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Improved Illinois-Type Methods for the Solution of Nonlinear Equations

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An approach for the construction of a class of derivative-free methods for the solution of a single nonlinear equation in one variable is described and several new methods are obtained. The prototype for the class is the "Illinois" Method, which itself is a variant of the classical method of Regula Falsi. These methods deal with the problem of "end-point retention" in Regula Falsi and the consequent failure to achieve superlinear convergence by modifying one of the function-values used in the linear interpolation. The results of numerical experiments on new and existing algorithms in the class are reported, indicating that the performance of some of the new methods obtained here is very promising. The local convergence of these methods is analyzed and the asymptotic orders of convergence and patterns of behavior are determined and compared with those of existing methods.

INTRODUCTION

This paper will be concerned with a class of methods, which are called Illinois-type methods, for the solution of the single nonlinear equation:

$$f(x) = 0 \,, \tag{1}$$

with a single real variable x. The need to solve such equations arises frequently in many areas of numerical computation. A typical situation occurs when a nonlinear equation must be solved as a sub-problem within an iterative method (say, to determine a parameter required later in the iteration; see, for example, [1]), when the need for efficiency and speed is evident. The name for this class of methods is derived from the first such method, described by Dowell and Jarratt [2]. Each member of the class may be viewed as a modification of the classical Regula Falsi method and, as such, is derivative-free and guaranteed to converge to a solution, on condition that f is continuous and given a bracket $[a, b]$ for which $f(a).f(b) < 0$. Other known members of the class are the Pegasus method [3] and the method of Anderson and Björck [4]. The construction of several other members of the class and analyses of their asymptotic behavior will be described. The results of numerical tests will also be presented and these will demonstrate the superiority, from a practical point of view, of some of the new methods over the known methods. It is noted that all of these methods are easily programmed and that existing code for any of the known methods may readily be modified in order to implement the new methods.

It will be assumed that the root x^* of interest in Equation 1 is simple. Given two estimates (r and s, say) of x^* such that x^* lies within the bracket $[r, s]$ and $f(r).f(s) < 0$, Regula Falsi computes a new estimate, t, by means of linear interpolation:

$$t = s - f(s)/f[s, r] \tag{2}$$

$$= \{sf(r) - rf(s)\}/\{f(r) - f(s)\}, \tag{3}$$

where $f[s, r]$ denotes the standard divided difference:

$$f[s, r] = \{f(s) - f(r)\}/\{s - r\}.$$

The next iteration is then carried out with a bracket constituted by t and either s (if $f(s).f(t) < 0$) or r. In this way, the root always lies within the current bracket and convergence is guaranteed. However, it is well-known that, for many functions, this process results ultimately in one end-point being permanently retained and, thus, only linear convergence. Illinois-type methods eliminate this "end-point retention" in the following way.

Given two estimates x_{i-1} and x_i of x^* with $f_{i-1}f_i < 0$, where:

$$f_j \stackrel{def}{=} f(x_j) \quad \forall j \,, \tag{4}$$

apply Equation 2 with $r = x_{i-1}$ and $s = x_i$, and call

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Table 1. Known methods corresponding to differing choices of γ.

Method	Value of γ
Illinois	0.5
Pegasus	$f_i/(f_i + f_{i+1})$
Anderson & Björck	$f[x_{i+1}, x_i]/f[x_i, x_{i-1}]$

the result x_{i+1}. If $f_{i+1}f_i < 0$, then $r := x_i$, $s := x_{i+1}$, and apply Equation 2 again. If $f_{i+1}f_i = 0$, then x_{i+1} is the desired root. Otherwise, $f_{i+1}f_i > 0$, so $s := x_{i+1}$ and the modified formula:

$$t = \{s[\gamma f(r)] - rf(s)\}/\{[\gamma f(r)] - f(s)\}, \qquad (5)$$

is applied instead of Equation 2, where r retains the value x_{i-1}. In other words, when $[x_i, x_{i+1}]$ does not contain x^*, the value f_{i-1} is scaled by the factor γ.

