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Maclaurin symmetric means for linguistic Z-numbers and their application to multiple-attribute decision-making

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Abstract. Linguistic Z-Numbers (LZNs), as a more rational extension of linguistic description, consider the fuzzy restriction of assessment information and take the reliability of the information into account. Maclaurin Symmetric Mean (MSM) operator has the advantage which can take account of the interrelationship of different attributes and there are a lot of research results on it. However, it has not been used to handle Multi-Attribute Decision-Making (MADM) problems expressed by LZNs. To summarize the advantages of LZNs and MSM, in this article, we propose the Linguistic Z-number MSM (LZMSM) and Linguistic Z-number Weight MSM (LZWMSM) operators respectively, and several characters and special cases of them are discussed. In addition, we propose a method to deal with some MADM problems using the LZWMSM operator. Finally, by comparing it with several existing methods, an example is given to illustrate the effectiveness and superiority of this newly proposed method.

1. Introduction

Zadeh [1] proposed Fuzzy Sets (FSs), which laid the foundation for fuzzy evaluation. However, the FSs only contain the membership function which is a crisp number in [0, 1]. So, when the fuzzy information is complicated, it is difficult to describe it by FSs. Later, Atanassov [2] introduced the Intuitionistic FS (IFS) which includes not only membership degree but also non-membership degree to express more complex fuzzy information. Descrijver and Kerre [3] further explored and defined Intuitionistic Fuzzy Numbers (IFNs). However, we found that these information expression methods do not consider the confidence degree of decision-making information. In fact since everyone is a bounded rational person, there are cultural differences, which can produce significant differences when evaluating the same thing. To solve this problem, Zadeh [4] first introduced the concept of Z-number. A Z-number is an ordered pair of Fuzzy Numbers (FNs) that are expressed as $Z = (A, B)$ representing constraint A and B reliability. Since then, there are many research fruits based on Z-numbers [5–9]. Aliev et al. [10] investigated the operational rules of Z-numbers, but the operations are also complicated. Further, to avoid the complexity of calculating regarding Z-numbers, Kang et al. [11] presented a method that can transform the Z-numbers to traditional FNs. Similarly, Yaakob and Gegov [6] converted the Z-numbers to trapezoidal FNs and proposed the interactive Z-TOPSIS method to solve Multi-Attribute Decision-

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Making (MADM) problems. Peng et al. [12] presented the Asymmetric Normal Z-value (ANZ) and gave an innovative multi-criteria game model. In addition, Saravi et al. [13] proposed Z-numbers DEA for location optimization of biomass plants based on agricultural residues. However, in many cases, people are more inclined to give linguistic evaluation information, so it is also necessary to apply Z-numbers to a linguistic evaluation environment.

In real life, there are great deals of qualitative information for MADM problems, which are easily expressed by Linguistic Terms (LTs). To facilitate the processing of language information, Zadeh [14–16] first introduced the definition of Language Variables (LVs). However, the original LVs proposed by Zadeh [14] is discrete and a lot of original information may be lost in the calculation process. To overcome the drawback, Xu [17] extended the discrete LTs set to a continuous one. Accordingly, LVs were extended to a series of types relating to fuzzy information, such as uncertain LVs [18,19], hesitant fuzzy linguistic information [20] and so on [21,22]. As an extension of quantitative research [23,24], based on the LVs and FSs, Chen and Liu [25] presented the Linguistic IFNs (LIFNs) that can integrate the advantages of IFNs and LVs. Then, there are many types of researches based on LIFNs, such as the partitioned Heronian Means (HMs) for LIFNs [26], preference relations for LIFNs [27]. In the same way, for combining the advantages of Z-number and LTs, Wang et al. [28] proposed the Linguistic Z-Numbers (LZNs). At the same time, they used the Language Scaling Function (LSF) to modify and define the rules of operation to overcome the limitation of using traditional operations to lose original information [6,11].

Moreover, compared with the traditional operations of Z-numbers in [4,29], the operations proposed by Wang et al. [28] is simpler and more flexible, because different LSFs can be selected according to different situations. Besides, Wang et al. [28] proposed the extended TODIM method by using Choquet integral with Z-numbers for Multi-Criteria Decision-Making (MCDM) problems. However, we found that Wang methods cannot handle the variable interrelationship between different attributes. In actual decision-making, many attributes affect each other. Therefore, we will consider the use of aggregation operators to deal with such problems in this article.

Many traditional aggregation operators [30] just can aggregate a crisp values set into one. Now, to deal with some special functions, the researchers have developed many extended aggregation operators. The Bonferroni Mean (BM) operator [31] presented by Bonferroni can capture the relationship between any pair of different attributes c_i and c_j $(i \neq j, i, j = 1, 2, ..., n)$ in the decision-making problems. However, it ignores the correlation between c_i and c_j when $i = j$. Then, the HM was proposed by Beliakov et al. [32], which can overcome this drawback. Based on these two operators, a lot of research has been done. For example, Liang et al. [33] proposed BM operators for Pythagorean FNs (PFNs), Yang and Pang [34] proposed the partitioned BM operators for q-Rung Orthopair FNs (q-ROFNs) and Wei et al. [35] introduced HM operators for picture FNs and so on. However, both the HM operator and the BM operator can only consider the relationship between any two attributes at most. The Maclaurin Symmetric Mean (MSM) operator was first presented by Maclaurin [36], and developed by Detemple and Robertson [37], which considered the interrelationship between different numbers of attributes by adjusting the variable parameter. Further, there are many achievements to solve MADM problems by using MSM operator: Wei et al. [38] presented the MSM operators for q-ROFNs; Yang and Pang [39] proposed MSM operators for PFNs; Peng [40] presented the MSM operators for single-valued neutrosophic numbers and so on [41].

To summarize the advantages of LZNs and MSM, we extended the MSM operator to LZNs, and developed the Linguistic Z-number MSM (LZMSM) operator and Language Z-Weighted MSM (LZWMSM) operator. The advantages of our presented operators are that they can not only take account of the merits of MSM by considering interrelationship among multi-attributes by a variable parameter, but also consider the reliability about the constraint A of Z-number in qualitative environment. So, the aims of our paper are given as follows: (1) investigate several new MSM operators for LZNs based on LSFs; (2) discuss the desired characters of the proposed operators and several special cases; (3) use our proposed operator to handle MADM problems under the circumstance of LZNs and to propose a novel decision method which can take interrelationship of multi-attributes into account; and (4) demonstrate the merits of this new method by comparing with several existing approaches.

The following sections are organized as follows. In Section 2 we give an outline of several basic notions of LSFs, MSM operator, LZNs and some new operational rules. In Section 3 we introduce LZMSM and LZWMSM operators and study some properties of them. In Section 4 we present a MADM approach according to our proposed LZWMSM operator; in Section 5 an example is presented to express the validity and superiority of our new method by comparing it with other methods. In Section 6 the conclusions are presented.

2. Preliminaries

In this part, we review some basic concepts and basic theories of Z-number, LZNS, LSF and MSM operators.

