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Estimation of general parameters under stratified adaptive cluster sampling based on dual use of auxiliary information

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Abstract. Auxiliary information is used mostly in conjunction with study variables to enhance the efficiency of estimators for population mean, total, and variance. Thompson introduced adaptive cluster sampling as an appropriate sampling scheme for rare and clustered populations. This paper presents difference-type and difference-cum-exponential-ratio-type estimators utilizing two auxiliary variables for estimating general parameters under stratified adaptive cluster sampling. The proposed estimators utilize auxiliary information in terms of ranks, variances, and means of auxiliary variables in hth stratum. Expressions for bias and mean square error of the proposed estimators are derived using first-order approximation. This numerical study aims to evaluate the performance of the proposed estimators.

1. Introduction

Thompson [1] introduced Adaptive Cluster Sampling (ACS) as an efficient sampling scheme under rare, hidden, and clustered population such as drug addicts, endangered species of animals, fisheries, contagious diseases, rare and precious plants, minerals, and natural resources. ACS begins by taking an initial sample using traditional sampling designs, e.g., simple random sampling with or without replacement, systematic and strip ACS, stratified sampling, inverse sampling, ranked set sampling, two-stage sampling, partial systematic sampling, double sampling, sampling via probability proportional to size, and simple Latin square sampling. Then, the sample using information of neighboring units, which satisfies the pre-specified condition, adaptively increases. Recent advances in the area of ACS include the works of Chutiman et al. [2], Gattone et al. [3], Yasmeen and Thompson [4], Qureshi et al. [5], Bak [6], Younis and Shabbir [7–10].

In case of a sufficient correlation between the study and auxiliary variables, auxiliary information is used to enhance the precision of estimators. Haq et al. [11] and Shabbir [12] propounded that ranks of auxiliary variables could also be used to increase the efficiency of estimators. In this article, difference-type and difference-cum-exponential-ratio-type estimators were presented utilizing two auxiliary variables to estimate general parameters under Stratified Adaptive Cluster Sampling (SACS). Estimators are proposed assuming that population parameters are known for one auxiliary variable (say z) and unknown for another auxiliary variable (say x). We adopt the two-phase sampling scheme using ACS as follows:

1. In Phase 1, a large sample of size n' is drawn and information on the auxiliary variables (x and z) is recorded;
2. In Phase 2, a sub sample of size n is drawn

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from phase one n' and information on the study variable (y) and the auxiliary variables (x and z) is accessible.

1.1. Symbols and Notations

Consider a finite population of N units partitioned into L strata such that:

$$\sum_{h=1}^{L} N_h = N.$$

Let y_{hi} and (x_{hi}, z_{hi}), be the observed values of the study variable y and the auxiliary variables (x, z), respectively, in the h^{th} stratum. Let $r(x_{hi})$, $r(z_{hi})$ be the ranks of two auxiliary variables (x, z) in the h^{th} stratum.

Let $\overline{w}_{yh}$, $\overline{w}_{xh}$, $\overline{w}_{zh}$, $\overline{r}(w_x)_h$, and $\overline{r}(w_z)_h$ be the sample means corresponding to population means $\overline{Y}_{wh}$, $\overline{X}_{wh}$, $\overline{Z}_{wh}$, $\overline{R}(w_x)_h$, and $\overline{R}(w_z)_h$, respectively, in the h^{th} stratum. Let $s^2_{w_{yh}}, s^2_{w_{xh}}$, and $s^2_{w_{zh}}$ be the sample variances corresponding to the population variances $S^2_{w_{yh}}, S^2_{w_{xh}}$, and $S^2_{w_{zh}}$, respectively, in the h^{th} stratum. Also, let $\hat{C}_{w_{yh}}, \hat{C}_{w_{xh}}$, and $\hat{C}_{w_{zh}}$ be the sample coefficients of variation corresponding to population coefficients of variation $C_{w_{yh}}, C_{w_{xh}}$, and $C_{w_{zh}}$, respectively, in the h^{th} stratum.

The following notations are used:

$$\overline{R}(w_u) = \sum_{h=1}^{L} \frac{N_h}{N} \overline{R}(w_u)_h, \overline{R}(w_u)_h = \frac{1}{N_h}\sum_{i=1}^{N_h} r(w_{ui})_h, \overline{r}(w_u)_h = \frac{1}{n_h}\sum_{i=1}^{n_h} r(w_{ui})_h,$$

$$S^2_{r(w_u)_h} = \frac{1}{N_h-1}\sum_{i=1}^{N_h}\left\{r(w_{ui})_h - \overline{R}(w_u)_h\right\}^2, C_{r(w_u)_h} = \frac{S_{r(w_u)_h}}{\overline{R}(w_u)_h}, \quad \forall u = x, z, y.$$

Error terms are defined as:

$$\zeta_{0h} = \frac{\overline{w}_{yh} - \overline{Y}_{wh}}{\overline{Y}_{wh}}, \qquad \zeta_{1h} = \frac{\overline{w}_{xh} - \overline{X}_{wh}}{\overline{X}_{wh}},$$

$$\zeta'_{1h} = \frac{\overline{w}'_{xh} - \overline{X}_{wh}}{\overline{X}_{wh}}, \qquad \zeta_{2h} = \frac{\overline{w}_{zh} - \overline{Z}_{wh}}{\overline{Z}_{wh}},$$

$$\zeta'_{2h} = \frac{\overline{w}'_{zh} - \overline{Z}_{wh}}{\overline{Z}_{wh}}, \qquad \zeta_{3h} = \frac{s^2_{w_{yh}} - S^2_{w_{yh}}}{S^2_{w_{yh}}},$$

$$\zeta_{4h} = \frac{s^2_{w_{xh}} - S^2_{w_{xh}}}{S^2_{w_{xh}}}, \qquad \zeta'_{4h} = \frac{s^{2'}_{w_{xh}} - S^2_{w_{xh}}}{S^2_{w_{xh}}},$$

$$\zeta_{5h} = \frac{s^2_{w_{zh}} - S^2_{w_{zh}}}{S^2_{w_{zh}}}, \qquad \zeta'_{5h} = \frac{s^{2'}_{w_{zh}} - S^2_{w_{zh}}}{S^2_{w_{zh}}},$$

$$\zeta_{6h} = \frac{\overline{r}(w_x)_h - \overline{R}(w_x)_h}{\overline{R}(w_x)_h}, \qquad \zeta'_{6h} = \frac{\overline{r}'(w_x)_h - \overline{R}(w_x)_h}{\overline{R}(w_x)_h},$$

$$\zeta_{7_h} = \frac{\overline{r}(w_z)_h - \overline{R}(w_z)_h}{\overline{R}(w_z)_h}, \qquad \zeta'_{7h} = \frac{\overline{r}'(w_z)_h - \overline{R}(w_z)_h}{\overline{R}(w_z)_h},$$

such that:

$$E(e_{ih}) = 0 \qquad \forall i = 0, 1, ..., 7$$

$$E(e'_{ih}) = 0 \qquad \forall\ i = 1, 2, 4, 5, 6, 7.$$

$$E(\zeta^2_{ih}) = \theta_h C^2_i, \qquad E(\zeta_i \zeta_j) = \theta_h C_{ijh},$$

$$\forall i = w_{yh}, w_{xh}, w_{zh}, r(w_{xh}), r(w_{zh}),$$

where $\zeta_{w_{yh}} = \zeta_{0h}, \zeta_{w_{xh}} = \zeta_{1h}, \zeta_{w_{zh}} = \zeta_{2h}, \zeta_{r(w_{xh})} = \zeta_{6h}, \zeta_{r(w_{xh})} = \zeta_{7_h}$. Also, we have:

$$E(\zeta^2_{3h}) = \theta_h \lambda^*_{40000h},$$

$$E(\zeta^2_{4h}) = \theta_h \lambda^*_{04000h},$$

$$E(\zeta^2_{5h}) = \theta_h \lambda^*_{00040h},$$

$$E(\zeta_{0h}\zeta_{3h}) = \theta_h C_{w_{yh}} \lambda_{30000h},$$

$$E(\zeta_{0h}\zeta_{4h}) = \theta_h C_{w_{yh}} \lambda_{12000h},$$

$$E(\zeta_{0h}\zeta_{5h}) = \theta_h C_{w_{yh}} \lambda_{10020h},$$

$$E(\zeta_{1h}\zeta_{3h}) = \theta_h C_{w_{xh}} \lambda_{21000h},$$

$$E(\zeta_{1h}\zeta_{4h}) = \theta_h C_{w_{xh}} \lambda_{03000h},$$

$$E(\zeta_{1h}\zeta_{5h}) = \theta_h C_{w_{xh}} \lambda_{01020h},$$

$$E(\zeta_{2h}\zeta_{3h}) = \theta_h C_{w_{zh}} \lambda_{20010h},$$

$$E(\zeta_{2h}\zeta_{4h}) = \theta_h C_{w_{zh}} \lambda_{02010h},$$

$$E(\zeta_{2h}\zeta_{5h}) = \theta_h C_{w_{zh}} \lambda_{00030h},$$

$$E(\zeta_{3h}\zeta_{4h}) = \theta_h \lambda^*_{22000h},$$

$$E(\zeta_{3h}\zeta_{5h}) = \theta_h \lambda^*_{20020h},$$

$$E(\zeta_{3h}\zeta_{6h}) = \theta_h C_{r(w_{xh})} \lambda_{20100h},$$

$$E(\zeta_{3h}\zeta_{7h}) = \theta_h C_{r(w_{zh})} \lambda_{20001h},$$

$$E(\zeta_{4h}\zeta_{5h}) = \theta_h \lambda^*_{02020h},$$

$$E(\zeta_{4h}\zeta_{6h}) = \theta_h C_{r(w_{xh})} \lambda_{02100h},$$

$$E(\zeta_{4h}\zeta_{7h}) = \theta_h C_{r(w_{zh})} \lambda_{02001h},$$

$$E(\zeta_{5h}\zeta_{6h}) = \theta_h C_{r(w_{xh})} \lambda_{00120h},$$

$$E(\zeta_{5h}\zeta_{7h}) = \theta_h C_{r(w_{zh})} \lambda_{00021h},$$

where:

$$E(\zeta'_{ih}\zeta'_{jh}) = E(\zeta_{ih}\zeta'_{jh}) = E(\zeta_{jh}\zeta'_{ih}) = \theta'_h E(\zeta_{jh}\zeta_{ih}),$$

$$E(\zeta'^2_{ih}) = E(\zeta_{ih}\zeta'_{ih}) = \theta'_h E(\zeta^2_{ih}), \forall i, j = 1, 2, 4, 5, 6, 7.$$

$$\theta_h = \left(\frac{1}{n_h} - \frac{1}{N_h}\right),$$

$$C_{w_y r(w_x)h} = C_{w_{yh}} C_{r(w_{xh})} \rho_{w_y r_{w_x} h},$$

$$C_{w_x r(w_z)h} = C_{w_{xh}} C_{r(w_{zh})} \rho_{w_x r_{w_z} h},$$

$$\theta'_h = \left(\frac{1}{n'_h} - \frac{1}{N_h}\right),$$

$$C_{w_y r(w_z)h} = C_{w_{yh}} C_{r(w_{zh})} \rho_{w_y r_{w_z} h},$$

$$C_{w_x r(w_x)h} = C_{w_{xh}} C_{r(w_{xh})} \rho_{w_x r_{w_x} h},$$

$$\theta''_h = \theta_h - \theta'_h,$$

$$C_{w_y w_x h} = C_{w_{yh}} C_{w_{xh}} \rho_{w_y w_x h},$$

$$C_{w_x w_z h} = C_{w_{xh}} C_{w_{zh}} \rho_{w_x w_z h},$$

$$\lambda_{rstlv_h} = \frac{\mu_{rstlv_h}}{\mu_{20000_h}^{\frac{r}{2}} \mu_{02000_h}^{\frac{s}{2}} \mu_{00200_h}^{\frac{t}{2}} \mu_{00020_h}^{\frac{l}{2}} \mu_{00002_h}^{\frac{v}{2}}},$$

$$C_{r(w_x)r(w_z)h} = C_{r(w_{xh})} C_{r(w_{zh})} \rho_{r_{w_x} r_{w_z} h},$$

$$\lambda^*_{rstlv_h} = \lambda_{rstlv_h} - 1,$$

$$C_{w_z r(w_z)h} = C_{w_{zh}} C_{r(w_{zh})} \rho_{w_z r_{w_z} h},$$

$$C_{w_z r(w_x)h} = C_{w_{zh}} C_{r(w_{xh})} \rho_{w_z r_{w_x} h},$$

$$\mu_{rstlv_h} = \frac{1}{N_h - 1} \sum_{i=1}^{N_h}$$

$$\left\{ \begin{matrix} (w_{yh_i} - \overline{Y}_{wh})^r (w_{xh_i} - \overline{X}_{wh})^s (r(w_{xi})_h - \overline{R}(w_x)_h)^t \\ (w_{zh_i} - \overline{Z}_{wh})^l (r(w_{zi})_h - \overline{R}(w_z)_h)^v \end{matrix} \right\}.$$

2. Existing estimators

2.1. Estimators for population mean

Some of the existing estimators for population mean under Simple Random Sampling (SRS) using two auxiliary variables are discussed in this section under SACS.

