also discussed the effects of length scale parameters via MCS theory to analyze the pull-in phenomenon of viscoelastic microbeams with electronic actuation [33]. Ghayesh and Farokhi studied the dynamics of Euler-Bernoulli viscoelastic microcantilevers such as stability and bifurcations [34,35], internal resonances and modal interactions [36], and large-amplitude oscillation [37,38] and investigated the resonant response of an imperfect Timoshenko microbeam considering extensibility [39]. They employed the MCS theory to evaluate the effects of length scale size in these studies and numerically solved equations. In another study, Ghayesh [40] numerically analyzed the nonlinear mechanics of an elastically supported viscoelastic FG microcantilever considering energy transfer via internal resonance and motion complexity. He also employed the MCS theory to investigate the size effects.

A review of the studies on the vibration of nonlinear microcantilevers clearly shows that strain gradient size effects are not considered in modeling the nonlinear viscoelastic microcantilevers and the resultant formulations of full nonlinear models based on the MCS theory are numerically solved. In the present study, the general nonlinear equation of motion for a viscoelastic microcantilever beam was expressed based on the modified strain gradient theory represented by Lam et al. [7] and solved analytically for the first time. The nonlinear curvature effect and inertia terms were also taken into account while modeling the shortening effect. The equation was analytically solved through multiple time scale method, and time-variant nonlinear first-mode frequency of the microcantilever was consequently derived. According to the obtained results, the significance of size effects on the vibration amplitude, damping time, and natural frequency of microcantilever became more highlighted.

2. Model development and analytical solution

A typical rectangular beam of width b, thickness h, and length L was considered in this study. For a nonlinear cantilever, the elongation of an element with length ds (as depicted in Figure 1) can be defined as follows [41]:

$$e=\frac{ds^{*}-ds}{ds}=\sqrt{(1+u')^2+v'^2}-1, \qquad (1)$$

where u and v are the longitudinal and lateral deflections of the cantilever, respectively, prime denotes derivative to x, and z represents the distance of the cross-section from the neutral axis.

An inextensible beam was taken into account to model the shortening effect [41]:

$$e=0\Rightarrow u'=\sqrt{1-v'^2}-1\approx-\frac{1}{2}v'^2$$

$$\Rightarrow u=-\int_0^x\frac{1}{2}v'^2dx, \qquad (2)$$

Referring to Figure 1 [41]:

$$\tan\theta=\frac{v'}{1+u'}. \qquad (3)$$

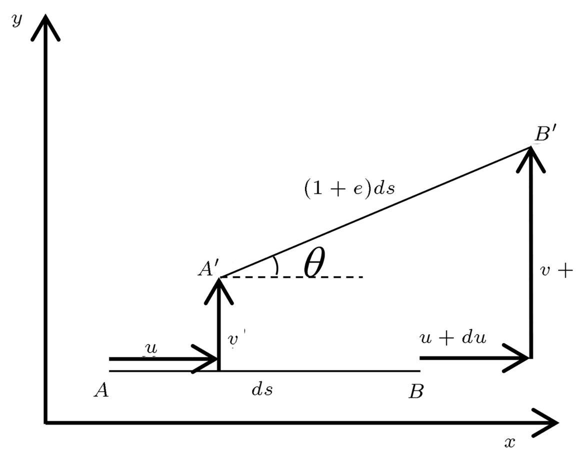

Figure 1. Coordinate system of a deflected microcantilever.

Differentiating Eq. (3) and using Eq. (2) would yield [41]:

$$\theta'=\frac{v''(1+u')-v'u''}{(1+u')^2+v'^2}\Rightarrow\theta'\approx v''+\frac{1}{2}v''v'^2. \qquad (4)$$

Hamilton's principle was employed to derive the equation of motion, according to which the kinetic and potential energy of vibrating homogenous microcantilever with a cross-section A (i.e., $A=bh$) and density ρ could be calculated.

