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Multiple attribute decision making based on Muirhead mean operators with 2-tuple linguistic Pythagorean fuzzy information

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Green supplier selection.

Abstract. This paper extends the Muirhead Mean (MM) operator and Dual MM (DMM) operator with 2-Tuple Linguistic Pythagorean Fuzzy Numbers (2TLPFNs) to define the 2-Tuple Linguistic Pythagorean Fuzzy MM (2TLPFMM) operator, 2-Tuple Linguistic Pythagorean Fuzzy Weighted MM (2TLPFWMM) operator, 2-Tuple Linguistic Pythagorean Fuzzy DMM (2TLPFDMM) operator, and 2-Tuple Linguistic Pythagorean Fuzzy Weighted DMM (2TLPFNWDMM) operator. Based on the proposed operators, two methods are developed to deal with the Multiple Attribute Decision Making (MADM) problems with 2TLPFNs and the validity and advantages of the proposed method are analyzed by comparison with some existing approaches. The methods proposed in this paper can effectively handle the MADM problems with 2TLPFNs. Finally, an example of green supplier selection is given to illustrate the viability of the proposed methods.

1. Introduction

Recently, Pythagorean Fuzzy Set (PFS) [1,2] has been proposed with the membership and non-membership degrees and the sum of squares is less than, or equal to 1. Zhang and Xu [3] designed the TOPSIS for Multiple Attribute Decision Making (MADM) with Pythagorean Fuzzy Numbers (PFNs). Peng and Yang [4] defined the superiority and inferiority ranking model to cope with Multiple Attribute Group Decision Making (MAGDM) with PFNs. Beliakov and James [5] investigated the "averaging" under PFNs. Reformat and Yager [6] handled the recommender system under PFNs. Gou et al. [7] researched the properties of continuous PFNs. Garg [8] developed the generalized Einstein operations with PFNs. Zeng et al. [9] defined the hybrid model to solve the MADM with PFNs. Garg [10] studied the accuracy function with Interval-Valued PFNs (IVPFNs). Ren et al. [11] extended TODIM to solve the MADM with PFNs. Wei and Lu [12] extended MSM (Maclaurin Symmetric Mean) operator [13] with PFNs. Wei [14] developed some interaction operators under PFNs. Wu and Wei [15] proposed Hamacher operators with PFNs. Wei and Lu [16] defined some Hamacher operators under dual hesitant PFNs. Lu et al. [17] proposed some Hamacher operators with hesitant PFNs. Wei et al. [18] presented the Pythagorean hesitant fuzzy hamacher operators. Gao et al. [19] proposed the interaction

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operators under PFNs in MADM. Garg [20] defined some generalized geometric interaction operators based on Einstein operations with PFNs. Wei and Wei [21] defined the similarity measures of PFNs based on the cosine function. Wei and Lu [22] developed power operators with PFNs in MADM. Garg [23] proposed a new decision-making model with probabilistic information and immediate probabilities to aggregate the PFNs. Liang et al. [24] presented the Bonferroni mean operators under PFNs. Garg [25] proposed novel correlation coefficients between PFNs. Wang et al. [26] defined the generalized Dice similarity measures to deal with MAGDM with PFNs. Tang et al. [27] defined some Muirhead Mean (MM) operators for green supplier selection with IVPFNs. Muhammad et al. [28] extended TOPSIS method on the basis of Choquet integral with IVPFNs. Wan et al. [29] used the mathematical programming method to solve MAGDM with PPFNs. Garg [30] defined some exponential operations for IVPFNs. Garg [31] defined the improved accuracy function of IVPFNs. Garg [32] proposed the improved score function of IVPFNs based on the TOPSIS method.

However, all the above methods and models are not useful in depicting information on the truth-membership degree and falsity-membership degree of an element to a set by 2-tuple linguistic variables according to the given linguistic term sets, which can reflect the confidence level of the decision-maker [33–37]. In order to overcome this issue, Deng et al. [38] proposed the 2-Tuple Linguistic Pythagorean Fuzzy Set (2TLPFSs) to solve this issue on the basis of the PFS [1,2] and 2-Tuple Linguistic Sets (2TLSs) [39,40]. Deng et al. [41] proposed some Hamy mean operators with 2-Tuple Linguistic Pythagorean Fuzzy Numbers (2TLPFNs). Moreover, MM operator [42] is a useful application to depict interrelationships among any number by a variable vector. Therefore, the MM operator can give a robust and flexible mechanism to aggregate information in MADM. Because the 2TLPFNs can easily describe the fuzzy and uncertain information and the MM can depict interrelationships among any number by a variable vector, it is quite necessary to extend the MM operator to deal with the 2TLPFNs.

The purpose of this work is to extend the MM operator to 2TLPFNs to study MADM problems more effectively. Thus, the main contribution of this paper is that:

1. The MADM problems are investigated with 2TLPFNs;
2. Some MM operator and Dual MM (DMM) operator are proposed with 2TLPFNs and some properties of these operators are analyzed;
3. Some novel algorithms are proposed to solve MADM problems based on these operators with 2TLPFNs;
4. A numerical case for green supplier selection is given to illustrate the advantages of the new method.

For the sake of clarity, the rest of this research is organized as follows. In Section 2, the concept of 2TLPFSs is proposed. In Section 3, some MM operators with 2TLPFNs are defined. In Section 4, an example is given for green supplier selection. Section 5 concludes this paper.

2. Preliminaries

The concept of 2TLSs, (PFSs), and 2TLPFSs are introduced in this section.

2.1. 2TLSs

Definition 1 [39,40]. Let $S=\{s_i|i=0,1,\cdots,t\}$ be a linguistic term set with odd cardinality. s_i denotes a possible value in a linguistic variable and S can be depicted as follows:

$$S=\left\{\begin{matrix} s_0=\text{extremely poor}, \quad s_1=\text{very poor}, \\ s_2=\text{poor}, \quad s_3=\text{medium}, \quad s_4=\text{good}, \\ s_5=\text{very good}, \quad s_6=\text{extremely good}. \end{matrix}\right\}.$$

2.2. PFSs

Let X be a space of points (objects) with a generic element in the fixed set X, denoted by x. PFSs A in X are shown in the following [1,2]:

$$A=\{\langle x,\underline{u}_A(x),v_A(x)\rangle\,|x\in X\},\tag{1}$$

where $u_A(x)$ and $v_A(x)$ denote the membership and non-membership degrees that satisfy $u_A(x):X\to[0,1]$, $v_A(x):X\to[0,1]$ and $(u_A(x))^2+(v_A(x))^2\leq 1$.

2.3. 2TLPFSs

Deng et al. [38] defined the 2TLPFSs.

Definition 2 [38]. Assume that $P=\{p_0,p_1,\cdots,p_t\}$ is a 2TLS with odd cardinality $t+1$. If $p=\{(s_\phi,\varphi),(s_\theta,\vartheta)\}$ is defined for $(s_\phi,\varphi),(s_\theta,\vartheta)\in P$ and $\varphi,\vartheta\in[0,t]$, where (s_ϕ,φ) and (s_θ,ϑ) depict independently the truth degree, indeterminacy degree, and falsity degree by 2TLSs, respectively; then, the definition of 2TLPFSs is defined as follows:

$$p_j=\{(s_{\phi_j},\varphi_j),(s_{\theta_j},\vartheta_j)\},\tag{2}$$

where $0\leq\Delta^{-1}(s_{\phi_j},\varphi_j)\leq t$, $0\leq\Delta^{-1}(s_{\theta_j},\vartheta_j)\leq t$, and $0\leq(\Delta^{-1}(s_{\phi_j},\varphi_j))^2+(\Delta^{-1}(s_{\theta_j},\vartheta_j))^2\leq t^2$.

Then, the score and accuracy functions of 2TLPFNs are given as follows:

Definition 3 [38]. Let $p_1 = \{(s_{\phi_1}, \varphi_1), (s_{\theta_1}, \vartheta_1)\}$ be a 2TLPFN in P. Then, the score and accuracy functions of p_1 are defined as follows:

$$S(p_1) = \Delta\left\{\frac{t}{2}\left(1+\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^2 - \left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^2\right)\right\}, \quad S(p_1)\in[0,t], \quad (3)$$

$$H(p_1) = \Delta\left\{t\left(\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^2\right)\right\}, \quad H(p_1)\in[0,t]. \quad (4)$$

Furthermore, Deng et al. [38] proposed the comparative laws between 2TLPFNs:

Definition 4 [38]. Let $p_1 = \{(s_{\phi_1}, \varphi_1), (s_{\theta_1}, \vartheta_1)\}$ and $p_2 = \{(s_{\phi_2}, \varphi_2), (s_{\theta_2}, \vartheta_2)\}$ be two 2TLPNs; then, we have:

1. If $S(p_1) < S(p_2)$, then $p_1 < p_2$;
2. If $S(p_1) > S(p_2)$, then $p_1 > p_2$;
3. If $S(p_1) = S(p_2)$, $H(p_1) < H(p_2)$, then $p_1 < p_2$;
4. If $S(p_1) = S(p_2)$, $H(p_1) > H(p_2)$, then $p_1 > p_2$;
5. If $S(p_1) = S(p_2)$, $H(p_1) = H(p_2)$, then $p_1 = p_2$.

Then, Deng et al. [38] defined some new operations on the 2TLPFNs.

Definition 5 [38]. Let $p_1 = \{(s_{\phi_1}, \varphi_1), (s_{\theta_1}, \vartheta_1)\}$ and $p_2 = \{(s_{\phi_2}, \varphi_2), (s_{\theta_2}, \vartheta_2)\}$ be two 2TLPFNs; then, we have equations shown in Box I.

2.4. MM operators

Muirhead [42] proposed the MM operator.