1. Usual sample mean in SACS is given by:

$$t_{S-1m} = \frac{1}{N} \sum_{h=1}^{L} N_h \overline{w}_{yh}. \quad (1)$$

The Mean Square Error (MSE) of t_{S-1m} to the first-order approximation is given by:

$$MSE(t_{S-1m}) \cong \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \theta_h \overline{Y}^2_{wh} C^2_{w_{yh}}. \quad (2)$$

2. Usual ratio estimator for population mean in SACS is given by:

$$t_{S-2m} = \frac{1}{N} \sum_{h=1}^{L} N_h \overline{w}_{yh} \left(\frac{\overline{w}'_{xh}}{\overline{w}_{xh}}\right) \left(\frac{\overline{Z}_{wh}}{\overline{w}'_{zh}}\right). \quad (3)$$

The bias and MSE of t_{S-2m} to first-order approximation are given by:

$$Bias(t_{S-2m}) \cong \frac{1}{N} \sum_{h=1}^{L} N_h \overline{Y}_{wh}$$

$$\left[\theta''_h (C^2_{w_{xh}} - C_{w_y w_x h}) + \theta'_h (C^2_{w_{zh}} - C_{w_y w_z h})\right], \quad (4)$$

and:

$$MSE(t_{S-2m}) \cong \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \overline{Y}^2_{wh}$$

$$\left[\begin{matrix} \theta_h C^2_{w_{yh}} + \theta'' (C^2_{w_{xh}} - 2C_{w_y w_x h}) \\ + \theta'_h (C^2_{w_{zh}} - 2C_{w_y w_z h}) \end{matrix} \right]. \quad (5)$$

3. Traditional exponential ratio-type estimator for population mean in SACS is given by:

$$t_{S-3m} = \frac{1}{N} \sum_{h=1}^{L} N_h \overline{w}_{yh} \exp\left(\frac{\overline{w}'_{xh} - \overline{w}_{xh}}{\overline{w}'_{xh} + \overline{w}_{xh}}\right)$$

$$\exp\left(\frac{\overline{Z}_{wh} - \overline{w}'_{zh}}{\overline{Z}_{wh} + \overline{w}'_{zh}}\right). \quad (6)$$

The bias and MSE of t_{S-3m} to first-order approximation are given by:

$$Bias(t_{S-3m}) \cong \frac{1}{N} \sum_{h=1}^{L} N_h \overline{Y}_{wh}$$

$$\left[\theta''_h \left(\frac{3C^2_{w_{xh}}}{8} - \frac{C_{w_y w_x h}}{2}\right) + \theta'_h \left(\frac{3C^2_{w_{zh}}}{8} - \frac{C_{w_y w_z h}}{2}\right)\right], \quad (7)$$

and:

$$MSE(t_{S-3m}) \cong \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \overline{Y}^2_{wh}$$

$$\left[\begin{matrix} \theta_h C^2_{w_{yh}} + \theta''_h \left(\frac{C^2_{w_{xh}}}{4} - C_{w_y w_x h}\right) \\ + \theta'_h \left(\frac{C^2_{w_{zh}}}{4} - C_{w_y w_z h}\right) \end{matrix} \right]. \quad (8)$$

4. Traditional difference-type estimator for population mean in SACS is given by:

$$t_{S-4m}=\frac{1}{N}\sum_{h=1}^{L}N_h\Big[\overline{w}_{yh}+k_{1h}\left(\overline{w}'_{xh}-\overline{w}_{xh}\right)+k_{2h}\left(\overline{Z}_{wh}-\overline{w}'_{zh}\right)\Big], \quad (9)$$

where k_{1h} and k_{2h} are constants. The MSE of t_{S-4m} to first-order approximation is given by:

$$MSE(t_{S-4m})_{\min}\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2\overline{Y}^2_{wh}C^2_{w_{yh}}\left(\theta_h-\theta''_h\rho^2_{w_yw_xh}-\theta'_h\rho^2_{w_yw_zh}\right), \quad (10)$$

where:

$$k_{1h_{opt}}=\frac{\overline{Y}_{wh}C_{w_{yh}}\rho_{w_yw_xh}}{\overline{X}_{wh}C_{w_{xh}}},$$

$$k_{2h_{opt}}=\frac{\overline{Y}_{wh}C_{w_{yh}}\rho_{w_yw_zh}}{\overline{Z}_{wh}C_{w_{zh}}}.$$

5. Based on Shabbir and Gupta [13], the exponential ratio-type estimator for population mean in SACS is given by:

$$t_{S-5m}=\frac{1}{N}\sum_{h=1}^{L}N_h\overline{w}_{yh}\exp\left(\frac{\xi_{3h}\left(\overline{w}'_{xh}-\overline{w}_{xh}\right)}{\overline{w}'_{xh}+(k_{3h}-1)\,\overline{w}_{xh}}\right)\exp\left(\frac{\xi_{4h}\left(\overline{w}'_{zh}-\overline{w}_{zh}\right)}{\overline{w}'_{zh}+(k_{4h}-1)\,\overline{w}_{zh}}\right)\exp\left(\frac{\xi_{5h}\left(\overline{w}'_{zh}-\overline{Z}_{wh}\right)}{\overline{w}'_{zh}+(k_{5h}-1)\,\overline{Z}_{wh}}\right), \quad (11)$$

where $(\xi_{3h},\xi_{4h},\xi_{5h})$ and (k_{3h},k_{4h},k_{5h}) are constants. The bias and minimum MSE of t_{S-5m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-5m})\cong\frac{1}{N}\sum_{h=1}^{L}N_h\overline{Y}_{wh}\Bigg[\theta'_h\left\{\frac{C_{w_yw_zh}}{k_{5h}}-\frac{C^2_{w_{zh}}}{2k^2_{5h}}\right\}+\theta''_h\Bigg\{\frac{C^2_{w_{xh}}}{k_{3h}}-\frac{C_{w_yw_xh}}{k_{3h}}-\frac{C^2_{w_{xh}}}{2k^2_{3h}}+\frac{C^2_{w_{zh}}}{k_{4h}}-\frac{C_{w_yw_zh}}{k_{4h}}-\frac{C^2_{w_{zh}}}{2k^2_{4h}}+\frac{C_{w_xw_zh}}{k_{3h}k_{4h}}\Bigg\}\Bigg], \quad (12)$$

and:

$$MSE(t_{S-5m})_{\min}\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2\overline{Y}^2_{wh}C^2_{w_{yh}}\Bigg[\theta_h-\theta'_h\rho^2_{w_yw_zh}-\frac{\theta''_h}{1-\rho^2_{w_xw_zh}}\Big\{\rho^2_{w_yw_xh}+\rho^2_{w_yw_zh}-2\rho_{w_yw_xh}\rho_{w_yw_zh}\rho_{w_xw_zh}\Big\}\Bigg], \quad (13)$$

where:

$$\left(\frac{\xi}{k}\right)_{3h_{opt}}=\frac{C_{w_{yh}}\left(\rho_{w_yw_xh}-\rho_{w_yw_zh}\rho_{w_xw_zh}\right)}{C_{w_{xh}}\left(1-\rho^2_{w_xw_zh}\right)},$$

$$\left(\frac{\xi}{k}\right)_{4h_{opt}}=\frac{C_{w_{yh}}\left(\rho_{w_yw_zh}-\rho_{w_yw_xh}\rho_{w_xw_zh}\right)}{C_{w_{zh}}\left(1-\rho^2_{w_xw_zh}\right)},$$

$$\left(\frac{\xi}{k}\right)_{5h_{opt}}=\frac{-C_{w_yw_zh}}{C^2_{w_{zh}}}.$$

6. Based on Gupta and Shabbir [14], the ratio-type estimator for population mean in SACS is given by:

$$t_{S-6m}=\frac{1}{N}\sum_{h=1}^{L}N_h\overline{w}_{yh}\left(\frac{\overline{w}'_{xh}}{\overline{w}_{xh}}\right)^{k_{6h}}\left(\frac{\overline{Z}_{wh}}{\overline{w}'_{zh}}\right)^{k_{7h}}, \quad (14)$$

where (k_{6h},k_{7h}) are constants. The bias and minimum MSE of t_{S-6m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-6m})\cong\frac{1}{N}\sum_{h=1}^{L}N_h\overline{Y}_{wh}\Bigg[\theta''_h\left\{\frac{k_{6h}\left(k_{6h}+1\right)}{2}C^2_{w_{xh}}-k_{6h}C_{w_yw_xh}\right\}+\theta'_h\left\{\frac{k_{7h}\left(k_{7h}+1\right)}{2}C^2_{w_{zh}}-k_{7h}C_{w_yw_zh}\right\}\Bigg], \quad (15)$$

and:

$$MSE(t_{S-6m})_{\min}\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2\overline{Y}^2_{wh}C^2_{w_{yh}}\left[\theta_h-\theta''_h\rho^2_{w_yw_xh}-\theta'_h\rho^2_{w_yw_zh}\right], \quad (16)$$

where:

$$k_{6h_{opt}}=\frac{C_{w_yw_xh}}{C^2_{w_{xh}}}\quad\text{and}\quad k_{7h_{opt}}=\frac{C_{w_yw_zh}}{C^2_{w_{zh}}}.$$

7. According to Singh et al. [15], the exponential ratio-type estimator for population mean in SACS is given by:

$$t_{S-7m}=\frac{1}{N}\sum_{h=1}^{L}N_h\overline{w}_{yh}\exp\left[k_{8h}\left(\frac{\overline{w}'_{xh}-\overline{w}_{xh}}{\overline{w}'_{xh}+\overline{w}_{xh}}\right)+k_{9h}\left(\frac{\overline{w}'_{zh}-\overline{Z}_{wh}}{\overline{w}'_{zh}+\overline{Z}_{wh}}\right)\right], \quad (17)$$

where (k_{8h},k_{9h}) are constants. The bias and minimum MSE of t_{S-7m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-7m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \overline{Y}_{wh}\Bigg[\theta''_h\Bigg\{\frac{k_{8h}^2}{8}C_{w_xh}^2 + \frac{k_{8h}}{4}C_{w_xh}^2 - \frac{k_{8h}}{2}C_{w_yw_xh}\Bigg\} + \theta'_h\Bigg\{\frac{k_{9h}^2}{8}C_{w_zh}^2 - \frac{k_{9h}}{4}C_{w_zh}^2 + \frac{k_{9h}}{2}C_{w_yw_zh}\Bigg\}\Bigg], \quad (18)$$

and:

$$MSE(t_{S-7m})_{\min} \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \overline{Y}_{wh}^2 C_{w_yh}^2 \left[\theta_h - \theta''_h\rho_{w_yw_xh}^2 - \theta'_h\rho_{w_yw_zh}^2\right], \quad (19)$$

where:

$$k_{8h_{opt}} = \frac{2C_{w_yw_xh}}{C_{w_xh}^2} \quad \text{and} \quad k_{9h_{opt}} = \frac{-2C_{w_yw_zh}}{C_{w_zh}^2}.$$

8. According to Choudhury and Singh [16], the ratio-type estimator for population mean in SACS is given by:

$$t_{S-8m} = \frac{1}{N}\sum_{h=1}^{L} N_h \overline{w}_{yh}\left[k_{10h}\left(\frac{\overline{w}'_{xh}}{\overline{w}_{xh}}\right)\left(\frac{\overline{Z}_{wh}}{\overline{w}'_{zh}}\right) + (1-k_{10h})\left(\frac{\overline{w}_{xh}}{\overline{w}'_{xh}}\right)\left(\frac{\overline{w}'_{zh}}{\overline{Z}_{wh}}\right)\right], \quad (20)$$

where k_{10h} is a constant. The bias and minimum MSE of t_{S-8m} at optimum value of constant to first order of approximation are given by:

$$Bias(t_{S-8m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \overline{Y}_{wh}\Bigg[\theta''_h\Bigg\{C_{w_yw_xh} + k_{10h}\left(C_{w_xh}^2 - 2C_{w_yw_xh}\right)\Bigg\} + \theta'_h\Bigg\{C_{w_yw_zh} + k_{10h}\left(C_{w_zh}^2 - 2C_{w_yw_zh}\right)\Bigg\}\Bigg], \quad (21)$$

and:

$$MSE(t_{S-8m})_{\min} \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \overline{Y}_{wh}^2\left(A_h - \frac{F_h^2}{B_h}\right), \quad (22)$$

where:

$$A_h = \theta_h C_{w_yh}^2 + F_h, \quad B_h = \theta''_h C_{w_xh}^2 + \theta'_h C_{w_zh}^2,$$

$$k_{10h_{opt}} = \frac{F_h}{2B_h},$$

$$F_h = \theta''_h\left(C_{w_yw_xh} + C_{w_xh}^2\right) + \theta'_h\left(C_{w_yw_zh} + C_{w_zh}^2\right).$$

9. Hamad et al. [17] presented the difference ratio-type estimator for population mean in SACS below:

$$t_{S-9m} = \frac{1}{N}\sum_{h=1}^{L} N_h\left[\overline{w}_{yh} + k_{11h}\left(\overline{w}'_{xh} - \overline{w}_{xh}\right)\right]\left[k_{12h}\frac{\overline{w}'_{zh}}{\overline{w}_{zh}} + (1-k_{12h})\frac{\overline{w}_{zh}}{\overline{w}'_{zh}}\right], \quad (23)$$

where (k_{11h}, k_{12h}) are constants. The bias and minimum MSE of t_{S-9m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-9m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h\Big[\overline{Y}_{wh}C_{w_yw_zh}\theta''_h + \overline{Y}_{wh}k_{12h}\theta''_h\left(C_{w_zh}^2 - 2C_{w_yw_zh}\right) - \overline{X}_{wh}k_{11h}\theta''_h C_{w_xw_zh} + 2k_{11h}k_{12h}\overline{X}_{wh}\theta''_h C_{w_xw_zh}\Big], \quad (24)$$

and:

$$MSE(t_{S-9m})_{\min} \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \overline{Y}_{wh}^2\left(A_h + \frac{F_hG_hH_h - D_hG_h^2 - B_hH_h^2}{4B_hD_h - F_h^2}\right), \quad (25)$$

where:

$$A_h = \theta_h C_{w_yh}^2 + \theta''_h\left(2C_{w_yw_zh} + C_{w_zh}^2\right),$$

$$B_h = 4\theta''_h C_{w_zh}^2, \quad F_h = 4\theta''_h C_{w_xw_zh},$$

$$D_h = \theta''_h C_{w_xh}^2, \quad G_h = 4\theta''_h\left(C_{w_yw_zh} - C_{w_zh}^2\right),$$

$$k_{11h_{opt}} = \frac{\overline{Y}_{w_h}\left(2H_hB_h - F_hG_h\right)}{\overline{X}_{wh}\left(4B_hD_h - F_h^2\right)},$$

$$H_h = 2\theta''_h\left(C_{w_yw_xh} - C_{w_xw_zh}\right),$$

$$k_{12h_{opt}} = \frac{2G_hD_h - F_hH_h}{4B_hD_h - F_h^2}.$$

10. Chutiman [18], Yadav et al. [19], and Qureshi et al. [20] proposed the ratio-type estimator for population mean in SACS as follows:

$$t_{S-10m_j} = \frac{1}{N}\sum_{h=1}^{L} N_h \overline{w}_{yh}\left(\frac{k_{13h_{xj}}\overline{w}'_{xh} + k_{14h_{xj}}}{k_{13h_{xj}}\overline{w}_{xh} + k_{14h_{xj}}}\right)\left(\frac{k_{13h_{zj}}\overline{Z}_{wh} + k_{14h_{zj}}}{k_{13h_{zj}}\overline{w}'_{zh} + k_{14h_{zj}}}\right), \quad (26)$$