Suppose $S = (s_g|g = 0, 1, \ldots l)$ is a finite and totally ordered discrete LT Set (LTS), and s_g denotes an LV; l is even number. In practice, l can be set to 4, 6, 8, etc. For example, when $l = 4$, it could be represented as: $S = (s_0, s_1, s_2, s_3, s_4) = (very\ low, low, fair, high, veryhigh)$.

2.1. Language Scaling Functions (LSFs)

Definition 1 [28,42]. If $\rho_g \in [0, 1]$ is a number, then the LSF F conducts the mapping from s_g to ρ_g $(g = 0, 1, ..., l)$ which can be expressed as:

$$F : s_g \to \rho_g \left(g = 0, 1, \ldots, l\right), \tag{1}$$

where $0 \leq \rho_0 < \rho_2 < \ldots < \rho_l$.

Based on the function, the symbols ρ_g can express the LT $s_g \in S$ which reflects the assessment information, and the semantics of the LTs are denoted by the function or value. Next, we introduce four useful LSFs.

$$F_1\left(s_g\right) = \rho_g = \frac{g}{l}\left(g = 0, 1, 2, \ldots, l\right), \tag{2}$$

$$F_2\left(s_g\right) = \rho_g = \left(\frac{g}{l}\right)^{\frac{l}{2}}\left(g = 0, 1, 2, \ldots, l\right), \tag{3}$$

$$F_3\left(s_g\right) = \rho_g = \left(\frac{g}{l}\right)^{\frac{2}{l}}\left(g = 0, 1, 2, \ldots, l\right), \tag{4}$$

$$F_4\left(s_g\right) = \rho_g = \begin{cases} \frac{\left(\frac{l}{2}\right)^a - \left(\frac{l}{2} - g\right)^a}{2\left(\frac{l}{2}\right)^a} \left(g = 0, 1, 2, \ldots, \frac{l}{2}\right) \\ \frac{\left(\frac{l}{2}\right)^b + \left(g - \frac{l}{2}\right)^b}{2\left(\frac{l}{2}\right)^b} \left(g = \frac{l}{2} + 1, \frac{l}{2} + 2, \ldots, l\right) \end{cases} \tag{5}$$

In Eq. (5), $a, b \in (0, 1]$, and if $a = b = 1$, then $\rho_g = \frac{g}{l}$. To simplify the calculation, let $a = b = 0.5$.

2.2. Linguistic Z-Numbers (LZNs)

Definition 2 [4]. A Z-number is an ordered pair of FNs (A, B) that is related to a real-valued uncertain variable X, where, A is a fuzzy restriction on the values that the variable X is allowed to take, and B is a measure of the certainty of A. In general, A and B are described in natural language.

To simplify the concept of Z-numbers, here is a simple explanation. For instance, we can use the Z-number to express "I am sure that it takes about 1 hour and 45 minutes to travel from Jinan to Beijing by a high-speed train", where "travel from Jinan to Beijing by high-speed train" is the uncertainty variable in the Z-number. Zadeh [4] noted that the underlying probability distribution in a Z-number is unknowable, and the Z-number processing method he gave is complicated. Based on the idea of Z-number, Wang et al. [28] proposed a more understandable Z-number subclass- LZNs.

Definition 3 [28]. Suppose Y is a universe of discourse, $S_1 = (s_i|i = 0, 1, \ldots T)$ and $S_2 = (s_j|j = 0, 1, \ldots, L)$ are two LTSs, and T and L are two even numbers. In practice, T and L can be set to 4, 6, 8, etc. For instance, when $T = 4$, it can be expressed by $S = (s_0, s_1, s_2, s_3, s_4) = (very\ bad, bad, general, good, verygood)$. Further, let $A_{\varsigma(y)} \in S_1$ and $B_{\varsigma(y)} \in S_2$, then, the set of LZNs Z in Y is given by:

$$Z = \left\{\left(y, A_{\varsigma(y)}, B_{\xi(y)}\right)|y \in Y\right\}, \tag{6}$$

in which $A_{\varsigma(y)}$ is the fuzzy linguistic measure of y, and $B_{\xi(y)}$ is the probability measure of $A_{\varsigma(y)}$ that can measure the reliability. In general, the LTSs S_1 and S_2 are different. To the given element x, each pair of $\left(A_{\varsigma(x)}, B_{\xi(x)}\right)$ in Z is referred to as a LZN. For convenience $z_x = \left(A_{\varsigma(x)}, B_{\xi(x)}\right)$ is used to describe a LZN, which meets $A_{\varsigma(x)} \in S_1$ and $B_{\xi(x)} \in S_2$.

Example 1. Let $Y = \{y_1, y_2, y_3\}$ be the universe of discourse, and LTSs: $A = \{A_0, A_1, \ldots, A_6\} = \{$*very bad*, *bad*, *almost bad*, *general*, *almost good*, *good*, *very good*$\}$ and $B = \{B_0, B_1, B_2\} = \{rarely, occasionally, usually\}$. A LZN can be expressed as $z_1 = (A_3, B_2)$, where $A_3 \in A$ is the evaluation information relating to the discourse given by the Decision Makers (DMs) and $B_2 \in B$ is used to express the reliability measure of the LZN. Then, the LZN $z_1 = (A_3, B_2)$ represents as $(fair, usually)$. For reducing the information distortion caused by information loss during calculation process, it is very necessary to extend the original discrete LTSs A and B to the continuous LTSs $\bar{A} = \{A_i|i \in [0, T]\}$ and $\bar{B} = \{B_j|\ j \in [0, L]\}$ as illustrated by Xu [17].

Definition 4 [28]. Let $z_1 = \left(A_{\varsigma(1)}, B_{\xi(1)}\right)$ be a LZN, then the score function LS of z_1 can be given as follows:

$$LS(z_1) = \psi^*\left(A_{\varsigma(1)}\right) \times h^*\left(B_{\xi(1)}\right), \tag{7}$$

where $\psi^*\left(A_{\varsigma(1)}\right)$ and $h^*\left(B_{\xi(1)}\right)$ are any two LSFs in Definition 1.

Definition 5 [28]. Suppose $z_1 = \left(A_{\varsigma(1)}, B_{\xi(1)}\right)$ and $z_2 = \left(A_{\varsigma(2)}, B_{\xi(2)}\right)$ are any two LZNs, then according to the score function, by comparing these LZNs we have:

1. If $A_{\varsigma(1)} > A_{\varsigma(2)}$ and $B_{\xi(1)} > B_{\xi(2)}$, then z_1 is strictly better than z_2, i.e., $z_1 \succ z_2$;
2. If $LS(z_1) > LS(z_2)$ or $LS(z_1) = LS(z_2)$, then $z_1 \succ z_2$ or $z_1 = z_2$;
3. If $LS(z_1) < LS(z_2)$, then $z_1 \prec z_2$.