The kinetic energy of the microcantilever can be obtained as shown in the following [41]:

$$T=\frac{1}{2}\int_0^L\rho A\left\{\dot{u}^2+\dot{v}^2\right\}dx. \qquad (5)$$

By substituting u into v' from Eq. (2) in Eq. (5), the following equation can be obtained [41]:

$$T=\frac{1}{2}\int_0^L\left\{\rho A\left[\frac{\partial}{\partial t}\int_0^x\frac{1}{2}v'^2dx\right]^2+\rho A\dot{v}^2\right\}dx. \qquad (6)$$

The density of strain energy of a linear elastic solid based on the MSG theory derived by Lam et al. [7] can be expressed as follows:

$$\varrho=\sigma_{ij}\varepsilon_{ij}+p_i\gamma_i+\tau_{ijk}^{(1)}\eta_{ijk}^{(1)}+m_{ij}^{s}\chi_{ij}^{s}, \qquad (7)$$

where the components of the classical strain tensor (ε), dilatation gradient vector (γ), deviatoric stretch gradient tensor ($\eta^{(1)}$), and symmetric part of the rotation gradient tensor (χ^s) are defined as [7]:

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right), \qquad (8)$$

$$\gamma_i=\frac{\partial}{\partial i}(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}), \qquad (9)$$

$$\eta_{ijk}^{(1)}=\frac{1}{3}\left(\varepsilon_{ij,k}+\varepsilon_{jk,i}+\varepsilon_{ki,j}\right)$$
$$-\frac{1}{15}\Big[\delta_{ij}\left(\varepsilon_{mm,k}+2\varepsilon_{mk,m}\right)$$
$$+\delta_{jk}\left(\varepsilon_{mm,i}+2\varepsilon_{mi,m}\right)$$
$$+\delta_{ki}\left(\varepsilon_{mm,j}+2\varepsilon_{mj,m}\right)\Big], \qquad (10)$$

$$\chi_{ij}^{s}=\frac{1}{2}\left(e_{ipq}\varepsilon_{qj,p}+e_{jpq}\varepsilon_{qi,p}\right), \qquad (11)$$

where u_i presents the components of the displacement vector u; and δ_{ij} and e_{ijk} denote Kronecker delta and alternating tensor, respectively. For the viscoelastic material, the classical Kelvin-Voigt model is used. In this regard, the stress tensor (σ) and the corresponding higher-order stresses ($p,\tau^{(1)},m^s$) relations are:

$$\sigma_{tot}=E\varepsilon+C\dot{\varepsilon}=\sigma+\sigma_{vis}, \tag{12}$$

$$p_{tot}=p+p_{vis}=2\mu_k l_0^2\gamma+2\mu_C l_0^2\dot{\gamma}, \tag{13}$$

$$\tau_{tot}^{(1)}=\tau^{(1)}+\tau_{vis}^{(1)}=2\mu_k l_1^2\eta^{(1)}+2\mu_C l_0^2\dot{\eta}^{(1)}, \tag{14}$$

$$m_{tot}^s=m^s+m_{vis}^s=2\mu_k l_2^2\chi^s+2\mu_C l_2^2\dot{\chi}^s, \tag{15}$$

where E is the elastic modulus, C is the damping coefficient, and l_0, l_1, l_2 are new material constants known as length scale parameters. Moreover, μ_k and μ_C are elastic and viscoelastic shear moduli related to Poisson's ratio (υ), as shown in the following:

$$\mu_k=\frac{E}{2(1+\upsilon)}, \quad \mu_C=\frac{C}{2(1+\upsilon)}. \tag{16}$$

The variation of works in terms of non-conservative forces can be formulated as:

$$\delta W_{vis}=-\textstyle\int(\sigma_{vis}\delta\varepsilon+p_{vis}\delta\gamma+\tau_{vis}\delta\eta+m_{vis}\delta\chi)dV. \tag{17}$$