Table 1. Combinations of constants for t_{S-10m_j}.

j	$k_{13h_{xj}}$	$k_{14h_{xj}}$	$k_{13h_{zj}}$	$k_{14h_{zj}}$
		[18]		
1	1	$C_{w_{xh}}$	1	$C_{w_{zh}}$
2	1	$\beta_{2(w_{xh})}$	1	$\beta_{2(w_{zh})}$
3	$\beta_{2(w_{xh})}$	$C_{w_{xh}}$	$\beta_{2(w_{zh})}$	$C_{w_{zh}}$
4	$C_{w_{xh}}$	$\beta_{2(w_{xh})}$	$C_{w_{zh}}$	$\beta_{2(w_{zh})}$
		[19]		
5	1	$\rho_{w_y w_x h}$	1	$\rho_{w_y w_z h}$
6	$\beta_{2(w_{xh})}$	$\beta_{1(w_{xh})}$	$\beta_{2(w_{zh})}$	$\beta_{1(w_{zh})}$
7	$\beta_{1(w_{xh})}$	$\beta_{2(w_{xh})}$	$\beta_{1(w_{zh})}$	$\beta_{2(w_{zh})}$
		[20]		
8	$MR_{(w_{xh})}$	$\beta_{1(w_{xh})}$	$MR_{(w_{zh})}$	$\beta_{1(w_{zh})}$
9	$MR_{(w_{xh})}$	$TM_{(w_{xh})}$	$MR_{(w_{zh})}$	$TM_{(w_{zh})}$
10	$HL_{(w_{xh})}$	$\beta_{1(w_{xh})}$	$HL_{(w_{zh})}$	$\beta_{1(w_{zh})}$
11	$HL_{(w_{xh})}$	$TM_{(w_{xh})}$	$HL_{(w_{zh})}$	$TM_{(w_{zh})}$

where $(k_{13h_{xj}}, k_{14h_{xj}}, k_{13h_{zj}}, k_{14h_{zj}})$ are the constants that assume different values for $j = 1, 2, ..., 11$ as given in Table 1.

The bias and MSE of t_{S-10m_j} to first-order approximation are given by:

$$Bias(t_{S-10m_j}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \overline{Y}_{wh} \begin{bmatrix} \theta''_h \left\{Q^2_{xh}C^2_{w_{xh}} - Q_{xh}C_{w_y w_x h}\right\} \\ +\theta'_h \left\{Q^2_{zh}C^2_{w_{zh}} - Q_{zh}C_{w_y w_z h}\right\} \end{bmatrix}, \quad (27)$$

and:

$$MSE(t_{S-10m_j}) \cong \overline{Y}^2_{wh}\Bigg[\theta_h C^2_{w_{yh}} + \theta''_h\Bigg\{Q^2_{xh}C^2_{w_{xh}} - 2Q_{xh}C_{w_y w_x h}\Bigg\} + \theta'_h\Bigg\{Q^2_{zh}C^2_{w_{zh}} - 2Q_{zh}C_{w_y w_z h}\Bigg\}\Bigg], \quad (28)$$

where:

$$Q_{xh} = \frac{k_{13h_{xj}}\overline{X}_{wh}}{k_{13h_{xj}}\overline{X}_{wh} + k_{14h_{xj}}},$$

$$Q_{zh} = \frac{k_{13h_{zj}}\overline{Z}_{wh}}{k_{13h_{zj}}\overline{Z}_{wh} + k_{14h_{zj}}}.$$

11. Vishwakarma and Gangele [21] presented the exponential ratio-type estimator for population mean in SACS:

$$t_{S-11m} = \frac{1}{N}\sum_{h=1}^{L} N_h \overline{w}_{yh} \exp \left(\frac{\frac{\overline{w}'_{xh}}{k_{15h}\overline{w}'_{zh}+k_{16h}}\left(k_{15h}\overline{Z}_{wh} + k_{16h}\right) - \overline{w}_{xh}}{\frac{\overline{w}'_{xh}}{k_{15h}\overline{w}'_{zh}+k_{16h}}\left(k_{15h}\overline{Z}_{wh} + k_{16h}\right) + \overline{w}_{xh}}\right), \quad (29)$$

where (k_{15h}, k_{16h}) are constants. The bias and minimum MSE of t_{S-11m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-11m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \overline{Y}_{wh} \Bigg[\theta''_h\left\{\frac{3}{8}Q^2_h C^2_{w_{xh}} - \frac{1}{2}Q_h C_{w_y w_x h}\right\} - \theta'_h\Bigg\{\frac{1}{8}Q^2_h C^2_{w_{zh}} - \frac{1}{2}Q_h C_{w_y w_z h} + \frac{1}{2}Q_h C_{w_x w_z h}\Bigg\}\Bigg], \quad (30)$$

and:

$$MSE(t_{S-11m}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N^2_h \overline{Y}^2_{wh}\Bigg[\theta_h C^2_{w_{yh}} - \theta'_h C^2_{w_{yh}}\rho^2_{w_y w_z h} + \theta''_h\Bigg(\frac{C^2_{w_{xh}}}{4} - C_{w_y w_x h}\Bigg)\Bigg], \quad (31)$$

where:

$$Q_h = \frac{k_{15h}\overline{Z}_{wh}}{k_{15h}\overline{Z}_{wh} + k_{16h}},$$

$$Q_{h_{opt}} = \frac{2C_{w_{yh}}\rho_{w_y w_z h}}{C_{w_{zh}}}.$$

12. Singh and Khalid [22] gave the exponential ratio-type estimator for population mean in SACS below:

$$t_{S-12m} = \frac{1}{N}\sum_{h=1}^{L} N_h \overline{w}_{yh} \exp \left(\frac{\frac{\overline{w}'_{xh}}{\overline{Z}_{wh}}\overline{Z}^*_{wh} - \overline{w}_{xh}}{\frac{\overline{w}'_{xh}}{\overline{Z}_{wh}}\overline{Z}^*_{wh} + \overline{w}_{xh}}\right), \quad (32)$$

where $\overline{Z}^*_{wh} = \frac{N_h\overline{Z}_{wh} - n'_h\overline{w}'_{zh}}{N_h - n'_h}$. The bias and MSE of t_{S-12m} to first-order approximation are given by:

$$Bias(t_{S-12m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \overline{Y}_{wh} \left[\theta''_h\left\{\frac{3}{8}C^2_{w_{xh}} + \frac{1}{2}C_{w_y w_x h}\right\} + \theta'_h\left\{\frac{a^2_h}{8}C^2_{w_{zh}} + \frac{a_h}{2}C_{w_y w_z h}\right\}\right], \quad (33)$$

and:

$$MSE(t_{S-12m}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \overline{Y}_{wh}^2 \left[\theta_h C_{w_{yh}}^2 + \theta_h'' \left(\frac{C_{w_{xh}}^2}{4} - C_{w_y w_x h}\right) + \theta_h' a_h \left(\frac{C_{w_{zh}}^2}{4} - C_{w_y w_x h}\right)\right], \tag{34}$$

where $a_h = \frac{n_h'}{N_h - n_h'}$.

13. Khan and Al-Hossain [23] proposed a difference-type estimator for population mean in SACS:

$$t_{S-13m} = \frac{1}{N}\sum_{h=1}^{L} N_h \left[\overline{w}_{yh} + k_{17h}\left(\frac{\overline{w}_{xh}'}{\overline{w}_{zh}'}\overline{Z}_{wh} - \overline{w}_{xh}\right) + k_{18h}\left(\frac{\overline{w}_{xh}'}{\overline{w}_{xh}}\overline{Z}_{wh} - \overline{w}_{zh}\right)\right], \tag{35}$$

where (k_{17h}, k_{18h}) are constants. The bias and minimum MSE of t_{S-13m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-13m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \overline{Y}_{wh} \left[k_{17h}\overline{X}_{wh}\theta_h' \left(C_{w_{zh}}^2 - C_{w_x w_z h}\right) + k_{18h}\overline{Z}_{wh}\theta_h'' C_{w_{xh}}^2\right], \tag{36}$$

and:

$$MSE(t_{S-13m}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \overline{Y}_{wh}^2 \left[\theta_h C_{w_{yh}}^2 - \frac{1}{A_h B_h - E_h^2} \left\{B_h C_h^2 + A_h D_h^2 - 2 D_h E_h C_h\right\}\right], \tag{37}$$

where:

$$A_h = \theta_h'' C_{w_{xh}}^2 + \theta_h' C_{w_{zh}}^2,$$

$$B_h = A_h + 2\theta_h'' C_{w_x w_z h},$$

$$E_h = A_h + \theta_h'' C_{w_x w_z h},$$

$$C_h = \theta_h'' C_{w_{yh}} + \theta_h' C_{w_y w_z h},$$

$$k_{17h_{opt}} = \frac{\overline{Y}_{wh}(B_h C_h - D_h E_h)}{\overline{X}_{wh}(A_h B_h - E_h^2)},$$

$$D_h = \theta_h'' C_{w_y w_x h} + \theta_h' C_{w_y w_z h},$$

$$k_{18h_{opt}} = \frac{\overline{Y}_{wh}(A_h D_h - C_h E_h)}{\overline{Z}_{wh}(A_h B_h - E_h^2)}.$$

14. Based on Khan [24], the exponential-type estimator for population mean in SACS is given by:

$$t_{S-14m} = \frac{1}{N}\sum_{h=1}^{L} N_h \left[\begin{array}{c} \overline{w}_{yh} \exp\left(\frac{\overline{w}_{xh}' - \overline{w}_{xh}}{\overline{w}_{xh}' + \overline{w}_{xh}}\right)^{k_{19h}} \\ + k_{20h}\left\{\overline{w}_{xh}' \exp\left(\frac{\overline{Z}_{wh} - \overline{w}_{zh}'}{\overline{Z}_{wh} + \overline{w}_{zh}'}\right) - \overline{w}_{xh}\right\} \end{array}\right], \tag{38}$$

where (k_{19h}, k_{20h}) are constants. The bias and minimum MSE of t_{S-14m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-14m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \overline{Y}_{wh} \left[\theta_h''\left\{k_{19h}\left(\frac{1}{4}C_{w_{xh}}^2 - \frac{1}{2}C_{w_y w_x h}\right) + \frac{1}{8}k_{19h}^2 C_{w_{xh}}^2\right\} + \theta_h' k_{20h}\overline{X}_{wh} \left\{\frac{3}{8}C_{w_{zh}}^2 + \frac{1}{2}C_{w_x w_z h}\right\}\right], \tag{39}$$

and:

$$MSE(t_{S-14m}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \overline{Y}_{wh}^2 \left[\theta_h C_{w_{yh}}^2 - \theta_h'' \rho_{w_y w_x h}^2 - \theta_h' \rho_{w_y w_z h}^2\right], \tag{40}$$

where:

$$k_{19h_{opt}} = \frac{4 C_{w_{yh}}}{C_{w_{xh}} C_{w_{zh}}}\left(\frac{1}{2}\rho_{w_y w_x h} C_{w_{yh}} + \rho_{w_y w_z h} C_{w_{xh}}\right),$$

$$k_{20h_{opt}} = \frac{2\rho_{w_y w_z h} C_{w_{yh}}}{\overline{X}_{wh} C_{w_{zh}}}.$$

15. According to Singh et al. [25], the ratio-type estimator for population mean in SACS is given by:

$$t_{S-15m_j} = \frac{1}{N}\sum_{h=1}^{L} N_h \overline{w}_{yh}\left(\frac{k_{21h_{xj}}\overline{w}_{xh}' + k_{22h_{xj}}\overline{w}_{xh}}{k_{23h_{xj}}\overline{w}_{xh} + k_{24h_{xj}}\overline{w}_{xh}'}\right) \left(\frac{k_{21h_{zj}}\overline{Z}_{wh} + k_{22h_{zj}}\overline{w}_{zh}'}{k_{23h_{zj}}\overline{w}_{zh}' + k_{24h_{zj}}\overline{Z}_{wh}}\right), \tag{41}$$

where $(k_{21h_{ij}}, k_{22h_{ij}}, k_{23h_{ij}}, k_{24h_{ij}}, \forall i = x, z)$ are constants that assume different values for $j = 1, 2, 3$, as given in Table 2.