As shown in Table 1, the various known methods of this type correspond to differing choices of the parameter γ.

A step utilizing Formula 3 is known as an unmodified step and is denoted by the letter U, whereas a step employing Formula 5 is said to be modified and will be denoted here by I, P or A, according to the method that is in use. It is possible for the expression for γ in Anderson and Björck's method to yield a nonnegative value, in which case $\gamma = 0.5$ is used and the step is denoted by M. In this method, however, $\gamma \sim 1$ asymptotically, so that M-steps are only to be expected far from the root, if at all.

Before listing properties of these methods, some standard notation is introduced. Let:

$$c_k = f^{(k)}(x^*)/k!, \; k = 1, 2, \ldots,$$

$$\beta = c_2/c_1, \quad K = \beta^2 - c_3/c_1 .$$

(Note that the condition $c_1 \neq 0$ may be inferred, from the assumption that x^* is simple.) Furthermore, the errors ε_i are defined by:

$$\varepsilon_i = x_i - x^* ,$$

recalling that (e.g., [2]), for an unmodified step, the successive errors are related by (ignoring the higher-order terms):

$$\varepsilon_{i+1} \sim \beta \varepsilon_i \varepsilon_{i-1}. \qquad (6)$$

Table 2. Results concerning asymptotic behavior patterns.

Method	Asymptotic Step Pattern	Order over Pattern	Efficiency Index
Illinois	IUU	3	1.44225
Pegasus	PPUU	7.27492	1.64232
A & B ($K > 0$)	AUU	5	1.70998
A & B ($K < 0$)	AAUU	8	1.68179

The following results (Table 2) concerning asymptotic behavior patterns and speed of convergence are known (see [2–4]) for the Illinois-type methods described above. The "Order over Pattern" column is obtained by relating the errors at the beginning and end of a complete pattern of steps; the efficiency index is that defined by Traub [5].

NEW METHODS

In order to develop new methods of "Illinois" type, it is instructive to examine a typical instance in which a modified step is required. Therefore the situation illustrated in Figure 1 is considered, where the linear interpolation between the points (x_{i-1}, f_{i-1}) and (x_i, f_i) has yielded an iterate x_{i+1} on the same side of x^* as x_i, necessitating a modified step. The ideal choice of γ would then cause the chord joining the points $(x_{i-1}, \gamma f_{i-1})$ and (x_{i+1}, f_{i+1}) to pass through the desired point $(x^*, 0)$. Examination of the slope of this chord yields:

$$\gamma = \{f_{i+1}/(x_{i+1} - x^*)\}.\{(x_{i-1} - x^*)/f_{i-1}\} . \qquad (7)$$

Evidently, this expression for γ is not computable but, by approximating the constituents of the expression with other, known, quantities, it is possible to obtain new (and, in some cases, better) methods. Recalling that x^* is a zero of f, Equation 7 can be written in the following form:

$$\gamma = f[x_{i+1}, x^*]/f[x_{i-1}, x^*]. \qquad (8)$$

Thus, it is apparent that the ideal value for γ is the ratio of the slopes of the chords joining, respectively, (x_{i-1}, f_{i-1}) with $(x^*, f(x^*))$ and (x_{i+1}, f_{i+1}) with $(x^*, f(x^*))$. It is then possible to appreciate why the method of Anderson and Björck is effective, since it amounts to approximating x^* in Equation 8 with x_i. It should be remarked, however, that Anderson and Björck derived their formula by other means, using a derivative-free variant of Newton's Method.