Example 2. Following Example 1, let $z_1 = (A_2, B_2)$ and $z_2 = (A_3, B_2)$ be two LZNs, and suppose $\psi^*\left(s_g\right) = F_4\left(s_g\right)$ and $h^*\left(s_g\right) = F_1\left(s_g\right)$, then we can get:

$$LS_1 = \frac{3^{0.5} - 1^{0.5}}{2 \times 3^{0.5}} \times \frac{2}{2} = 0.21,$$

$$LS_2 = \frac{3^{0.5} - 0^{0.5}}{2 \times 3^{0.5}} \times \frac{2}{2} = 0.5.$$

So, $LS_1 < LS_2$, i.e., $z_1 \prec z_2$.

Definition 6 [28]. Let $z_1 = \left(A_{\varsigma(i)}, B_{\xi(i)}\right)$ and $z_2 = \left(A_{\varsigma(j)}, B_{\xi(j)}\right)$ be any two LZNs; ψ^* and h^* be the possible functions of $F_1(s_g), F_2(s_g), F_3(s_g)$ and $F_4(s_g)$, $\lambda \geq 0$, and suppose that the operational rules of LZNs are as follows:

$$neg(z_1) = \Big(\psi^{*-1}\left(\psi^*\left(A_{\varsigma(T)}\right) - \psi^*\left(A_{\varsigma(i)}\right)\right), \quad h^{*-1}\left(h^*\left(B_{\xi(L)}\right) - h^*\left(B_{\xi(i)}\right)\right)\Big). \tag{8}$$

Eq. (9) is shown in Box I.

$$\lambda z_1 = \left(\psi^{*-1}\left(\lambda\psi^*\left(A_{\varsigma(i)}\right)\right), B_{\xi(i)}\right) \tag{10}$$

$$z_1 \otimes z_2 = \left(\psi^{*-1}\left(\psi^*\left(A_{\varsigma(i)}\right)\psi^*\left(A_{\varsigma(j)}\right)\right)\right), \quad h^{*-1}\left(h^*\left(B_{\xi(i)}\right)h^*\left(B_{\xi(j)}\right)\right). \tag{11}$$

$$z_1{}^{\lambda} = \left(\psi^{*-1}\left(\psi^*\left(A_{\varsigma(i)}\right)^{\lambda}\right), h^{*-1}\left(h^*\left(B_{\xi(i)}\right)^{\lambda}\right)\right). \tag{12}$$

Theorem 1 [28]. Suppose $z_1 = \left(A_{\varsigma(i)}, B_{\xi(i)}\right)$ and $z_2 = \left(A_{\varsigma(j)}, B_{\xi(j)}\right)$ are two LZNs, and $\lambda, \lambda_1, \lambda_2 > 0$, then:

$$z_1 \oplus z_2 = z_2 \oplus z_1, \tag{13}$$

$$z_1 \otimes z_2 = z_2 \otimes z_1, \tag{14}$$

$$\lambda(z_1 \oplus z_2) = \lambda z_1 \oplus \lambda z_2, \tag{15}$$

$$(\lambda_1 + \lambda_2)z_1 = \lambda_1 z_1 \oplus \lambda_2 z_1, \tag{16}$$

$$(z_1 \otimes z_2)^{\lambda} = z_1{}^{\lambda} \otimes z_2{}^{\lambda}, \tag{17}$$

$$z_1{}^{\lambda_1} \otimes z_1{}^{\lambda_2} = z_1{}^{(\lambda_1+\lambda_2)}. \tag{18}$$

Example 3. Following Example 2, we can get:

$$z_1 \oplus z_2 = \Big(\psi^{*-1}(0.21 + 0.5), \quad h^{*-1}\left(\frac{0.21 \times 1 + 0.5 \times 1}{0.21 + 0.5}\right)\Big)$$

$$= \left(A_{(0.71\times2\times3^{0.5}-3^{0.5})^2+3}, B_{1\times2}\right)$$

$$= (A_{3.5}, B_2),$$

$$z_1 \otimes z_2 = \left(\psi^{*-1}(0.21 \times 0.5), h^{*-1}(1 \times 1)\right)$$

$$= \left(A_{3-(3^{0.5}-(0.105\times2\times3^{0.5}))^2}, B_{1\times2}\right)$$

$$= (A_{1.13}, B_2).$$

2.3. The MSM operator

MSM operator is a helpful technique proposed by Maclaurin [36], which can take the interrelationship of different numbers of attributes into account.

Definition 7 [36]. Let $q_i\,(i = 1, 2, \cdots, n)$ be a set of nonnegative real numbers, and $\partial = 1, 2, \cdots, n$, then MSM can be defined as:

$$MSM^{(\partial)}(q_1, q_2, \cdots, q_n) = \left(\frac{\sum_{1\leq i_1<\cdots<i_\partial\leq n}\prod_{j=1}^{\partial} q_{i_j}}{C_n^{\partial}}\right)^{1/\partial}, \tag{19}$$

where C_n^{∂} is the binomial coefficient and $(i_1, i_2, ..., i_\partial)$ traverses all the $\partial - tuple$ combinations of $(1, 2, ..., n)$.

Apparently, MSM operator has several properties as follows:

$$MSM^{(\partial)}(0, 0, \cdots, 0) = 0,$$

$$MSM^{(\partial)}(q, q, \cdots, q) = q;$$

$$MSM^{(\partial)}(q_1, q_2, \cdots, q_n) \leq MSM^{(\partial)}(p_1, p_2, \cdots, p_n),$$

$$\text{if } \; q_i \leq p_i \; \text{ for all } \; i;$$

$$\min_i \{q_i\} \leq MSM^{(\partial)}(q_1, q_2, \cdots, q_n) \leq \max_i \{q_i\}.$$

$$z_1 \oplus z_2 = \left(\psi^{*-1}\left(\psi^*\left(A_{\varsigma(i)}\right) + \psi^*\left(A_{\varsigma(j)}\right)\right)\right). \quad h^{*-1}\left(\frac{\psi^*\left(A_{\varsigma(i)}\right) \times h^*\left(B_{\xi(i)}\right) + \psi^*\left(A_{\varsigma(j)}\right) \times h^*\left(B_{\xi(j)}\right)}{\psi^*\left(A_{\varsigma(i)}\right) + \psi^*\left(A_{\varsigma(j)}\right)}\right). \tag{9}$$

Box I

3. Some MSM operators for LZNs

In this part, according to the operational laws of LZNs presented by Wang et al. [28], we propose the LZMSM operator and LZWMSM operator, and then we will investigate several characters and special cases.

3.1. LZMSM operator

Definition 8. Let $z_i\,(i=1,2,\cdots,n)$ be a set of LZNs, and $\partial=1,...,n$, then LZMSM operator is a mapping $LZMSM:\Phi^n\to\Phi$ which can be defined as:

$$LZMSM^{(\partial)}(z_1,z_2,\cdots,z_n)$$

$$=\left(\frac{\underset{1\le i_1<\cdots<i_\partial\le n}{\oplus}\overset{\partial}{\underset{j=1}{\otimes}} z_{i_j}}{C_n^\partial}\right)^{1/\partial}, \tag{20}$$

where Φ is a collection of all LZNs, C_n^∂ is the binomial coefficient and $(i_1,i_2,...,i_\partial)$ traverses all the $\partial-tuple$ combinations of $(1,2,...,n)$.

Based on operation rules of LZNs, we can deduce the result presented in the Theorem 2.