The bias and MSE of t_{S-15m_j} to the first-order approximation are given by:

Table 2. Combinations of constants for t_{S-15m_j}.

j	$k_{21h_{xj}}$	$k_{22h_{xj}}$	$k_{23h_{xj}}$	$k_{24h_{xj}}$	$k_{21h_{zj}}$	$k_{22h_{zj}}$	$k_{23h_{zj}}$	$k_{24h_{zj}}$
1	$C^2_{w_xh}$	$-\rho_{w_yw_xh}$	$C^2_{w_xh}$	$-\rho_{w_yw_xh}$	$C^2_{w_zh}$	$-\rho_{w_yw_zh}$	$C^2_{w_zh}$	$-\rho_{w_yw_zh}$
2	$\beta_{2(w_xh)}$	$-C_{w_yh}$	$\beta_{2(w_xh)}$	$-C_{w_yh}$	$\beta_{2(w_zh)}$	$-C_{w_yh}$	$\beta_{2(w_zh)}$	$-C_{w_yh}$
3	C_{w_yh}	C_{w_xh}	C_{w_yh}	C_{w_xh}	C_{w_yh}	C_{w_zh}	C_{w_yh}	C_{w_zh}

$$Bias(t_{S-15m_j}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h\overline{Y}_{wh} \Big[-\theta'_h C_{w_yw_xh}\delta_{1h} - \theta'_h C_{w_yw_zh}\delta_{3h} - \theta_h C_{w_yw_xh} q_{x3h} + \theta'_h C_{w_xw_zh} \{q_{x3h}q_{z3h} + q_{x4h}q_{z3h} - q_{x1h}q_{z3h} - q_{x3h}q_{z2h} - q_{x4h}q_{z2h} + q_{x1h}q_{z2h}\} + \theta'_h C^2_{w_xh}\{q^2_{x4h} - q_{x1h}q_{x4h} - q_{x2h}q_{x4h} - q_{x1h}q_{x3h} + 2q_{x3h}q_{x4h}\} + \theta C^2_{w_xh} \{q^2_{x3h} - q_{x2h}q_{x3h}\} + \theta'_h C^2_{w_zh} \{q^2_{z3h} - q_{z2h}q_{z3h}\}\Big], \tag{42}$$

and:

$$MSE(t_{S-15m_j}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2\overline{Y}^2_{wh} \Big[\theta_h\left\{C^2_{w_yh} + \delta^2_{2h}C^2_{w_xh} - 2\delta_{2h}C_{w_yw_xh}\right\} + \theta'_h\Big\{C^2_{w_xh}\left(\delta^2_{1h} + 2\delta_{1h}\delta_{2h}\right) + C^2_{w_zh}\delta^2_{3h} - 2\delta_{1h}C_{w_yw_xh} - 2\delta_{3h}C_{w_yw_zh} + C_{w_xw_zh} (2\delta_{1h}\delta_{3h} + 2\delta_{2h}\delta_{3hh})\Big\}\Big], \tag{43}$$

where:

$$q_{x1h} = \frac{k_{21h_{xj}}}{k_{21h_{xj}} + k_{22h_{xj}}}, \quad q_{x2h} = \frac{k_{22h_{xj}}}{k_{21h_{xj}} + k_{22h_{xj}}},$$

$$q_{x3h} = \frac{k_{23h_{xj}}}{k_{23h_{xj}} + k_{24h_{xj}}}, \quad q_{x4h} = \frac{k_{24h_{xj}}}{k_{23h_{xj}} + k_{24h_{xj}}},$$

$$q_{z2h} = \frac{k_{22h_{zj}}}{k_{21h_{zj}} + k_{22h_{zj}}}, \quad q_{z3h} = \frac{k_{23h_{zj}}}{k_{23h_{zj}} + k_{24h_{zj}}},$$

$$\delta_{1h} = q_{x4h} - q_{x1h}, \quad \delta_{2h} = q_{x3h} - q_{x2h},$$

$$\delta_{3h} = q_{z3h} - q_{z2h}.$$

16. Shabbir and Gupta [26] presented the difference-cum-exponential ratio-type estimator for population mean in SACS as follows:

$$t_{S-16m} = \frac{1}{N}\sum_{h=1}^{L} N_h \Big[\{k_{25h}\overline{w}_{yh} + k_{26h} (\overline{w}'_{xh} - \overline{w}_{xh}) + k_{27h}\left(\overline{Z}_{wh} - \overline{w}'_{zh}\right)\} \exp\left(\frac{\overline{w}'_{xh} - \overline{w}_{xh}}{\overline{w}'_{xh} + \overline{w}_{xh}}\right)\Big], \tag{44}$$

where $(k_{25h}, k_{26h}, k_{27h})$ are constants. The bias and minimum MSE of t_{S-16m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-16m}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \Big[k_{25h}\overline{Y}_{wh}\theta''_h\left\{\frac{3}{8}C^2_{w_xh} - \frac{1}{2}C_{w_yw_xh}\right\} + k_{26h}\overline{X}_{wh}\frac{\theta''_h}{2}C^2_{w_xh} + \overline{Y}_{wh}(k_{25h} - 1)\Big], \tag{45}$$

and:

$$MSE\left(t_{S-16m}\right)_{\min} \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2\frac{\overline{Y}^2_{wh}}{L_h} \left(L_h - H_h^2 - \frac{E_h^2 L_h}{B_h}\right), \tag{46}$$

where:

$$A_h = 1 + \theta_h C^2_{w_yh} + \theta''_h\left(C^2_{w_xh} - 2C_{w_yw_xh}\right),$$

$$F_h = \theta''_h\left(C^2_{w_xh} - C_{w_yw_xh}\right),$$

$$E_h = \frac{\theta''_h}{2}C^2_{w_xh}, \qquad k_{25h_{opt}} = \frac{H_h}{L_h},$$

$$k_{26h_{opt}} = \frac{\overline{Y}_{wh}\left(E_hL_h - F_hH_h\right)}{\overline{X}_{wh}B_hL_h},$$

$$B_h = \theta''_h C^2_{w_xh},$$

$$D_h = 1 + \theta''_h\left(\frac{3}{8}C^2_{w_xh} - \frac{1}{2}C_{w_yw_xh}\right),$$

$$H_h = D_h - \frac{E_h F_h}{B_h}, \qquad C_h = \theta'_h C^2_{w_z h},$$

$$G_h = \theta'_h C_{w_y w_z h}, \qquad L_h = A_h - \frac{F^2_h}{B_h} - \frac{G^2_h}{C_h},$$

$$k_{27h_{opt}} = \frac{\overline{Y}_{wh} G_h H_h}{\overline{Z}_{wh} C_h L_h}.$$

17. According to Muneer et al. [27], difference-cum-exponential estimators for population mean in SACS are given by:

$$t_{S-17m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[\left\{ k_{28h} \overline{w}_{yh} + k_{29h} (\overline{w}'_{xh} - \overline{w}_{xh}) \right\} \left\{ 2 - \exp \left(\frac{\overline{w}'_{zh} - \overline{Z}_{wh}}{\overline{w}'_{zh} + \overline{Z}_{wh}} \right) \right\} \right], \tag{47}$$

and:

$$t_{S-18m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[\left\{ k_{30h} \overline{w}_{yh} + k_{31h} (\overline{w}'_{xh} - \overline{w}_{xh}) \right\} \exp \left(\frac{\overline{Z}_{wh} - \overline{w}'_{zh}}{\overline{Z}_{wh} + \overline{w}'_{zh}} \right) \right], \tag{48}$$

where $(k_{28h}, k_{29h}, k_{30h}, k_{31h})$ are constants. The bias and minimum MSE of t_{S-17m} and t_{S-18m} at optimum values of constants to first-order of approximation are given by:

$$Bias(t_{S-17m}) \cong \frac{1}{N} \sum_{h=1}^{L} N_h \left[\overline{Y}_{wh} (k_{28h} - 1) + k_{28h} \overline{Y}_{wh} \theta'_h \left(\frac{1}{8} C^2_{w_z h} - \frac{1}{2} C_{w_y w_z h} \right) \right], \tag{49}$$

$$Bias(t_{S-18m}) \cong \frac{1}{N} \sum_{h=1}^{L} N_h \left[\overline{Y}_{wh} (k_{30h} - 1) + k_{30h} \overline{Y}_{wh} \theta'_h \left(\frac{3}{8} C^2_{w_z h} - \frac{1}{2} C_{w_y w_z h} \right) \right], \tag{50}$$

$$MSE(t_{S-17m})_{\min} \cong \sum_{h=1}^{L} N^2_h \left[\overline{Y}^2_{wh} \frac{M_{1h} - B^2_{1h}}{M_{1h}} \right], \tag{51}$$

and:

$$MSE(t_{S-18m})_{\min} \cong \frac{1}{N^2} \sum_{h=1}^{L} N^2_h \left[\overline{Y}^2_w \frac{M_{2h} - B^2_{2h}}{M_{2h}} \right], \tag{52}$$

where:

$$A_{1h} = 1 + \theta_h C^2_{w_y h} + \theta'_h \left(\frac{1}{2} C^2_{w_z h} - 2 C_{w_y w_z h} \right),$$

$$B_{1h} = 1 + \theta'_h \left(\frac{1}{8} C^2_{w_z h} - \frac{1}{2} C_{w_y w_z h} \right),$$

$$k_{28h_{opt}} = \frac{B_{1h}}{M_{1h}},$$

$$A_{2h} = 1 + \theta_h C^2_{w_y h} + \theta'_h \left(C^2_{w_z h} - 2 C_{w_y w_z h} \right),$$

$$B_{2h} = 1 + \theta'_h \left(\frac{3}{8} C^2_{w_z h} - \frac{1}{2} C_{w_y w_z h} \right),$$

$$D_h = C^2_{w_x h},$$

$$F_h = C_{w_y w_x h}, \qquad k_{30h_{opt}} = \frac{B_{2h}}{M_{2h}},$$

$$M_{1h} = A_{1h} - \theta''_h \frac{F^2_h}{D_h},$$

$$k_{29h_{opt}} = \frac{\overline{Y}_{wh} F_h B_{1h}}{\overline{X}_{wh} D_h M_{1h}},$$

$$M_{2h} = A_{2h} - \theta''_h \frac{F^2_h}{D_h},$$

$$k_{31h_{opt}} = \frac{\overline{Y}_{wh} F_h B_{2h}}{\overline{X}_{wh} D_h M_{2h}}.$$

18. Shabbir [12] found the difference-type estimator for population mean in SACS below:

$$t_{S-19m} = \frac{1}{N} \sum_{h=1}^{L} N_h \Big[\overline{w}_{yh} + k_{32h} \{ \overline{w}'_{xh} - \overline{w}_{xh} \} + k_{33h} \{ \overline{r}'(w_x)_h - \overline{r}(w_x)_h \} + k_{34h} \{ \overline{Z}_{wh} - \overline{w}'_{zh} \} + k_{35h} \{ \overline{R}(w_z)_h - \overline{r}'(w_z)_h \} \Big], \tag{53}$$

where $(k_{32h}, k_{33h}, k_{34h}, k_{35h})$ are constants. The bias and minimum MSE of t_{S-19m} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-19m}) = 0, \tag{54}$$

and:

$$MSE(t_{S-19m})_{\min} \cong \frac{1}{N^2} \sum_{h=1}^{L} N^2_h \overline{Y}^2_{wh} C^2_{w_y h} \left[\theta_h - \theta''_h R_{xh} - \theta'_h R_{zh} \right], \tag{55}$$

where:

$$R_{xh} = \frac{\rho^2_{w_y w_x h} + \rho^2_{w_y r(w_x)h} - 2\rho_{w_y w_x h}\rho_{w_y r(w_x)h}\rho_{w_x r(w_x)h}}{1-\rho^2_{w_x r(w_x)h}},$$

$$R_{zh} = \frac{\rho^2_{w_y w_z h} + \rho^2_{w_y r(w_z)h} - 2\rho_{w_y w_z h}\rho_{w_y r(w_z)h}\rho_{w_z r(w_z)h}}{1-\rho^2_{w_z r(w_z)h}},$$

$$k_{32h_{opt}} = \frac{S_{w_y h}\left\{\rho_{w_y w_x h} - \rho_{w_x r(w_x)h}\rho_{w_y r(w_x)h}\right\}}{S_{w_y h}\left(1-\rho^2_{w_x r(w_x h}\right)},$$

$$k_{33h_{opt}} = \frac{S_{w_y h}\left\{\rho_{w_y r(w_x)h} - \rho_{w_x r(w_x)h}\rho_{w_y w_x h}\right\}}{S_{r(w_x)h}\left(1-\rho^2_{w_x r(w_x)h}\right)},$$

$$k_{34h_{opt}} = \frac{S_{w_y h}\left\{\rho_{w_y w_z h} - \rho_{w_z r(w_z)h}\rho_{w_y r(w_z)h}\right\}}{S_{w_z h}\left(1-\rho^2_{w_z r(w_z)h}\right)},$$

$$k_{35h_{opt}} = \frac{S_{w_y h}\left\{\rho_{w_y r(w_z)h} - \rho_{w_z r(w_z)h}\rho_{w_y w_z h}\right\}}{S_{r(w_z)h}\left(1-\rho^2_{w_z r(w_z)h}\right)}.$$

2.2. Estimators for population variance

Some of the existing estimators for population variance using two auxiliary variables are discussed in this section under SACS and SRS. Usual sample variance in SACS is given by:

1. Usual sample variance in SACS is given by:

$$t_{S-1v} = \frac{1}{N}\sum_{h=1}^{L} N_h s^2_{w_y h}. \tag{56}$$

The MSE of t_{S-1v} to first-order approximation is given by:

$$MSE(t_{S-1v}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \theta_h S^4_{w_y h}\lambda^*_{400h}. \tag{57}$$

2. Usual ratio estimator (using variance of auxiliary variables) for population variance in SACS is given as follows:

$$t_{S-2v} = \frac{1}{N}\sum_{h=1}^{L} N_h s^2_{w_y h}\left(\frac{s^{2\prime}_{w_x h}}{s^2_{w_x h}}\right)\left(\frac{S^2_{w_z h}}{s^{2\prime}_{w_z h}}\right). \tag{58}$$

The bias and MSE of t_{S-2v} to first-order approximation are given by:

$$Bias(t_{S-2v}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h S^2_{w_y h}$$

$$\left[\theta''_h\left(\lambda^*_{040h} - \lambda^*_{220h}\right) + \theta'_h\left(\lambda^*_{004h} - \lambda^*_{202h}\right)\right], \tag{59}$$

and:

$$MSE(t_{S-2v}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 S^4_{w_y h}$$

$$\left[\begin{array}{c}\theta_h\lambda^*_{400h} + \theta''_h\left(\lambda^*_{040h} - 2\lambda^*_{220h}\right)\\ +\theta'_h\left(\lambda^*_{004h} - 2\lambda^*_{202h}\right)\end{array}\right]. \tag{60}$$

3. Usual ratio estimator (using mean of auxiliary variables) for population variance in SACS is given by:

$$t_{S-3v} = \frac{1}{N}\sum_{h=1}^{L} N_h s^2_{w_y h}\left(\frac{\overline{w}'_{xh}}{\overline{w}_{xh}}\right)\left(\frac{\overline{Z}_{wh}}{\overline{w}'_{zh}}\right). \tag{61}$$