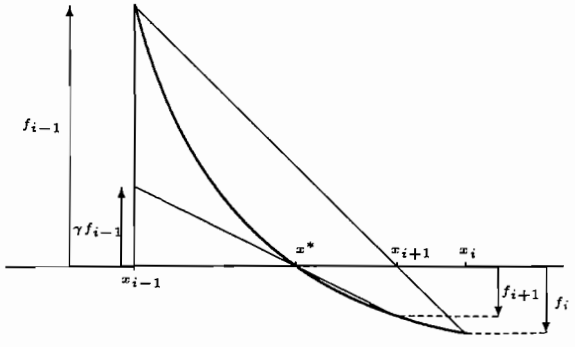

Figure 1. The ideal value of γ.

Once it is recognized that the essential problem is one of obtaining two slopes, it is possible to derive many new methods by employing and combining various estimates. A valuable tool in this respect is the expression for the derivative of the quadratic (q, say) interpolating f at x_{i-1}, x_i and x_{i+1}. Traub [5] shows that if $\{j,k,l\}$ is some permutation of $\{i-1,i,i+1\}$, then the derivative of the interpolating quadratic at x_j is given by the expression $\{f[x_j,x_k]+f[x_j,x_l]-f[x_k,x_l]\}$. Thus, for example, the numerator in Equation 8 can be estimated through replacement of x^* with the most recent approximation, x_{i+1}, and use of the derivative of q:

$$\begin{aligned} f[x_{i+1},x^*] &\approx f[x_{i+1},x_{i+1}] \\ &= f'(x_{i+1}) \text{ (formally)} \\ &\approx q'(x_{i+1}) \\ &= f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}] \\ &\quad - f[x_i,x_{i-1}] \ . \end{aligned}$$

Therefore, as shown in Table 3, four new methods have been defined. In each case, γ is the ratio of the two stated approximations. As with Anderson and Björck's method, it is possible for these expressions for γ to yield a non-negative value, in which case $\gamma = 0.5$ is used. A fifth method, defined by the expression:

$$\gamma = f[x_{i+1},x_i]/\{f[x_{i+1},x_{i-1}] + f[x_i,x_{i-1}] - f[x_{i+1},x_i]\},$$

was discarded after initial numerical tests showed its performance to be no better than the method of Anderson and Björck and, often, significantly worse.

It is clear that the methods outlined above by no means exhaust the list of possible choices for γ and for example, suitably-chosen convex combinations of appropriate terms might well be considered. However, here consideration is restricted to the methods that have been numbered 1 to 4.

NUMERICAL TESTS

Before embarking upon an examination of the local convergence properties of the new methods, the results of numerical experiments are presented. Tests have been carried out on a wide variety of functions: a representative sample of the results is presented here. (Results from a more extensive set of experiments are reported by Ford [6].) Some of the functions used were drawn from the literature on solving nonlinear equations: Function 4 is taken from Anderson and Björck [4], Function 2 is studied by Hopgood and McKee [7], and Function 3 comes from the paper by Shacham and Kehat [8]. Other test functions were specifically designed to test the capabilities of the methods on different types of functions. To reduce the possibility of freak results, a selection of different initial brackets was employed for each function. The tests were carried out in double precision (about 16 decimal places). A tolerance, τ, provided by the user is converted to a "program tolerance", ε, by means of the relation:

$$\varepsilon = \tau + 2^{-53}\max(|\ a\ |,|\ b\ |,1)\ ,$$

where $[a,b]$ is the initial bracket. Convergence was assumed when a function-value less than ε in modulus was obtained or when the length of the current bracket was less than 0.95ε. In all the experiments, τ was 10^{-14} and a limit of 200 iterations was imposed.

The Test Functions

1. $f(x) \equiv \sin(0.01/x) - 0.01$;
   $x^* = 0.99998333286109.$
2. $f(x) \equiv 10^{-8}(x-1)\prod_{i=1}^{10}(x^2+x+i)$;
   $x^* = 1.0.$
3. $f(x) \equiv \exp(21,000/x)/(1.11\times 10^{11}x^2) - 1$;
   $x^* = 551.77382493033.$
4. $f(x) \equiv 2x\exp(-20) + 1 - 2\exp(-20x)$;
   $x^* = 0.034657358821882.$
5. $f(x) \equiv \exp(x^{-1} - 25) - 1$;
   $x^* = 0.04.$

In the following table of results (Table 4), the number of iterations required by each method to locate the root to within the specified tolerance is given. The

Table 3. Four new methods.