The dimensionless parameters presented below are utilized to derive the normalized form of the differential equations:

$$\bar{x}=\frac{x}{L}, \quad \bar{v}=\frac{v}{L}, \quad \theta=\sqrt{\frac{EI}{\rho A L^4}},$$

$$\tau=\theta t, \quad \eta=\frac{C\theta}{E}, \quad \xi=\frac{\eta\omega}{2},$$

$$S=EI\tilde{S}, \quad K=EIL^2\tilde{K},$$

$$S_C=CI\tilde{S}, \quad K_C=CIL^2\tilde{K},$$

$$\tilde{S}=1+\frac{6}{1+\upsilon}\left(2\left(\frac{l_0}{h}\right)^2+\frac{120}{225}\left(\frac{l_1}{h}\right)^2+\left(\frac{l_2}{h}\right)^2\right),$$

$$\tilde{K}=\frac{1}{2(1+\upsilon)}\left(2\left(\frac{l_0}{L}\right)^2+\frac{4}{5}\left(\frac{l_1}{L}\right)^2\right), \tag{18}$$

where l is the area moment of inertia. The Hamilton principle can be used to achieve the normalized form of the nonlinear differential equation of motion and the related boundary conditions. It can be observed that due to the coupling effects of large deformation, strain gradient theory, and viscoelasticity, different nonlinearities are incorporated.

$$\begin{aligned}
&\frac{\partial^2\bar{v}}{\partial\tau^2}+\tilde{S}\frac{\partial^4\bar{v}}{\partial\bar{x}^4}+\eta\tilde{S}\frac{\partial^5\bar{v}}{\partial\bar{x}^4\partial\tau}-\eta\tilde{K}\frac{\partial^7\bar{v}}{\partial\bar{x}^6\partial\tau}-\tilde{K}\frac{\partial^6\bar{v}}{\partial\bar{x}^6}\\
&\quad+\tilde{S}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\right)\right)\\
&\quad+\frac{\partial}{\partial\bar{x}}\left[\frac{\partial\bar{v}}{\partial\bar{x}}\textstyle\int_1^{\bar{x}}\int_0^s\left(\left(\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\tau}\right)^2+\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^3\bar{v}}{\partial\bar{x}\partial\tau^2}\right)dsds\right]\\
&\quad-\eta\tilde{K}\left[\frac{\partial^4}{\partial\bar{x}^3\partial\tau}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\right)\right)\right.\\
&\quad-\frac{\partial}{\partial\bar{x}}\left(\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\tau}\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^5\bar{v}}{\partial\bar{x}^5}\right)-\frac{\partial^2}{\partial\bar{x}\partial\tau}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\frac{\partial^4\bar{v}}{\partial\bar{x}^4}\right)\\
&\quad\left.-\frac{\partial^3}{\partial\bar{x}^2\partial\tau}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\frac{\partial^3\bar{v}}{\partial\bar{x}^3}\right)\right]\\
&\quad+\eta\tilde{S}\left[\frac{\partial^2}{\partial\bar{x}\partial\tau}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\right)\right)\right.\\
&\quad\left.-\frac{\partial}{\partial\bar{x}}\left(\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\tau}\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^3\bar{v}}{\partial\bar{x}^3}\right)\right]\\
&\quad-\tilde{K}\left[\frac{\partial^3}{\partial\bar{x}^3}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\right)\right)\right.\\
&\quad\left.-\frac{\partial^2}{\partial\bar{x}^2}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\frac{\partial^3\bar{v}}{\partial\bar{x}^3}\right)-\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}\frac{\partial^4\bar{v}}{\partial\bar{x}^4}\right)\right]\\
&\quad=0
\end{aligned} \tag{19}$$

$$\begin{bmatrix} (\bar{v}'''+\bar{v'''}\bar{v'}^2+\bar{v}'\bar{v''}^2)\delta\bar{v}|_0^1=0 \\ (\bar{v''}+\bar{v''}\bar{v'}^2)\delta\bar{v'}|_0^1=0 \end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} \bar{x}=0\rightarrow\bar{v}=\bar{v}'=0 \\ \bar{x}=1\rightarrow\bar{v}''=\bar{v}'''=0 \end{bmatrix}. \tag{20}$$

The Galerkin method was employed so that the following mode shapes could be considered for the cantilever beam:

$$\bar{v}=\sum_{m=1}^{\infty}\varphi_m(\bar{x})\,q_m(\tau), \tag{21}$$

$$\begin{aligned}\varphi_m(\bar{x})&=\cosh(z_m\bar{x})-\cos(z_m\bar{x})\\&\quad+\frac{\cos(z_m)+\cosh(z_m)}{\sin(z_m)+\sinh(z_m)}(\sin(z_m\bar{x})-\sinh(z_m\bar{x})),\end{aligned} \tag{22}$$

$$\cos(z_m)\cosh(z_m)+1=0. \tag{23}$$

The shape functions are normalized with the orthogonality condition of Eq. (24):

$$<\varphi_m(\bar{x}),\varphi_n(\bar{x})>=\int_0^1\varphi_m(\bar{x})\varphi_n(\bar{x})d\bar{x}=\delta_{mn}. \tag{24}$$