The bias and MSE of t_{S-3v} to first-order approximation are given by:

$$Bias(t_{S-3v}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h S^2_{w_y h}$$

$$\left[\theta''_h\left(C^2_{w_x h} - C_{w_x h}\lambda_{210h}\right) + \theta'_h\left(C^2_{w_z h} - C_{w_z h}\lambda_{201h}\right)\right], \tag{62}$$

and:

$$MSE(t_{S-3v}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 S^4_{w_y h}$$

$$\left[\begin{array}{c}\theta_h\lambda^*_{400h} + \theta''_h\left(C^2_{w_x h} - 2C_{w_x h}\lambda_{210h}\right)\\ +\theta'_h\left(C^2_{w_z h} - 2C_{w_z h}\lambda_{201h}\right)\end{array}\right]. \tag{63}$$

4. Traditional exponential ratio-type estimator for population variance in SACS is given by:

$$t_{S-4v} = \frac{1}{N}\sum_{h=1}^{L} N_h s^2_{w_y h}\exp\left(\frac{s^{2\prime}_{w_x h} - s^2_{w_x h}}{s^{2\prime}_{w_x h} + s^2_{w_x h}}\right)\exp\left(\frac{S^2_{w_z h} - s^{2\prime}_{w_z h}}{S^2_{w_z h} + s^{2\prime}_{w_z h}}\right). \tag{64}$$

The bias and MSE of t_{S-4v} to first-order approximation are given by:

$$Bias(t_{S-4v}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h S^2_{w_y h}$$

$$\left[\theta''_h\left(\frac{3\lambda^*_{040h}}{8} - \frac{\lambda^*_{220h}}{2}\right) + \theta'_h\left(\frac{3\lambda^*_{004h}}{8} - \frac{\lambda^*_{202h}}{2}\right)\right], \tag{65}$$

and:

$$MSE(t_{S-4v}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 S_{w_{yh}}^4 \left[\begin{array}{c}\theta_h\lambda_{400h}^* + \theta_h''\left(\frac{\lambda_{040h}^*}{4} - \lambda_{220h}^*\right) \\ +\theta_h'\left(\frac{\lambda_{004h}^*}{4} - \lambda_{202h}^*\right)\end{array}\right]. \quad (66)$$

5. Traditional difference-type estimator for population variance in SACS is given by:

$$t_{S-5v} = \frac{1}{N}\sum_{h=1}^{L} N_h\left[s_{w_{yh}}^2 + p_{1h}\left(s_{w_{xh}}^{2\prime} - s_{w_{xh}}^2\right) + p_{2h}\left(S_{w_{zh}}^2 - s_{w_{zh}}^{2\prime}\right)\right], \quad (67)$$

where p_{1h} and p_{2h} are constants. The minimum MSE of t_{S-5v} at optimum values of constants to first-order approximation is given by:

$$MSE(t_{S-5v})_{\min} \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 S_{w_{yh}}^4 \left[\theta_h\lambda_{400h}^* - \theta_h''\left(\frac{\lambda_{220h}^{*2}}{\lambda_{040h}^*}\right) - \theta_h'\left(\frac{\lambda_{202h}^{*2}}{\lambda_{004h}^*}\right)\right], \quad (68)$$

where:

$$p_{1h_{opt}} = \frac{S_{w_{yh}}^2\lambda_{220h}^*}{S_{w_{xh}}^2\lambda_{040h}^*}, \qquad p_{2h_{opt}} = \frac{S_{w_{yh}}^2\lambda_{202h}^*}{S_{w_{zh}}^2\lambda_{004h}^*}.$$

6. Singh et al. [28] presented the exponential ratio-type estimator for population variance in SACS below:

$$t_{S-6v} = \frac{1}{N}\sum_{h=1}^{L} N_h s_{w_{yh}}^2\left[p_{3h}\exp\left(\frac{s_{w_{xh}}^{2\prime} - s_{w_{xh}}^2}{s_{w_{xh}}^{2\prime} + s_{w_{xh}}^2}\right) + (1-p_{3h})\exp\left(\frac{S_{w_{zh}}^2 - s_{w_{zh}}^{2\prime}}{S_{w_{zh}}^2 + s_{w_{zh}}^{2\prime}}\right)\right], \quad (69)$$

where p_{3h} is a constant. The bias and minimum MSE of t_{S-6v} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-6v}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h S_{w_{yh}}^2 \left[p_{3h}\left\{\theta_h''\left(\frac{3}{8}\lambda_{04000h}^* - \frac{1}{2}\lambda_{22000h}^*\right) - \theta_h'\left(\frac{3}{8}\lambda_{00040h}^* - \frac{1}{2}\lambda_{20020h}^*\right)\right\} + \theta_h'\left(\frac{3}{8}\lambda_{00040h}^* - \frac{1}{2}\lambda_{20020h}^*\right)\right], \quad (70)$$

and:

$$MSE(t_{S-6v})_{\min} \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 S_{w_{yh}}^4\left(D_h - \frac{B_h^2}{A_h}\right), \quad (71)$$

where:

$$A_h = \frac{\theta_h''}{4}\lambda_{04000h}^* + \frac{\theta_h'}{4}\lambda_{00040h}^*, \qquad p_{3h_{opt}} = \frac{-B_h}{A_h}$$

$$B_h = \theta_h'\left(\frac{1}{2}\lambda_{20020h}^* - \frac{1}{4}\lambda_{00040h}^*\right) - \frac{\theta_h''}{2}\lambda_{22000h}^*,$$

$$D_h = \theta_h\lambda_{40000h}^* + \theta_h'\left(\frac{1}{4}\lambda_{00040h}^* - \lambda_{20020h}^*\right).$$

7. As proposed by Singh and Solanki [29], the ratio-type estimator for population variance in SACS is given by:

$$t_{S-7v_j} = \frac{1}{N}\sum_{h=1}^{L} N_h s_{w_{yh}}^2\left(\frac{p_{4h_{xj}}s_{w_{xh}}^{2\prime} + p_{5h_{xj}}}{p_{4h_{xj}}s_{w_x}^2 + p_{5h_{xj}}}\right)\left(\frac{p_{4h_{zj}}S_{w_zh}^2 + p_{5h_{zj}}}{p_{4h_{zj}}s_{w_z}^{2\prime} + p_{5h_{zj}}}\right), \quad (72)$$

where $(p_{4h_{xj}}, p_{5h_{xj}}, p_{4h_{zj}}, p_{5h_{zj}})$ are constants that assume different values for $j = 1, 2, 3, 4$ as given in Table 3.

The bias and MSE of t_{S-7v_j} to first-order approximation are given by:

$$Bias(t_{S-7v_j}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h S_{w_{yh}}^2 \left[\theta_h''\left\{Q_{xh}^2\lambda_{04000h}^* - Q_{xh}\lambda_{22000h}^*\right\} + \theta_h'\left\{Q_{zh}^2\lambda_{00040h}^* - Q_{zh}\lambda_{20020h}^*\right\}\right], \quad (73)$$

and:

$$MSE(t_{S-7v_j}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 S_{w_{yh}}^4 \left[\theta_h''\left\{Q_{xh}^2\lambda_{04000h}^* - 2Q_{xh}\lambda_{22000h}^*\right\}\right.$$

Table 3. Combinations of constants for t_{S-7v_j}.

j	1	2	3	4
$p_{4h_{xj}}$	1	1	$C_{w_{xh}}$	$\beta_{2(w_{xh})}$
$p_{5h_{xj}}$	$C_{w_{xh}}$	$\beta_{2(w_{xh})}$	$\beta_{2(w_{xh})}$	$C_{w_{xh}}$
$p_{4h_{zj}}$	1	1	$C_{w_{xh}}$	$\beta_{2(w_{zh})}$
$p_{5h_{zj}}$	$C_{w_{zh}}$	$\beta_{2(w_{zh})}$	$\beta_{2(w_{xh})}$	$C_{w_{zh}}$

$$+\theta_h\lambda^*_{40000h}+\theta'_h\left\{Q^2_{zh}\lambda^*_{00040h}-2Q_{zh}\lambda^*_{20020h}\right\}\Bigg], \quad (74)$$

where:

$$Q_{xh}=\frac{p_{4h_{xj}}S^2_{w_{xh}}}{p_{4h_{xj}}S^2_{w_{xh}}+p_{5h_{xj}}},$$

$$Q_{zh}=\frac{p_{4h_{zj}}S^2_{w_{zh}}}{p_{4h_{zj}}S^2_{w_{zh}}+p_{5h_{zj}}}.$$

8. Olufadi and Kadilar [30] presented the ratio-type estimator for population variance in SACS as follows:

$$t_{S-8v}=\frac{1}{N}\sum_{h=1}^{L}N_h s^2_{w_{yh}}\left(\frac{s^{2'}_{w_{xh}}}{s^2_{w_{xh}}}\right)^{p_{6h}}\left(\frac{S^2_{w_{zh}}}{s^{2'}_{w_{zh}}}\right)^{p_{7h}}, \quad (75)$$

where p_{6h} and p_{7h} are constants. The bias and minimum MSE of t_{S-8v} at optimum values of constants to first-order approximation are given by:

$$Bias(t_{S-8v})\cong\frac{1}{N}\sum_{h=1}^{L}N_h S^2_{w_{yh}}\left[\theta''_h\left\{\frac{p_{6h}(p_{6h}+1)}{2}\lambda^*_{04000h}-p_{6h}\lambda^*_{22000h}\right\}+\theta'_h\left\{\frac{p_{7h}(p_{7h}+1)}{2}\lambda^*_{00040h}-p_{7h}\lambda^*_{20020h}\right\}\right], \quad (76)$$

and:

$$MSE(t_{S-8v})_{\min}\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2 S^4_{w_{yh}}\left[\theta_h\lambda^*_{40000h}-\theta''_h\frac{\lambda^{*2}_{22000h}}{\lambda^*_{04000h}}-\theta'_h\frac{\lambda^{*2}_{20020h}}{\lambda^*_{00040h}}\right], \quad (77)$$

where:

$$p_{6h_{opt}}=\frac{\lambda^*_{22000h}}{\lambda^*_{04000h}},\qquad p_{7h_{opt}}=\frac{\lambda^*_{20020h}}{\lambda^*_{00040h}}.$$

9. Amin et al. [31] presented the ratio-type estimators for population variance in SACS below:

$$t_{S-9v}=\frac{1}{N}\sum_{h=1}^{L}N_h s^2_{w_{yh}}\left\{\frac{s^{2'}_{w_{xh}}}{p_{8h}s^{2'}_{w_{xh}}+(1-p_{8h})s^2_{w_{xh}}}\right\}^{p_{9h}}\left\{\frac{S^2_{w_{zh}}}{p_{10h}S^2_{w_{zh}}+(1-p_{10h})s^{2'}_{w_{zh}}}\right\}^{p_{11h}}, \quad (78)$$

and:

$$t_{S-10v}=\frac{1}{N}\sum_{h=1}^{L}N_h p_{12h}s^2_{w_{yh}}\left(\frac{s^{2'}_{w_{xh}}}{s^2_{w_{xh}}}\right)\left(\frac{S^2_{w_{zh}}}{s^{2'}_{w_{zh}}}\right). \quad (79)$$

The minimum MSE values for t_{S-9v} and t_{S-10v} at optimum values of constants to first-order approximation are given by:

$$MSE(t_{S-9v})_{\min}\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2 S^4_{w_{yh}}\left[\theta_h\lambda^*_{40000h}-\theta''_h\frac{\lambda^{*2}_{22000h}}{\lambda^*_{04000h}}-\theta'_h\frac{\lambda^{*2}_{20020h}}{\lambda^*_{00040h}}\right], \quad (80)$$

and:

$$MSE(t_{S-10v})_{\min}\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2 S^4_{w_{yh}}\left(1-\frac{M^2_{1h}}{L_{1h}}\right), \quad (81)$$

where:

$$p_{9h}(1-p_{8h})_{opt}=\frac{\lambda^*_{22000h}}{\lambda^*_{04000h}},$$

$$p_{11h}(1-p_{10h})_{opt}=\frac{\lambda^*_{20020h}}{\lambda^*_{00040h}},$$

$$L_{1h}=1+\theta_h\lambda^*_{40000h}+\theta''_h(3\lambda^*_{04000h}-4\lambda^*_{22000h})+\theta'_h(3\lambda^*_{00040h}-4\lambda^*_{20020h}),$$

$$M_{1h}=1+\theta''_h(\lambda^*_{04000h}-\lambda^*_{22000h})+\theta'_h(\lambda^*_{00040h}-\lambda^*_{20020h}),$$

$$p_{12h_{opt}}=\frac{M_{1h}}{L_{1h}}.$$

3. Proposed estimators

3.1. Difference-type estimator

The following difference-type estimator is proposed for general parameters under SACS:

$$\hat{\tau}_{S-(\alpha,\beta)yP1}=\frac{1}{N}\sum_{h=1}^{L}N_h\Big[\hat{\tau}_{(\alpha,\beta)yh}+f_{1h}\{\overline{w}'_{xh}-\overline{w}_{xh}\}+f_{2h}\{s^{2'}_{w_{xh}}-s^2_{w_{xh}}\}+\{\overline{r}'(w_x)_h-\overline{r}(w_x)_h\}+f_{3h}\{\overline{Z}_{wh}-\overline{w}'_{zh}\}+f_{4h}\{S^2_{w_{zh}}-s^{2'}_{w_{zh}}\}+\{\overline{R}(w_z)_h-\overline{r}'(w_z)_h\}\Big], \quad (82)$$

where $f_{ih}(i=1,2,3,4)$ are constants whose values are to be determined. Estimators for population mean (t_{S-P1m}) and variance (t_{S-P1v}) can be obtained by substituting ($\alpha=1,\beta=0$) and ($\alpha=0,\beta=2$) in Eq. (82), respectively. Rewriting Eq. (82) in terms of errors, we get:

$$\hat{\tau}_{S-(\alpha,\beta)yP1} - \tau_{(\alpha,\beta)y} \cong \frac{1}{N}\sum_{h=1}^{L} N_h \tau_{(\alpha,\beta)yh} \Bigg[\Bigg\{\alpha\zeta_{0h} + \frac{\beta}{2}\zeta_{3h} + \frac{\alpha(\alpha-1)}{2}\zeta_{0h}^2 + \frac{\beta(\beta-2)}{8}\zeta_{3h}^2 + \frac{\alpha\beta}{2}\zeta_{0h}\zeta_{3h}\Bigg\} + f_{1h}\overline{X}_{wh}(\zeta'_{1h}-\zeta_{1h}) + f_{2h}S^2_{w_{xh}}(\zeta'_{4h}-\zeta_{4h}) + \overline{R}(w_x)_h(\zeta'_{6h}-\zeta_{6h}) - f_{3h}\overline{Z}'_{wh}\zeta_{2h} - f_{4h}S^2_{w_{zh}}\zeta'_{5h} - \overline{R}(w_z)_h\zeta'_{7h}\Bigg]. \quad (83)$$