Method	Approximation to $f[x_{i+1},x^*]$	Approximation to $f[x_{i-1},x^*]$
1	$f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}]-f[x_i,x_{i-1}]$	$f[x_{i-1},x_{i+1}]+f[x_{i-1},x_i]-f[x_{i+1},x_i]$
2	$f[x_{i+1},x_i]$	$f[x_{i-1},x_{i+1}]$
3	$f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}]-f[x_i,x_{i-1}]$	$f[x_{i-1},x_{i+1}]$
4	$f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}]-f[x_i,x_{i-1}]$	$f[x_{i-1},x_i]$

Table 4. Comparison of old and new methods.

Function (root)	Initial Bracket	Illinois	Pegasus	Anderson & Björck	Method 1	Method 2	Method 3	Method 4
1	[0.5, 2]	10	6	5	5	8	7	8
(0.999983)	[0.2, 6]	12	10	5	10	9	9	10
	[0.004, 200]	21	18	7	13	10	13	15
2	[0.9, 1.1]	8	7	7	8	8	7	7
(1.0)	[0.5, 1.5]	15	14	12	20	12	17	17
	[-5, 10]	70	103	45	65	51	65	59
3	[550, 560]	7	5	5	6	6	5	6
(551.774)	[400, 600]	29	27	18	15	20	12	12
	[350, 850]	44	42	200+	15	200+	15	19
4	[0, 1]	9	10	11	10	14	9	10
(0.034657)	[-0.1, 1.5]	15	14	43	11	46	12	12
	[-0.5, 2]	33	31	200+	16	200+	14	13
5	[0.035, 0.05]	14	14	18	9	21	12	10
(0.04)	[0.03, 0.09]	27	26	200+	15	200+	7	13
	[0.025, 0.5]	49	46	200+	19	200+	17	15

notation "200+" indicates that the method required more than 200 iterations (in many cases, substantially more than 200) in order to converge.

DISCUSSION OF NUMERICAL RESULTS

In the discussion of numerical performance, it should first be emphasized that, for "simple" functions, the behavior of all the methods may be expected to be broadly similar, as computational experience indeed indicates. On functions of greater difficulty, however, it does appear to be possible to draw some firm conclusions about relative performance. It is first observed that the behavior of Method 2 is very similar to that of Anderson and Björck's method. Only on rare occasions is there a significant difference in the figures returned and therefore further discussion of Method 2 will be omitted.

Perhaps the most noticeable feature of the results is the frequent inability of Anderson and Björck's method to converge within the limit of 200 iterations, whereas none of the other methods (except Method 2) experienced this difficulty. It appears that Anderson and Björck's method may have this difficulty when the initial bracket is (relatively) large. Typically, the method will almost immediately commence the asymptotic "AUU" pattern of behavior, but makes extremely slow progress. Notwithstanding this difficulty, Anderson and Björck's method is noticeably better than all the other methods, on occasion; Function 1 probably provides the best demonstration of this. Conversely, there are also functions for which Anderson and Björck's method is clearly worse. The Illinois and Pegasus methods are reliable but sometimes comparatively slow (for example, Function 3).

With regard to the new methods, it would appear to be the case that, overall, Method 1 is not quite as effective as Methods 3 and 4, although it may outperform them on occasion. Methods 3 and 4 are rather more difficult to separate; more extensive testing has indicated that Method 3 may have a slight overall advantage. It is, however, certainly fair to conclude from the results presented here and in [6] that all three of the new methods have performed better, in general, than the older methods.