By substituting Eq. (21) into Eq. (19) and taking the inner product of the resulting equation with φ_1 (simply mentioned as φ), the first-mode shape time-dependent equation can be expressed as the following differential equation:

$$\begin{aligned}&\ddot{q}+2\xi\omega\dot{q}+\omega^2q+g_1q^3+g_2q\dot{q}^2+g_3q^2\ddot{q}+\frac{2\xi g_4}{\omega}\dot{q}q^2\\&\quad=0,\end{aligned} \tag{25}$$

where the coefficients ω and g_i can be obtained as:

$$\omega^2 = \tilde{S}a_1 - \tilde{K}a_2,$$

$$g_1 = \tilde{S}a_4 - \tilde{K}a_5,$$

$$g_2 = g_3 = a_3,$$

$$g_4 = \tilde{S}a_6 - \tilde{K}a_7, \quad (26)$$

where a_i is derived as Eq. (27):

$$a_1 = \int_0^1 \varphi\varphi^{IV} d\bar{x},$$

$$a_2 = \int_0^1 \varphi\varphi^{VI} d\bar{x},$$

$$a_3 = \int_0^1 \varphi\left(\varphi' \int_1^s \int_0^s \varphi'^2 ds ds\right)' d\bar{x},$$

$$a_4 = \int_0^1 \varphi\left(\varphi''^3 + 4\varphi'\varphi''\varphi''' + \varphi'^2\varphi^{IV}\right) d\bar{x},$$

$$a_5 = \int_0^1 \varphi\left[\varphi'^2\varphi^V + 4\varphi'\varphi''\varphi^{IV} + 6\varphi''^2\varphi''' + 3\varphi'\varphi'''^2\right] d\bar{x},$$

$$a_6 = 3a_4 - \int_0^1 \varphi\left[2\varphi'\varphi''\varphi''' + \varphi'^2\varphi^{IV}\right] d\bar{x},$$

$$a_7 = 3a_5 - \int_0^1 \varphi\left[2\varphi'\varphi''\varphi^V + \varphi'^2\varphi^{VI}\right] d\bar{x}. \quad (27)$$

The multiscale perturbation method was employed to solve Eq. (25). For weak nonlinearities in the microcantilever beam, Eq. (25) can be rewritten as:

$$\ddot{q} + 2\varepsilon\xi\omega\dot{q} + \omega^2 q + \varepsilon N[q(\tau)] = 0, \quad (28)$$

where N is the summation of nonlinear terms:

$$N = g_1 q^3 + g_2 q\dot{q}^2 + g_3 q^2\ddot{q} + \frac{2\xi g_4}{\omega}\dot{q}q^2. \quad (29)$$

To calculate the periodic solution of q, the expansion in terms of ε can be written as:

$$q = q_0(T_0, T_1) + \varepsilon q_1(T_0, T_1), \quad (30)$$

where $T_0 = \tau$ and $T_1 = \varepsilon\tau$ are fast and slow time scales, respectively, that characterize the motions happening at the natural frequencies and related shifts due to nonlinearities. Then, in terms of the power of ε, Eq. (28) can be rewritten as follows:

$$\varepsilon^0 \rightarrow \frac{\partial^2 q_0}{\partial T_0^2} + \omega^2 q_0 = 0, \quad (31)$$

$$\varepsilon^1 \rightarrow \frac{\partial^2 q_1}{\partial T_0^2} + \omega^2 q_1 = -2\frac{\partial^2 q_0}{\partial T_0 \partial T_1} - 2\xi\omega\frac{\partial q_0}{\partial T_0} - g_1 {q_0}^3 - g_2 q_0\left(\frac{\partial q_0}{\partial T_0}\right)^2 - g_3 q_0^2\frac{\partial^2 q_0}{\partial T_0^2} - \frac{2\xi g_4}{\omega}q_0^2\frac{\partial q_0}{\partial T_0}. \quad (32)$$