Definition 6 [42]. Let $a_j (j = 1, 2, \cdots, n)$ be a set of nonnegative real numbers and $P = (p_1, p_2, \cdots, p_n) \in R^n$ be a vector of parameters. If:

$$MM^P(a_1,a_2,\cdots,a_n) = \left(\frac{1}{n!}\sum_{\sigma\in S_n}\prod_{j=1}^{n} a_{\sigma(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \quad (5)$$

where $\sigma(j)(j = 1, 2, \cdots, n)$ is a permutation of $\{1, 2, \cdots, n\}$ and S_n is a set of all permutations of $\{1, 2, \cdots, n\}$.

3. Some MM operators with 2TLPFNs

3.1. The 2-Tuple Linguistic Pythagorean Fuzzy MM (2TLPFMM) operator

This section proposes some MM operators and DMM operators with 2TLPFNs.

Definition 7. Let $p_j = \{(s_{\phi_j}, \varphi_j), (s_{\theta_j}, \vartheta_j)\}$ be a group of 2TLPFNs. The 2-Tuple Linguistic Pythagorean Fuzzy MM (2TLPFMM) operator is:

$$2TLPFMM^{\lambda}(p_1,p_2,\cdots,p_n) = \left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}. \quad (6)$$

Theorem 1. Let $p_j = \{(s_{\phi_j}, \varphi_j), (s_{\theta_j}, \vartheta_j)\}$ be a group of 2TLPFNs. The fused value using 2TLPFMM operators is also a 2TLPFN where 2TLPFMM is obtained by Eq. (7) shown in Box II.

Proof.

$$p_{\sigma(j)}^{\lambda_j} = \left\{\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{\lambda_j}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}}\right)\right\}. \quad (8)$$

$$p_1\oplus p_2 = \left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^2\right)\left(1-\left(\frac{\Delta^{-1}(s_{\phi_2},\varphi_2)}{t}\right)^2\right)}\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\bullet\frac{\Delta^{-1}(s_{\theta_2},\vartheta_2)}{t}\right)\right)\right\},$$

$$p_1\otimes p_2 = \left\{\Delta\left(t\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\bullet\frac{\Delta^{-1}(s_{\phi_2},\varphi_2)}{t}\right)\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^2\right)\left(1-\left(\frac{\Delta^{-1}(s_{\theta_2},\vartheta_2)}{t}\right)^2\right)}\right)\right\},$$

$$\lambda p_1 = \left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^2\right)^{\lambda}}\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^{\lambda}\right)\right\},$$

$$(p_1)^{\lambda} = \left\{\Delta\left(t\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^{\lambda}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^2\right)^{\lambda}}\right)\right\}.$$

Box I

$$2TLPFMM^{\lambda}(p_1, p_2, \cdots, p_n) = \left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$= \left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}. \quad (7)$$

Box II

Thus:

$$\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j} = \left\{\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{\lambda_j}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}}\right)\right\}. \quad (9)$$

Thereafter Eqs. (10) and (11), shown in Box III, are obtained and therefore we get Eq. (12) shown in Box IV. Hence, Eq. (7) is kept.

Then, we prove that Eq. (7) is a 2TLPFN. So, we shall prove these two conditions:

① $0 \le \Delta^{-1}(s_{\phi_j}, \varphi_j) \le t$, $0 \le \Delta^{-1}(s_{\theta_j}, \vartheta_j) \le t$;

② $0 \le (\Delta^{-1}(s_{\phi_j}, \varphi_j))^2 + (\Delta^{-1}(s_{\theta_j}, \vartheta_j))^2 \le t^2$.

Let we have equations shown in Box V.

Proof.

① Since $0 \le \frac{\Delta^{-1}(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)})}{t} \le 1$, we get:

$$0 \le \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j} \le 1,$$

and:

$$0 \le 1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j} \le 1. \quad (13)$$

Then:

$$0 \le \left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}} \le 1, \quad (14)$$

Eq. (15) is shown in Box VI. That means $0 \le \Delta^{-1}(s_{\phi_j}, \varphi_j) \le t$; therefore, ① is kept. Similarly, we can have $0 \le \Delta^{-1}(s_{\theta_j}, \vartheta_j) \le t$;

② Since $\left(\frac{\Delta^{-1}(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)})}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)})}{t}\right)^2 \le 1$, we have the inequality shown in Box VII. That means $0 \le (\Delta^{-1}(s_{\phi_j}, \varphi_j))^2 + (\Delta^{-1}(s_{\theta_j}, \vartheta_j))^2 \le t^2$; thus, ② is maintained.

Example 1. Let $\{(s_3, 0.4), (s_2, -0.3)\}$, $\{(s_2, 0.3), (s_1, 0.2)\}$; and $\{(s_5, 0.3), (s_3, -0.2)\}$ be three 2TLPFNs and $\lambda = (0.2, 0.3, 0.5)$; then, according to Eq. (7), we have the equation shown in Box VIII. Then, we shall discuss some properties of 2TLPFMM operator.

Property 1 (idempotency). If $p_j = \{(s_{\phi_j}, \varphi_j), (s_{\theta_j}, \vartheta_j)\}$ $(j = 1, 2, \cdots, n)$ are equal, then:

$$2TLPFNMM^{\lambda}(p_1, p_2, \cdots, p_n) = p. \quad (16)$$

Proof. Since $p_{\sigma(j)} = p = \{(s_{\phi_j}, \varphi_j), (s_{\theta_j}, \vartheta_j)\}$, then the equation shown in Box IX is obtained.

Property 2 (monotonicity). Let $p_{x_i} = \{(s_{\phi_{x_i}}, \varphi_{x_i}), (s_{\theta_{x_i}}, \vartheta_{x_i})\}$ $(i = 1, 2, \cdots, n)$ and $p_{y_i} = \{(s_{\phi_{y_i}}, \varphi_{y_i}), (s_{\theta_{y_i}}, \vartheta_{y_i})\}$ $(i = 1, 2, \cdots, n)$ be two sets of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{x_i}}, \varphi_{x_i}) \le \Delta^{-1}(s_{\phi_{y_i}}, \varphi_{y_i})$, and $\Delta^{-1}(s_{\theta_{x_i}}, \vartheta_{x_i}) \ge \Delta^{-1}(s_{\theta_{y_i}}, \vartheta_{y_i})$ hold for all i, then:

$$2TLPFMM^{\lambda}(p_{x_1}, p_{x_2}, \cdots, p_{x_n}) \le 2TLPFMM^{\lambda}(p_{y_1}, p_{y_2}, \cdots, p_{y_n}). \quad (17)$$

Proof. Let:

$$\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)=\left\{\Delta\left(t\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)}\right),\right.$$

$$\left.\Delta\left(t\sqrt{\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)}\right)\right\}. \tag{10}$$

$$\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)\right)=\left\{\Delta\left(t\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right),\right.$$

$$\left.\Delta\left(t\left(\sqrt{\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)}\right)^{\frac{1}{n!}}\right)\right\}. \tag{11}$$

Box III

$$2TLPFMM^{\lambda}(p_1,p_2,\cdots,p_n)=\left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}. \tag{12}$$

Box IV

$$\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t}=\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}},$$

$$\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}=\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}.$$

Box V

$$0 \leq \left(\sqrt{1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right) \right)^{\frac{1}{n!}}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \leq 1. \tag{15}$$

Box VI

$$\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^2$$

$$= \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}$$

$$+ \left(1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \right)$$

$$\leq \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}$$

$$+ \left(1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \right) = 1,$$

i.e.,

$$0 \leq \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^2 \leq 1.$$

Box VII

$$2TLPFMM^{\lambda}(p_{x_1}, p_{x_2}, \cdots, p_{x_n})$$
$$= \{(s_{\phi_{x_i}}, \varphi_{x_i}), (s_{\theta_{x_i}}, \vartheta_{x_i})\}(i = 1, 2, \cdots, n),$$

and:

$$2TLPFMM^{\lambda}(p_{y_1}, p_{y_2}, \cdots, p_{y_n})$$
$$= \{(s_{\phi_{y_i}}, \varphi_{y_i}), (s_{\theta_{y_i}}, \vartheta_{y_i})\}(i = 1, 2, \cdots, n),$$

given that $\Delta^{-1}(s_{\phi_{x_i}}, \varphi_{x_i}) \leq \Delta^{-1}(s_{\phi_{y_i}}, \varphi_{y_i})$, we can obtain:

$$\prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}}, \varphi_{x_i}\right)}{t} \right)^{2\lambda_j} \leq \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}}, \varphi_{y_i}\right)}{t} \right)^{2\lambda_j}, \tag{18}$$

$$1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}}, \varphi_{x_i}\right)}{t} \right)^{2\lambda_j}$$
$$\geq 1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}}, \varphi_{y_i}\right)}{t} \right)^{2\lambda_j}. \tag{19}$$

$$2TLPFMM^{(0.2,0.3,0.5)}\begin{pmatrix}\{(s_3,0.4),(s_2,-0.3)\},\\ \{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,-0.2)\}.\end{pmatrix}=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\right.,$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)}^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\right\}$$

$$=\left\{\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}\left(1-\left(\frac{3.4}{6}\right)^{0.4}\times\left(\frac{2.3}{6}\right)^{0.6}\times\left(\frac{5.3}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{3.4}{6}\right)^{0.4}\times\left(\frac{5.3}{6}\right)^{0.6}\times\left(\frac{2.3}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{2.3}{6}\right)^{0.4}\times\left(\frac{3.4}{6}\right)^{0.6}\times\left(\frac{5.3}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{2.3}{6}\right)^{0.4}\times\left(\frac{5.3}{6}\right)^{0.6}\times\left(\frac{3.4}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{5.3}{6}\right)^{0.4}\times\left(\frac{3.4}{6}\right)^{0.6}\times\left(\frac{2.3}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{5.3}{6}\right)^{0.4}\times\left(\frac{2.3}{6}\right)^{0.6}\times\left(\frac{3.4}{6}\right)^{1}\right)\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.3+0.5}}\right)\right.,$$

$$\left.\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.5}\right)\end{array}\right)^{\frac{1}{3!}}\right)}^{\frac{1}{0.2+0.3+0.5}}\right),\right\}$$

$$=\{(s_3,0.5036).(s_5,0.5351)\}.$$

Box VIII

$$2TLPFMM^{\lambda}(p_1,p_2,\cdots,p_n)\left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}p_{\sigma(j)}^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\left(\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{n!}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}$$

$$=\Delta\left\{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right),\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)\right\}=p.$$

Box IX

Thereafter:

$$1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}$$

$$\leq 1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}. \quad (20)$$

Furthermore Eq. (21) shown in Box X is obtained.

