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Difference-type-exponential estimators based on dual auxiliary information under simple random sampling

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Abstract. Auxiliary information plays a vital role in parameter selection and estimation to achieve efficient estimates of unknown population parameters. Dual use of auxiliary information, i.e., "original" and "ranked" auxiliary variables, helps increase the efficiency of estimators. In this paper, the performance of difference-type-exponential estimators was proposed and evaluated based on dual auxiliary information for population mean under simple random sampling. Mathematical expressions for the bias and the mean squared error of the proposed estimators were obtained. Three real-life data sets and Monte Carlo simulation studies were carried out for illustration. The results of empirical and simulation studies indicate that the proposed estimators outperformed their counterparts in terms of mean square errors and percentage relative efficiency.

1. Introduction

Over the last few decades, survey sampling has evolved into an extensive body of theories, methods, and operations used daily all over the world. It is broadly used in agriculture, business management, demography, economics, education, engineering, industry, medical sciences, political science, social sciences, and many others. In sample surveys, there are many estimators of finite population mean under simple random sampling that rely on auxiliary information. In fact, proper use of auxiliary information in probability sampling results in a considerable reduction in the variance of the estimator of unknown population parameter(s). The existing estimation procedures are based only on the original form of the supplementary information provided by auxiliary variable(s).

Recently, Haq et al. [1] initiated the idea of utilizing additional information of auxiliary variable along with its original information to boost the efficiency of estimators. This additional information is in the form of ranks of the auxiliary variable, called ranked auxiliary variable. This study is motivated to explore more efficient estimators by using the dual auxiliary information. It proposes two difference-type-exponential estimators based on the original and ranked auxiliary information for efficient estimation of population mean under simple random sampling scheme.

Consider a sample of size n drawn using Simple Random Sampling Without Replacement (SRSWOR) scheme from a population of size N, for $i = 1, 2, 3, ..., N$. Let x_i, y_i, and $r_{x,i}$ denote the observations on the auxiliary variable, study variable, and ranked auxiliary variable, respectively, for the ith unit of the population. Some useful measures are explained in Table 1.

To obtain bias, Mean Square Error (MSE), and

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Table 1. Some useful measures.

Measures	Study variable (y)	Auxiliary variable (x)	Ranked auxiliary variable (r_x)
Sample mean	$\overline{y} = n^{-1} \sum_{i=1}^{n} y_i$	$\overline{x} = n^{-1} \sum_{i=1}^{n} x_i$	$\overline{r_x} = n^{-1} \sum_{i=1}^{n} r_{x,i}$
Population mean	$\overline{Y} = N^{-1} \sum_{i=1}^{N} y_i$	$\overline{X} = N^{-1} \sum_{i=1}^{N} x_i$	$\overline{R_x} = N^{-1} \sum_{i=1}^{N} r_{x,i}$
Population variance	$S_y^2 = (N-1)^{-1} \sum_{i=1}^{N} \left(y_i - \overline{Y}\right)^2$	$S_x^2 = (N-1)^{-1} \sum_{i=1}^{N} \left(x_i - \overline{X}\right)^2$	$S_{rx}^2 = (N-1)^{-1} \sum_{i=1}^{N} \left(r_{x,i} - \overline{R}_x\right)^2$
Population coefficient of variation	$C_y^2 = \left(\overline{Y}^2\right)^{-1} S_y^2$	$C_x^2 = \left(\overline{X}^2\right)^{-1} S_x^2$	$C_r^2 = \left(\overline{R}_x{}^2\right)^{-1} S_{rx}^2$

minimum MSE of the proposed estimators, the following relative error terms and their expectations are defined:

$$\xi_0 = \frac{\overline{y} - \overline{Y}}{\overline{Y}}, \qquad \xi_1 = \frac{\overline{r_x} - \overline{R_x}}{\overline{R_x}}, \qquad \text{and}$$

$$\xi_2 = \frac{\overline{x} - \overline{X}}{\overline{X}}, \tag{1}$$

such that:

$$E(\xi_0) = E(\xi_1) = E(\xi_2) = 0,$$

$$E(\xi_0^2) = \psi C_y^2, \qquad E(\xi_1^2) = \psi C_r^2,$$

$$E(\xi_2^2) = \psi C_x^2, \qquad E(\xi_0 \xi_1) = \psi \rho_{yr_x} C_y C_r,$$

$$E(\xi_1 \xi_2) = \psi \rho_{xr_x} C_x C_r, \qquad E(\xi_0 \xi_2) = \psi \rho_{yx} C_y C_x,$$

where:

$$\psi = \left(\frac{1}{n} - \frac{1}{N}\right), \qquad \rho_{yx} = (S_y S_x)^{-1} S_{yx},$$

$$\rho_{yr_x} = (S_y S_{r_x})^{-1} S_{yr_x}, \qquad \rho_{xr_x} = (S_x S_{r_x})^{-1} S_{xr_x},$$

$$S_{yx} = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \overline{Y})(x_i - \overline{X}),$$

$$S_{yr_x} = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \overline{Y})(r_{x,i} - \overline{R}_x),$$

$$S_{xr_x} = (N-1)^{-1} \sum_{i=1}^{N} (x_i - \overline{X})(r_{x,i} - \overline{R}_x).$$

Some important expressions used in upcoming sections are defined below:

$$R = \frac{\overline{Y}}{\overline{X}}, \qquad \overset{'}{R} = \frac{\overline{X}}{\overline{Y}}, \qquad \overset{'}{R}_x = \frac{\overline{R_x}}{\overline{Y}},$$

$$\gamma = \frac{\alpha \overline{X}}{\alpha \overline{X} + \beta}, \qquad \kappa = \rho_{yx} \frac{C_y}{C_x}, \qquad \varphi_1 = \psi C_x^2,$$

$$\varphi_2 = 1 + \psi C_y^2, \qquad \varphi_3 = \psi C_r^2, \qquad \varphi_4 = \psi C_x^2 (\kappa - 1),$$

$$\varphi_5 = \psi C_x^2 \left(\kappa - \frac{1}{2}\right), \qquad \varphi_6 = \psi \rho_{yr_x} C_y C_r,$$

$$\varphi_7 = \psi \rho_{xr_x} C_x C_r, \qquad \varphi_8 = \psi C_x^2 (2\kappa - 1),$$

$$\varphi_9 = \frac{\psi C_x^2}{2}.$$

2. Traditional and existing exponential-type estimators

Several authors have used ratio, product, and regression-type estimators to estimate population mean when both study and auxiliary variables are directly observable. Readers are referred to the studies in [2–18], etc., for more details.

This section gives a brief introduction of traditional estimators: unbiased, ratio, product, and regression as well as the well-known exponential-type estimators of population mean under simple random sampling.