Taking expectations of both sides, we get:

$$Bias(\hat{\tau}_{S-(\alpha,\beta)yP1}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \tau_{(\alpha,\beta)y}\theta_h \Bigg\{\frac{\alpha(\alpha-1)}{2}C^2_{w_{yh}} + \frac{\beta(\beta-2)}{8}\lambda^*_{40000h} + \frac{\alpha\beta}{2}C_{w_{yh}}\lambda_{30000h}\Bigg\}. \quad (84)$$

Bias of the proposed estimator for population mean (t_{S-P1m}) and variance (t_{S-P1v}) can be obtained by substituting ($\alpha = 1, \beta = 0$) and ($\alpha = 0, \beta = 2$) in Eq. (84), respectively. Squaring Eq. (83) and considering first-order approximation, we get:

$$[\hat{\tau}_{S-(\alpha,\beta)yP1} - \tau_{(\alpha,\beta)y}]^2 \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \Bigg[\tau_{(\alpha,\beta)yh}\left(\alpha\zeta_{0h} + \frac{\beta}{2}\zeta_{3h}\right) + f_{1h}\overline{X}_{wh}(\zeta'_{1h}-\zeta_{1h}) + f_{2h}S^2_{w_{xh}}(\zeta'_{4h}-\zeta_{4h}) + \overline{R}(w_x)_h(\zeta'_{6h}-\zeta_{6h}) - f_{3h}\overline{Z}'_{wh}\zeta_{2h} - f_{4h}S^2_{w_{zh}}\zeta'_{5h} - \overline{R}(w_z)_h\zeta'_{7h}\Bigg]^2. \quad (85)$$

Taking expectations of both sides, we get:

$$MSE(\hat{\tau}_{S-(\alpha,\beta)yP1}) \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2\tau^2_{(\alpha,\beta)yh}\theta_h A_{yh} + \theta'_h\{f^2_{3h}A_{zh} + f^2_{4h}B_{zh} - 2f_{3h}D_{zh} - 2f_{4h}E_{zh} + 2f_{3h}f_{4h}F_{zh} + G_{zh}\} + \theta''_h\{f^2_{1h}A_{xh} + f^2_{2h}B_{xh} - 2f_{1h}D_{xh} - 2f_{2h}E_{xh} + 2f_{1h}f_{2h}F_{xh} + G_{xh}\}, \quad (86)$$

where:

$$A_{yh} = \alpha^2 C^2_{w_{yh}} + \frac{\beta^2}{4}\lambda^*_{40000h} + \alpha\beta C_{w_{yh}}\lambda_{30000h},$$

$$A_{xh} = \overline{X}^2_{wh}C^2_{w_{xh}},$$

$$B_{xh} = S^4_{w_{xh}}\lambda^*_{04000h},$$

$$F_{xh} = \overline{X}_{wh}S^2_{w_{xh}}C_{w_{xh}}\lambda_{03000h},$$

$$D_{xh} = \tau_{(\alpha,\beta)yh}\overline{X}_{wh}\left(\alpha C_{w_y w_x h} + \frac{\beta}{2}C_{w_{xh}}\lambda_{21000h}\right) - \overline{X}_{wh}\overline{R}(w_x)_h C_{w_x r(w_x)h},$$

$$E_{xh} = \tau_{(\alpha,\beta)yh}S^2_{w_{xh}}\left(\alpha C_{w_{yh}}\lambda_{12000h} + \frac{\beta}{2}\lambda^*_{22000h}\right) - S^2_{w_{xh}}\overline{R}(w_x)_h C_{r(w_{xh})}\lambda_{02100h},$$

$$G_{xh} = \overline{R}(w_x)^2_h C^2_{r(w_{xh})} - 2\tau_{(\alpha,\beta)yh}\overline{R}(w_x)_h\left(\alpha C_{w_y r(w_x)h} + \frac{\beta}{2}C_{r(w_{xh})}\lambda_{20100h}\right),$$

$$A_{zh} = \overline{Z}^2_{wh}C^2_{w_{zh}},$$

$$B_{zh} = S^4_{w_{zh}}\lambda^*_{00040h},$$

$$F_{zh} = \overline{Z}_{wh}S^2_{w_{zh}}C_{w_{zh}}\lambda_{00030h},$$

$$D_{zh} = \tau_{(\alpha,\beta)yh}\overline{Z}_{wh}\left(\alpha C_{w_y w_z h} + \frac{\beta}{2}C_{w_{zh}}\lambda_{20010h}\right) - \overline{Z}_{wh}\overline{R}(w_z)_h C_{w_z r(w_z)h},$$

$$E_{zh} = \tau_{(\alpha,\beta)yh}S^2_{w_{zh}}\left(\alpha C_{w_{yh}}\lambda_{10020h} + \frac{\beta}{2}\lambda^*_{20020h}\right) - S^2_{w_{zh}}\overline{R}(w_z)_h C_{r(w_{zh})}\lambda_{00021h},$$

$$G_{zh} = \overline{R}(w_z)^2_h C^2_{r(w_{zh})} - 2\tau_{(\alpha,\beta)yh}\overline{R}(w_z)_h\left(\alpha C_{w_y r(w_z)h} + \frac{\beta}{2}C_{r(w_{zh})}\lambda_{20001h}\right).$$

From (Eq. 86), the optimum values of $f_{ih}(i = 1, 2, 3, 4)$ are:

$$f_{1h_{opt}} = \frac{B_{xh}D_{xh} - E_{xh}F_{xh}}{A_{xh}B_{xh} - F^2_{xh}},$$

$$f_{2h_{opt}} = \frac{A_{xh}E_{xh} - F_{xh}D_{xh}}{A_{xh}B_{xh} - F^2_{xh}},$$

$$f_{3h_{opt}} = \frac{B_{zh}D_{zh} - E_{zh}F_{zh}}{A_{zh}B_{zh} - F_{zh}^2},$$

$$f_{4h_{opt}} = \frac{A_{zh}E_{zh} - F_{zh}D_{zh}}{A_{zh}B_{zh} - F_{zh}^2}.$$

By substituting optimum values of $f_{ih}(i = 1, 2, 3, 4)$ in Eq. (86), the minimum MSE of the proposed difference-type estimator for general parameters is obtained as follows:

$$MSE(\hat{\tau}_{S-(\alpha,\beta)yP1})_{\min} \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \left[\tau^2_{(\alpha,\beta)yh}\theta_h A_{yh} + \theta''_h Q_{xh} + \theta'_h Q_{zh}\right], \quad (87)$$

where:

$$Q_{ih} = G_{ih} + \frac{2D_{ih}E_{ih}F_{ih} - B_{ih}D_{ih}^2 - A_{ih}E_{ih}^2}{A_{ih}B_{ih} - F_{ih}^2}, \quad \forall i = x, z.$$

Minimum MSE of the proposed estimator for population mean (t_{S-P1m}) and variance (t_{S-P1v}) can be obtained by substituting ($\alpha = 1, \beta = 0$) and ($\alpha = 0, \beta = 2$) in Eq. (87), respectively.

3.2. *Difference-cum-exponential-ratio-type estimator*

The following difference-cum-exponential-ratio-type estimator is proposed for general parameters under SACS:

$$\hat{\tau}_{S-(\alpha,\beta)yP2} = \frac{1}{N}\sum_{h=1}^{L} N_h \Bigg[\{g_{1h}\hat{\tau}_{(\alpha,\beta)yh} + g_{2h}\left(\overline{w}'_{xh} - \overline{w}_{xh}\right) + g_{3h}\left(\overline{Z}_{wh} - \overline{w}'_{zh}\right)\} \exp\left(\frac{s^{2'}_{w_{xh}} - s^2_{w_{xh}}}{s^{2'}_{w_{xh}} + s^2_{w_{xh}}}\right) \exp\left(\frac{\overline{r}'(w_x)_h - \overline{r}(w_x)_h}{\overline{r}'(w_x)_h + \overline{r}(w_x)_h}\right)\Bigg], \quad (88)$$

where $g_{ih}(i = 1, 2, 3)$ are constants whose values are to be determined. Estimators for population mean (t_{S-P2m}) and population variance (t_{S-P2v}) can be obtained by substituting ($\alpha = 1, \beta = 0$) and ($\alpha = 0, \beta = 2$) in Eq. (88), respectively. Rewriting Eq. (88) in terms of errors and considering first order approximation, we get:

$$\hat{\tau}_{S-(\alpha,\beta)yP2} \cong \frac{1}{N}\sum_{h=1}^{L} N_h \Bigg[\Bigg\{g_{1h}\tau_{(\alpha,\beta)yh} \left(1 + \alpha\zeta_{0h} + \frac{\beta}{2}\zeta_{3h} + \frac{\alpha(\alpha-1)}{2}\zeta_{0h}^2 + \frac{\alpha\beta}{2}\zeta_{0h}\zeta_{3h} + \frac{\beta(\beta-2)}{8}\zeta_{3h}^2\right) + g_{2h}\overline{X}_{wh}\left[\zeta'_{1h} - \zeta_{1h}\right] - g_{3h}\overline{Z}'_{wh}\zeta_{2h}\Bigg\} \left\{1 + \frac{\zeta'_{4h} - \zeta_{4h}}{2} + \frac{3\left(\zeta^2_{4h} - \zeta^{2'}_{4h}\right)}{8}\right\} \left\{1 + \frac{\zeta'_{6h} - \zeta_{6h}}{2} + \frac{3\left(\zeta^2_{6h} - \zeta^{2'}_{6h}\right)}{8}\right\}\Bigg]. \quad (89)$$

Taking expectations of both sides, we get:

$$Bias(\hat{\tau}_{S-(\alpha,\beta)yP2}) \cong \frac{1}{N}\sum_{h=1}^{L} N_h \left[g_{1h}\tau_{(\alpha,\beta)yh}\left(\theta''_h Q_{1h} + \theta_h T_{1h}\right) + g_{2h}\overline{X}_{wh}\frac{\theta''_h}{2}U_{1h} + \tau_{(\alpha,\beta)yh}\left(g_{1h} - 1\right)\right], \quad (90)$$

where:

$$Q_{1h} = \frac{3}{8}\lambda^*_{04000h} + \frac{3}{8}C^2_{r(w_{xh})} - \frac{\alpha}{2}\left(C_{w_{yh}}\lambda_{12000h} + C_{w_y r(w_x)h}\right) + \frac{1}{4}C_{r(w_{xh})}\lambda_{02100h} - \frac{\beta}{4}\left(\lambda^*_{22000h} + C_{r(w_{xh})}\lambda_{20100h}\right),$$

$$T_{1h} = \frac{\alpha(\alpha-1)}{2}C^2_{w_{yh}} + \frac{\beta(\beta-2)}{8}\lambda^*_{40000h} + \frac{\alpha\beta}{2}C_{w_{yh}}\lambda_{30000h},$$

$$U_{1h} = C_{w_{xh}}\lambda_{03000h} + C_{w_x r(w_x)h}.$$

Bias of the estimator for population mean (t_{S-P2m}) and population variance (t_{S-P2v}) can be obtained by substituting ($\alpha = 1, \beta = 0$) and ($\alpha = 0, \beta = 2$) in Eq. (90), respectively. Squaring Eq. (89) and considering first-order approximation, we have:

$$[\hat{\tau}_{S-(\alpha,\beta)yP2} - \tau_{(\alpha,\beta)y}]^2 \cong \frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \Bigg[g^2_{1h}\tau^2_{(\alpha,\beta)yh}\left\{\frac{\zeta'_{4h} - \zeta_{4h}}{2} + \frac{\zeta'_{6h} - \zeta_{6h}}{2} + \alpha\zeta_{0h} + \frac{\beta}{2}\zeta_{3h}\right\}^2 - 2g_h g_{3h}\overline{X}_{wh}\overline{Z}_{wh}$$

$$\{\zeta'_{1h}\zeta'_{2h}-\zeta'_{1h}\zeta_{2h}\}+g_{2h}^2\overline{X}_{wh}^2\{\zeta'_{1h}-\zeta_{1h}\}^2$$

$$+g_{3h}^2\overline{Z}_{wh}^2\zeta_{4h}^{2'}+\tau_{(\alpha,\beta)yh}^2(g_{1h}-1)^2$$

$$-2g_{1h}g_{3h}\tau_{(\alpha,\beta)yh}\overline{Z}_{wh}\{\alpha\zeta'_{0h}\zeta_{2h}$$

$$+\frac{\beta}{2}\zeta_{3h}\zeta'_{2h}\}+2g_{1h}g_{2h}\overline{X}_{wh}\tau_{(\alpha,\beta)yh}$$

$$\left\{\frac{\zeta_{1h}\zeta_{4h}-\zeta_{4h}\zeta'_{1h}}{2}+\frac{\zeta_{1h}\zeta_{6h}-\zeta_{6h}\zeta'_{1h}}{2}\right.$$

$$+\alpha\left(\zeta_{0h}\zeta'_{1h}-\zeta_{0h}\zeta_{1h}\right)+\frac{\beta}{2}$$

$$\left.\left(\zeta_{3h}\zeta'_{1h}-\zeta_{3h}\zeta_{1h}\right)\right\}+2g_{1h}\left(g_{1h}-1\right)$$

$$\tau_{(\alpha,\beta)yh}^2\left\{\frac{3\left(\zeta_{4h}^2-\zeta_{4h}^{2'}\right)}{8}+\frac{\alpha}{2}\right.$$

$$\left(\zeta_{0h}\zeta'_{4h}-\zeta_{0h}\zeta_{4h}+\zeta_{0h}\zeta'_{6h}-\zeta_{0h}\zeta_{6h}\right)$$

$$+\frac{\beta}{4}\left(\zeta_{3h}\zeta'_{4h}-\zeta_{3h}\zeta_{4h}+\zeta_{3h}\zeta'_{6h}-\zeta_{3h}\zeta_{6h}\right)$$

$$+\frac{\alpha(\alpha-1)}{2}\zeta_{0h}^2+\frac{3\left(\zeta_{6h}^2-\zeta_{6h}^{2'}\right)}{8}$$

$$+\frac{\beta(\beta-2)}{8}\zeta_{3h}^2+\frac{\alpha\beta}{2}\zeta_{0h}\zeta_{3h}$$

$$\left.+\frac{\zeta_{4h}\zeta_{6h}-\zeta_{4h}\zeta'_{6h}}{4}\right\}+2g_{2h}\left(g_{1h}-1\right)$$

$$\overline{X}_{wh}\tau_{(\alpha,\beta)yh}\left\{\frac{\zeta_{1h}\zeta_{4h}-\zeta_{1h}\zeta'_{4h}}{2}\right.$$

$$\left.\left.+\frac{\zeta_{1h}\zeta_{6h}-\zeta_{1h}\zeta'_{6h}}{2}\right\}\right]. \quad (91)$$