Theorem 2. Suppose $z_i=(A_{\varsigma(i)},B_{\xi(i)})$ $(i=1,2,\cdots,n)$ is a collection of LZNs and $\partial=1,2,...,n$, then the result aggregated from Eq. (20) is also a LZN:

$$LZMSM^{(\partial)}(z_1,z_2,\cdots,z_n)$$

$$=\left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^\partial}\right)^{\frac{1}{\partial}},\right.$$

$$\left.h^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j}^{\partial}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)^{\frac{1}{\partial}}}{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)\right). \tag{21}$$

Proof. We first calculate $\overset{\partial}{\underset{j=1}{\otimes}} z_{i_j}$, and get:

$$\overset{\partial}{\underset{j=1}{\otimes}} z_{i_j}=\left(\psi^{*-1}\prod_{j=1}^{\partial}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\right),\right.$$

$$\left.h^{*-1}\prod_{j=1}^{\partial}\left(h^*\left(B_{\xi(i_j)}\right)\right)\right),$$

and:

$$\underset{1\le i_1<\cdots<i_\partial\le n}{\oplus}\left(\overset{\partial}{\underset{j=1}{\otimes}} z_{i_j}\right)=$$

$$\left(\psi^{*-1}\left(\sum_{1\le i_1<\cdots<i_\partial\le n}\prod_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)\right),h^{*-1}\right.$$

$$\left.\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)\right),$$

then we get:

$$\frac{1}{C_n^\partial}\left(\underset{1\le i_1<\cdots<i_\partial\le n}{\oplus}\left(\overset{\partial}{\underset{j=1}{\otimes}} z_{i_j}\right)\right)=$$

$$\left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^\partial}\right),h^{*-1}\right.$$

$$\left.\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)\right).$$

Therefore, we have:

$$\left(\frac{\underset{1\le i_1<\cdots<i_\partial\le n}{\oplus}\overset{\partial}{\underset{j=1}{\otimes}} z_{i_j}}{C_n^\partial}\right)^{1/\partial}=$$

$$\left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^\partial}\right)^{1/\partial},h^{*-1}\right.$$

$$\left.\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)^{1/\partial}}{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)\right),$$

so, Theorem 2 is proved.

Next, several properties and special cases of the LZMSM will be explored.

Theorem 3 (Idempotency). Suppose $z_i=(A_i,B_i)$ $(i=1,2,...,n)$ is a collection of LZNs, if $z_i=z_1=\left(A_{\varsigma(1)},B_{\xi(1)}\right)$, $i=1,2,\cdots,n$, then:

$$LZMSM^{(\partial)}(z_1,z_2,\cdots z_n)=z_1=\left(A_{\varsigma(1)},B_{\xi(1)}\right). \tag{22}$$

Proof. Since $z_i=z_1=\left(A_{\varsigma(1)},B_{\xi(1)}\right)$ $(i=1,2,\cdots,n)$, we have:

$$\prod_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)=\psi^*\left(A_{\varsigma(1)}\right)^\partial,$$

$$\prod_{j=1}^{\partial}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)=\psi^*\left(A_{\varsigma(1)}\right)^\partial h^*\left(B_{\xi(1)}\right)^\partial,$$

$$\sum_{1\le i_1<\cdots<i_\partial\le n}\prod_{j=1}^{\partial}\psi^*\left(A_{\varsigma(i_j)}\right)=C_n^\partial\psi^*\left(A_{\varsigma(1)}\right)^\partial.$$

So, we can get:

$$LZMSM^{\partial)}(z_1, z_2, \ldots, z_n) = \left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n} \prod\limits_{j=1}^{\partial} \psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^\partial} \right)^{\frac{1}{\partial}}, h^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n} \prod\limits_{j=1}^{\partial} \left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1<\cdots<i_\partial\le n} \prod\limits_{j=1}^{\partial} \psi^*\left(A_{\varsigma(i_j)}\right)} \right)^{\frac{1}{\partial}} \right)$$

$$= \left(A_{\varsigma(1)}, B_{\xi(1)}\right) = z_1.$$

Theorem 4 (Commutativity). Suppose $z_i = (A_{\varsigma(i)}, B_{\xi(i)})$ $(i = 1, 2, ..., n)$ is a set of LZNs, and $z_i' = \left({A'}_{\varsigma(i)}, {B'}_{\xi(i)}\right)$ be any permutation of z_i, then:

$$LZMSM^{(\partial)}\left(z_1, z_2, \cdots, z_n\right) = LZMSM^{(\partial)}\left({z'}_1, {z'}_2, \cdots, {z'}_n\right). \tag{23}$$

Proof. According to Eq. (20), we obtain:

$$LZMSM^{(\partial)}\left(z_1, z_2, \cdots, z_n\right) = \left(\frac{\bigoplus\limits_{1\le i_1<\cdots<i_\partial\le n} \bigotimes\limits_{j=1}^{\partial} z_{i_j}}{C_n^\partial} \right)^{1/\partial},$$

$$LZMSM^{(\partial)}\left({z'}_1, {z'}_2, \cdots, {z'}_n\right) = \left(\frac{\bigoplus\limits_{1\le i_1<\cdots<i_\partial\le n} \bigotimes\limits_{j=1}^{\partial} {z'}_{i_j}}{C_n^\partial} \right)^{\frac{1}{\partial}}.$$

Since $z_i' = \left({A'}_{\varsigma(i)}, {B'}_{\xi(i)}\right)$ can be any permutation of z_i, then:

$$\left(\frac{\bigoplus\limits_{1\le i_1<\cdots<i_\partial\le n} \bigotimes\limits_{j=1}^{\partial} z_{i_j}}{C_n^\partial} \right)^{1/\partial} = \left(\frac{\bigoplus\limits_{1\le i_1<\cdots<i_\partial\le n} \bigotimes\limits_{j=1}^{\partial} {z'}_{i_j}}{C_n^\partial} \right)^{1/\partial}.$$

Thus:

$$LZMSM^{(\partial)}\left(z_1, z_2, \cdots, z_n\right) = LZMSM^{(\partial)}\left({z'}_1, {z'}_2, \cdots, {z'}_n\right).$$

Now, we will explore several special cases about LZMSM by adjusting values of ∂.

1. When $\partial = 1$, according to LZMSM operator from Eq. (21), we have Eq. (24) shown in Box II;
2. When $\partial = 2$, the LZMSM operator may be obtained by Eq. (25) as shown in Box III;
3. When $\partial = n$, on the basis of the Eq. (21), we can obtain Eq. (26) as shown in Box IV.