Commonly used unbiased, ratio, product, and regression estimators of the population mean $\overline{Y}$ are as follows:

$$\widehat{\overline{Y}} = \overline{y}, \tag{2}$$

$$\widehat{\overline{Y_R}} = \overline{y} \left(\frac{\overline{X}}{\overline{x}}\right), \qquad \overline{x} \neq 0, \tag{3}$$

$$\widehat{\overline{Y_P}} = \overline{y}\left(\frac{\overline{x}}{\overline{X}}\right), \tag{4}$$

$$\widehat{\overline{Y}}_{REG} = \overline{y} + b\left(\overline{X} - \overline{x}\right),$$

where:

$$b = \frac{\rho_{yx}S_y}{S_x}, \tag{5}$$

is the slope coefficient.

The expressions for the biases of $\widehat{\overline{Y_R}}$ and $\overline{Y_P}$ are given by:

$$Bias\left(\widehat{\overline{Y_R}}\right) \cong \psi\overline{Y}C_x\left(C_x - \rho_{yx}C_y\right), \tag{6}$$

$$Bias\left(\widehat{\overline{Y_P}}\right) \cong \psi\overline{Y}C_x\left(C_x + \rho_{yx}C_y\right). \tag{7}$$

The following ratio and product exponential-type estimators were suggested by Bahl and Tuteja [19]:

$$\widehat{\overline{Y}}_{BT,R} = \overline{y}\exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right), \tag{8}$$

$$\widehat{\overline{Y}}_{BT,P} = \overline{y}\exp\left(\frac{\overline{x} - \overline{X}}{\overline{x} + \overline{X}}\right). \tag{9}$$

Average of Eq. (8) and Eq. (9) can be written as follows:

$$\widehat{\overline{Y}}_{BT,Avg} = \frac{\overline{y}}{2}\left[\exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right) + \exp\left(\frac{\overline{x} - \overline{X}}{\overline{x} + \overline{X}}\right)\right].$$

Haq and Shabbir [8] proposed three improved exponential-type estimators based on original auxiliary information given by:

$$\widehat{\overline{Y}}_{HS1} = \left[\frac{\lambda_1}{2}\overline{y}\left(\frac{\overline{X}}{\overline{x}} + \frac{\overline{x}}{\overline{X}}\right) + \lambda_2\left(\overline{X} - \overline{x}\right)\right] \exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right). \tag{10}$$

$$\widehat{\overline{Y}}_{HS2} = \left[\lambda_3\widehat{\overline{Y}}_{BT,Avg} + \lambda_4\left(\overline{X} - \overline{x}\right)\right] \exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right). \tag{11}$$

$$\widehat{\overline{Y}}_{HS3} = \left[\frac{\lambda_5}{2}\widehat{\overline{Y}}_{BT,Avg}\left(\frac{\overline{X}}{\overline{x}} + \frac{\overline{x}}{\overline{X}}\right) + \lambda_6\left(\overline{X} - \overline{x}\right)\right] \exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right). \tag{12}$$

Expressions for biases of $\widehat{\overline{Y}}_{HS1}, \widehat{\overline{Y}}_{HS2}$, and $\widehat{\overline{Y}}_{HS3}$ are given by:

$$Bias\left(\widehat{\overline{Y}}_{HS1}\right) \cong \frac{1}{8}\Big[-8\overline{Y} + \overline{Y}\Big\{8 + \psi C_x \left(7C_x - 4\rho_{yx}C_y\right)\Big\}\lambda_1 + 4\overline{X}\psi C_x^2\lambda_2\Big], \tag{13}$$

$$Bias\left(\widehat{\overline{Y}}_{HS2}\right) \cong \frac{1}{2}\Big[-2\overline{Y} + \overline{Y}\Big\{2 + \psi C_x \left(C_x - \rho_{yx}C_y\right)\Big\}\lambda_3 + \overline{X}\psi C_x^2\lambda_4\Big], \tag{14}$$

$$Bias\left(\widehat{\overline{Y}}_{HS3}\right) \cong \frac{1}{2}\Big[-2\overline{Y} + \overline{Y}\Big\{2 + \psi C_x \left(2C_x - \rho_{yx}C_y\right)\Big\}\lambda_5 + \overline{X}\psi C_x^2\lambda_6\Big]. \tag{15}$$

Ekpenyong and Enang [10] proposed the following two efficient exponential ratio estimators:

$$\widehat{\overline{Y}}_{JI1} = \lambda_7\overline{y} + \lambda_8\left(\overline{X} - \overline{x}\right)\exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right), \tag{16}$$

$$\widehat{\overline{Y}}_{JI2} = \lambda_9\overline{y} + \lambda_{10}\left(\overline{X} - \overline{x}\right)\exp\left(\frac{2\left(\overline{X} - \overline{x}\right)}{\overline{X} + \overline{x}}\right). \tag{17}$$

Biases of $\widehat{\overline{Y}}_{JI1}$ and $\widehat{\overline{Y}}_{JI2}$ are given below:

$$Bias\left(\widehat{\overline{Y}}_{JI1}\right) \cong \overline{Y}\left[(\lambda_7 - 1) + \lambda_8 R'\psi\frac{C_x^2}{2}\right], \tag{18}$$

$$Bias\left(\widehat{\overline{Y}}_{JI2}\right) \cong \overline{Y}\left[(\lambda_9 - 1) + \lambda_{10} R'\psi C_x^2\right]. \tag{19}$$

Haq et al. [1] suggested an improved class of estimators following the lines of Shabbir and Gupta [20] and Grover and Kaur [5,6]. This class is based on the original and ranked auxiliary information.

$$\widehat{\overline{Y}}_{HA} = \left[\lambda_{11}\overline{y} + \lambda_{12}\left(\overline{X} - \overline{x}\right) + \lambda_{13}\left(\overline{R}_x - \overline{r}_x\right)\right] \exp\left(\frac{\alpha(\overline{X} - \overline{x})}{\alpha(\overline{X} - \overline{x}) + 2\beta}\right), \tag{20}$$

where α and β may be any constant values or functions of the known parameters of the auxiliary variable.

Expression for bias of $\widehat{\overline{Y}}_{HA}$ is stated as follows:

$$Bias\left(\widehat{\overline{Y}}_{HA}\right) \cong \frac{1}{8}\Big[-8\overline{Y} + 4\psi\gamma C_x\big(\overline{X}C_x\lambda_{12} + \overline{R}_x C_r\lambda_{13}\rho_{yx}\big) + \overline{Y}\lambda_{11}\Big\{8 + \psi\gamma C_x \left(3\gamma C_x - 4C_y\rho_{yx}\right)\Big\}\Big]. \tag{21}$$

Remark 2.1. As is given, λ_i, $i = 1, 2, ..., 13$ appearing in the above equations are the unknown weights determined such that the MSEs are minimized. Therefore, the optimal values of λ_i are obtained using the following condition:

$$\frac{\partial MSE(*)}{\partial \lambda_i} = 0;$$

$$* = \widehat{\overline{Y}}_{HS1}, \widehat{\overline{Y}}_{HS2}, \widehat{\overline{Y}}_{HS3}, \widehat{\overline{Y}}_{JI1}, \widehat{\overline{Y}}_{JI2}, \widehat{\overline{Y}}_{HA}$$

$$(i = 1, 2, 3, \cdots, 13).$$

On finding a solution, we have, equations shown in Box I.