That means $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i}) \leq \Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$. Similarly, we can have $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i}) \geq \Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$.

If $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i}) < \Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$ and $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i}) \geq \Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$:

$$2TLPNMM^{\lambda}\left(p_{x_1},p_{x_2},\cdots,p_{x_n}\right)$$

$$<2TLPNMM^{\lambda}\left(p_{y_1},p_{y_2},\cdots,p_{y_n}\right).$$

If $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i}) = \Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$ and $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i}) = \Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$:

$$2TLPNMM^{\lambda}\left(p_{x_1},p_{x_2},\cdots,p_{x_n}\right)$$

$$=2TLPNMM^{\lambda}\left(p_{y_1},p_{y_2},\cdots,p_{y_n}\right).$$

Thus, Property 2 is right.

Property 3 (boundedness). Let $p_i=\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}(i=1,2,\cdots,n)$ be a group of 2TLPFNs. If:

$$p^{+}=\left(\max_i\left(S_{\phi_i},\varphi_i\right),\min_i\left(S_{\theta_i},\vartheta_i\right)\right),$$

and:

$$p^{-}=\left(\min_i\left(S_{\phi_i},\varphi_i\right),\max_i\left(S_{\theta_i},\vartheta_i\right)\right),$$

then:

$$p^{-}\leq 2TLPFMM^{\lambda}(p_1,p_2,\cdots,p_n)\leq p^{+}. \quad (22)$$

From Property 1:

$$2TLPFMM^{\lambda}(p_1^{-},p_2^{-},\cdots,p_n^{-})=p^{-},$$

$$2TLPFMM^{\lambda}(p_1^{+},p_2^{+},\cdots,p_n^{+})=p^{+}.$$

$$t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$\leq t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}. \tag{21}$$

Box X

$$\mathit{2TLPFWMM}_{nw}^{\lambda}(p_1,p_2,\cdots,p_n)=\left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}. \tag{24}$$

Box XI

From Property 2:

$$p^{-}\leq \mathit{2TLPFMM}^{\lambda}(p_1,p_2,\cdots,p_n)\leq p^{+}.$$

3.2. The 2-Tuple Linguistic Pythagorean Fuzzy Weighted MM (2TLPFWMM) operator

In real MADM, it is very important to pay attention to attribute weights. The 2-Tuple Linguistic Pythagorean Number Weighted MM (2TLPFWMM) operator is defined in this section.

Definition 8. Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}(i=1,2,\cdots,n)$ be a group of 2TLPFNs with their weight vector being $w_i=(w_1,w_2,\cdots,w_n)^T$ and satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$ and let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)\in R^n$ be a vector of parameters. Let:

$$\mathit{2TLPFWMM}_{nw}^{\lambda}(p_1,p_2,\cdots,p_n)$$

$$=\left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}, \tag{23}$$

where $\sigma(j)$ $(j=1,2,\cdots,n)$ is a permutation of $\{1,2,\cdots,n\}$ and S_n is a set of all permutations of $\{1,2,\cdots,n\}$.

Theorem 2. Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}(i=1,2,\cdots,n)$ be a group of 2TLPFNs. The fused value by 2TLPFWMM operators is also a 2TLPFN where $\text{2TLPFWMM}_{nw}^{\lambda}(p_1,p_2,...,p_n)$ is equal to what can be seen in Box XI (Eq. (24)).

Proof:

$nw_{\sigma(j)}p_{\sigma(j)}$

$$=\left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}}\right),\right.$$

$$\left.\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{nw_{\sigma(j)}}\right)\right\}. \tag{25}$$

$$\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}=\left\{\Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)\right\}. \tag{26}$$

$$\underset{j=1}{\overset{n}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}=\left\{\Delta\left(t\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)\right\}. \tag{27}$$

Box XII

Thus Eq. (26) and consequently Eq. (27) as shown in Box XII are obtained. Thereafter Eq. (28) and then Eq. (29) as shown in Box XIII are yield, and therefore, Eq. (30) shown in Box XIV is obtained. Hence, Eq. (24) is kept.

Then, we shall prove that Eq. (24) is a 2TLPFN.

① $0\leq\Delta^{-1}(s_{\phi_j},\varphi_j)\leq t$, $0\leq\Delta^{-1}(s_{\theta_j},\vartheta_j)\leq t$,

② $0\leq(\Delta^{-1}(s_{\phi_j},\varphi_j))^2+(\Delta^{-1}(s_{\theta_j},\vartheta_j))^2\leq t^2$.

Proof. Let $\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}$ and $\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}$ be calculated by the equations shown in Box XV.

① Since $0\leq\frac{\Delta^{-1}(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)})}{t}\leq 1$, we have:

$$0\leq\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\leq 1$$

and:

$$0\leq\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\leq 1. \tag{31}$$

Then Eqs. (32)–(34), shown in Box XVI, are obtained. That means $0\leq\Delta^{-1}(s_{\phi_j},\varphi_j)\leq t$; therefore, ① is kept, similarly, we can have $0\leq\Delta^{-1}(s_{\theta_j},\vartheta_j)\leq t$.

② Since $\left(\frac{\Delta^{-1}(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)})}{t}\right)^2+\left(\frac{\Delta^{-1}(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)})}{t}\right)^2\leq 1$, we can have the following inequality:

$$\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2$$

$$=\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$+\left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)$$

$$\leq\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)=\left\{\Delta\left(t\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)}\right),\right.$$

$$\left.\Delta\left(t\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)\right\}. \tag{28}$$

$$\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)\right)$$

$$=\left\{\Delta\left(t\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right),\right.$$

$$\left.\Delta\left(t\left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)^{\frac{1}{n!}}\right)\right\}. \tag{29}$$

Box XIII

$$2TLPFWMM_{nw}^{\lambda}(p_1,p_2,\cdots,p_n)=\left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}. \tag{30}$$

Box XIV

$$\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}=\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}=\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}.$$

Box XV

$$0 \leq 1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j} \leq 1, \tag{32}$$

$$0 \leq \left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}} \leq 1, \tag{33}$$

$$0 \leq \left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \leq 1. \tag{34}$$

Box XVI

$$+\left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)$$

$$=1,$$

i.e.:

$$0 \leq \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2} + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2} \leq 1.$$

That means $0 \leq (\Delta^{-1}(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}))^2 + (\Delta^{-1}(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}))^2 \leq t^2$; therefore, ② is maintained.

Example 2. Let $\{(s_3,0.4),(s_2,-0.3)\}\{(s_2,0.3),(s_1,0.2)\}$ and $\{(s_5,0.3),(s_3,-0.2)\}$ be three 2TLPFNs, and $\lambda=(0.2,0.3,0.5)$, $w=(0.4,0.2,0.4)$; then, according to Eq. (24), we have the equation shown in Box XVII. Then, we shall analyze some properties of 2TLPFWMM operator.

Property 4 (monotonicity). Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}(i=1,2,\cdots,n)$ and $p_{y_i}=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}$ $(i=1,2,\cdots,n)$ be two sets of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i}) \leq \Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$, and $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i}) \geq \Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$ hold for all i, then:

$$2TLPFWMM_{nw}^{\lambda}(p_{x_1},p_{x_2},\cdots,p_{x_n}) \leq 2TLPFWMM_{nw}^{\lambda}(p_{y_1},p_{y_2},\cdots,p_{y_n}). \tag{35}$$

The proof is similar to 2TLPFMM.

Property 5 (boundedness). Let $p_i=\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}(i=1,2,\cdots,n)$ be a set of 2TLPFNs. If:

$$p^{+}=\left(\max_i(S_{\phi_i},\varphi_i),\min_i(S_{\theta_i},\vartheta_i)\right),$$

and

$$p^{-}=\left(\min_i(S_{\phi_i},\varphi_i),\max_i(S_{\theta_i},\vartheta_i)\right),$$

then:

$$p^{-} \leq 2TLPFWMM_{nw}^{\lambda}(p_1,p_2,\cdots,p_n) \leq p^{+}. \tag{36}$$

From Theorem 2, we get Eqs. (37) and (38) shown in Box XVIII. From Property 4, we get:

$$p^{-} \leq 2TLPFWMM_{nw}^{\lambda}(p_1,p_2,\cdots,p_n) \leq p^{+}. \tag{39}$$

It is obvious that 2TLPFWMM operator lacks the property of idempotency.

3.3. The 2-Tuple Linguistic Pythagorean Fuzzy DMM (2TLPFDMM) operator

Qin and Liu [43] presented the DMM operator.

Definition 9 [43]. Let a_i $(i=1,2,\cdots,n)$ be a group of nonnegative real numbers and $P=(p_1,p_2,\cdots,p_n)\in R^n$ be a vector of parameters. If:

$$DMM^{P}(a_1,a_2,\cdots,a_n) = \frac{1}{\sum_{j=1}^{n}p_j}\left(\prod_{\sigma\in S_n}\sum_{j=1}^{n}p_j a_{\sigma(j)}\right)^{\frac{1}{n!}}, \tag{40}$$

where $\sigma(j)(j=1,2,\cdots,n)$ is a permutation of $\{1,2,\cdots,n\}$ and S_n is a set of all permutations of $\{1,2,\cdots,n\}$.