3.2. LZWMSM operator

We realize that the LZMSM operator does not consider the weights of very important attributes. In some practical cases, particularly in MADM problems, the weights have a significant impact on decision-making results. To overcome the limitations of the LZMSM operator, we developed the LZWMSM operator below.

$$LZMSM^{(1)}\left(z_1, z_2, \cdots, z_n\right) = \left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n} \prod\limits_{j=1}^{1} \psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^1} \right)^{\frac{1}{1}}, h^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n} \prod\limits_{j=1}^{1} \left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1\le n} \prod\limits_{j=1}^{1} \psi^*\left(A_{\varsigma(i_j)}\right)} \right)^{\frac{1}{1}} \right)$$

$$= \left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n} \psi^*\left(A_{\varsigma(i_j)}\right)}{n} \right), h^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n} \left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1\le n} \psi^*\left(A_{\varsigma(i_j)}\right)} \right) \right) \text{(let } i_1 = j)$$

$$= \left(\psi^{*-1}\left(\frac{\sum\limits_{1\le j\le n} \psi^*\left(A_{\varsigma(j)}\right)}{n} \right), h^{*-1}\left(\frac{\sum\limits_{1\le j\le n} \left(\psi^*\left(A_{\varsigma(j)}\right)\cdot h^*\left(B_{\xi(j)}\right)\right)}{\sum\limits_{1\le j\le n} \psi^*\left(A_{\varsigma(j)}\right)} \right) \right). \tag{24}$$

Box II

$$LZMSM^{(2)}\left(z_1,z_2,\cdots,z_n\right)=$$
$$\left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1\le i_2\le n}\prod\limits_{j=1}^{2}\psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^2}\right)^{\frac{1}{2}},h^{*-1}\left(\frac{\sum\limits_{1\le i_1\le i_2\le n}\prod\limits_{j=1}^{2}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1\le i_2\le n}\prod\limits_{j=1}^{2}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)^{\frac{1}{2}}\right)$$
$$=\left(\psi^{*-1}\left(\frac{2}{n\left(n-1\right)}\times\frac{1}{2}\times\sum_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)\right)^{\frac{1}{2}},\right.$$
$$\left.h^{*-1}\left(\frac{\frac{1}{2}\times\left(\sum\limits_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot h^*\left(B_{\xi(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)\cdot h^*\left(B_{\xi(i_2)}\right)\right)}{\frac{1}{2}\times\sum\limits_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)}\right)^{\frac{1}{2}}\right)$$
$$=\left(\psi^{*-1}\left(\frac{1}{n\left(n-1\right)}\times\sum_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)\right)^{\frac{1}{2}},\right.$$
$$\left.h^{*-1}\left(\frac{\left(\sum\limits_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot h^*\left(B_{\xi(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)\cdot h^*\left(B_{\xi(i_2)}\right)\right)}{\sum\limits_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)}\right)^{\frac{1}{2}}\right).\tag{25}$$

Box III

$$LZMSM^{(n)}\left(z_1,z_2,\cdots,z_n\right)=$$
$$\left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1\le\cdots\le i_n\le n}\prod\limits_{j=1}^{n}\psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^n}\right)^{\frac{1}{n}},h^{*-1}\left(\frac{\sum\limits_{1\le i_1\le\cdots\le i_n\le n}\prod\limits_{j=1}^{n}\left(\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1\le\cdots\le i_n\le n}\prod\limits_{j=1}^{n}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)^{\frac{1}{n}}\right)$$
$$(\text{let } i_j=j)=\left(\psi^{*-1}\left(\prod_{j=1}^{n}\psi^*\left(A_{\varsigma(j)}\right)\right)^{\frac{1}{n}},h^{*-1}\left(\prod_{j=1}^{n}h^*\left(B_{\xi(j)}\right)\right)^{\frac{1}{n}}\right).\tag{26}$$

Box IV

Definition 9. Let $\partial = 1, 2, ..., n$ and $z_i (i = 1, 2, ..., n)$ is a set of LZNs, and $w = (w_1, \cdots, w_n)^T$ is the weight vector of z_i. Then, LZWMSM operator is a mapping LZWMSM : $\Phi^n \to \Phi$, which could be expressed as follows:

$$LZWMSM^{(\partial)}(z_1, z_2, \cdots, z_n) = \left(\frac{\bigoplus_{\substack{1 \leq i_1 < \cdots \\ < i_\partial \leq n}} \bigotimes_{j=1}^{\partial} \left(w_{i_j} z_{i_j} \right)}{C_n^\partial} \right)^{1/\partial}, \tag{27}$$

where Φ is the set of all LZNs, C_n^∂ is the binomial coefficient and $(i_1, i_2, ..., i_\partial)$ traverses all the $\partial - tuple$ combination of $(1, 2, ..., n)$.

According to the operation rules of the LZNs, the aggregation result from Eq. (27) can be deduced as follows.

Theorem 5. Let $\partial = 1, 2, ..., n$, and $z_i = \left(A_{\varsigma(i)}, B_{\xi(i)} \right)$ $(i = 1, 2, ..., n)$ is a collection of LZNs, then the result aggregated from Eq. (27) is also an LZN. Then Eq. (28) shown in Box V.

Proof. We first calculate $w_{i_j} z_{i_j}$, and get:

$$w_{i_j} z_{i_j} = \left(\psi^{*-1} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \right), B_{\xi(i_j)} \right) \quad \text{and}$$

$$\bigotimes_{j=1}^{\partial} w_{i_j} z_{i_j} = \left(\psi^{*-1} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \right), h^{*-1} \prod_{j=1}^{\partial} \left(h^* \left(A_{\xi(i_j)} \right) \right) \right).$$

Then, we get the equations shown in Box VI. Finally, we obtain the equation shown in Box VII, which is also an LZN. So, Theorem 5 is proved.

By taking different values of parameter ∂, we can explore several special cases of LZWMSM operator.

$$LZWMSM^{(\partial)}(z_1, z_2, \cdots, z_n) = \left(\psi^{*-1} \left(\frac{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right)}{C_n^\partial} \right)^{\frac{1}{\partial}}, h^{*-1} \left(\frac{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \cdot h^* \left(B_{\xi(i_j)} \right) \right)}{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right)} \right)^{\frac{1}{\partial}} \right). \tag{28}$$

Box V

$$\bigoplus_{1 \leq i_1 < \cdots < i_\partial \leq n} \left(\bigotimes_{j=1}^{\partial} w_{i_j} z_{i_j} \right) = \left(\psi^{*-1} \left(\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \right) \right), \right.$$

$$\left. h^{*-1} \left(\frac{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \cdot h^* \left(B_{\xi(i_j)} \right) \right)}{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \right)} \right) \right),$$

$$\frac{1}{C_n^\partial} \left(\bigoplus_{1 \leq i_1 < \cdots < i_\partial \leq n} \left(\bigotimes_{j=1}^{\partial} w_{i_j} z_{i_j} \right) \right) = \left(\psi^{*-1} \left(\frac{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \right)}{C_n^\partial} \right), \right.$$

$$\left. h^{*-1} \left(\frac{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \cdot h^* \left(B_{\xi(i_j)} \right) \right)}{\sum_{1 \leq i_1 < \cdots < i_\partial \leq n} \prod_{j=1}^{\partial} \left(w_{i_j} \psi^* \left(A_{\varsigma(i_j)} \right) \right)} \right) \right).$$