Remark 2.2. MSEs and minimum MSEs at optimal values of λ_i, $i = 1, 2, ..., 13$ of the estimators presented in Eq. (2) to Eq. (20) are given below:

$$MSE\left(\widehat{\overline{Y}}\right) = V\left(\widehat{\overline{Y}}\right) = \psi \overline{Y}^2 C_y^2, \quad (22)$$

$$MSE\left(\widehat{\overline{Y}_R}\right) \cong \psi \overline{Y}^2 \left[C_y^2 + C_x^2 - 2\rho_{yx} C_y C_x\right], \quad (23)$$

$$MSE\left(\widehat{\overline{Y}_P}\right) \cong \psi \overline{Y}^2 \left[C_y^2 + C_x^2 + 2\rho_{yx} C_y C_x\right], \quad (24)$$

$$MSE\left(\widehat{\overline{Y}}_{REG}\right) \cong \psi \overline{Y}^2 C_y^2 \left[1 - \rho_{yx}^2\right], \quad (25)$$

$$MSE\left(\widehat{\overline{Y}}_{BT,R}\right) \cong \frac{\psi}{4} \overline{Y}^2 \left[4C_y^2 + C_x^2 - 4\rho_{yx} C_y C_x\right], \quad (26)$$

$$MSE\left(\widehat{\overline{Y}}_{BT,P}\right) \cong \frac{\psi}{4} \overline{Y}^2 \left[4C_y^2 + C_x^2 + 4\rho_{yx} C_y C_x\right], \quad (27)$$

$$MSE_{\min}\left(\widehat{\overline{Y}}_{HS1}\right) = \frac{\psi \overline{Y}^2 \left[-25\psi C_x^4 + 16(-1 + \rho_{yx}^2)\left(-4 + \psi C_x^2\right) C_y^2\right]}{64[1 + \psi C_x^2 + \psi C_y^2(1 - \rho_{yx}^2)]}, \quad (28)$$

$$MSE_{\min}\left(\widehat{\overline{Y}}_{HS2}\right) = \frac{\psi \overline{Y}^2 \left[-\psi C_x^4 + 4(-1 + \rho_{yx}^2)\left(-4 + \psi C_x^2\right) C_y^2\right]}{4[4 + \psi C_x^2 - 4\psi C_y^2(-1 + \rho_{yx}^2)]}, \quad (29)$$

$$MSE_{\min}\left(\widehat{\overline{Y}}_{HS3}\right) = \frac{\psi \overline{Y}^2 \left[-9\psi C_x^4 + 4(-1 + \rho_{yx}^2)\left(-4 + \psi C_x^2\right) C_y^2\right]}{4[4 + 5\psi C_x^2 - 4\psi C_y^2(-1 + \rho_{yx}^2)]}, \quad (30)$$

$$MSE_{\min}\left(\widehat{\overline{Y}}_{JI1}\right) = \overline{Y}^2 \left(1 - \frac{\varphi_1 + 2\varphi_5\varphi_9 + \varphi_2\varphi_9^2}{\varphi_1\varphi_2 - \varphi_5^2}\right), \quad (31)$$

$$MSE_{\min}\left(\widehat{\overline{Y}}_{JI2}\right) = \overline{Y}^2 \left(1 - \frac{\varphi_1 + 2\varphi_1\varphi_4 + \varphi_2\varphi_1^2}{\varphi_1\varphi_2 - \varphi_4^2}\right), \quad (32)$$

$$MSE_{\min}\left(\widehat{\overline{Y}}_{HA}\right) = \frac{\psi \overline{Y}^2 \left[64C_y^2(1 - R_{y.xr_x}^2) - \psi\gamma^4 C_x^4 - 16\psi\gamma^2 C_x^2 C_y^2 (1 - R_{y.xr_x}^2)\right]}{64[1 + \psi C_y^2(1 - R_{y.xr_x}^2)]}. \quad (33)$$

where:

$$R_{y.xr_x}^2 = \frac{\rho_{yx}^2 + \rho_{yr_x}^2 - 2\rho_{yx}\rho_{yr_x}\rho_{xr_x}}{1 - \rho_{xr_x}^2}.$$

3. Proposed estimators

In this section, two new difference-type-exponential estimators are proposed for population mean under SRSWOR. These estimators are based on the dual use of auxiliary information. (1) The auxiliary variable uses original/actual measurements of the auxiliary variable; (2) The ranked auxiliary variable uses the ranks of the auxiliary variable. Mathematical properties such as bias, MSE, and minimum MSE of the proposed estimators are derived up to the first order of approximation. The bias of an estimator is the difference between the estimator's expected value and the true value of the parameter being estimated, i.e., $\text{Bias}\left(\widehat{\overline{Y}}\right) = E\left(\widehat{\overline{Y}} - \overline{Y}\right)$, and MSE can be defined as the divergence of the estimator values from the true parameter value, i.e., $MSE\left(\widehat{\overline{Y}}\right) = E\left(\widehat{\overline{Y}} - \overline{Y}\right)^2$.

3.1. First proposed estimator

$$\widehat{\overline{Y}}_{P1} = \frac{\lambda_{14}}{2}\left(\frac{\overline{X}}{\overline{x}} + \frac{\overline{x}}{\overline{X}}\right)\widehat{\overline{Y}}_{BT,Avg} + \lambda_{15}\left(\overline{R_x} - \overline{r_x}\right) + \lambda_{16}\left(\overline{X} - \overline{x}\right)\exp\left(\frac{\overline{X} - \overline{x}}{\overline{X} + \overline{x}}\right), \quad (34)$$

where $\lambda_{14}, \lambda_{15}$, and λ_{16} are the properly chosen constants.