While solving Eq. (31), q_0 can be determined as follows:

$$q_0 = A(T_1)e^{i\omega T_0} + \bar{A}(T_1)e^{-i\omega T_0}. \quad (33)$$

Therefore, we have:

$$q_0^3 = A^3 e^{3i\omega T_0} + 3A^2\bar{A}e^{i\omega T_0} + CC,$$

$$q_0\left(\frac{\partial q_0}{\partial T_0}\right)^2 = \left(-\omega^2\right) \times \left(A^3 e^{3i\omega T_0} - A^2\bar{A}e^{i\omega T_0} + CC\right),$$

$$q_0^2\frac{\partial^2 q_0}{\partial T_0^2} = \left(-\omega^2\right) \times \left(A^3 e^{3i\omega T_0} + 3A^2\bar{A}e^{i\omega T_0} + CC\right),$$

$$q_0^2\frac{\partial q_0}{\partial T_0} = (i\omega) \times \left(A^3 e^{3i\omega T_0} + A^2\bar{A}e^{i\omega T_0} + CC\right), \quad (34)$$

where CC denotes Complex Conjugate. By substituting Eq. (34) into Eq. (32), we have:

$$\frac{\partial^2 q_1}{\partial T_0^2} + \omega^2 q_1 = -\left[g_1 - \omega^2(g_2 + g_3) + 2i\xi g_4\right]A^3 e^{3i\omega T_0} - \left\{\begin{matrix} 2i\omega A' + 2i\xi\omega^2 A + \\ \left[3g_1 + \omega^2(g_2 - 3g_3) + 2i\xi g_4\right]A^2\bar{A} \end{matrix}\right\} e^{i\omega T_0} + CC. \quad (35)$$

In case the secular terms of Eq. (35) are equal to zero, q_1 can be calculated as:

$$q_1 = \frac{g_1 - \omega^2(g_2 + g_3) + 2i\xi g_4}{8\omega^2}A^3 e^{3i\omega T_0} + CC. \quad (36)$$

If the polar form of A is used to omit secular terms of Eq. (35), we have:

$$A = \frac{1}{2}\alpha(T_1)e^{i\beta(T_1)} \rightarrow \left\{\begin{matrix} \alpha' + \xi\omega\alpha + \frac{\xi g_4}{4\omega}\alpha^3 = 0 \\ \frac{\partial\beta}{\partial T_1} = \frac{3g_1 + \omega^2(g_2 - 3g_3)}{8\omega}\alpha^2 \end{matrix}\right\}. \quad (37)$$

Calculating α and β from Eq. (37) would result in:

$$\begin{cases} \alpha^2 = \frac{4\omega^2\alpha_0^2}{(4\omega^2 + g_4\alpha_0^2)e^{2\xi\omega\varepsilon\tau} - g_4\alpha_0^2} \\ \beta = \frac{3g_1 + \omega^2(g_2 - 3g_3)}{4\xi g_4}\left(\ln\left(\begin{matrix} (4\omega^2 + g_4\alpha_0^2)e^{2\xi\omega\varepsilon\tau} \\ -g_4\alpha_0^2 \end{matrix}\right) - 2\xi\omega\varepsilon\tau - \ln(4\omega^2)\right). \end{cases} \quad (38)$$

By substituting Eq. (38) into Eq. (33) and Eq. (36), with consideration of Eq. (30) and Eq. (21), the nonlinear frequency and response of a viscoelastic microcantilever based on the MSG theory may be expressed as shown in the following:

$$\begin{cases} \omega_{NL} = \omega + \frac{\beta}{\tau} \\ v = \varphi \Big[\alpha \cos(\omega_{NL}\tau) - \frac{\varepsilon \xi g_4 \alpha^3}{16\omega^2} \sin(3\omega_{NL}\tau) \\ \qquad + \frac{\varepsilon \alpha^3}{32\omega^2} \left[\left(g_1 - \omega^2 (g_2 + g_3) \right) \cos(3\omega_{NL}\tau) \right] \Big] \end{cases} \tag{39}$$