Wang et al. [44] defined some picture fuzzy dual MM operators for evaluating the financial investment risk. Hong et al. [45] proposed some hesitant fuzzy

$$2TLPFWMM^{(0.2,0.3,0.5)}_{(0.4,0.2,0.4)}\begin{pmatrix}\{(s_3,0.4),(s_2,-0.3)\},\\ \{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,-0.2)\}.\end{pmatrix}$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}$$

$$=\left\{\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}\left(1-\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{1.2}\right)^{0.5}\right)\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.3+0.5}}\right),\right.$$

$$\left.\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.5}\right)\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.2}\right.\\ \left.\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.5}\right)\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.3}\right.\\ \left.\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.5}\right)\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.5}\right)\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.2}\right.\\ \left.\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.5}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.3+0.5}}}\right)\right\}$$

$$=\{(s_3,0.3821),(s_4,0.2002)\}.$$

Box XVII

$$2TLPFWMM_{nw}^{\lambda}\left(p_1^-, p_2^-, \cdots, p_n^-\right)$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\min\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\max\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}, \tag{37}$$

$$2TLPFWMM_{nw}^{\lambda}\left(p_1^+, p_2^+, \cdots, p_n^+\right)$$

$$=\left\{\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\max\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\min\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right\}. \tag{38}$$

Box XVIII

dual MM operators in MADM. This section proposes the DMM operator for 2TLPNs as follows.

Definition 10. Let $p_{x_i} = \{(s_{\phi_{x_i}}, \varphi_{x_i}), (s_{\theta_{x_i}}, \vartheta_{x_i})\}$ $(i = 1, 2, \cdots, n)$ be a group of 2TLPFNs and let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in R^n$ be a vector of parameters. Let:

$$2TLPFDMM^{\lambda}(p_1, p_2, \cdots, p_n)$$

$$= \frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}\right)\right)\right)^{\frac{1}{n!}}. \tag{41}$$

Then, we call $2TLPFDMM^{\lambda}$ the 2TLPFDMM operator, where $\sigma(j)$ $(j = 1, 2, \cdots, n)$ is a permutation of $\{1, 2, \cdots, n\}$ and S_n is a set of all permutations of $\{1, 2, \cdots, n\}$.

Theorem 3. Let $p_{x_i} = \{(s_{\phi_{x_i}}, \varphi_{x_i}), (s_{\theta_{x_i}}, \vartheta_{x_i})\}(i = 1, 2, \cdots, n)$ be a set of 2TLPFNs. The fused value using 2TLPFDMM operators is also a 2TLPN where 2TLPFDMM $(p_1, p_2, ..., p_n)$ can be calculated by Eq. (42) shown in Box XIX.

Proof:

$\lambda_j p_{\sigma(j)}$

$$=\left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{\lambda_j}\right)\right\}. \tag{43}$$

Thus:

$$\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}\right) =$$

$$\left\{\Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{\lambda_j}\right)\right\}. \tag{44}$$

$$2TLPFDMM^{\lambda}(p_1, p_2, \cdots, p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j} \left(\underset{\sigma \in S_n}{\otimes} \left(\overset{n}{\underset{j=1}{\oplus}} \left(\lambda_j p_{\sigma(j)} \right) \right) \right)^{\frac{1}{n!}}$$

$$= \left\{ \Delta \left(t \sqrt{1 - \left(1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}} \right), \right.$$

$$\left. \Delta \left(t \left(\sqrt{1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{n!}}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \right) \right\}. \tag{42}$$

Box XIX

Therefore:

$$\underset{\sigma \in S_n}{\otimes} \left(\overset{n}{\underset{j=1}{\oplus}} \left(\lambda_j p_{\sigma(j)} \right) \right) = \left\{ \Delta \left(t \prod_{\sigma \in S_n} \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j}} \right), \right.$$

$$\left. \Delta \left(t \sqrt{1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right)} \right) \right\}. \tag{45}$$

Furthermore Eq. (46) shown in Box XX is obtained. Therefore we get Eq. (47) as shown in Box XXI. Thus, Eq. (42) is kept.

In addition, we can prove that Eq. (42) is a 2TLPFN.

① $0 \leq \Delta^{-1}(s_{\phi_j}, \varphi_j) \leq t$, $0 \leq \Delta^{-1}(s_{\theta_j}, \vartheta_j) \leq t$;

② $0 \leq (\Delta^{-1}(s_{\phi_j}, \varphi_j))^2 + (\Delta^{-1}(s_{\theta_j}, \vartheta_j))^2 \leq t^2$.

Let $\frac{\Delta^{-1}\left(s_{\phi_j}, \varphi_j\right)}{t}$ and $\frac{\Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right)}{t}$ be calculated by the equations shown in Box XXII.

① Since $0 \leq \frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \leq 1$, we get:

$$0 \leq \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 \leq 1, \qquad \text{and:}$$

$$0 \leq 1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j} \leq 1, \tag{48}$$

$$0 \leq 1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j} \right)^{\frac{1}{n!}} \leq 1. \tag{49}$$

$$\left(\underset{\sigma \in S_n}{\otimes} \left(\overset{n}{\underset{j=1}{\oplus}} \left(\lambda_j p_{\sigma(j)} \right) \right) \right)^{\frac{1}{n!}} = \left\{ \Delta \left(t \left(\prod_{\sigma \in S_n} \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^2 \right)^{\lambda_j}} \right)^{\frac{1}{n!}} \right), \right.$$

$$\left. \Delta \left(t \sqrt{1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{n!}}} \right) \right\}. \tag{46}$$

Box XX

$$2TLPFDMM^{\lambda}(p_1,p_2,\cdots,p_n)=\frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}\right)\right)\right)^{\frac{1}{n!}}$$

$$=\left\{\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right),\right.$$

$$\left.\Delta\left(t\left(\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\right\}. \tag{47}$$

Box XXI

$$\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t}=\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}$$

$$\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}=\left(\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}.$$

Box XXII

Then Eq. (50), shown in Box XXIII, is obtained. That means $0\leq\Delta^{-1}(s_{\phi_j},\varphi_j)\leq t$; therefore, ① is maintained; similarly, we can have $0\leq\Delta^{-1}(s_{\theta_j},\vartheta_j)\leq t$.

② Since $\left(\frac{\Delta^{-1}(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)})}{t}\right)^2+\left(\frac{\Delta^{-1}(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)})}{t}\right)^2\leq 1$, we have the following inequality:

$$\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2$$

$$=\left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)+\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$0\leq\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\leq 1. \tag{50}$$

Box XXIII

$$\leq \left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)+\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$=1.$$

That means $0\leq(\Delta^{-1}(s_{\phi_j},\varphi_j))^2+(\Delta^{-1}(s_{\theta_j},\vartheta_j))^2\leq t^2$; therefore, ② is maintained.

Example 3. Let $\{(s_3,0.4),(s_2,-0.3)\},\{(s_2,0.3),(s_1,0.2)\}$ and $\{(s_5,0.3),(s_3,-0.2)\}$ be three *2TLPFNs* and $\lambda=(0.2,0.3,0.5)$; then, according to Eq. (42), we have the equation shown in Box XXIV. Similar to *2TLPFMM* operator, we can have the properties as follows.

Property 6 (idempotency). If $p_i=\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}$ $(i=1,2,\cdots,n)$ are equal, then:

$$2TLPFDMM^{\lambda}(p_1,p_2,\cdots,p_n)=p. \tag{51}$$

Property 7 (monotonicity). Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ $(i=1,2,\cdots,n)$ and $p_{y_i}=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}(i=1,2,\cdots,n)$ be two sets of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})\leq\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$ and $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i})\geq\Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$ hold for all i , then:

$$2TLPFDMM^{\lambda}(p_{x_1},p_{x_2},\cdots,p_{x_n})\leq 2TLPFDMM^{\lambda}(p_{y_1},p_{y_2},\cdots,p_{y_n}). \tag{52}$$

Property 8 (boundedness). Let $p_i=\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}(i=1,2,\cdots,n)$ be a set of 2TLPFNs. If:

$$p^{+}=\left(\max_i\left(S_{\phi_i},\varphi_i\right),\min_i\left(S_{\theta_i},\vartheta_i\right)\right),$$

and:

$$p^{-}=\left(\min_i\left(S_{\phi_i},\varphi_i\right),\max_i\left(S_{\theta_i},\vartheta_i\right)\right),$$

then:

$$p^{-}\leq 2TLPFDMM^{\lambda}(p_1,p_2,\cdots,p_n)\leq p^{+}. \tag{53}$$

3.4. The 2TLPFWDMM operator

In real MADM, it is essential to consider attribute weights. Thus, this section proposes the 2TLPFWDMM operator.

Definition 11. Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ $(i=1,2,\cdots,n)$ be a set of 2TLPFNs with weight vector being $w_i=(w_1,w_2,\cdots,w_n)^T$ and satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, and let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)\in R^n$ be a vector of parameters. If:

$$2TLPFWDMM^{\lambda}_{nw}(p_1,p_2,\cdots,p_n)=\frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\bigotimes_{\sigma\in S_n}\left(\bigoplus_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}, \tag{54}$$

where $\sigma(j)(j=1,2,\cdots,n)$ is a permutation of $\{1,2,\cdots,n\}$ and S_n is a set of all permutations of $\{1,2,\cdots,n\}$.

Theorem 4. Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ $(i=1,2,\cdots,n)$ be a set of 2TLPFNs. The fused value of 2TLPFWDMM operators is also a 2TLPFN; $2TLPFDMM^{\lambda}_{nw}$ $(p_1,p_2,...,p_n)$ can be calculated by Eq. (55), shown in Box XXV.