Here, we use Eq. (1) to rewrite the estimator in Eq. (34), subtract $\overline{Y}$ from both sides, and get the expression up to the first order of approximation in this way:

$$\widehat{\overline{Y}}_{P1} - \overline{Y} = \left[\lambda_{14}\overline{Y} + \frac{5\lambda_{14}\overline{Y}\xi_2^2}{8} + \lambda_{14}\overline{Y}\xi_0 - \lambda_{15}\overline{R_x}\xi_1 - \lambda_{16}\overline{X}\xi_2 + \frac{\lambda_{16}\overline{X}\xi_2^2}{2} - \overline{Y}\right]. \quad (35)$$

Applying expectation on both sides of Eq. (35), we get the bias of the proposed estimator as follows:

$$Bias\left(\widehat{\overline{Y}}_{HA}\right) \cong \frac{1}{8}\Bigg[-8\overline{Y}+4\psi\gamma C_x\left(\overline{X}C_x\lambda_{12}+\overline{R}_xC_r\lambda_{13}\rho_{yx}\right)+\overline{Y}\lambda_{11}\Big\{8+\psi\gamma C_x\left(3\gamma C_x-4C_y\rho_{yx}\right)\Big\}\Bigg],$$

$$\lambda_1=\frac{8+3\psi C_x^2}{8[1+\psi C_x^2+\psi C_y^2(1-\rho_{yx}^2)]},$$

$$\lambda_2=\frac{\overline{Y}\left[8C_y\rho_{yx}+C_x\left\{-4+\psi(C_x^2+3\rho_{yx}C_yC_x-4(-1+\rho_{yx}^2)C_y^2)\right\}\right]}{8\overline{X}C_x[1+\psi C_x^2+\psi C_y^2(1-\rho_{yx}^2)]},$$

$$\lambda_3=\frac{4}{4+\psi C_x^2-4\psi C_y^2(-1+\rho_{yx}^2)},$$

$$\lambda_4=\frac{\overline{Y}}{2\overline{X}}\left[1+\frac{-8C_x+8C_y\rho_{yx}}{C_x[4+\psi C_x^2-4\psi C_y^2(-1+\rho_{yx}^2)]}\right],$$

$$\lambda_5=\frac{4+2\psi C_x^2}{4+5\psi C_x^2-4\psi C_y^2(-1+\rho_{yx}^2)},$$

$$\lambda_6=\frac{\overline{Y}\left[8C_y\rho_{yx}+C_x\left\{-4+\psi(C_x^2+4\rho_{yx}C_yC_x-4C_y^2(-1+\rho_{yx}^2))\right\}\right]}{2\overline{X}C_x[4+5\psi C_x^2-4\psi C_y^2(-1+\rho_{yx}^2)]},$$

$$\lambda_7=\frac{\varphi_1+\varphi_5\varphi_9}{\varphi_1\varphi_2-\varphi_5^2},$$

$$\lambda_8=R\left(\frac{\varphi_5+\varphi_2\varphi_9}{\varphi_1\varphi_2-\varphi_5^2}\right),$$

$$\lambda_9=\frac{\varphi_1+\varphi_1\varphi_4}{\varphi_1\varphi_2-\varphi_4^2},$$

$$\lambda_{10}=R\left(\frac{\varphi_4+\varphi_1\varphi_2}{\varphi_1\varphi_2-\varphi_4^2}\right),$$

$$\lambda_{11}=\frac{8-\psi\gamma^2C_x^2}{8\left[1+\psi C_y^2(1-R_{y.xr_x}^2)\right]},$$

$$\lambda_{12}=\frac{\overline{Y}\left[\psi\gamma^3C_x^3\left(-1+\rho_{xr_x}^2\right)+\left(-8C_y+\psi\gamma^2C_x^2C_y\right)\left(\rho_{yx}-\rho_{xr_x}\rho_{yr_x}\right)+4\gamma C_x(-1+\rho_{xr_x}^2)\left\{-1+\psi C_y^2\left((1-R_{y.xr_x}^2)\right)\right\}\right]}{8\overline{X}C_x(-1+\rho_{xr_x}^2)\left[1+\psi C_y^2(1-R_{y.xr_x}^2)\right]},$$

$$\lambda_{13}=\frac{\overline{Y}\left(8-\psi\gamma^2C_x^2\right)C_y\left(\rho_{xr_x}\rho_{yx}-\rho_{yr_x}\right)}{8\overline{R_x}C_r(-1+\rho_{xr_x}^2)\left[1+\psi C_y^2(1-R_{y.xr_x}^2)\right]}.$$

Box I

$$Bias\left(\hat{\hat{\overline{Y}}}_{P1}\right) \cong (\lambda_{14}-1)\,\overline{Y}+\frac{\varphi_1}{2}\left(\lambda_{16}\overline{X}+\frac{5\lambda_{14}\overline{Y}}{4}\right). \tag{36}$$

Squaring both sides of Eq. (35) up to the first order of approximation, we have:

$$\left(\hat{\hat{\overline{Y}}}_{P1}-\overline{Y}\right)^2 \cong \Big[\overline{Y}^2+\lambda_{14}^2\overline{Y}^2+\lambda_{14}^2\overline{Y}^2\xi_0^2+\lambda_{15}^2\overline{R}_x^2\xi_1^2$$
$$+\lambda_{16}^2\overline{X}^2\xi_2^2+\frac{5}{4}\lambda_{14}^2\overline{Y}^2\xi_2^2+\lambda_{14}\lambda_{16}\overline{XY}\xi_2^2-2\lambda_{14}\overline{Y}^2$$
$$-\frac{5}{4}\lambda_{14}\overline{Y}^2\xi_2^2-2\lambda_{14}\lambda_{15}\overline{YR_x}\xi_0\xi_1-2\lambda_{14}\lambda_{16}\overline{XY}\xi_0\xi_2$$
$$+2\lambda_{15}\lambda_{16}\overline{XR_x}\xi_1\xi_2-\lambda_{16}\overline{XY}\xi_2^2\Big]. \tag{37}$$

The MSE of $\hat{\hat{\overline{Y}}}_{P1}$ is obtained by taking expectation of both sides of Eq. (36)

$$MSE\left(\hat{\hat{\overline{Y}}}_{P1}\right) \cong \overline{Y}^2\Big[1+\lambda_{14}^2\varphi_2+\frac{5}{4}\lambda_{14}^2\varphi_1+\lambda_{15}^2 R_x'^2\varphi_3$$
$$+\lambda_{16}^2 R'^2\varphi_1-2\lambda_{14}-\frac{5}{4}\lambda_{14}\varphi_1-\lambda_{16}R'\varphi_1$$
$$-2\lambda_{14}\lambda_{15}R_x'\varphi_6+2\lambda_{15}\lambda_{16}R'R_x'\varphi_7-\lambda_{14}\lambda_{16}R'\varphi_8\Big]. \tag{38}$$

Now, we have to choose the weights of $\lambda_{14},\lambda_{15}$, and λ_{16} such that the resulting MSE of $\hat{\hat{\overline{Y}}}_{P1}$ is minimized. Thus, the optimal weights of $\lambda_{14},\lambda_{15}$ and λ_{16} are selected with the help of the following equations.