Box VI

$$\left(\frac{\underset{1\le i_1<\cdots<i_\partial\le n}{\oplus}\overset{\partial}{\underset{j=1}{\otimes}} w_{i_j} z_{i_j}}{C_n^\partial}\right)^{1/\partial} = \left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\left(w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)\right)}{C_n^\partial}\right)^{1/\partial},\right.$$

$$\left. h^{*-1}\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\left(w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1<\cdots<i_\partial\le n}\prod\limits_{j=1}^{\partial}\left(w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)\right)}\right)^{1/\partial}\right).$$

Box VII

$$LZWMSM^{(1)}(z_1, z_2, \cdots, z_n) =$$

$$\left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n}\prod\limits_{j=1}^{1} w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^1}\right)^{\frac{1}{1}}, h^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n}\prod\limits_{j=1}^{1}\left(w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1\le n}\prod\limits_{j=1}^{1} w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)^{\frac{1}{1}}\right)$$

$$= \left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n} w_{i_1}\psi^*\left(A_{\varsigma(i_1)}\right)}{n}\right), h^{*-1}\left(\frac{\sum\limits_{1\le i_1\le n}\left(w_{i_1}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot h^*\left(B_{\xi(i_1)}\right)\right)}{\sum\limits_{1\le i_1\le n} w_{i_1}\psi^*\left(A_{\varsigma(i_1)}\right)}\right)\right)$$

$$(\text{let`}i_1 = j) = \left(\psi^{*-1}\left(\frac{\sum\limits_{1\le j\le n} w_j\psi^*\left(A_{\varsigma(j)}\right)}{n}\right),\right.$$

$$\left. h^{*-1}\left(\frac{\sum\limits_{1\le j\le n}\left(w_j\psi^*\left(A_{\varsigma(j)}\right)\cdot h^*\left(B_{\xi(j)}\right)\right)}{\sum\limits_{1\le j\le n} w_j h^*\left(A_{\varsigma(j)}\right)}\right)\right). \tag{29}$$

Box VIII

1. When $\partial = 1$, Eq. (28) will become Eq. (29) as shown in Box VIII;
2. When $\partial = 2$, Eq. (28) will become Eq. (30) as shown in Box IX;
3. When $\partial = n$, Eq. (28) will become Eq. (31) as shown in Box X.

4. A MADE method on the basis of LZWMSM operator

In this part, the LZWMSM operator is used to deal with the MADM problems. Suppose that the set of alternatives is $\{\delta_1, \delta_2, \cdots, \delta_m\}$, and the set of attributes is $\{c_1, c_2, \cdots, c_n\}$ with the weight vector $w = (w_1, \cdots, w_n)^T$ which satisfies $w_j \ge 0$ and $\sum\limits_{j=1}^{n} w_j = 1\,(j = 1, 2, \cdots, n)$. Suppose $\tilde{Z} = \left[\tilde{z}_{ij}\right]_{m\times n}$ is the decision matrix of MADM problems, and:

$$\tilde{z}_{ij} = \left(A_{\varsigma(ij)}, B_{\xi(ij)}\right)$$

$$(i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n),$$

is the assessment information given by the DM in regard to alternative δ_i for attribute c_j, and is expressed by LZNs, where $A_{\varsigma(ij)} \in \{A_i | i \in [0, T]\}$ and $B_{\xi(ij)} \in \{B_j | j \in [0, L]\}$. Then, our aim is to obtain the ranking results and choose the best alternatives.

$$LZWMSM^{(2)}(z_1,z_2,\cdots,z_n)=\left(\left(\psi^{*-1}\left(\frac{2}{n(n-1)}\times\frac{1}{2}\times\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} w_{i_1}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot w_{i_2}\psi^*\left(A_{\varsigma(i_2)}\right)\right)\right)^{\frac{1}{2}},\right.$$

$$\left.h^{*-1}\left(\frac{\frac{1}{2}\times\left(\sum\limits_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} w_{i_1}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot h^*\left(B_{\xi(i_1)}\right)\cdot w_{i_2}\psi^*\left(A_{\varsigma(i_2)}\right)\cdot h^*\left(B_{\xi(i_2)}\right)\right)}{\frac{1}{2}\times\sum\limits_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} w_{i_1}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot w_{i_2}\psi^*\left(A_{\varsigma(i_2)}\right)}\right)^{\frac{1}{2}}\right)$$

$$=\left(\left(\psi^{*-1}\left(\frac{1}{n(n-1)}\times\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} w_{i_1}w_{i_2}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)\right)\right)^{\frac{1}{2}},\right.$$

$$\left.h^{*-1}\left(\frac{\left(\sum\limits_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} w_{i_1}w_{i_2}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot h^*\left(B_{\xi(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)\cdot h^*\left(B_{\xi(i_2)}\right)\right)}{\sum\limits_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} w_{i_1}w_{i_2}\psi^*\left(A_{\varsigma(i_1)}\right)\cdot\psi^*\left(A_{\varsigma(i_2)}\right)}\right)^{\frac{1}{2}}\right). \tag{30}$$

Box IX

$$LZWMSM^{(n)}(z_1,z_2,\cdots,z_n)=\left(\psi^{*-1}\left(\frac{\sum\limits_{1\le i_1\le\cdots\le i_n\le n}\prod\limits_{j=1}^{n} w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)}{C_n^n}\right)^{\frac{1}{n}},\right.$$

$$\left.h^{*-1}\left(\frac{\sum\limits_{1\le i_1\le\cdots\le i_n\le n}\prod\limits_{j=1}^{n}\left(w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)\cdot h^*\left(B_{\xi(i_j)}\right)\right)}{\sum\limits_{1\le i_1\le\cdots\le i_n\le n}\prod\limits_{j=1}^{n} w_{i_j}\psi^*\left(A_{\varsigma(i_j)}\right)}\right)^{\frac{1}{n}}\right)\ (\text{let } i_j=j)$$

$$=\left(\psi^{*-1}\left(\prod_{j=1}^{n} w_j\psi^*\left(A_{\varsigma(j)}\right)\right)^{\frac{1}{n}}, h^{*-1}\left(\frac{\prod\limits_{j=1}^{n}\left(w_j\psi^*\left(A_{\varsigma(j)}\right)\cdot h^*\left(B_{\xi(j)}\right)\right)}{\prod\limits_{j=1}^{n} w_j\psi^*\left(A_{\varsigma(j)}\right)}\right)^{\frac{1}{n}}\right)$$

$$=\left(\psi^{*-1}\left(\prod_{j=1}^{n} w_j\psi^*\left(A_{\varsigma(j)}\right)\right)^{\frac{1}{n}}, h^{*-1}\left(\prod_{j=1}^{n} h^*\left(B_{\xi(j)}\right)\right)^{\frac{1}{n}}\right). \tag{31}$$

Box X

Next, we will propose a MADM method based on LZWMSM operator and give its rational decision-making process:

Step 1. Standardize the evaluation information of attributes. In general, the attributes can be divided into two types, i.e., cost attributes and benefit attributes. In order to eliminate the impact of different types of attribute evaluation information on decision-making results, they need to be transformed into the same type. Usually, we are accustomed to converting the cost type to the benefit type. Suppose $\tilde{z}_{ij} = \left(A_{\varsigma(ij)}, B_{\xi(ij)}\right)$ is cost type of evaluation information, and we can standardize it to benefit type as (the standardized evaluation information is also represented by $\tilde{z}_{ij}$):

$$\tilde{z}_{ij} = \left(A_{T-\varsigma(ij)}, B_{\xi(ij)}\right). \tag{32}$$

Step 2. Utilize the LZWMSM operator to aggregate the decision-making information of all attributes to a comprehensive evaluation value of each alternative.

$$r_i = LZWMSM^{(\partial)}(Z_{i1}, Z_{i2}, \cdots, Z_{in}). \tag{33}$$

Step 3. Compute the score function $LS(r_i)$ of the comprehensive values r_i, and then rank all alternatives $\{\delta_1, \delta_2, \cdots, \delta_m\}$.