Proof:

$$p_{\sigma(j)}^{nw_{\sigma(j)}}=\left\{\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{nw_{\sigma(j)}}\right),\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}}\right)\right\}. \tag{56}$$

Then:

$$\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}=\left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right),\Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right)\right\}. \tag{57}$$

Thus:

$$\bigoplus_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)=\left\{\Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right),\Delta\left(t\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right)\right\}. \tag{58}$$

Therefore, we have Eq. (59) and then Eq. (60), shown in

$$2TLPFDMM^{(0.2,0.3,0.5)}\begin{pmatrix}\{(s_3,0.4),(s_2,-0.3)\},\\ \{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,-0.2)\}\end{pmatrix}=\left\{\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right),\right.$$

$$\left.\Delta\left(t\left(\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\right\}$$

$$=\left\{\Delta\left(6\times\sqrt{1-\left(1-\begin{pmatrix}\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.5}\right)\end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.3+0.5}}}\right),\right.$$

$$\left.\Delta\left(6\times\left(\sqrt{1-\begin{pmatrix}\left(1-\left(\frac{1.7}{6}\right)^{0.4}\times\left(\frac{1.2}{6}\right)^{0.6}\times\left(\frac{2.8}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{1.7}{6}\right)^{0.4}\times\left(\frac{2.8}{6}\right)^{0.6}\times\left(\frac{1.2}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{1.2}{6}\right)^{0.4}\times\left(\frac{1.7}{6}\right)^{0.6}\times\left(\frac{2.8}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{1.2}{6}\right)^{0.4}\times\left(\frac{2.8}{6}\right)^{0.6}\times\left(\frac{1.7}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{2.8}{6}\right)^{0.4}\times\left(\frac{1.7}{6}\right)^{0.6}\times\left(\frac{1.2}{6}\right)^{1}\right)\\ \times\left(1-\left(\frac{2.8}{6}\right)^{0.4}\times\left(\frac{1.2}{6}\right)^{0.6}\times\left(\frac{1.7}{6}\right)^{1}\right)\end{pmatrix}^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.3+0.5}}\right)\right\}$$

$$=\{(s_4,0.1781),(s_1,0.8043)\}.$$

Box XXIV

$$2TLPFWDMM_{nw}^{\lambda}(p_1,p_2,\cdots,p_n)=\frac{1}{\sum\limits_{j=1}^{n}\lambda_j}\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}$$

$$=\left\{\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right),\right.$$

$$\left.\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\right\}. \quad (55)$$

Box XXV

$$\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)=\left\{\Delta\left(t\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)}\right)\right\}. \quad (59)$$

$$\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}=\left\{\Delta\left(t\left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)^{\frac{1}{n!}}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)\right\}. \quad (60)$$

Box XXVI

Box XXVI. Therefore Eq. (61), shown in Box XXVII, is obtained. Thus, Eq. (55) is kept.

Then, we shall prove that Eq. (55) is a 2TLPFN.

① $0\le\Delta^{-1}(s_{\phi_j},\varphi_j)\le t$, $0\le\Delta^{-1}(s_{\theta_j},\vartheta_j)\le t$;

② $0\le(\Delta^{-1}(s_{\phi_j},\varphi_j))^2+(\Delta^{-1}(s_{\theta_j},\vartheta_j))^2\le t^2$.

Proof: Let $\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}$ and $\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}$ be calculated by the equations shown in Box XXVIII.

① Since $0\le\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\le 1$, we have:

$$0\le\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\le 1,$$

and:

$$0\le 1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\le 1. \quad (62)$$

$$2TLPFWDMM_{nw}^{\lambda}(p_1,p_2,\cdots,p_n)=\frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}$$

$$=\left\{\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right),\right.$$

$$\left.\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\right\}. \quad (61)$$

Box XXVII

$$\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t}=\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}$$

$$\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}=\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}.$$

Box XXVIII

Then Eqs. (63) and (64), shown in Box XXIX, are obtained. That means $0\le\Delta^{-1}(s_{\phi_j},\varphi_j)\le t$; thus, ① is maintained; similarly, we can have $0\le\Delta^{-1}(s_{\theta_j},\vartheta_j)\le t$.

② Since $\left(\frac{\Delta^{-1}(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)})}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\le 1$, we have the inequality shown in Box XXX. That means $0\le(\Delta^{-1}(s_{\phi_j},\varphi_j))^2+(\Delta^{-1}(s_{\theta_j},\vartheta_j))^2\le t^2$. Thus, ② is maintained.

Example 4. Let $\{(s_3,0.4),(s_2,-0.3)\}$ $\{(s_2,0.3),(s_1,0.2)\}$, and $\{(s_5,0.3),(s_3,-0.2)\}$ be three 2TLPNs, and $\lambda=(0.2,0.3,0.5)$, $w=(0.4,0.2,0.4)$; then, according

$$0\le\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\le 1, \quad (63)$$

$$0\le\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\le 1. \quad (64)$$

Box XXIX

$$\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}+\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}$$

$$=\left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)$$

$$+\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$\leq\left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)$$

$$+\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}=1.$$

Box XXX

to Eq. (55), we have the equation shown in Box XXXI. Then, we analyze the properties of 2TLPFDWMM operator.

Property 9 (monotonicity). Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),\ (s_{\theta_{x_i}},\vartheta_{x_i})\}(i=1,2,\cdots,n)$ and $p_{y_i}=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}(i=1,2,\cdots,n)$ be two sets of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})\leq\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$, and $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i})\geq\Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$ hold for all i, then:

$$\mathit{2TLPFWDMM}_{nw}^{\lambda}(p_{x_1},p_{x_2},\cdots,p_{x_n})$$
$$\leq\mathit{2TLPFWDMM}_{nw}^{\lambda}(p_{y_1},p_{y_2},\cdots,p_{y_n}).\qquad(65)$$

Property 10 (boundedness). Let $p_i=\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}(i=1,2,\cdots,n)$ be a set of 2TLPNs. If:

$$p^{+}=\left(\max_i\left(S_{\phi_i},\varphi_i\right),\min_i\left(S_{\theta_i},\vartheta_i\right)\right),$$

and:

$$p^{-}=\left(\min_i\left(S_{\phi_i},\varphi_i\right),\max_i\left(S_{\theta_i},\vartheta_i\right)\right),$$

then:

$$p^{-}\leq\mathit{2TLPFWDMM}_{nw}^{\lambda}(p_1,p_2,\cdots,p_n)\leq p^{+}.\qquad(66)$$

From Theorem 4, we have Eqs. (67) and (68) shown in Box XXXII. From Property 9,

$$p^{-}\leq\mathit{2TLPFWDMM}_{nw}^{\lambda}(p_1,p_2,\cdots,p_n)\leq p^{+}.\qquad(69)$$

4. Numerical example and comparative analysis

4.1. Numerical example

MADM is the process of ranking a finite set of alternatives with respect to a list of attributes. It has been extensively studied and applied in various areas such as human resource selection [46], transportation management [47], military affair [48], construction engineering project risk assessment [49,50], potential evaluation of emerging technology commercialization [51–56], and strategic supplier selection [57]. With the rapid development of growing enterprise competition and economic globalization, the competition among modern enterprises has become the competition among the supply chains [58–62]. Therefore, the supplier selection problem has paid great attention in practical production management and supply chain management theory. In actual supplier selection problems, there are a large number of uncertainties, fuzziness, and risk in the whole supply chains [63–67]. These factors are of

$$2TLPFWDMM^{(0.2,0.3,0.5)}_{(0.4,0.3,0.3)}\begin{pmatrix}\{(s_3,0.4),(s_2,-0.3)\},\\ \{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,-0.2)\}.\end{pmatrix}$$

$$=\left\{\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right),\right.$$

$$\left.\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\right\}$$

$$=\left(\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.5}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.3+0.5}}}\right),\right.$$

$$\left.\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}\left(1-\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.5}\right)\\ \times\left(1-\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.3+0.5}}\right),\right)$$

$$=\{(s_4,0.1744),(s_1,0.1173)\}.$$

Box XXXI

$$2TLPFWDMM_{nw}^{\lambda}(p_1^-, p_2^-, \cdots, p_n^-)$$

$$= \left\{ \Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\min \Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \right), \right.$$

$$\left. \Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\max \Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \right) \right\}, \tag{67}$$

$$2TLPFWDMM_{nw}^{\lambda}(p_1^+, p_2^+, \cdots, p_n^+)$$

$$= \left\{ \Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\max \Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \right), \right.$$

$$\left. \Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\min \Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \right) \right\}. \tag{68}$$

Box XXXII

Table 1. 2-Tuple Linguistic Pythagorean Fuzzy Number (2TLPFN) decision matrix (R_1).

	G_1	G_2	G_3	G_4
A_1	$<(s_4,0),(s_2,0)>$	$<(s_4,0),(s_1,0)>$	$<(s_3,0),(s_4,0)>$	$<(s_3,0),(s_2,0)>$
A_2	$<(s_5,0),(s_1,0)>$	$<(s_5,0),(s_2,0)>$	$<(s_5,0),(s_3,0)>$	$<(s_5,0),(s_2,0)>$
A_3	$<(s_4,0),(s_3,0)>$	$<(s_2,0),(s_4,0)>$	$<(s_4,0),(s_3,0)>$	$<(s_5,0),(s_2,0)>$
A_4	$<(s_1,0),(s_4,0)>$	$<(s_2,0),(s_5,0)>$	$<(s_3,0),(s_4,0)>$	$<(s_1,0),(s_2,0)>$
A_5	$<(s_4,0),(s_2,0)>$	$<(s_4,0),(s_3,0)>$	$<(s_3,0),(s_1,0)>$	$<(s_3,0),(s_2,0)>$

great significance in actual assessment and selection. In this section, we shall give an example to show green supplier selection under 2TLPFNs. Suppose that there are five possible green suppliers; $A_i (i = 1, 2, 3, 4, 5)$ are assessed according to four attributes:

① G_1 is the environmental factors;

② G_2 is the product quality factor;

③ G_3 is the price factors;

④ G_4 is the delivery factor.