$$\frac{\partial MSE\left(\hat{\hat{\overline{Y}}}_{P1}\right)}{\partial\lambda_{14}}=(8\varphi_2+10\varphi_1)\,\lambda_{14}-8\lambda_{15}R_x'\varphi_6$$
$$-4\lambda_{16}R'\varphi_8-8-5\varphi_1,$$

$$\frac{\partial MSE\left(\hat{\hat{\overline{Y}}}_{P1}\right)}{\partial\lambda_{15}}=2\lambda_{15}R_x'^2\varphi_3-2\lambda_{14}R_x'\varphi_6$$
$$+2\lambda_{16}R'R_x'\varphi_7,$$

$$\frac{\partial MSE\left(\hat{\hat{\overline{Y}}}_{P1}\right)}{\partial\lambda_{16}}=2\lambda_{16}R'^2\varphi_1+2\lambda_{15}R'R_x'\varphi_7$$
$$-\lambda_{14}R'\varphi_8-R'\varphi_1.$$

Setting $\frac{\partial MSE\left(\hat{\hat{\overline{Y}}}_{P1}\right)}{\partial\lambda_i}=0$, $i=14,15,16$ and solving simultaneously, we get:

$$\lambda_{14(opt)}=\frac{E_1E_2-2E_3\varphi_1\varphi_3}{E_2E_4-2E_3^2},$$

$$\lambda_{15(opt)}=\frac{2\varphi_6\,(E_1E_2-2E_3\varphi_1\varphi_3)+\varphi_7\,(E_1E_3-\varphi_1\varphi_3E_4)}{2\varphi_3\,R_x'\,(E_2E_4-2E_3^2)},$$

$$\lambda_{16(opt)}=\frac{\varphi_1\varphi_3E_4-E_1E_3}{2\,R'\,(E_2E_4-2E_3^2)},$$

where:

$$E_1=8\varphi_3+5\varphi_1\varphi_3,\qquad E_2=\varphi_1\varphi_3-\varphi_7^2$$

$$E_3=2\varphi_6\varphi_7-\varphi_3\varphi_8,\quad E_4=8\varphi_2\varphi_3+10\varphi_1\varphi_3-8\varphi_6^2.$$

Inserting optimal weights of $\lambda_{14},\lambda_{15}$, and λ_{16} in Eq. (38), we get the minimum MSE of the proposed estimator as follows:

$$MSE_{\min}\left(\hat{\hat{\overline{Y}}}_{P1}\right)=\frac{\overline{Y}^2}{4\varphi_3F_1^2}$$
$$\Big[4\varphi_3F_1^2+\left(4\varphi_2\varphi_3-4\varphi_6^2+5\varphi_1\varphi_3\right)$$
$$F_2^2+\left(\varphi_1\varphi_3-\varphi_7^2\right)F_3^2-(8+5\varphi_1)\,\varphi_3F_1F_2$$
$$+2\,(2\varphi_6\varphi_7-\varphi_3\varphi_8)\,F_2F_3-2\varphi_1\varphi_3F_1F_3\Big], \tag{39}$$

where:

$$F_1=E_2E_4-2E_3^2,$$

$$F_2=E_1E_2-2\varphi_1\varphi_3E_3,$$

$$F_3=\varphi_1\varphi_3E_4-E_1E_3.$$

3.2. Second proposed estimator

$$\hat{\hat{\overline{Y}}}_{P2}=\lambda_{17}\overline{y}+\lambda_{18}\left(\overline{R_x}-\overline{r_x}\right)$$
$$+\lambda_{19}\left(\overline{X}-\overline{x}\right)\exp\left(\frac{2\left(\overline{X}-\overline{x}\right)}{\overline{X}+\overline{x}}\right), \tag{40}$$

where $\lambda_{17},\lambda_{18}$, and λ_{19} are the properly chosen constants.

Following the same procedure mentioned in Section 3.1, we have the following expressions:

$$\hat{\hat{\overline{Y}}}_{P2}-\overline{Y}=$$
$$\left[\lambda_{17}\overline{Y}+\lambda_{17}\overline{Y}\xi_0-\lambda_{18}\overline{R_x}\xi_1-\lambda_{19}\overline{X}\xi_2+\lambda_{19}\overline{X}\xi_2^2-\overline{Y}\right], \tag{41}$$

$$Bias\left(\hat{\hat{\overline{Y}}}_{P2}\right)\cong\overline{Y}\left[(\lambda_{17}-1)+\lambda_{19}\varphi_1R'\right], \tag{42}$$

$$\left(\hat{\hat{\overline{Y}}}_{P2}-\overline{Y}\right)^2\cong\Big[\overline{Y}^2+\lambda_{17}^2\overline{Y}^2+\lambda_{17}^2\overline{Y}^2\xi_0^2+\lambda_{18}^2\overline{R}_x^2\xi_1^2$$
$$+\lambda_{19}^2\overline{X}^2\xi_2^2+2\lambda_{17}\lambda_{19}\overline{XY}\xi_2^2-2\lambda_{17}\overline{Y}^2$$
$$-2\lambda_{17}\lambda_{18}\overline{YR_x}\xi_0\xi_1-2\lambda_{17}\lambda_{19}\overline{XY}\xi_0\xi_2$$
$$+2\lambda_{18}\lambda_{19}\overline{XR_x}\xi_1\xi_2-2\lambda_{19}\overline{XY}\xi_2^2\Big], \tag{43}$$

$$MSE\left(\widehat{\widehat{Y}}_{P2}\right) \cong \overline{Y}^2$$

$$\Big[1 + \lambda_{17}^2\varphi_2 + \lambda_{18}^2 \overset{'}{R}{}_x^2 \varphi_3 + \lambda_{19}^2 \overset{'}{R}{}^2 \varphi_1 - 2\lambda_{17}$$

$$-2\lambda_{19} \overset{'}{R} \varphi_1 + 2\lambda_{18}\lambda_{19} \overset{'}{R} \overset{'}{R}_x \varphi_7$$

$$-2\lambda_{17}\lambda_{18} \overset{'}{R}_x \varphi_6 - 2\lambda_{17}\lambda_{19} \overset{'}{R} \varphi_4\Big], \quad (44)$$

The optimal weights of $\lambda_{17}, \lambda_{18}$, and λ_{19} are obtained as follows:

$$\frac{\partial MSE\left(\widehat{\widehat{Y}}_{P2}\right)}{\partial\lambda_{17}} = 2\left(\lambda_{17}\varphi_2 - 1 - \lambda_{19} \overset{'}{R} \varphi_4 - \lambda_{18} \overset{'}{R}_x \varphi_6\right),$$

$$\frac{\partial MSE\left(\widehat{\widehat{Y}}_{P2}\right)}{\partial\lambda_{18}} = 2\left(\lambda_{18} \overset{'}{R}{}_x^2 \varphi_3 + \lambda_{19} \overset{'}{R} \overset{'}{R}_x \varphi_7 - \lambda_{17} \overset{'}{R}_x \varphi_6\right),$$

$$\frac{\partial MSE\left(\widehat{\widehat{Y}}_{P2}\right)}{\partial\lambda_{19}} = 2\left(\lambda_{19} \overset{'}{R}{}^2 \varphi_1 + \lambda_{18} \overset{'}{R} \overset{'}{R}_x \varphi_7 - \overset{'}{R} \varphi_1 - \lambda_{17} \overset{'}{R} \varphi_4\right).$$