ANALYSIS OF CONVERGENCE

The local convergence of Methods 1, 3 and 4 will be examined by studying the application of one or more modified steps in each case, on the assumption that the preceding step (i.e., the step producing x_{i+1}) was unmodified. It then follows that the latest error, ε_{i+2}, is related to the previous errors by (compare Equations 2 and 3):

$$\varepsilon_{i+2} = \{\gamma f_{i-1}\varepsilon_{i+1} - f_{i+1}\varepsilon_{i-1}\}/\{\gamma f_{i-1} - f_{i+1}\}$$

$$= \frac{\varepsilon_{i-1}\varepsilon_{i+1}}{\gamma f_{i-1} - f_{i+1}}\left(\gamma\frac{f_{i-1}}{\varepsilon_{i-1}} - \frac{f_{i+1}}{\varepsilon_{i+1}}\right). \qquad (9)$$

In all three of the new methods, it is easy to show that $\gamma \sim 1$ asymptotically. Hence, γ will be expressed in the following form:

$$\gamma = 1 + \nu/\eta, \qquad (10)$$

where it is assumed (and will be demonstrated in each case) that:

$$\nu = O(\varepsilon);\ \eta = O(1), \qquad (11)$$

where:

$$\varepsilon = \max(|\ \varepsilon_{i-1}\ |, |\ \varepsilon_i\ |). \tag{12}$$

It follows that the bracketed term in Equation 9 may be expressed in the following form:

$$\psi = c_2(\varepsilon_{i-1} - \varepsilon_{i+1}) + c_3\varepsilon_{i-1}^2 + \frac{\nu}{\eta}(c_1 + c_2\varepsilon_{i-1}) + O(\varepsilon^3), \tag{13}$$

since:

$$\varepsilon_{i+1} = \beta\varepsilon_{i-1}\varepsilon_i + O(\varepsilon^3) = O(\varepsilon^2), \tag{14}$$

from Relation 6. Relation 14 implies that, asymptotically,

$$\mathrm{sgn}(\varepsilon_{i+1}) = -\mathrm{sgn}(\beta)\ , \tag{15}$$

and, since a modified step was necessary to produce ε_{i+2}, it is evident that:

$$\mathrm{sgn}(\varepsilon_i) = -\mathrm{sgn}(\beta); \quad \mathrm{sgn}(\varepsilon_{i-1}) = \mathrm{sgn}(\beta). \tag{16}$$

Method 1

It is not difficult to show, from the expression for γ in this method, that:

$$\nu = 2\{c_2(\varepsilon_{i+1} - \varepsilon_{i-1}) - c_3\varepsilon_{i-1}(\varepsilon_i + \varepsilon_{i-1}) + O(\varepsilon^3)\};$$

$$\eta = c_1 + 2c_2\varepsilon_{i-1} + c_3\varepsilon_{i-1}(2\varepsilon_{i-1} + \varepsilon_i) + O(\varepsilon^3)\ .$$

Thus, from Equation 13,

$$\psi = -c_2\varepsilon_{i-1} + O(\varepsilon^2)\ ,$$

and, substituting in Equation 9, the following relation is obtained:

$$\varepsilon_{i+2} = -\beta\varepsilon_{i-1}\varepsilon_{i+1} + O(\varepsilon^4). \tag{17}$$

From Equations 15 and 16, it is deduced that, in the neighbourhood of the root,

$$\mathrm{sgn}(\varepsilon_{i+2}) = -\mathrm{sgn}(\varepsilon_{i+1}),$$

so that the next step will be unmodified. Consequently, if a modified step for this method is denoted by B, the asymptotic step pattern will be "BUU". To compute the order of the method over the complete composite step BUU, the following relation may be drived:

$$\varepsilon_{i+4} \sim \beta^2\varepsilon_{i+1}^4\varepsilon_{i-2}^{-1}.$$

Thus, the order is $2 + \sqrt{3} \approx 3.73205$ and the efficiency index is:

$$(2 + \sqrt{3})^{1/3} \approx 1.55113. \tag{18}$$

Method 3

Returning to Equation 13, it can be shown that ν and η for this method are given by:

$$\nu = c_2(\varepsilon_{i+1} - \varepsilon_{i-1}) - c_3\varepsilon_{i-1}(\varepsilon_i + \varepsilon_{i-1}) + O(\varepsilon^3)\ ;$$

$$\eta = c_1 + c_2(\varepsilon_{i+1} + \varepsilon_{i-1}) + c_3\varepsilon_{i-1}^2 + O(\varepsilon^3)\ .$$

It follows that:

$$\psi = -c_3\varepsilon_{i-1}\varepsilon_i + O(\varepsilon^3).$$

Thus, from Equation 9, the following relation is obtained:

$$\varepsilon_{i+2} = -(c_3/c_1)\varepsilon_{i-1}\varepsilon_i\varepsilon_{i+1} + O(\varepsilon^5). \tag{19}$$

(It is interesting to note that the same result, to $O(\varepsilon^5)$, is obtained when the Müller-Traub method is used; see [4], for instance.) It is apparent that the properties of Method 3 will depend on the sign of (c_3/c_1). If it is negative, then Equations 15 and 16 imply that:

$$\mathrm{sgn}(\varepsilon_{i+2}) = \mathrm{sgn}(\beta) = -\mathrm{sgn}(\varepsilon_{i+1})\ ,$$

and the next step will be unmodified. Denoting a modified step for this method by E, it is concluded that the asymptotic step sequence, in the case when $(c_3/c_1) < 0$, will be "EUU" and the order over this sequence may be derived from the relation:

$$\varepsilon_{i+4} \sim (c_3/c_1)^2\varepsilon_{i+1}^5.$$

Hence, the order is 5 and the efficiency index is:

$$5^{1/3} \approx 1.70998. \tag{20}$$

Turning to the case when $(c_3/c_1) > 0$, it is clear that Equation 19 then implies that:

$$\mathrm{sgn}(\varepsilon_{i+2}) = -\mathrm{sgn}(\beta) = \mathrm{sgn}(\varepsilon_{i+1}), \tag{21}$$

so that a further modified step will be required. To analyze a second such step, some relations will be required (valid whatever value for γ is used) which have been derived by Anderson and Björck [4]:

$$\frac{\tilde{f}_{i-1} - f_{i+1}}{x_{i-1} - x_{i+1}} = \tilde{f}[x_{i-1}, x_{i+1}] = \frac{f_{i+1}}{(\varepsilon_{i+1} - \varepsilon_{i+2})} \tag{22}$$

$$\gamma f_{i-1} = \tilde{f}_{i-1} = \{(\varepsilon_{i-1} - \varepsilon_{i+2})/(\varepsilon_{i+1} - \varepsilon_{i+2})\}f_{i+1} \tag{23}$$

$$= \theta f_{i+1}, \text{say.} \tag{24}$$

The "γ - factor" required for scaling $\tilde{f}_{i-1}$ for the second modified step will therefore be:

$$\hat{\gamma} = (f[x_{i+2}, x_{i+1}] + \tilde{f}[x_{i+2}, x_{i-1}] - \tilde{f}[x_{i+1}, x_{i-1}])/\tilde{f}[x_{i+2}, x_{i-1}]$$
$$= \{f_{i+1} - (1+\theta^{-1})f_{i+2}\}/\{f_{i+1} - \theta^{-1}f_{i+2}\}.$$

Thus, the scaled function-value to be used in the new step is:

$$\hat{f}_{i-1} = \hat{\gamma}\tilde{f}_{i-1} = \theta\{f_{i+1} - (1+\theta^{-1})f_{i+2}\}f_{i+1} /\{f_{i+1} - \theta^{-1}f_{i+2}\}. \quad (25)$$