These five green suppliers A_i $(i = 1, 2, 3, 4, 5)$ are evaluated by 2TLPFNs under four attributes by three experts. The evaluation results are listed in Tables 1–3. The attribute weight vector is $\omega = (0.16, 0.27, 0.29, 0.28)$ and the expert weight vector is $\omega = (0.2, 0.6, 0.2)$. In the following, the developed methods and models are employed to select green suppliers.

Step 1. In accordance with the 2TLPFNs r_{ij} $(i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4)$, we can fuse the 2TLPFNs with 2TLPFWAA (2TLPFWGA) operator to have the 2TLPFNs A_i $(i = 1, 2, 3, 4, 5)$ of the green suppliers A_i. Then, the calculated values are shown in Table 4.

Table 2. 2-Tuple Linguistic Pythagorean Fuzzy Number (2TLPFN) decision matrix (R_2).

	G_1	G_2	G_3	G_4
A_1	$<(s_3,0),(s_2,0)>$	$<(s_2,0),(s_1,0)>$	$<(s_1,0),(s_2,0)>$	$<(s_5,0),(s_2,0)>$
A_2	$<(s_5,0),(s_2,0)>$	$<(s_4,0),(s_2,0)>$	$<(s_4,0),(s_3,0)>$	$<(s_5,0),(s_2,0)>$
A_3	$<(s_3,0),(s_2,0)>$	$<(s_4,0),(s_1,0)>$	$<(s_3,0),(s_1,0)>$	$<(s_2,0),(s_2,0)>$
A_4	$<(s_1,0),(s_4,0)>$	$<(s_2,0),(s_5,0)>$	$<(s_3,0),(s_4,0)>$	$<(s_2,0),(s_3,0)>$
A_5	$<(s_3,0),(s_2,0)>$	$<(s_4,0),(s_3,0)>$	$<(s_1,0),(s_2,0)>$	$<(s_5,0),(s_2,0)>$

Table 3. 2-Tuple Linguistic Pythagorean Fuzzy Number (2TLPFN) decision matrix (R_3).

	G_1	G_2	G_3	G_4
A_1	$<(s_3,0),(s_2,0)>$	$<(s_4,0),(s_1,0)>$	$<(s_2,0),(s_3,0)>$	$<(s_2,0),(s_3,0)>$
A_2	$<(s_5,0),(s_1,0)>$	$<(s_4,0),(s_2,0)>$	$<(s_4,0),(s_3,0)>$	$<(s_5,0),(s_3,0)>$
A_3	$<(s_3,0),(s_2,0)>$	$<(s_2,0),(s_2,0)>$	$<(s_3,0),(s_1,0)>$	$<(s_1,0),(s_2,0)>$
A_4	$<(s_1,0),(s_4,0)>$	$<(s_2,0),(s_5,0)>$	$<(s_1,0),(s_2,0)>$	$<(s_3,0),(s_2,0)>$
A_5	$<(s_4,0),(s_1,0)>$	$<(s_1,0),(s_2,0)>$	$<(s_3,0),(s_1,0)>$	$<(s_1,0),(s_3,0)>$

Table 4. The calculation results by the 2TLPFWA operator.

	G_1	G_2
A_1	$<(s_3,0.2516),(s_2,0.0000)>$	$<(s_3,0.0797),(s_1,0.0000)>$
A_2	$<(s_5,0.0000),(s_2,-0.4843)>$	$<(s_4,0.2725),(s_2,0.0000)>$
A_3	$<(s_3,0.2516),(s_2,0.1689)>$	$<(s_3,0.4443),(s_2,-0.4843)>$
A_4	$<(s_1,0.0000),(s_4,0.0000)>$	$<(s_2,0.0000),(s_5,0.0000)>$
A_5	$<(s_3,0.4719),(s_2,-0.2589)>$	$<(s_4,-0.3079),(s_3,-0.2337)>$
	G_3	G_4
A_1	$<(s_2,-0.1668),(s_2,0.4915)>$	$<(s_4,0.4387),(s_2,0.1689)>$
A_2	$<(s_4,0.2725),(s_3,0.0000)>$	$<(s_5,0.0000),(s_2,0.1689)>$
A_3	$<(s_3,0.2516),(s_1,0.2457)>$	$<(s_3,0.1123),(s_2,0.0000)>$
A_4	$<(s_3,-0.2502),(s_3,0.4822)>$	$<(s_2,0.1236),(s_3,-0.4492)>$
A_5	$<(s_2,0.1097),(s_2,-0.4843)>$	$<(s_4,-0.4054),(s_2,0.1689)>$

Definition 12 [38]. Let $p_j=\{(s_{\phi_j},\varphi_j),(s_{\theta_j},\vartheta_j)\}$ $(j=1,2,\cdots,n)$ be a group of 2TLPFNs with weight vector being $w_i=(w_1,w_2,\cdots,w_n)^T$ and satisfying $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$; then, we shall obtain:

$$2TLPFWA(p_1,p_2,\cdots,p_n)=\sum_{j=1}^n w_jp_j$$

$$=\left\{\Delta\left(t\sqrt{1-\prod_{j=1}^n\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t}\right)^2\right)^{w_j}}\right),\right.$$

$$\left.\Delta\left(t\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{w_j}\right)\right\},\qquad(70)$$

$$2TLPFWG(p_1,p_2,\cdots,p_n)=\prod_{j=1}^n(l_j)^{w_j}$$

$$=\left\{\Delta\left(t\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t}\right)^{w_j}\right),\right.$$

$$\left.\Delta\left(t\sqrt{1-\prod_{j=1}^n\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{w_j}}\right)\right\}.\qquad(71)$$

Step 2. According to Table 4, we shall fuse the 2TLPFNsr_{ij} with 2TLPFWMM (2TLPFWDMM) operator to achieve the overall 2TLPFNs A_i $(i=1,2,3,4,5)$. Let $P=(1,1,0,0)$; then, the calculated results are listed in Table 5.

Step 3. According to the calculation results in Table 2, the scores are listed in Table 6.

Step 4. According to Table 6, the ordering is listed in Table 7 where the best green supplier is A_2.

Table 5. The calculation results by the 2TLPFWMM (2TLPFWDMM) operator.

	2TLPFWMM	**2TLPFWDMM**
$\boldsymbol{A_1}$	$<(s_3, 0.1613), (s_2, 0.0510)>$	$<(s_3, 0.3350), (s_2, -0.0559)>$
$\boldsymbol{A_2}$	$<(s_5, -0.3914), (s_2, 0.2446)>$	$<(s_5, -0.3441), (s_2, 0.2356)>$
$\boldsymbol{A_3}$	$<(s_3, 0.2479), (s_2, -0.1613)>$	$<(s_3, 0.3146), (s_2, -0.2980)>$
$\boldsymbol{A_4}$	$<(s_2, 0.0874), (s_4, -0.1043)>$	$<(s_2, 0.0152), (s_4, -0.2420)>$
$\boldsymbol{A_5}$	$<(s_3, 0.4362), (s_2, 0.1589)>$	$<(s_4, -0.4135), (s_2, 0.0710)>$

Table 6. The score of the green suppliers.

	2TLPFWMM	**2TLPFWDMM**
$\boldsymbol{A_1}$	$(s_3, 0.4822)$	$(s_4, -0.3881)$
$\boldsymbol{A_2}$	$(s_4, 0.3501)$	$(s_4, 0.3900)$
$\boldsymbol{A_3}$	$(s_4, -0.4027)$	$(s_4, -0.3259)$
$\boldsymbol{A_4}$	$(s_2, -0.0984)$	$(s_2, 0.1615)$
$\boldsymbol{A_5}$	$(s_3, -0.4044)$	$(s_4, -0.2855)$

Table 7. Ordering of the green suppliers.

	Ordering
2TLPFWMM	$A_2 > A_3 > A_5 > A_1 > A_4$
2TLPFWDMM	$A_2 > A_5 > A_3 > A_1 > A_4$

4.2. Influence analysis of the parameter

In order to illustrate the effects of altering parameters of P in the 2TLPFWMM (2TLPFWDMM) operators on the ranking results, the results are listed in Tables 8 and 9.

Table 10. Order of the green suppliers.

	Order
LPFWA operator [68]	$A_2 > A_5 > A_1 > A_3 > A_4$
LPFWG operator [68]	$A_2 > A_3 > A_5 > A_1 > A_4$

4.3. Comparative analysis

Then, we shall compare our methods with LPFWA and LPFWG operators [68]. The comparative results are depicted in Table 10.

According to the above, it appears that LPFWA and LPFWG operators are not correlated in terms of the relationship between the discussed arguments. The proposed 2TLPFWMM and 2TLPFWDMM operators consider the relationship among the aggregated arguments.

5. Conclusion

2-Tuple Linguistic Pythagorean Fuzzy Number (2TLPFNs) are characterized by the advantages of 2-tuple linguistic term sets and Pythagorean fuzzy

Table 8. Ranking results of different parameters for 2-Tuple Linguistic Pythagorean Weighted MM (2TLPFWMM) operator.