Setting $\frac{\partial MSE\left(\widehat{\widehat{Y}}_{P2}\right)}{\partial\lambda_i} = 0, i = 17, 18, 19$ and solving simultaneously, we get:

$$\lambda_{17(opt)} = \frac{\varphi_2\varphi_3E_5 - (\varphi_2\varphi_7 + \varphi_6E_7)E_6 + \varphi_4E_7E_8}{\varphi_2\left(E_5E_8 - E_6^2\right)},$$

$$\lambda_{18(opt)} = \frac{\varphi_6E_5 - E_6E_7}{R_x'\left(E_5E_8 - E_6^2\right)},$$

$$\lambda_{19(opt)} = \frac{E_7E_8 - \varphi_6E_6}{R'\left(E_5E_8 - E_6^2\right)},$$

where:

$$E_5 = \varphi_1\varphi_2 - \varphi_4^2, \qquad E_6 = \varphi_2\varphi_7 - \varphi_4\varphi_6,$$

$$E_7 = \varphi_1\varphi_2 + \varphi_4, \qquad E_8 = \varphi_2\varphi_3 - \varphi_6^2.$$

Inserting optimal weights of $\lambda_{17}, \lambda_{18}$, and λ_{19} in Eq. (44), we get the minimum MSE of the proposed estimator as follows:

$$MSE_{\min}\left(\widehat{\widehat{Y}}_{P2}\right) = \frac{\overline{Y}^2}{\varphi_2F_4^2}\Big[\varphi_2F_4^2 + \varphi_1\varphi_2F_5^2 + \varphi_2\varphi_3F_6^2$$

$$+F_7^2 - 2(\varphi_1F_4 - \varphi_7F_6)\varphi_2F_5$$

$$-2(\varphi_6F_6 + \varphi_4F_5 + F_4)F_7\Big], \quad (45)$$

where:

$$F_4 = E_5E_8 - E_6^2, \quad F_5 = E_7E_8 - \varphi_6E_6,$$

$$F_6 = \varphi_6E_5 - E_6E_7,$$

$$F_7 = \varphi_2\varphi_3E_5 - \varphi_2\varphi_7E_6 - \varphi_6E_6E_7 + \varphi_4E_7E_8.$$

Remark 3.1. Of note, the parameters ρ_{yx}, ρ_{yr_x}, ρ_{xr_x}, C_y, C_x, and C_r appearing in the expressions of optimal weights and the minimum MSEs are generally unknown. However, these parameters can be estimated quite accurately from the preliminary data or from the repeated surveys based on sampling over several occasions. Many different authors have dealt with utilization of prior information of parameters at the estimation stage including Singh and Singh [21] and Vishwakarma and Kumar [22].

4. Applications

In this section, three real-life datasets are used to evaluate the performance of the proposed estimators as compared to the existing estimators in terms of Percentage Absolute Relative Bias (PARB), MSE, and Percentage Relative Efficiencies (PRE). For more details of these measures, see Rao et al. [23], Silva and Skinner [24], Nidhi et al. [25], etc. MSEs are calculated using the expressions defined in Sections 2 and 3. PARB and PRE of an estimator can be computed through the following expressions:

$$PARB(*) = \left|\frac{(*) - \overline{Y}}{\overline{Y}}\right| \times 100, \quad (46)$$

$$PRE = \frac{MSE\left(\widehat{\widehat{Y}}\right)}{MSE(*)} \times 100, \quad (47)$$

where $* = \widehat{\widehat{Y}}, \widehat{\widehat{Y}}_{REG}, \widehat{\widehat{Y}}_{HS1}, \widehat{\widehat{Y}}_{HS2,}, \widehat{\widehat{Y}}_{HS3}, \widehat{\widehat{Y}}_{JI1}, \widehat{\widehat{Y}}_{JI2}, \widehat{\widehat{Y}}_{HA}, \widehat{\widehat{Y}}_{P1}, \widehat{\widehat{Y}}_{P2}$.

Population 1 [26].

$$N = 69, \quad n = 12, \quad \overline{Y} = 135.2608, \quad \overline{X} = 345.7536,$$

$$C_y = 0.8422, \quad C_x = 0.8479, \quad C_r = 0.5747,$$

$$\beta_{2(x)} = 7.2159, \quad \overline{R}_x = 34.9565, \quad \rho_{yx} = 0.9224,$$

$$\rho_{yr_x} = 0.7136, \quad \rho_{xr_x} = 0.8185.$$

Population 2 [10].

$N=923, \quad n=180, \quad \overline{Y}=436.4345, \quad \overline{X}=11440.5,$

$C_y = 1.7183, \quad C_x = 1.8645, \quad C_r = 0.577,$

$\beta_{2(x)} = 18.7208, \quad \overline{R}_x = 461.9642, \quad \rho_{yx} = 0.9543,$

$\rho_{yr_x} = 0.6442, \quad \rho_{xr_x} = 0.6306.$

Population 3 [27].

$N = 854, \quad n = 290, \quad \overline{Y} = 2930.12,$

$\overline{X} = 37600.11, \quad C_y = 5.8379, \quad C_x = 3.8509,$

$C_r = 0.1883, \quad \beta_{2(x)} = 312.0651,$

$\overline{R}_x = 426.8747, \quad \rho_{yx} = 0.9165,$

$\rho_{yr_x} = 0.2585, \quad \rho_{xr_x} = 0.3458.$

Table 2 shows the PARB of all the estimators; and Table 3 gives the MSEs and PREs.

5. Simulation study based on real data sets

Monte Carlo simulation study is carried out to check the potential of the proposed estimators over the competing estimators through R software.

A step-by-step approach to the simulation study is as follows:

1. Select an SRSWOR of size n from the population of size N;
2. Use sample data from Step 1 to find PARB and MSE of all the estimators under study;
3. Repeat 50,000 times Steps 1 and 2;
4. Obtain 50,000 values for PARB and MSE of all the estimators;
5. Average of 50,000 values obtained in Step 4 in the PARB and MSE;
6. Calculate PARB and PREs (with respect to sample mean $\widehat{\overline{Y}}$) of all the estimators.

6. Important findings

- From Table 2, the PARB of the proposed estimators is the least among all the competing estimators for all three data sets;
- From Table 3, it is imperative to mention that the proposed estimators $\widehat{\overline{Y}}_{P1}$ and $\widehat{\overline{Y}}_{P2}$ have minimum MSEs and maximum PREs compared to all the traditional and existing estimators in all populations;
- From Table 4, simulation study indicates that PARB of the proposed estimators is lower than the com-

Table 2. Numerical comparison of percentage absolute relative bias.