In the absence of damping, $\xi = 0$, the outcome of Eq. (37) and Eq. (38) would be:

$$\begin{cases} \alpha = \alpha_0 \\ \beta = \frac{3g_1 + \omega^2 (g_2 - 3g_3)}{8\omega} \alpha_0^2 \varepsilon \tau \end{cases} \tag{40}$$

if $l_0 = l_1 = l_2 = 0$, the obtained equation would be the same as that reported by Nayfeh and Nayfeh [42] for elastic microcantilever via classical beam theory.

$$\begin{cases} \omega_{NL} = \omega + \frac{\beta}{\tau} \\ v = \varphi \Big\{ \alpha \cos(\omega_{NL}\tau) + \frac{\varepsilon \alpha^3}{32\omega^2} \left(g_1 - \omega^2 (g_2 + g_3) \right) \\ \qquad \cos(3\omega_{NL}\tau) \Big\} \end{cases} \tag{41}$$

3. Results and discussion

In this section, the first-mode nonlinear frequency and time response of the viscoelastic microcantilever beam are investigated considering the dimensionless parameters, namely $\eta = 0.01$, $\nu = 0.17$, $\omega_1 = 3.5160$, and $h/l = 5$. Constants a_i were evaluated for the first-mode shape of the microcantilever beam, and for MSG formulation, equal length scale parameters are considered (i.e., $l_0 = l_1 = l_2 = l$) [7,43].

The numerical results obtained from Eq. (28) were compared with the analytical findings from Eq. (38) plotted in Figure 2. The outcomes indicate that the two-term expansion of Eq. (38) models the time response fairly with the maximum error of 5%.

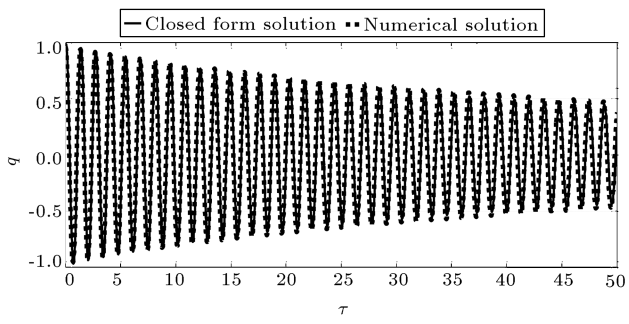

Figure 2. Numerical and two-term expansion model response of a viscoelastic microcantilever at the free end.

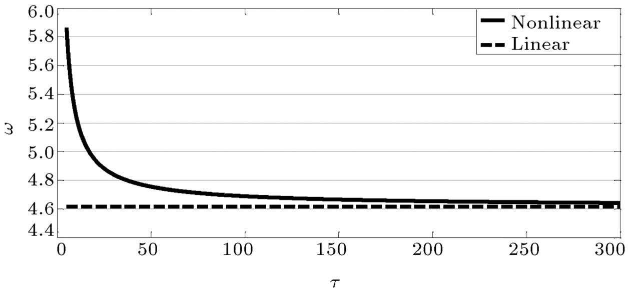

Figure 3. Linear and nonlinear frequencies of a viscoelastic microcantilever.

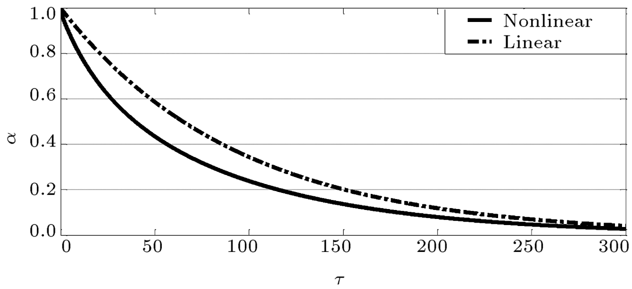

Figure 4. Linear and nonlinear maximum amplitudes of free vibration of a viscoelastic microcantilever.