$\boldsymbol{P}$	$\boldsymbol{s(A_1)}$	$\boldsymbol{s(A_2)}$	$\boldsymbol{s(A_3)}$	$\boldsymbol{s(A_4)}$	$\boldsymbol{s(A_5)}$	**Ordering**
(1,0,0,0)	$(s_4, -0.3067)$	$(s_4, 0.3979)$	$(s_4, -0.3333)$	$(s_2, 0.3187)$	$(s_4, -0.2537)$	$A_2 > A_5 > A_1 > A_3 > A_4$
(1,1,0,0)	$(s_3, 0.4822)$	$(s_4, 0.3501)$	$(s_4, -0.4027)$	$(s_2, 0.0984)$	$(s_4, -0.4044)$	$A_2 > A_3 > A_5 > A_1 > A_4$
(1,1,1,0)	$(s_3, 0.4049)$	$(s_4, 0.3341)$	$(s_4, -0.4453)$	$(s_2, -0.0208)$	$(s_4, -0.4732)$	$A_2 > A_3 > A_5 > A_1 > A_4$
(1,1,1,1)	$(s_3, 0.3470)$	$(s_4, 0.3247)$	$(s_4, -0.4792)$	$(s_2, -0.1261)$	$(s_3, 0.4758)$	$A_2 > A_3 > A_5 > A_1 > A_4$
(2,2,2,2)	$(s_3, 0.3470)$	$(s_4, 0.3247)$	$(s_4, -0.4792)$	$(s_2, -0.1261)$	$(s_3, 0.4758)$	$A_2 > A_3 > A_5 > A_1 > A_4$
(2,0,0,0)	$(s_4, -0.1833)$	$(s_4, 0.4123)$	$(s_4, -0.3159)$	$(s_2, 0.4430)$	$(s_4, -0.1637)$	$A_2 > A_5 > A_1 > A_3 > A_4$
(3,0,0,0)	$(s_4, -0.0694)$	$(s_4, 0.4284)$	$(s_4, -0.2989)$	$(s_3, -0.4552)$	$(s_4, -0.0815)$	$A_2 > A_1 > A_5 > A_3 > A_4$

Table 9. Ranking results of different parameters for 2TLPFWDMM operator.

$\boldsymbol{P}$	$\boldsymbol{s(A_1)}$	$\boldsymbol{s(A_2)}$	$\boldsymbol{s(A_3)}$	$\boldsymbol{s(A_4)}$	$\boldsymbol{s(A_5)}$	**Ordering**
(1,0,0,0)	$(s_3, 0.3898)$	$(s_4, 0.2906)$	$(s_4, -0.3602)$	$(s_2, -0.0033)$	$(s_4, -0.4952)$	$A_2 > A_3 > A_5 > A_1 > A_4$
(1,1,0,0)	$(s_4, -0.3881)$	$(s_4, 0.3900)$	$(s_4, -0.3259)$	$(s_2, -0.1615)$	$(s_4, -0.2855)$	$A_2 > A_5 > A_3 > A_1 > A_4$
(1,1,1,0)	$(s_4, -0.3047)$	$(s_4, 0.4535)$	$(s_4, -0.3066)$	$(s_2, -0.2167)$	$(s_4, -0.2249)$	$A_2 > A_5 > A_3 > A_1 > A_4$
(1,1,1,1)	$(s_4, -0.2488)$	$(s_5, -0.4833)$	$(s_4, -0.2916)$	$(s_2, 0.2570)$	$(s_4, -0.1859)$	$A_2 > A_5 > A_3 > A_1 > A_4$
(2,2,2,2)	$(s_4, -0.2488)$	$(s_5, -0.4833)$	$(s_4, -0.2916)$	$(s_2, 0.2570)$	$(s_4, -0.1859)$	$A_2 > A_5 > A_3 > A_1 > A_4$
(2,0,0,0)	$(s_3, 0.3041)$	$(s_4, 0.1821)$	$(s_4, -0.3813)$	$(s_2, -0.1017)$	$(s_3, 0.4148)$	$A_2 > A_3 > A_5 > A_1 > A_4$
(3,0,0,0,)	$(s_3, 0.2311)$	$(s_4, 0.0847)$	$(s_4, -0.4027)$	$(s_2, -0.1978)$	$(s_3, 0.3292)$	$A_2 > A_3 > A_5 > A_1 > A_4$

numbers. They can flexibly express cognitive information as well as effectively characterize the reliability of information. Therefore, it is of great significance to study Multiple Attribute Decision Making (MADM) methods with 2TLPFNs. This study investigated the MADM problems with 2TLPFNs. Then, the Muirhead Mean (MM) operator and Dual MM (DMM) operator were extended to propose some MM operators with 2TPFNs. The main characteristics of these operators were analyzed. Then, the 2-Tuple Linguistic Pythagorean Fuzzy Weighted MM (2TLPFWMM) and 2-Tuple Linguistic Pythagorean Fuzzy Weighted DMM (2TLPFWDMM) operators were applied to MADM problems with 2TPFNs. Finally, a practical example with green supplier selection was employed to show the developed methods. In future works, the extension and application of 2TPFNs need to be investigated in the other uncertain and fuzzy domains.

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References

1. Yager, R.R. "Pythagorean fuzzy subsets", *Proc. Joint IFSA World Congress NAFIPS Annual Meet.*, **1**, Edmonton, Canada, pp. 57–61 (2013).
2. Yager, R.R. "Pythagorean membership grades in multicriteria decision making", *IEEE Transactions on Fuzzy Systems*, **22**(4), pp. 958–965 (2013).
3. Zhang, X.L. and Xu, Z.S. "Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets", *International Journal of Intelligent Systems*, **29**(12), pp. 1061–1078 (2014).
4. Peng, X. and Yang, Y. "Some results for Pythagorean fuzzy sets", *International Journal of Intelligent Systems*, **30**(11), pp. 1133–1160 (2015).
5. Beliakov, G. and James, S. "Averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs", *FUZZ-IEEE*, **2014**, pp. 298–305 (2014).
6. Reformat, M.Z. and Yager, R.R. "Suggesting recommendations using Pythagorean fuzzy sets illustrated using Netflix movie data", *IPMU*, **2014**(1), pp. 546–556 (2014).
7. Gou, X.J., Xu, Z.S., and Ren, P.J. "The properties of continuous Pythagorean fuzzy information", *International Journal of Intelligent Systems*, **31**(5), pp. 401–424 (2016).
8. Garg, H. "A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making", *International Journal of Intelligent Systems*, **31**(9), pp. 886–920 (2016).
9. Zeng, S.Z., Chen, J.P., and Li, X.S. "A hybrid method for Pythagorean fuzzy multiple-criteria decision making", *International Journal of Information Technology and Decision Making*, **15**(2), pp. 403–422 (2016).
10. Garg, H. "A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem", *Journal of Intelligent and Fuzzy Systems*, **31**(1), pp. 529–540 (2016).
11. Ren, P.J., Xu, Z.S., and Gou, X.J. "Pythagorean fuzzy TODIM approach to multi-criteria decision making", *Applied Soft Computing*, **42**, pp. 246–259 (2016).
12. Wei, G.W. and Lu, M. "Pythagorean fuzzy Maclaurin symmetric mean operators in multiple attribute decision making", *International Journal of Intelligent Systems*, **33**(5), pp. 1043–1070 (2018).
13. Maclaurin, C. "A second letter to Martin Folkes, Esq.; concerning the roots of equations, with demonstration of other rules of algebra", *Philos Trans Roy Soc London Ser A*, **36**, pp. 59–96 (1729).
14. Wei, G.W. "Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making", *Journal of Intelligent and Fuzzy Systems*, **33**(4), pp. 2119–2132 (2017).
15. Wu, S.J. and Wei, G.W. "Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making", *International Journal of Knowledge-based and Intelligent Engineering Systems*, **21**(3), pp. 189–201 (2017).
16. Wei, G.W. and Lu, M. "Dual hesitant Pythagorean fuzzy Hamacher aggregation operators in multiple attribute decision making", *Archives of Control Sciences*, **27**(3), pp. 365–395 (2017).
17. Lu, M., Wei, G.W., Alsaadi, F.E., Hayat, T., and Alsaedi, A. "Hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making", *Journal of Intelligent and Fuzzy Systems*, **33**(2), pp. 1105–1117 (2017).
18. Wei, G.W., Lu, M, Tang, X.Y., and Wei, Y. "Pythagorean hesitant fuzzy hamacher aggregation operators and their application to multiple attribute decision making", *International Journal of Intelligent Systems*, **33**(6), pp. 1197–1233 (2018).
19. Gao, H., Lu, M, Wei, G.W., and Wei, Y. "Some novel Pythagorean fuzzy interaction aggregation operators in multiple attribute decision making", *Fundamenta Informaticae*, **159**(4), pp. 385–428 (2018).
20. Garg, H. "Generalized Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations and their application to decision making", *Journal of Experimental & Theoretical Artificial Intelligence*, **132**(4), pp. 763–794 (2018).
21. Wei, G.W. and Wei, Y. "Similarity measures of Pythagorean fuzzy sets based on cosine function and their applications", *International Journal of Intelligent Systems*, **33**(3), pp. 634–652 (2018).

22. Wei, G.W. and Lu, M. "Pythagorean fuzzy power aggregation operators in multiple attribute decision making", *International Journal of Intelligent Systems*, **33**(1), pp. 169–186 (2018).

23. Garg, H. "Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment", *International Journal of Intelligent Systems*, **33**(4), pp. 687–712 (2018).

24. Liang, D.C., Zhang, Y.R.J., Xu, Z.S., and Darko, A.P. "Pythagorean fuzzy Bonferroni mean aggregation operator and its accelerative calculating algorithm with the multithreading", *International Journal of Intelligent Systems*, **33**(3), pp. 615–633 (2018).

25. Garg, H. "A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision-making processes", *International Journal of Intelligent Systems*, **31**(12), pp. 1234–1252 (2016).

26. Wang, J., Gao, H., and Wei, G.W. "The generalized dice similarity measures for Pythagorean fuzzy multiple attribute group decision making", *International Journal of Intelligent Systems*, **34**(6), pp. 1158–1183 (2019).

27. Tang, X.Y., Wei, G.W., and Gao, H. "Models for multiple attribute decision making with interval-valued Pythagorean fuzzy muirhead mean operators and their application to green suppliers selection", *Informatica*, **30**(1), pp. 153–186 (2019).