Estimators	Population 1	Population 2	Population 3
$\widehat{\overline{Y}}$	–	–	–
$\widehat{\overline{Y}}_{REG}$	–	–	–
$\widehat{\overline{Y}}_{HS1}$	0.5901	0.1062	1.1345
$\widehat{\overline{Y}}_{HS2}$	0.6905	0.1153	1.1993
$\widehat{\overline{Y}}_{HS3}$	0.5439	0.1017	1.1069
$\widehat{\overline{Y}}_{JI1}$	0.6399	0.1115	1.1195
$\widehat{\overline{Y}}_{JI2}$	0.4642	0.0961	0.9136
$\widehat{\overline{Y}}^{(1)}_{HA}\,(\alpha = 1, \beta = C_x)$	0.6857	0.1129	1.1855
$\widehat{\overline{Y}}^{(2)}_{HA}\,(\alpha = 1, \beta = \beta_{2(x)})$	0.6863	0.1130	1.1857
$\widehat{\overline{Y}}^{(3)}_{HA}\,(\alpha = \beta_{2(x)}, \beta = C_x)$	0.6857	0.1129	1.1855
$\widehat{\overline{Y}}^{(4)}_{HA}\,(\alpha = C_x, \beta = \beta_{2(x)})$	0.6865	0.1130	1.1856
$\widehat{\overline{Y}}^{(5)}_{HA}\,(\alpha = 1, \beta = \rho_{yx})$	0.6857	0.1130	1.1855
$\widehat{\overline{Y}}^{(6)}_{HA}\,(\alpha = C_x, \beta = \rho_{yx})$	0.6858	0.1129	1.1855
$\widehat{\overline{Y}}^{(7)}_{HA}\,(\alpha = \rho_{yx}, \beta = C_x)$	0.6857	0.1129	1.1855
$\widehat{\overline{Y}}^{(8)}_{HA}\,(\alpha = \beta_{2(x)}, \beta = \rho_{yx})$	0.6857	0.1130	1.1855
$\widehat{\overline{Y}}^{(9)}_{HA}\,(\alpha = \rho_{yx}, \beta = \beta_{2(x)})$	0.6863	0.1129	1.1858
$\widehat{\overline{Y}}^{(10)}_{HA}\,(\alpha = 1, \beta = N\overline{X})$	0.6981	0.1138	1.1974
$\widehat{\overline{Y}}_{P1}$	0.3343	0.0845	0.9117
$\widehat{\overline{Y}}_{P2}$	0.3588	0.0938	0.8730

Table 3. Mean Square Error (MSE) and Percentage Relative Efficiencies (PRE)of the existing and proposed estimators.

Estimators	Population 1		Population 2		Population 3	
	MSE's	PRE's	MSE's	PRE's	MSE's	PRE's
$\hat{\bar{Y}}$	893.344	100.000	2515.074	100.000	666353.000	100.000
$\hat{\bar{Y}}_{REG}$	133.267	670.342	224.611	1119.747	106635.000	624.891
$\hat{\bar{Y}}_{HS1}$	107.982	827.308	202.369	1242.816	97410.320	684.068
$\hat{\bar{Y}}_{HS2}$	126.334	707.129	219.747	1144.532	102974.400	647.105
$\hat{\bar{Y}}_{HS3}$	99.528	897.581	193.846	1297.460	95034.750	701.168
$\hat{\bar{Y}}_{JI1}$	117.484	760.396	212.505	1183.536	96116.890	693.274
$\hat{\bar{Y}}_{JI2}$	84.936	1051.785	183.133	1373.359	78444.970	849.453
$\hat{\bar{Y}}_{HA}^{(1)}$	125.473	711.981	215.232	1168.541	101787.500	654.651
$\hat{\bar{Y}}_{HA}^{(2)}$	125.577	711.391	215.239	1168.503	101806.400	654.529
$\hat{\bar{Y}}_{HA}^{(3)}$	125.460	712.055	215.231	1168.546	101787.300	654.652
$\hat{\bar{Y}}_{HA}^{(4)}$	125.597	711.278	215.235	1168.525	101792.300	654.620
$\hat{\bar{Y}}_{HA}^{(5)}$	125.474	711.975	215.232	1168.541	101787.300	654.652
$\hat{\bar{Y}}_{HA}^{(6)}$	125.477	711.958	215.232	1168.541	101787.300	654.652
$\hat{\bar{Y}}_{HA}^{(7)}$	125.474	711.975	215.232	1168.541	101787.500	654.651
$\hat{\bar{Y}}_{HA}^{(8)}$	125.460	712.055	215.231	1168.546	101787.300	654.652
$\hat{\bar{Y}}_{HA}^{(9)}$	125.587	711.335	215.239	1168.503	101808.100	654.519
$\hat{\bar{Y}}_{HA}^{(10)}$	127.734	699.378	216.793	1160.127	102806.400	648.162
$\hat{\bar{Y}}_{P1}$	61.170	1460.428	161.068	1561.498	78281.210	851.229
$\hat{\bar{Y}}_{P2}$	65.656	1360.643	178.828	1406.421	74957.120	888.979

Table 4. Percentage absolute relative bias of estimators with respect to $\hat{\bar{Y}}$ based on simulation study.