Taking expectations of both sides, we have:

$$MSE(\hat{\tau}_{S-(\alpha,\beta)yP2})\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2$$

$$\left[\tau_{(\alpha,\beta)yh}^2+g_{1h}^2A_{1h}+g_{2h}^2B_{1h}+g_{3h}^2D_{1h}\right.$$

$$-2g_{1h}E_{1h}-2g_{2h}F_{1h}+2g_{1h}g_{2h}H_{1h}$$

$$\left.-2g_{1h}g_{3h}J_{1h}\right], \quad (92)$$

where:

$$A_{1h}=\tau_{(\alpha,\beta)yh}^2\left(1+\theta_h a_{xh}+\theta''_h a_{zh}\right),$$

$$E_{1h}=\tau_{(\alpha,\beta)yh}^2\left(1+\theta_h e_{xh}+\theta''_h e_{zh}\right),$$

$$a_{xh}=\left(2\alpha^2-\alpha\right)C_{w_{yh}}^2+\frac{\left(\beta^2-\beta\right)}{2}\lambda_{04000h}^*$$

$$+2\alpha\beta C_{w_{yh}}\lambda_{30000h},$$

$$B_{1h}=\theta''_h b_{xh},$$

$$a_{zh}=C_{r(w_{xh})}\lambda_{02100h}-2\alpha(C_{w_{yh}}\lambda_{12000h}+C_{w_{yh}r(w_{xh})})$$

$$\lambda_{04000h}^*-\beta\left(\lambda_{22000h}^*+C_{r(w_{xh})}\lambda_{20100h}\right)+C_{r(w_{xh})}^2,$$

$$J_{1h}=\theta'_h j_{zh},$$

$$e_{xh}=\frac{\alpha(\alpha-1)}{2}C_{w_{yh}}^2+\frac{\beta(\beta-2)}{8}\lambda_{40000h}^*$$

$$+\frac{\alpha\beta}{2}C_{w_{yh}}\lambda_{30000h},$$

$$b_{xh}=\overline{X}_{wh}^2C_{w_{xh}}^2,$$

$$j_{zh}=\tau_{(\alpha,\beta)yh}\overline{Z}_{wh}\left(\alpha C_{w_yw_zh}+\frac{\beta}{2}C_{w_{zh}}\lambda_{20010h}\right),$$

$$D_{1h}=\theta'_h d_{zh},\qquad d_{zh}=\overline{Z}_{wh}^2C_{w_{zh}}^2,$$

$$e_{zh}=\frac{3}{8}\lambda_{04000h}^*+\frac{1}{4}C_{r(w_{xh})}\lambda_{02100h}$$

$$-\frac{\alpha}{2}(C_{w_{yh}}\lambda_{12000h}+C_{w_yr(w_x)h})$$

$$+\frac{3}{8}C_{r(w_{xh})}^2-\frac{\beta}{4}\left(\lambda_{22000h}^*+C_{r(w_{xh})}\lambda_{20100h}\right),$$

$$h_{xh}=\tau_{(\alpha,\beta)yh}\overline{X}_{wh}\left(C_{w_{xh}}\lambda_{03000h}+C_{w_xr(w_x)h}\right.$$

$$\left.-\alpha C_{w_yw_xh}-\frac{\beta}{2}C_{w_{xh}}\lambda_{21000h}\right),$$

$$f_{xh}=\frac{\tau_{(\alpha,\beta)yh}\overline{X}_{wh}}{2}\left(C_{w_{xh}}\lambda_{03000h}+C_{w_xr(w_x)h}\right),$$

$$F_{1h}=\theta''_h f_{xh},\qquad H_{1h}=\theta''_h h_{xh}.$$

From Eq. (92), the optimum values of $g_{ih},(i=1,2,3)$ are as follows:

$$g_{1h}=\frac{L_{1h}}{M_{1h}},\qquad g_{2h}=\frac{F_{1h}M_{1h}-H_{1h}L_{1h}}{B_{1h}M_{1h}},$$

$$g_{3h}=\frac{J_{1h}L_{1h}}{D_{1h}M_{1h}},$$

where:

$$M_{1h}=A_{1h}-\frac{H_{1h}^2}{B_{1h}}-\frac{J_{1h}^2}{D_{1h}},$$

$$L_{1h}=E_h-\frac{F_{1h}H_{1h}}{B_{1h}}.$$

By substituting optimum values of $g_{ih}(i=1,2,3)$ in Eq. (92), the minimum MSE of the proposed exponential ratio-type estimator for general parameters is as follows:

$$MSE(\hat{\tau}_{S-(\alpha,\beta)yP2})_{\min}\cong\frac{1}{N^2}\sum_{h=1}^{L}N_h^2$$

$$\left[\tau^2_{(\alpha,\beta)yh}-\frac{1}{M_{1h}}\left\{L_{1h}^2+\frac{F_{1h}^2M_{1h}}{B_{1h}}\right\}\right]. \quad (93)$$

The minimum MSE of the proposed exponential ratio-type estimator for population mean (t_{S-P2m}) and variance (t_{S-P2v}) can be obtained by substituting $(\alpha=1,\beta=0)$ and $(\alpha=0,\beta=2)$ in Eq. (93), respectively.

4. Numerical study

Data of teals from Smith et al. [32] are considered to make a numerical comparison between the existing and the proposed estimators. The data of Blue-winged teal is used as a study variable for stratum 1 and data of Green-winged teal is used as a study variable for stratum 2. Auxiliary variables (x_h and z_h) are generated using the concept given by Dryver and Chao [33] and Chao et al. [34] as follows:

$$x_i=\begin{cases}y_i*Poi(600)+\epsilon_i & \text{if } y_i<100\\ y_i & \text{otherwise}\end{cases} \quad (94)$$

where $\epsilon_i\sim N(0,y_i)$ and Poi represents random generation from Poisson distribution. Data statistics at different levels of correlation are given below:

1. $N=400$, $N_1=200$, $N_2=200$, $n_1=20$, $n_2=20$, $n_1'=50$, $n_2'=50$, $E(v_1)=38$, $\rho_{w_yw_x1}=0.42$, $\rho_{w_yw_x2}=0.47$, $\rho_{w_yw_z1}=0.40$, $\rho_{w_yw_z2}=0.41$, $\rho_{w_xw_z1}=0.998$, $\rho_{w_xw_z2}=0.987$, $\overline{Y}_{w1}=70.60485$, $\overline{Y}_{w2}=12.01$, $S^2_{w_{y1}}=130872.4$, $S^2_{w_{y2}}=12816.53$, $E(v_2)=22$, $\overline{X}_{w1}=367.81$, $\overline{X}_{w2}=47.64$, $S^2_{w_x1}=1473602$, $S^2_{w_x2}=116807.1$, $\overline{Z}_{w1}=391.035$, $\overline{Z}_{w2}=59.24$, $S^2_{w_z1}=1713164$, $S^2_{w_z2}=172156$.
2. $\rho_{w_yw_x1}=0.66$, $\rho_{w_yw_x2}=0.61$, $\rho_{w_yw_z1}=0.59$, $\rho_{w_yw_z2}=0.58$, $\rho_{w_xw_z1}=0.993$, $\rho_{w_xw_z2}=0.989$, $\overline{Y}_{w1}=70.60485$, $\overline{Y}_{w2}=12.01$, $S^2_{w_{y1}}=130872.4$, $S^2_{w_{y2}}=12816.53$, $\overline{X}_{w1}=208.79$, $\overline{X}_{w2}=35.33$, $S^2_{w_x1}=439839.1$, $S^2_{w_x2}=55797.76$, $\overline{Z}_{w1}=232.095$, $\overline{Z}_{w2}=36.255$, $S^2_{w_z1}=563897.9$, $S^2_{w_z2}=60042$.
3. $\rho_{w_yw_x1}=0.88$, $\rho_{w_yw_x2}=0.83$, $\rho_{w_yw_z1}=0.84$, $\rho_{w_yw_z2}=0.78$, $\rho_{w_xw_z1}=0.995$, $\rho_{w_xw_z2}=0.996$, $\overline{Y}_{w1}=70.60485$, $\overline{Y}_{w2}=12.01$, $S^2_{w_{y1}}=130872.4$, $S^2_{w_{y2}}=12816.53$, $\overline{X}_{w1}=125.165$, $\overline{X}_{w2}=22.22$, $S^2_{w_x1}=191574.7$, $S^2_{w_x2}=22789.27$, $\overline{Z}_{w1}=138.9$, $\overline{Z}_{w2}=24.545$, $S^2_{w_z1}=216293.5$, $S^2_{w_z2}=27145$.
4. $\rho_{w_yw_x1}=0.92$, $\rho_{w_yw_x2}=0.94$, $\rho_{w_yw_z1}=0.89$, $\rho_{w_yw_z2}=0.83$, $\rho_{w_xw_z1}=0.997$, $\rho_{w_xw_z2}=0.964$, $\overline{Y}_{w1}=70.60485$, $\overline{Y}_{w2}=12.01, S^2_{w_{y1}}=130872.4, S^2_{w_{y2}}=12816.53$, $\overline{X}_{w1}=115.055$, $\overline{X}_{w2}=18.485$, $S^2_{w_x1}=174073.3, S^2_{w_x2}=17558.42$, $\overline{Z}_{w1}=121.36$, $\overline{Z}_{w2}=20.845, S^2_{w_z1}=186673.5, S^2_{w_z2}=21254$.

For the data sets discussed above, Absolute Relative Bias (ARB) and Percent Relative Efficiency (PRE) are calculated for the existing and proposed estimators. Results of ARB and PRE of the existing and proposed estimators for population mean are presented in Tables 4 and 5. Similarly, results of ARB and Relative Efficiency (RE) for the existing and proposed estimators for population variance are given in Tables 6 and 7. Results presented in Tables 4–7 reveal that for the proposed difference cum exponential-ratio-type estimator for population mean and variance, ARB decreases upon an increase in the correlation between the study and auxiliary variables. The proposed difference-type estimator population mean (t_{S-P1m}) and for population variance (t_{S-P1v}) is unbiased. Thus, ARB remains zero at all correlation levels.

The proposed difference-cum-exponential-ratio-type estimator for population mean (t_{S-P2m}) is more efficient when the correlation between the study and auxiliary variables is low or moderate. The proposed difference type estimator for population mean (t_{S-P1m}) outperforms all other estimators when the correlation is high. When the correlation between the study and auxiliary variables is low or moderate, t_{S-16m} is most efficient among all existing estimators. Thus, the comparison between t_{S-P2m} and t_{S-16m} is given in Figure 1(a). At high correlation levels, t_{S-19m} outperforms all existing estimators. Thus, the comparison between t_{S-P1m} and t_{S-19m} is given in Figure 1(b). Finally, the comparison between t_{S-P1m} and t_{S-P2m} is given in Figure 1(c).

The proposed difference-cum-exponential-ratio-type estimator for population variance t_{S-P2v} is more efficient when the correlation between the study and auxiliary variables is low or moderate. The proposed difference-type estimator for population variance t_{S-P1v} outperforms all other estimators when the correlation is high. When the correlation between the study and the auxiliary variables is low or moderate, t_{S-10v} is most efficient among all existing estimators. Thus, the comparison between t_{S-P2v} and t_{S-10v} is given in Figure 2(a). At high levels of correlation, t_{S-9v} performs better than all existing estimators. Thus, the comparison between t_{S-P1v} and t_{S-9v} is given in

Table 4. Absolute Relative Bias (ARB) of different estimators for population mean.