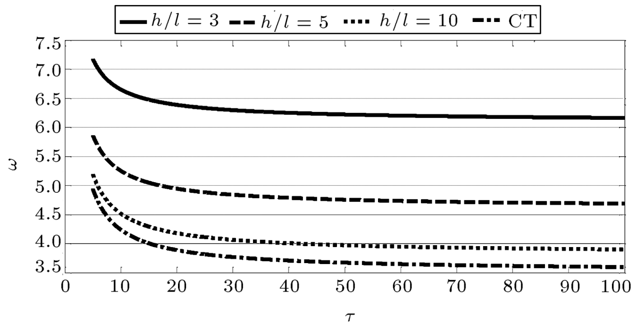

Figure 5. Nonlinear frequency of viscoelastic microcantilever for different values of h/l and Classical Theory (CT).

The first mode linear and nonlinear natural frequencies of a viscoelastic microcantilever with respect to the MSG theory are depicted in Figure 3 which shows that nonlinear frequency approaches the linear one asymptotically over time. The dimensionless maximum amplitudes of linear and nonlinear vibrations are compared the results of which are shown in Figure 4. As observed, the nonlinear phenomena have a lower amplitude at a fixed frequency, implying that nonlinear effect causes higher strength in the structure.

The effects of length scale parameters on the first-mode frequency and the amplitude of viscoelastic microcantilever beam vibration are presented in Figures 5 and 6. As the thickness of the microcantilever cross-section approaches the length scale parameters, the nonlinear frequency evaluated by MSG theory would increase (Figure 5). The same phenomenon for this

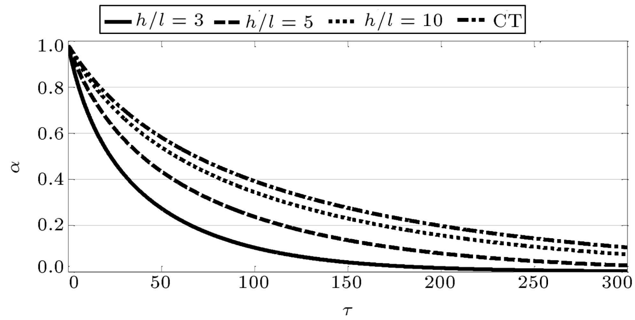

Figure 6. Nonlinear maximum amplitude for different values of h/l and Classical Theory (CT).

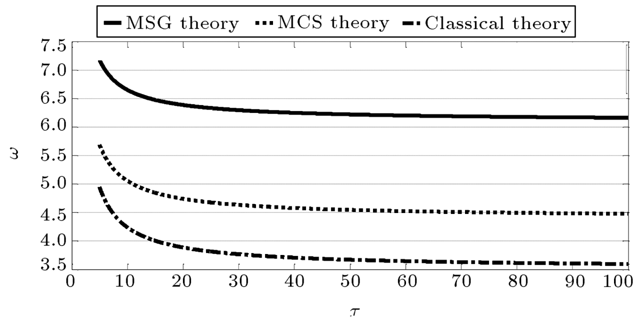

Figure 7. Nonlinear frequency of a viscoelastic microcantilever by Modified Coupled Stress (MCS) theory, Modified Strain Gradient (MSG) theory, and classical theory.

amplitude is depicted in Figure 6. The results indicated that as thickness approached the length scale, the vibration amplitude drastically decreased. Moreover, the hardening effect of considering the length scale parameters on the microcantilever behavior would lead to higher nonlinear frequency of the microcantilever and lower amplitude of the vibration. Figures 5 and 6 indicate that the effect of the length scale increased as the thickness-to-length scale parameter ratio of the beam reduced; therefore, the difference between the results of the derived model and predictions of the classical one cannot be ignored.

In Figures 7 and 8, the frequencies and maximum amplitudes of viscoelastic microcantilever beam versus time are compared based on the modified strain gradient, MCS, and classical theories. It can be observed that both MCS and MSG theories estimate higher stiffness for the beam than classical estimations; however, the modified strain gradient theory could improve the hardness much more than the MCS theory. In other words, the values of natural frequency calculated by the MSG theory are higher than those modeled by MCS theory, and the values estimated by MCS theory are higher than those estimated by the classical theory. However, the maximum amplitude of the free vibration was reversed, implying that the classical method could calculate the maximum amplitude of the vibration