Step 4. Rank all the alternatives and choose the most suitable one(s). Rank all the alternatives $\{\delta_1, \delta_2, \cdots, \delta_m\}$ and choose the most suitable one(s) by calculating the score function $LS(r_i)$.

Step 5. End.

5. An illustrative example

For the sake of showing the utilization of the approach proposed in Section 4, we will give an example (adapted from [28]) related to the apps evaluation for five medical inquiry apps $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$. The DM evaluates all the alternatives based on the following four attributes: application platform (c_1), user experience (c_2), visual foreground (c_3) and network background (c_4). The DM uses the two LTSs $A = \{A_0, A_1, A_2, A_3, A_4, A_5, A_6\} = \{$*very poor*, *poor*, slightly poor, fair, *slightly good*, *good*, *very good*$\}$ and $B = \{B_0, B_1, B_2, B_3, B_4\} = \{$*uncertain*, *slightly uncertain*, *medium*, *slightlysure*, *sure*$\}$ to give the evaluation value $z_{ij} = \left(A_{\varsigma(ij)}, B_{\xi(ij)}\right)$ ($i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4$). The decision-making information given by the DM about alternatives with respect to attribute can be obtained from the LT in and the reliability about the decision-making information can be obtained from LTs in B. Then all of the evaluation results constitute a decision matrix $R = \left[r_{ij}\right]_{5\times 4}$ listed in Table 1, where r_{ij} is represented by LZN $\left(A_{\varsigma(ij)}, B_{\xi(ij)}\right)$. $w = (0.2, 0.25, 0.25, 0.3)^T$ is the weight vector of c_j. Our aim is to choose the best app for our daily life.

Table 1. Linguistic Z-numbers decision matrix R given by D.

	c_1	c_2	c_3	c_4
δ_1	(A_4, B_4)	(A_5, B_2)	(A_5, B_4)	(A_5, B_3)
δ_2	(A_3, B_3)	(A_6, B_3)	(A_4, B_2)	(A_4, B_3)
δ_3	(A_4, B_2)	(A_3, B_4)	(A_5, B_3)	(A_5, B_2)
δ_4	(A_5, B_3)	(A_4, B_4)	(A_4, B_2)	(A_3, B_4)
δ_5	(A_5, B_3)	(A_3, B_3)	(A_3, B_4)	(A_4, B_3)

5.1. Decision-making steps

Next, the process of obtaining the best alternatives is given:

Step 1. Standardize the evaluation information of attributes. All of the attributes are benefit type, so we can omit the normalization.

Step 2. Utilize the LZWMSM operator to aggregate the decision-making information of all attributes to a comprehensive evaluation value of each alternative. (suppose $\partial = 2$, $\psi^*(s_g) = F_4(s_g)$ and $h^*(s_g) = F_1(s_g)$), and get:

$$r_1 = (2.058, 3.164), \qquad r_2 = (1.869, 2.729),$$

$$r_3 = (1.867, 2.619), \qquad r_4 = (1.780, 3.181),$$

$$r_5 = (1.653, 3.201).$$

Step 3. Compute $LS(r_i)$ of the comprehensive values r_i, and obtain:

$$LS(r_1) = 0.1739, \qquad LS(r_2) = 0.1317,$$

$$LS(r_3) = 0.1262, \qquad LS(r_4) = 0.1441,$$

$$LS(r_5) = 0.1320.$$

Step 4. Rank all the alternatives and choose the most suitable one.

On the basis of the score functions $LS(r_i)$, we can rank the alternatives $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ shown as follows:

$$\delta_1 \succ \delta_4 \succ \delta_5 \succ \delta_2 \succ \delta_3.$$

So, alternative δ_1 will be chosen.

5.2. The analysis of the effect of the values of the parameter ∂ on final ranking results of the example

To discuss the effect of parameter ∂ on the evaluation results, we will change different values of parameter ∂ during the calculation process. The ranking results are given in Table 2.

Table 2. Ranking results by using different parameter values.

∂	Score function $LS(\tilde{z}_g)$	Ranking
$\partial = 1$	$LS(r_1) = 0.1757, LS(r_2) = 0.1346,$ $LS(r_3) = 0.1276, LS(r_4) = 0.1455,$ $LS(r_5) = 0.1331$	$\delta_1 \succ \delta_4 \succ \delta_2 \succ \delta_5 \succ \delta_3$
$\partial = 2$	$LS(r_1) = 0.1738, LS(r_2) = 0.1317,$ $LS(r_3) = 0.1262, LS(r_4) = 0.1441,$ $LS(r_5) = 0.1320$	$\delta_1 \succ \delta_4 \succ \delta_5 \succ \delta_2 \succ \delta_3$
$\partial = 3$	$LS(r_1) = 0.1719, LS(r_2) = 0.1284,$ $LS(r_3) = 0.1247, LS(r_4) = 0.1426,$ $LS(r_5) = 0.1309$	$\delta_1 \succ \delta_4 \succ \delta_5 \succ \delta_2 \succ \delta_3$
$\partial = 4$	$LS(r_1) = 0.1698, LS(r_2) = 0.1252,$ $LS(r_3) = 0.1230, LS(r_4) = 0.1412,$ $LS(r_5) = 0.1297$	$\delta_1 \succ \delta_4 \succ \delta_5 \succ \delta_2 \succ \delta_3$

According to Table 2, we can easily get that the best choice is δ_1 and the ranking results by taking different values of δ are a little different. This is because when $\partial = 1$, the LZWMSM operator does not take account of the interrelationship between different attributes. For the same alternative, the score function $LS(\tilde{z}_i)$ becomes smaller as the value of parameter ∂ increases. Qin and Liu [43] pointed out that parameter can be regarded as the risk preference of the DM. In different realistic MADM problems, DMs can select suitable values of ∂ according to different risk preference. The risk preference DM can choose a lager value of ∂, or a smaller value. In the application, we generally use the round function [ ] to obtain the ∂ value as $[n/2]$, where n is the number of aggregated elements. Qin and Liu [43] explained that when $\partial = [n/2]$, the DM remains neutral which could be rational.