This step will yield an error ε_{i+3} defined by:

$$\varepsilon_{i+3} = (\hat{f}_{i-1}\varepsilon_{i+2} - f_{i+2}\varepsilon_{i-1})/(\hat{f}_{i-1} - f_{i+2}) = \phi_1/\phi_2, \text{ say}, \quad (26)$$

where, using Equation 25,

$$\phi_1 = \theta\varepsilon_{i+2}f_{i+1}\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - \varepsilon_{i-1}f_{i+2}\{f_{i+1} - \theta^{-1}f_{i+2}\},$$

and:

$$\phi_2 = \theta f_{i+1}\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - f_{i+2}\{f_{i+1} - \theta^{-1}f_{i+2}\}.$$

Observing, from Equations 23 and 24, that $\theta = O(\varepsilon^{-1})$, it may then be shown that

$$\phi_1 = c_2\varepsilon_{i-1}\varepsilon_{i+1}^2\varepsilon_{i+2}f_{i+1}/(\varepsilon_{i+1} - \varepsilon_{i+2}) + O(\varepsilon^{10}),$$

and:

$$\phi_2 = \theta f_{i+1}^2 + O(\varepsilon^5).$$

Thus, in Equation 26,

$$\varepsilon_{i+3} = \beta\varepsilon_{i+1}\varepsilon_{i+2} + O(\varepsilon^7). \quad (27)$$

It is deduced, by means of Equation 21, that

$$\text{sgn}(\varepsilon_{i+3}) = \text{sgn}(\beta) = -\text{sgn}(\varepsilon_{i+2}),$$

so that the next step will be unmodified. Therefore, it has been shown that, asymptotically, the pattern of steps in the case when $(c_3/c_1) > 0$ will be "EEUU" with corresponding order 8, derived from the relation:

$$\varepsilon_{i+5} \sim -\beta(c_3/c_1)^3\varepsilon_{i+1}^8.$$

Finally, the efficiency index is:

$$8^{1/4} \approx 1.68179. \quad (28)$$

Method 4

Returning to Equation 13 again, it may first be shown, for Method 4, that:

$$\nu = c_2(2\varepsilon_{i+1} - \varepsilon_{i-1} - \varepsilon_i) - c_3(\varepsilon_{i-1}^2 + \varepsilon_{i-1}\varepsilon_i + \varepsilon_i^2) + O(\varepsilon^3);$$
$$\eta = c_1 + c_2(\varepsilon_{i-1} + \varepsilon_i) + c_3(\varepsilon_{i-1}^2 + \varepsilon_{i-1}\varepsilon_i + \varepsilon_i^2) + O(\varepsilon^3).$$

Thus,

$$\psi = -c_2\varepsilon_i + O(\varepsilon^2),$$

and, by Equation 9,

$$\varepsilon_{i+2} = -\beta\varepsilon_i\varepsilon_{i+1} + O(\varepsilon^4). \quad (29)$$

Hence, Equations 15 and 16 show that, as the root is approached,

$$\text{sgn}(\varepsilon_{i+2}) = -\text{sgn}(\beta) = \text{sgn}(\varepsilon_{i+1}), \quad (30)$$

so that a further modified step will be necessary. To analyze the effect of such a step, it is first noted that Equations 22, 23 and 24 still hold and that, therefore, the new "γ - factor" will be:

$$\hat{\gamma} = (f[x_{i+2}, x_{i+1}] + \tilde{f}[x_{i+2}, x_{i-1}] - \tilde{f}[x_{i+1}, x_{i-1}])/\tilde{f}[x_{i+1}, x_{i-1}]$$
$$= \{f_{i+1} - (1+\theta^{-1})f_{i+2}\}/f_{i+1},$$

which implies that the scaled version of $\tilde{f}_{i-1}$ is given by:

$$\hat{f}_{i-1} = \theta\{f_{i+1} - (1+\theta^{-1})f_{i+2}\}, \quad (31)$$

by virtue of Equation 24. The second modified step will thus yield an error determined by:

$$\varepsilon_{i+3} = (\hat{f}_{i-1}\varepsilon_{i+2} - f_{i+2}\varepsilon_{i-1})/(\hat{f}_{i-1} - f_{i+2}) = \phi_1/\phi_2, \text{ say},$$

where, from Equation 31,

$$\phi_1 = \theta\varepsilon_{i+2}\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - \varepsilon_{i-1}f_{i+2}$$

and:

$$\phi_2 = \theta\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - f_{i+2}.$$

Since $\theta = O(\varepsilon^{-1})$ again, it follows, using Equation 29, that:

$$\phi_1 = c_2\varepsilon_{i+1}^2\varepsilon_{i+2}(\varepsilon_{i-1} + 2\varepsilon_i)/(\varepsilon_{i+1} - \varepsilon_{i+2}) + O(\varepsilon^7);$$

$$\phi_2 = \theta f_{i+1} + O(\varepsilon^3),$$

and hence that

$$\varepsilon_{i+3} = \beta\varepsilon_{i+1}\varepsilon_{i+2}(1 + 2\varepsilon_i/\varepsilon_{i-1}) + O(\varepsilon^6).$$

Therefore, as long as $\varepsilon_i/\varepsilon_{i-1}$ remains less than $1/2$ in magnitude (in practice, it would be anticipated that $\varepsilon_i/\varepsilon_{i-1} \to 0$), it may be deduced that, asymptotically,

$$\varepsilon_{i+3} \sim \alpha\varepsilon_{i+1}\varepsilon_{i+2}, \tag{32}$$

for some constant α with the same sign as β. From Equation 30, it is inferred that ε_{i+3} and ε_{i+2} will have opposite sign, so that the next step will be unmodified. The sequence of iterations as the root is approached, therefore, may be expected to be "FFUU" (where F denotes a modified step for this method) and the order may be determined from the relation:

$$\varepsilon_{i+9} \sim -\alpha\beta^4\varepsilon_{i+5}^7\varepsilon_{i+1}^{-1} .$$

The order is thus the dominant zero of the quadratic $\rho^2 - 7\rho + 1$, which is $\tau^4 \approx 6.85410$ (where $\tau = (1 + \sqrt{5})/2$, the Golden Section), while the efficiency index is, therefore,

$$\tau \approx 1.61803.$$

For the sake of completeness, it may be stated here that Method 2 exhibits the asymptotic pattern "CUU" (where C denotes a modified step for this method), has the order $\tau^3 \approx 4.23607$ over this pattern and an efficiency index of τ.

Finally, it may be noted that the numerical experiments reported in the third section and in [6] confirm the conclusions of this section regarding the patterns of steps as the root is approached.

SUMMARY AND CONCLUSIONS

A technique has been developed for the construction of methods of Illinois-type for the solution of single-variable nonlinear equations. Three of the new methods thus constructed have been shown, empirically, to exhibit improved numerical performance and their asymptotic properties have been determined.

The analysis presented here for the local convergence of the new methods has shown that only Method 3 is competitive with Anderson and Björck's method insofar as efficiency indices are concerned. In that practical experience hitherto appears to indicate that (c_3/c_1) and K are both more likely to be positive than negative, it may be reckoned that, locally, Method 3 is at a very slight disadvantage when compared with Anderson and Björck's method. It has already been observed, however, that, globally, the picture is very different, with Methods 1, 3 and 4 often out-performing Anderson and Björck's method by a considerable margin. The behavior of the new methods demonstrates clearly that such measures as order and efficiency index by no means tell the whole story and that other factors must be taken into account in assessing the relative merits of methods. Such behavior may also be taken as a strong indication that attempts to develop better methods in this class should devote at least as much attention to the issue of global behavior as to improving the order of convergence.

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