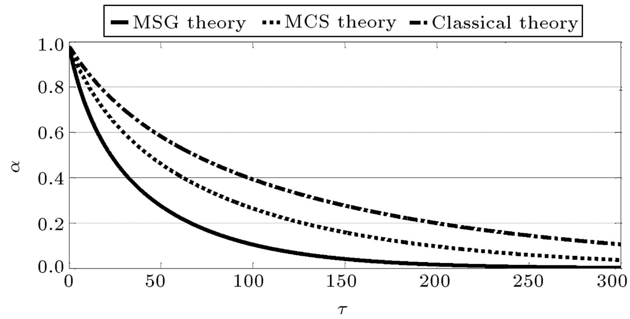

Figure 8. Maximum amplitude of nonlinear free vibration of a viscoelastic microcantilever by Modified Coupled Stress (MCS) theory, Modified Strain Gradient (MSG) theory, and classical theory.

more precisely than MCS theory, and MCS theory could calculate it more accurately than MSG theory.

4. Conclusion

A number of investigations have been conducted to model the nonlinear dynamics of microcantilevers. However, the size-dependent behavior of viscoelastic microcantilevers in the framework of strain gradient theory has not been analytically studied so far. In this paper, the nonlinear dynamics of the viscoelastic Euler-Bernoulli microcantilever beam via modified strain gradient theory was investigated using centerline inextensibility. To this end, the shortening effect theory was utilized to develop the equation of motion for the microcantilever beam. The Kelvin-Voigt scheme was also employed to model the viscoelastic behavior of the beam materials. The equation of motion was derived using Hamilton's principle, and the regular mode shape functions of a microcantilever were used to derive time response equation through the Galerkin method.

Through the perturbation method, i.e., the multiple time scales method, both nonlinear natural frequency and time response of the microcantilever were analytically obtained based on the modified strain gradient theory. The resultant formulations showed time dependency of the vibration amplitudes and nonlinear natural frequencies. This formulation allows modeling any nonlinearity arising from inertia, damping, and stiffness and directly derives analytical solutions for the first time.

The numerical results were measured to validate the findings. The first-mode nonlinear frequency and maximum amplitude of free vibration were compared using different methods such as the modified strain gradient, modified couple stress, and classical theories for various length scale parameters. The obtained results indicated that when the thickness of the beam was in order of the material length scale, the difference between the results of the classical and non-classical theories was considerable. However, as the thickness-to-material length scale ratio of the beam increased, the results of the non-classical assumption converged to those of the classical theory. These findings highlighted the significance of the size effect in analyzing the mechanical behavior of the small-scale structures.

Acknowledgement

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Nomenclature

A	Microbeam cross-section area
b	Microbeam width
C	Damping coefficient
ds	Microbeam element length
dV	Microbeam element volume
E	Elastic modulus
e	Microbeam element elongation
e_{ijk}	Components of alternating tensor
h	Microbeam thickness
l	Area moment of inertia
L	Microbeam length
l_0, l_1, l_2	Length scale parameters
m_{ij}^s	Components of higher-order stress due to the rotation gradient tensor
p_i	Components of higher-order stress due to the dilatation gradient vector
$q(\tau)$	The time part of the microbeam response
T	The kinetic energy of the microbeam
T_0, T_1	Fast and Slow dimensionless time scales
u	Longitudinal deflection of the microbeam
v	Lateral deflection of the microbeam
x	Distance from the clamped end of the microbeam
y	The lateral axis of coordinate system
z	The distance of the cross-section from the neutral axis
χ_{ij}^s	Components of the symmetric part of the rotation gradient tensor
δ_{ij}	Components of Kronecker delta tensor
δW_{vis}	Variation of works due to non-conservative forces
ε_{ij}	Components of classical strain tensor
$\varphi(\bar{x})$	Shape function of the microbeam
γ_i	Components of dilatation gradient vector
$\eta_{ijk}^{(1)}$	Components of deviatoric stretch gradient tensor
μ_c	Viscoelastic shear modulus
μ_k	Elastic shear modulus
ρ	Microbeam material density
σ_{ij}	Components of the classical stress tensor
$\tau_{ijk}^{(1)}$	Components of higher-order stress due to deviatoric stretch gradient tensor
v	Poisson's ratio
ω	Dimensionless frequency
θ	The density of strain energy of the microbeam
Dot	Differentiation with respect to time
Prime	Differentiation with respect to x

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