5.3. The validity

To prove the validity of this method, we will use the method presented by Wang et al. [28] (it is worth noting that in this example, the attribute weights are deterministic and additive, i.e. $\sum_{j=1}^{4} w_j = 1$) to deal with the same example. To make the results more convincing, we will use the same linguistic scaled model as the method proposed by Wang et al. [28] to deal with the evaluation information $A_{\varsigma(ij)}$ of LZNs as:

$$F_5\left(s_g\right) = \rho_g = \begin{cases} \frac{a^{\frac{T}{2}} - a^{\frac{T}{2}-g}}{2a^{\frac{T}{2}}-2} \left(g = 0, 1, 2, \ldots, \frac{T}{2}\right) \\ \frac{a^{\frac{T}{2}} + a^{g-\frac{T}{2}} - 2}{2a^{\frac{T}{2}}-2} \left(g = \frac{T}{2}+1, \frac{T}{2}+2, \ldots, T\right) \end{cases}$$

and then we can get the result through LZWMSM operators. Table 3 shows the ranking results of the two different methods.

As is clear from Table 3, when $\delta = 1$, our ranking result and those obtained by Wang et al. [28] are the same, because, in our method ($\partial = 1$) and that of Wang et al. [28] the interrelationship between different arguments is not considered. This can be taken as a strong evidence to prove the validity and effectiveness of our method. In addition, when $\partial = 3$, the ranking results of the two methods are different. The reason is that Wang method proposed in [28] used the extended TODIM approach and cannot take account of the interrelationship between different multi-attributes. However, the method based on LZWMSM operator can consider the interrelationship of multiple attributes by defining different value of ∂. In real MADM problems, there are more or fewer connections between different attributes, so it is not accurate to consider them independently in many cases. For example, when we choose a travel mode, we could consider two factors including weather and traffic conditions, but snowy days could cause traffic congestion. Therefore, the method using the LZWMSM operator has a wider range of applications.

5.4. Further comparison analysis

From the above analysis, we have proved the usability of our proposed method based on LZWMSM operator. Next, to better illustrate the superiority of our proposed method, we will use the method proposed by Qiao et al. to deal with the example in Subsection 5.1. [44]. The final comparison results are shown in Table 4.

As is clear from Table 4, when $\partial = 1$, the optimal solution obtained by Qiao method [44] and our method

Table 3. Ranking results by using Wang's method and method based on LZWMSM operator.

Method	Score values LS or priority weight P	Ranking
Method by Wang et al. [28] based on $Z-TODIM$	$\zeta(r_1)=1, \zeta(r_2)=0.350,$ $\zeta(r_3)=0, \zeta(r_4)=0.558,$ $\zeta(r_5)=0.262$	$\delta_1 \succ \delta_4 \succ \delta_2 \succ \delta_5 \succ \delta_3$
LZWMSM ($\partial=1$)	$LS(r_1)=0.147, LS(r_2)=0.119,$ $LS(r_3)=0.112, LS(r_4)=0.124,$ $LS(r_5)=0.118$	$\delta_1 \succ \delta_4 \succ \delta_2 \succ \delta_5 \succ \delta_3$
LZWMSM ($\partial=3$)	$LS(r_1)=0.143, LS(r_2)=0.113,$ $LS(r_3)=0.109, LS(r_4)=0.121,$ $LS(r_5)=0.118$	$\delta_1 \succ \delta_4 \succ \delta_5 \succ \delta_2 \succ \delta_3$

Table 4. Ranking results by using Qiao's method and method based on LZWMSM operator.

Method	Score values LS or Priority weight P	Ranking
Method by Qiao et al. [44]	$\zeta(r_1)=1.148, \zeta(r_2)=-0.572,$ $\zeta(r_3)=-0.248, \zeta(r_4)=0.089,$ $\zeta(r_5)=-0.417$	$\delta_1 \succ \delta_4 \succ \delta_3 \succ \delta_5 \succ \delta_2$
LZWMSM ($\partial=1$)	$LS(r_1)=0.1757, LS(r_2)=0.1346,$ $LS(r_3)=0.1276, LS(r_4)=0.1455,$ $LS(r_5)=0.1331$	$\delta_1 \succ \delta_4 \succ \delta_2 \succ \delta_5 \succ \delta_3$

is the same. However, the ranking results of the two methods are slightly different. The reason is that when the expert evaluates the attribute by using qualitative linguistic form, Qiao method [44] needs to convert the linguistic information into Triangular Fuzzy Numbers (TFNs) in the calculation process, which will result in a certain degree of information loss. The method we proposed and the method proposed by Wang et al. [28] both use LSFs to directly process the LZNs evaluation information, which can reduce the degree of information loss. Therefore, the method proposed in this paper may get more practical results.

In short, compared with some existing methods, the MADM method based on the LZWMSM operator could be more convenient and flexible, because the models such as [6] which are based on the traditional algorithm or rules are so complicated. In addition, our method is more general due to the consideration of the interrelationships between multiple attributes based on different situations.

Through the above comparison, we can summarize the advantages of our method as follows:

1. We use operations presented by Wang et al. [28] and choose different LSFs to calculate the result, which are easier and more flexible than the method in [44]. Besides, the method in [44] needs to convert linguistic assessment information into TFNs for calculation which could lead to the loss of the original information. We omit the intermediate conversion steps which can reduce the loss of the original data, therefore our results could be more realistic;
2. Compared with the method in [28], the MSM operator we use can take account of interrelationship between different numbers of attributes and can also reflect the attitude of the DMs, so that we can select different values of ∂ according to different actual scenarios. Obviously, in real life, the method which uses LZWMSM operator could be convenient and has a wider range of applications than Wang method.
3. As an extension sub-class of Z-number, LZN proposed in [28] uses the LVs to represent Z-number two components that can combine the advantages of Z-numbers and LTs. Many of the specific MADM issues are urgent and vague. Compared with quantitative evaluation, DM is more likely to give some simple qualitative evaluations, so that LZNs are more flexible and practical in that situation.

Through the comparisons and analysis of the above issues, it could be concluded that the proposed method which is based on the LZWMSM operator could be more general than some other methods of aggregating LZN.

6. Conclusions

We extend Maclaurin Symmetric Mean (MSM) operator to handle Linguistic Z-Number (LZNs) by using the new operations introduced by Wang et al. [28] which are easier and more flexible than the traditional operations in which the Language Scaling Functions (LSFs) are used. Then we propose the Linguistic Z-Number MSM (LZMSM) operator and Linguistic Z-Number Weight MSM (LZWMSM) operator, and explore several properties of them. Moreover, several special cases are also investigated, and a Multi-Attribute Decision-Making (MADM) method is given using LZWMSM operator. Compared with some existing methods, our presented method could be more general and flexible. The significant advantage is that our MADM method can consider the interrelationship between different numbers of parameters with flexibility by considering different values of parameter ∂. Besides, the method based on LZWMSM operator uses LZNs for evaluation, which can not only consider the reliability of the constraint $A_{\varsigma(x)}$, but also is more flexible in many specific environments.

In future research, we should expand the Z-number more deeply. In this regard, the topics such as exploring the more rational operations, uncertainty of $A_{\varsigma(x)}$ and $B_{\xi(x)}$ in LZNs can be mentioned as examples. In addition, we should combine some operators to make better use of LZN to deal with realistic MADM problems [45–48].

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