ARB	$(\rho_{wywxh1}, \rho_{wywzh1}$: $\rho_{wywxh2}, \rho_{wywzh2})$			
Estimators	$(0.42, 0.40:$ $0.47, 0.41)$	$(0.66, 0.59:$ $0.61, 0.58)$	$(0.88, 0.84:$ $0.83, 0.78)$	$(0.92, 0.89:$ $0.94, 0.83)$
t_{S-1m}	0.0000	0.0000	0.0000	0.0000
t_{S-2m}	0.5515	0.1493	0.2145	0.3068
t_{S-3m}	0.1023	0.0807	0.2696	0.3321
t_{S-4m}	0.0000	0.0000	0.0000	0.0000
t_{S-5m}	1.3852	1.1827	0.6127	1.0998
t_{S-6m}	0.1654	0.0622	0.1285	0.1892
t_{S-7m}	0.1654	0.0622	0.1285	0.1892
t_{S-8m}	0.6076	0.2005	0.3638	0.5285
t_{S-9m}	27.8491	16.4717	82.6546	88.8825
$t_{S-10m_{j=1}}$	0.3695	0.0134	0.3697	0.4766
$t_{S-10m_{j=2}}$	0.0173	0.1951	0.3524	0.3556
$t_{S-10m_{j=3}}$	0.5488	0.1454	0.2200	0.3122
$t_{S-10m_{j=4}}$	0.2729	0.0480	0.3832	0.4903
$t_{S-10m_{j=5}}$	0.5371	0.1295	0.2454	0.3470
$t_{S-10m_{j=6}}$	0.5482	0.1451	0.2203	0.3127
$t_{S-10m_{j=7}}$	0.3110	0.0370	0.3789	0.4862
$t_{S-10m_{j=8}}$	0.5514	0.1492	0.2151	0.3072
$t_{S-10m_{j=9}}$	0.5515	0.1493	0.2148	0.3068
$t_{S-10m_{j=10}}$	0.0000	0.0000	0.0000	0.0000
t_{S-11m}	0.1004	0.1621	0.5210	0.6334
t_{S-12m}	0.6897	0.7432	0.9310	1.0603
t_{S-13m}	3.7853	1.0862	10.7265	14.3532
t_{S-14m}	3534.36	5727.64	10347.7	14189.8
$t_{S-15m_{j=1}}$	0.5934	0.2024	0.1532	0.2412
$t_{S-15m_{j=2}}$	2.8973	7.2896	3.1225	1.4372
$t_{S-15m_{j=3}}$	0.3445	0.4902	0.7428	0.8782
t_{S-16m}	0.4559	0.4245	0.2804	0.1944
t_{S-17m}	0.6559	0.6233	0.5304	0.4702
t_{S-18m}	0.6000	0.5664	0.4558	0.3608
t_{S-19m}	0.0000	0.0000	0.0000	0.0000
$\mathbf{t_{S-P1m}}$	0.0000	0.0000	0.0000	0.0000
$\mathbf{t_{S-P2m}}$	0.2978	0.3798	0.2527	0.1442

Figure 2(b). Thus, the comparison between t_{S-P1v} and t_{S-P2v} is given in Figure 2(c).

5. Conclusion

In this article, difference-type and difference-cum-exponential-ratio-type estimators were recommended for general parameters under stratified adaptive cluster sampling. Estimators were proposed using two auxiliary variables. The proposed estimators utilized auxiliary information in terms of mean, variance, and ranks of auxiliary variates in the hth stratum. Based on the numerical study, it became clear that the proposed estimators for population mean were more efficient than the usual mean, ratio, exponential-ratio, difference estimator, and estimators of Gupta and Shabbir [14], Singh et al. [15], Choudhury and Singh [16], Hamad et al. [17], Chutiman [18], Yadav et al. [19], Vishwakarma and Gangele [21], Singh and Khalid [22], Khan and Al-Hossain [23], Khan [24],

Table 5. Percent Relative Efficiency (PRE) of different estimators for population mean.

PRE	$(\rho_{wywxh1}, \rho_{wywzh1} : \rho_{wywxh2}, \rho_{wywzh2})$			
Estimators	$(0.42, 0.40 : 0.47, 0.41)$	$(0.66, 0.59 : 0.61, 0.58)$	$(0.88, 0.84 : 0.83, 0.78)$	$(0.92, 0.89 : 0.94, 0.83)$
t_{S-1m}	165.4901	165.4901	165.4901	165.4901
t_{S-2m}	187.4728	276.0105	584.5970	786.8074
t_{S-3m}	198.6005	236.4534	314.0641	344.9164
t_{S-4m}	200.5705	277.1917	674.2123	983.4634
t_{S-5m}	216.6821	406.8355	1261.3110	1707.0440
t_{S-6m}	200.5705	277.1917	674.2123	983.4634
t_{S-7m}	200.5705	277.1917	674.2123	983.4634
t_{S-8m}	300.6597	834.2741	*	*
t_{S-9m}	107.9679	108.2773	89.7860	81.7906
$t_{S-10m_{j=1}}$	188.8910	276.3462	561.1049	726.3032
$t_{S-10m_{j=2}}$	190.9492	271.1515	477.9477	550.4525
$t_{S-10m_{j=3}}$	187.5180	276.0182	583.6020	784.7066
$t_{S-10m_{j=4}}$	189.5292	276.3007	554.8144	706.2154
$t_{S-10m_{j=5}}$	187.6124	276.0827	580.1901	775.3147
$t_{S-10m_{j=6}}$	187.5244	276.0207	583.5318	784.4774
$t_{S-10m_{j=7}}$	189.3103	276.3136	556.6877	713.5036
$t_{S-10m_{j=8}}$	187.4735	276.0113	584.5723	786.7348
$t_{S-10m_{j=9}}$	187.4728	276.0105	584.5970	786.8074
$t_{S-10m_{j=10}}$	165.4901	165.4901	165.4901	165.4901
t_{S-11m}	199.0499	245.0164	376.5730	433.6570
t_{S-12m}	190.5514	217.4764	264.4360	279.9786
t_{S-13m}	176.5711	195.3811	201.5196	200.5391
t_{S-14m}	200.5705	277.1917	674.2123	983.4634
$t_{S-15m_{j=1}}$	182.4002	274.0209	651.5288	922.5695
$t_{S-15m_{j=2}}$	102.7571	81.9119	256.3006	458.5207
$t_{S-15m_{j=3}}$	182.8062	197.0932	209.2839	208.3950
t_{S-16m}	460.1971	545.3373	1036.8820	1402.9160
t_{S-17m}	402.2027	455.2807	642.7281	735.3240
t_{S-18m}	420.8380	477.4118	683.2319	790.3534
t_{S-19m}	252.2743	284.3129	1046.1300	1812.2380
$\mathbf{t_{S-P1m}}$	303.3198	348.0587	**5152.0570**	**17687.0400**
$\mathbf{t_{S-P2m}}$	**488.7737**	**547.2316**	1219.2100	1823.2250

Shabbir and Gupta [13], Singh et al. [25], Shabbir and Gupta [26], Muneer et al. [27], Shabbir [12], and Qureshi et al. [20] under Stratified Adaptive Cluster Sampling (SACS).

Likewise, the proposed estimators for population variance were found more efficient than usual variance, ratio, exponential-ratio, difference estimator, and estimators of Singh et al. [28], Singh and Solanki [29], Olufadi and Kadilar [30], and Noor-Ul-Amin et al. [31] under SACS.

At a low or moderate correlation, the proposed difference-cum-exponential-ratio-type estimator was the most efficient one; in addition, at a high correlation between the study and auxiliary vari-

Table 6. Absolute Relative Bias (ARB) of different estimators for population variance.

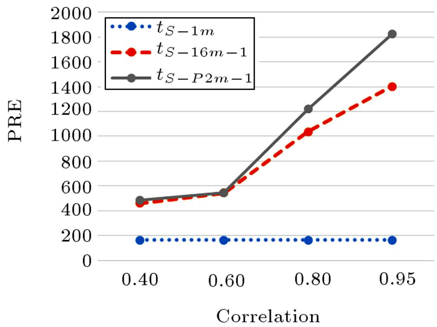
ARB	$(\rho_{wywxh1}, \rho_{wywzh1} : \rho_{wywxh2}, \rho_{wywzh2})$			
Estimators	$(0.42, 0.40 : 0.47, 0.41)$	$(0.66, 0.59 : 0.61, 0.58)$	$(0.88, 0.84 : 0.83, 0.78)$	$(0.92, 0.89 : 0.94, 0.83)$
t_{S-1v}	0.0000	0.0000	0.0000	0.0000
t_{S-2v}	1.9450	0.6879	0.2080	0.2778
t_{S-3v}	0.6095	0.1882	0.2036	0.3146
t_{S-4v}	0.6790	0.1487	0.2942	0.3701
t_{S-5v}	0.0000	0.0000	0.0000	0.0000
t_{S-6v}	0.3290	0.0590	0.2392	0.3182
$t_{S-7v_{j=1}}$	1.9448	0.6878	0.2084	0.2784
$t_{S-7v_{j=2}}$	1.9422	0.6858	0.2106	0.2826
$t_{S-7v_{j=3}}$	1.9446	0.6876	0.2084	0.2785
$t_{S-7v_{j=4}}$	1.9450	0.6880	0.2081	0.2778
t_{S-8v}	0.1664	0.1730	0.1234	0.1610
$\mathbf{t_{S-P1v}}$	0.0000	0.0000	0.0000	0.0000
$\mathbf{t_{S-P2v}}$	0.3023	0.3897	0.2751	0.1722

Table 7. Relative Efficiency (RE) of different estimators for population variance.

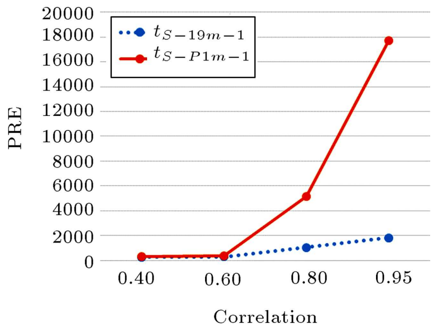
RE	$(\rho_{wywxh1}, \rho_{wywzh1} : \rho_{wywxh2}, \rho_{wywzh2})$			
Estimators	$(0.42, 0.40 : 0.47, 0.41)$	$(0.66, 0.59 : 0.61, 0.58)$	$(0.88, 0.84 : 0.83, 0.78)$	$(0.92, 0.89 : 0.94, 0.83)$
t_{S-1v}	21.37727	21.37727	21.37727	21.37727
t_{S-2v}	15.71694	30.27605	176.42810	347.89820
t_{S-3v}	23.25597	34.30783	72.82164	96.14209
t_{S-4v}	20.77428	28.58016	51.40028	59.05843
t_{S-5v}	21.92377	31.33513	263.27930	603.49040
t_{S-6v}	21.88700	26.52829	36.26399	38.08119
$t_{S-7v_{j=1}}$	15.71700	30.27619	176.41870	347.85300
$t_{S-7v_{j=2}}$	15.71754	30.27718	176.37890	347.60100
$t_{S-7v_{j=3}}$	15.71704	30.27624	176.41580	347.83580
$t_{S-7v_{j=4}}$	15.71694	30.27605	176.42760	347.89630
t_{S-8v}	21.92377	31.33513	263.27930	603.49040
t_{S-9v}	21.92377	31.33513	263.27930	603.49040
t_{S-10v}	33.84675	61.00437	189.48520	402.68180
$\mathbf{t_{S-P1v}}$	45.67044	37.93838	**764.39190**	**2077.81700**
$\mathbf{t_{S-P2v}}$	**58.36229**	**65.44724**	141.49770	203.16210

ables, the proposed difference-type estimator was of highest Percent Relative Efficiency (PRE) among all other estimators. Thus, the proposed difference-type and difference-cum-exponential-ratio-type estimators are suggested to make an efficient estimation of population mean and variance under rare and clustered populations like pollution concentration, drug addiction, and epidemiological studies of AIDS and HIV.

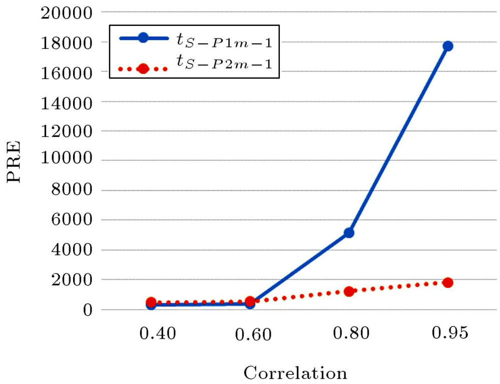
The present study considered ranks of the auxiliary variables for efficient estimation of general parameters under SACS design. Zamanzade and Vock [35] found that when actual quantification of the concomi-



(a) t_{S-P2m}

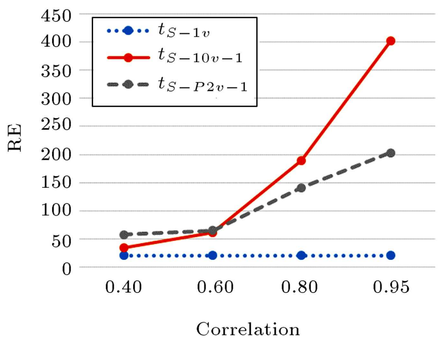


(b) t_{S-P1m}

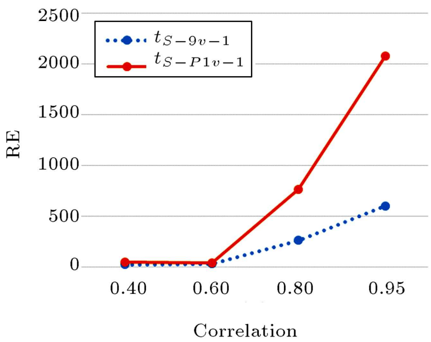


(c) Comparison of the proposed estimators for mean

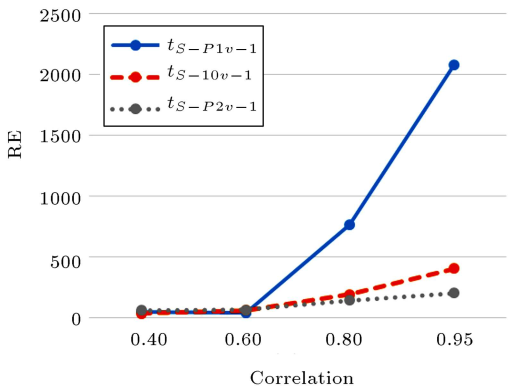
Figure 1. Percent Relative Efficiency (PRE) of estimators for mean in Stratified Adaptive Cluster Sampling (SACS).



(a) t_{S-P2v}



(b) t_{S-P1v}



(c) Comparison of the proposed estimators for variance

Figure 2. Relative Efficiency (RE) of estimators for variance in Stratified Adaptive Cluster Sampling (SACS).

tant variable was available, the ranked set sampling would be more efficient than usual double sampling. A rewarding area for further study is to incorporate ranked set sampling under Adaptive Cluster Sampling (ACS) and SACS designs.

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