Estimators	Population 1			Population 2			Population 3		
	$n = 12$	$n = 14$	$n = 16$	$n = 180$	$n = 200$	$n = 230$	$n = 290$	$n = 310$	$n = 330$
$\hat{\bar{Y}}$	—	—	—	—	—	—	—	—	—
$\hat{\bar{Y}}_{REG}$	—	—	—	—	—	—	—	—	—
$\hat{\bar{Y}}_{HS1}$	0.5140	0.4522	0.3982	0.0967	0.0873	0.0747	0.9971	0.9283	0.8595
$\hat{\bar{Y}}_{HS2}$	0.6106	0.5190	0.4469	0.1056	0.0942	0.0795	1.0573	0.9771	0.8997
$\hat{\bar{Y}}_{HS3}$	0.4688	0.4207	0.3753	0.0923	0.0839	0.0723	0.9703	0.9065	0.8415
$\hat{\bar{Y}}_{JI1}$	0.5639	0.4870	0.4236	0.1019	0.0913	0.0775	0.9871	0.9196	0.8522
$\hat{\bar{Y}}_{JI2}$	0.3864	0.3654	0.3356	0.0867	0.0796	0.0694	0.7930	0.7621	0.7223
$\hat{\bar{Y}}_{HA}^{(1)}$	0.5430	0.4661	0.4055	0.1024	0.0913	0.0770	1.0094	0.9322	0.8654
$\hat{\bar{Y}}_{HA}^{(2)}$	0.5387	0.4646	0.4051	0.1025	0.0906	0.0774	1.0246	0.9345	0.8645
$\hat{\bar{Y}}_{HA}^{(3)}$	0.5361	0.4689	0.4108	0.1026	0.0910	0.0771	1.0217	0.9388	0.8603
$\hat{\bar{Y}}_{HA}^{(4)}$	0.5305	0.4658	0.4042	0.1030	0.0914	0.0766	1.0184	0.9236	0.8598
$\hat{\bar{Y}}_{HA}^{(5)}$	0.5353	0.4633	0.4062	0.1029	0.0912	0.0771	1.0113	0.9283	0.8589
$\hat{\bar{Y}}_{HA}^{(6)}$	0.5400	0.4706	0.4057	0.1024	0.0914	0.0767	1.0237	0.9407	0.8521
$\hat{\bar{Y}}_{HA}^{(7)}$	0.5311	0.4678	0.4061	0.1024	0.0904	0.0776	1.0153	0.9345	0.8636
$\hat{\bar{Y}}_{HA}^{(8)}$	0.5373	0.4666	0.4083	0.1027	0.0915	0.0769	1.0172	0.9308	0.8786
$\hat{\bar{Y}}_{HA}^{(9)}$	0.5368	0.4658	0.4054	0.1021	0.0911	0.0767	1.0168	0.9285	0.8590
$\hat{\bar{Y}}_{HA}^{(10)}$	0.5462	0.4721	0.4071	0.1039	0.0910	0.0773	1.0277	0.9446	0.8663
$\hat{\bar{Y}}_{P1}$	0.2253	0.1965	0.1880	0.0744	0.0697	0.0619	0.7530	0.7232	0.6876
$\hat{\bar{Y}}_{P2}$	0.2347	0.1769	0.1574	0.0834	0.0767	0.0669	0.7210	0.6971	0.6664

Table 5. Percentage Relative Efficiencies (PRE) of estimators with respect to $\widehat{\overline{Y}}$ based on simulation study.

	Population 1			**Population 2**			**Population 3**		
Estimators	$\boldsymbol{n=12}$	$\boldsymbol{n=14}$	$\boldsymbol{n=16}$	$\boldsymbol{n=180}$	$\boldsymbol{n=200}$	$\boldsymbol{n=230}$	$\boldsymbol{n=290}$	$\boldsymbol{n=310}$	$\boldsymbol{n=330}$
$\widehat{\overline{Y}}$	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
$\widehat{\overline{Y}}_{REG}$	766.586	749.207	729.755	1206.283	1193.833	1178.324	676.225	618.575	556.940
$\widehat{\overline{Y}}_{HS1}$	987.347	913.926	858.046	1347.447	1313.015	1274.462	734.719	678.189	617.416
$\widehat{\overline{Y}}_{HS2}$	812.755	785.375	758.909	1234.139	1217.667	1197.828	696.543	639.131	577.600
$\widehat{\overline{Y}}_{HS3}$	1098.132	990.408	914.524	1411.218	1365.695	1315.974	753.201	697.281	637.084
$\widehat{\overline{Y}}_{JI1}$	888.710	842.743	803.538	1279.251	1255.959	1228.712	740.498	683.856	622.881
$\widehat{\overline{Y}}_{JI2}$	1382.002	1165.650	1032.976	1502.994	1440.106	1372.636	898.397	850.142	798.285
$\widehat{\overline{Y}}^{(1)}_{HA}$	908.289	868.816	833.734	1272.931	1257.712	1236.909	722.773	663.921	600.093
$\widehat{\overline{Y}}^{(2)}_{HA}$	907.008	869.154	832.204	1269.818	1254.615	1234.438	744.068	723.019	717.034
$\widehat{\overline{Y}}^{(3)}_{HA}$	905.622	866.220	839.286	1261.567	1244.772	1233.626	736.168	722.732	714.374
$\widehat{\overline{Y}}^{(4)}_{HA}$	905.313	849.367	834.382	1263.093	1260.664	1242.487	733.023	725.365	710.147
$\widehat{\overline{Y}}^{(5)}_{HA}$	903.575	864.829	836.822	1262.015	1259.152	1240.036	732.665	716.237	707.388
$\widehat{\overline{Y}}^{(6)}_{HA}$	911.161	868.626	832.772	1268.188	1250.123	1245.323	730.110	724.998	717.704
$\widehat{\overline{Y}}^{(7)}_{HA}$	897.334	865.880	848.572	1262. 963	1252.886	1244.779	726.805	717.746	713.321
$\widehat{\overline{Y}}^{(8)}_{HA}$	904.338	862.441	840.217	1259. 486	1253.556	1240.967	733.072	719.673	714.668
$\widehat{\overline{Y}}^{(9)}_{HA}$	907.359	878.727	838.201	1281.329	1251.296	1234.161	729.270	724.919	708.541
$\widehat{\overline{Y}}^{(10)}_{HA}$	894.414	853.038	839.442	1263.584	1247.958	1232.534	725.638	715.839	707.742
$\widehat{\overline{Y}}_{P1}$	3258.479	1952.065	1524.815	1748.603	1642.411	1536.096	937.697	889.288	836.048
$\widehat{\overline{Y}}_{P2}$	3055.020	1903.397	1464.315	1561.014	1493.770	1 423.147	968.348	922.812	872.949

peting estimators for all three data sets used in our study;

- From Table 5, the simulation study again reveals that $\widehat{\overline{Y}}_{P1}$ and $\widehat{\overline{Y}}_{P2}$ have maximum gain in PREs than all other estimators under competition. This phenomenon is observed in all populations under study.

Therefore, the above findings confirmed that the proposed estimators outperformed the competitors under study.

7. Concluding remarks

This study considered the improved estimation of finite population mean under Simple Random Sampling Without Replacement (SRSWOR). Efficient utilization of auxiliary information can play a vital role in this regard. Therefore, the purpose was achieved by using the original information of the auxiliary variable and the ranks of auxiliary variable. Some new difference-type-exponential estimators based on the above idea of dual auxiliary information were proposed. Mathematical properties including bias, Mean Square Error (MSE), and minimum MSE of the proposed estimators were derived up to the first degree of approximation. To assess the potentiality of the proposed estimators over the competing estimators, real data analysis and simulation study were carried out. Three natural data sets were used for the empirical study. The outcome of this comparison indicated that the proposed estimators were more efficient and less biased than the traditional and other well-known existing estimators. Thus, the researchers are encouraged to use the proposed estimators for estimating the population mean under SRSWOR. The present work could be extended to estimate: (1) The finite population mean under other sampling designs like stratified random sampling and two-phase sampling, etc.; (2) Other unknown finite population parameters including median, variance, proportions, etc.; and (3) Population mean of a sensitive variable in the presence of non-sensitive auxiliary information.

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