28. Muhammad, S.A.K., Saleem, A., Muhammad, Y.A., Iqtadar, H., and Muhammad, F. "Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment", *Journal of Intelligent and Fuzzy Systems*, **34**(1), pp. 267–282 (2018).

29. Wan, S.P., Jin, Z., and Dong, J.Y. "Pythagorean fuzzy mathematical programming method for multi-attribute group decision making with Pythagorean fuzzy truth degrees", *Knowledge and Information Systems*, **55**(2), pp. 437–466 (2018).

30. Garg, H. "New exponential operational laws and their aggregation operators for interval-valued Pythagorean fuzzy multicriteria decision-making", *International Journal of Intelligent Systems*, **33**(3), pp. 653–683 (2018).

31. Garg, H. "A novel improved accuracy function for interval valued Pythagorean fuzzy sets and its applications in the decision-making process", *International Journal of Intelligent Systems*, **32**(12), pp. 1247–1260 (2017).

32. Garg, H. "A new improved score function of an interval-valued pythagorean fuzzy set based TOPSIS method", *International Journal for Uncertainty Quantification*, **7**(5), pp. 463–474 (2017).

33. Wang, P., Wang, J., Wei, G.W., and Wei, C. "Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications", *Mathematics*, **7**(4), p. 340 (2019).

34. Tang, M., Wang, J., Lu, J.P., Wei, G.W., Wei, C., and Wei, Y. "Dual hesitant Pythagorean fuzzy Heronian mean operators in multiple attribute decision making", *Mathematics*, **7**(4), p. 344 (2019).

35. Tang, X.Y. and Wei, G.W. "Multiple attribute decision-making with dual hesitant Pythagorean fuzzy information", *Cognitive Computation*, **11**(2), pp. 193–211 (2019).

36. Tang, X.Y. and Wei, G.W. "Models for green supplier selection in green supply chain management with Pythagorean 2–tuple linguistic information", *IEEE Access*, **6**, pp. 18042–18060 (2018).

37. Zhang, S.Q., Gao, H., Wei, G.W., Wei, Y., and Wei, C. "Evaluation based on distance from average solution method for multiple criteria group decision making under picture 2-tuple Linguistic environment", *Mathematics*, **7**(3), p. 243 (2019).

38. Deng, X.M., Wei, G.W., Gao, H., and Wang, J. "Models for safety assessment of construction project with some 2-tuple linguistic Pythagorean fuzzy Bonferroni mean operators", *IEEE Access*, **6**, pp. 52105–52137 (2018).

39. Herrera, F. and Martinez, L. "The 2-tuple linguistic computational model. Advantages of its linguistic description, accuracy and consistency", *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems*, **9**, pp. 33–48 (2001).

40. Herrera, F. and Martinez, L. "A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making", *IEEE Transactions on Systems Man and Cybernetics Part B-Cybernetics*, **31**, pp. 227–234 (2001).

41. Deng, X.M., Wang, J., Wei, G.W., and Lu, M. "Models for multiple attribute decision making with some 2-tuple linguistic Pythagorean fuzzy hamy mean operators", *Mathematics*, **6**(11), p. 236 (2018).

42. Muirhead, R.F. "Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters", *Proceedings of the Edinburgh Mathematical Society*, **21**(3), pp. 144–162 (1902).

43. Qin, J.Q. and Liu, X.W. "2-tuple linguistic Muirhead mean operators for multiple attribute group decision making and its application to supplier selection", *Kybernetes*, **45**(1), pp. 2–29 (2017).

44. Wang, R., Wang, J., Gao, H., and Wei, G.W. "Methods for MADM with picture fuzzy muirhead mean operators and their application for evaluating the financial investment risk", *Symmetry*, **11**(1), p. 6 (2019).

45. Hong, Z.Y., Rong, Y., Qin, Y., and Liu, Y. "Hesitant fuzzy dual Muirhead mean operators and its application to multiple attribute decision making", *Journal of Intelligent & Fuzzy Systems*, **35**(2), pp. 2161–2172 (2018).
46. Verma, M. and Rajasankar, J. "A thermodynamical approach towards group multi-criteria decision making (GMCDM) and its application to human resource selection", *Applied Soft Computing*, **52**, pp. 323–332 (2016).
47. Samanta, S. and Jana, D.K. "A multi-item transportation problem with mode of transportation preference by MCDM method in interval type-2 fuzzy environment", *Neural Computing and Applications*, **31**, pp. 605–617 (2017).
48. Sánchez-Lozano, J.M., Serna, J., and Dolón-Payán, A. "Evaluating military training aircrafts through the combination of multi-criteria decision making processes with fuzzy logic. A case study in the Spanish Air Force Academy", *Aerospace Science and Technology*, **42**, pp. 58–65 (2015).
49. Wu, S., Wang, J., Wei, G.W., and Wei, Y. "Research on construction engineering project risk assessment with some 2-tuple Linguistic neutrosophic hamy mean operators", *Sustainability*, **10**(5), p. 1536 (2018).
50. Lu, J.P., Tang, X.Y., Wei, G.W., Wei, C., and Wei, Y. "Bidirectional project method for dual hesitant Pythagorean fuzzy multiple attribute decision-making and their application to performance assessment of new rural construction", *International Journal of Intelligent Systems*, **34**(8), pp. 1920–1934 (2019).
51. Wei, G.W., Wei, C., Wang, J., Gao, H., and Wei, H. "Some q-rung orthopair fuzzy maclaurin symmetric mean operators and their applications to potential evaluation of emerging technology commercialization", *International Journal of Intelligent Systems*, **34**(1), pp. 50–81 (2019).
52. Wang, J., Wei, G.W., and Lu, M. "An extended VIKOR method for multiple criteria group decision making with triangular fuzzy neutrosophic numbers", *Symmetry-Basel*, **10**(10), p. 497 (2018).
53. Wei, G.W., Wei, C., Wu, J., and Wang, H.J. "Supplier selection of medical consumption products with a probabilistic linguistic MABAC method", *International Journal of Environmental Research and Public Health*, **16**(24), p. 5082 (2019).
54. He, T.T., Wei, G.W., Lu, J.P., Wei, C., and Lin, R. "Pythagorean 2-tuple Linguistic VIKOR method for evaluating human factors in construction project management", *Mathematics*, **7**(12), p. 1149 (2019).
55. Lu, J.P., Wei, C., Wu, J., and Wei, G.W. "TOPSIS method for probabilistic linguistic MAGDM with entropy weight and its application to supplier selection of new agricultural machinery products", *Entropy*, **21**(10), p. 953 (2019).
56. Wang, J., Wei, G.W., and Lu, M. "An extended VIKOR method for multiple criteria group decision making with triangular fuzzy neutrosophic numbers", *Symmetry*, **10**(10), p. 497 (2018).
57. Lu, J.P., Tang, X.Y., Wei, G.W., Wei, C., and Wei, Y. "Bidirectional project method for dual hesitant Pythagorean fuzzy multiple attribute decision-making and their application to performance assessment of new rural construction", *International Journal of Intelligent Systems*, **34**(8), pp. 1920–1934 (2019).
58. Wang, J., Gao, H., Wei, G.W., and Wei, Y. "Methods for multiple-attribute group decision making with q-rung interval-valued orthopair fuzzy information and their applications to the selection of green suppliers", *Symmetry*, **11**(1), p. 56 (2019).
59. Wang, J., Wei, G.W., and Gao, H. "Approaches to multiple attribute decision making with interval-valued 2-tuple linguistic pythagorean fuzzy information", *Mathematics*, **6**(10), p. 201 (2018).
60. He, T.T., Wei, G.W., Lu, J.P., Wei, C., and Lin, R. "Pythagorean 2-tuple Linguistic taxonomy method for supplier selection in medical instrument industries", *International Journal of Environmental Research and Public Health*, **16**(23), p. 4875 (2019).
61. Li, J. and Wang, J.Q. "An extended QUALIFLEX method under probability Hesitant fuzzy environment for selecting green suppliers", *International Journal of Fuzzy Systems*, **19**, pp. 1866–1879 (2017).
62. Liao, H.C., Wu, D., Huang, Y.L., Ren, P.J., Xu, Z.S., and Verma, M. "Green logistic provider selection with a hesitant fuzzy linguistic thermodynamic method integrating prospect theory and PROMETHEE", *Sustainability*, **10**(4), p. 1291 (2018).
63. Li, Z.X., Wei, G.W., Wang, R., Wu, J., Wei, C., and Wei, Y. "EDAS method for multiple attribute group decision making under q-rung orthopair fuzzy environment", *Technological and Economic Development of Economy*, **26**(1), pp. 86–102 (2020).
64. Wu, L.P., Wei, G.W., Wu, J., and Wei, C. "Some interval-valued intuitionistic fuzzy Dombi Heronian mean operators and their application for evaluating the ecological value of forest ecological tourism demonstration areas", *International Journal of Environmental Research and Public Health*, **17**(3), p. 829 (2020).
65. Gao, H., Ran, L.G., Wei, G.W., Wei, C., and Wu, J. "VIKOR method for MAGDM based on Q-rung interval-valued orthopair fuzzy information and its application to supplier selection of medical consumption products", *International Journal of Environmental Research and Public Health*, **17**(2), p. 525 (2020).
66. Wu, L.P., Wang, J., and Gao, H. "Models for competiveness evaluation of tourist destination with some interval-valued intuitionistic fuzzy Hamy mean operators", *Journal of Intelligent and Fuzzy Systems*, **36**(6), pp. 5693–5709 (2019).
67. Zhang, S.Q., Wei, G.W., Gao, H., Wei, C., and Wei, Y. "EDAS method for multiple criteria group decision making with picture fuzzy information and its application to green suppliers selections", *Technological and*

Economic Development of Economy, **25**(6), pp. 1123–1138 (2019).

68. Garg, H. "Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision making process", *International Journal of Intelligent Systems*, **33**(6), pp. 1234–1263 (2018).

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