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Invited Paper

The nontrivial zeros of completed zeta function and Riemann hypothesis

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Abstract. Based on the completed zeta function, this paper addresses that the real part of every non-trivial zero of the Riemann's zeta function $\zeta(s) = \zeta(\sigma + ib) = \sum_{n=1}^{\infty} n^{-(\sigma+ib)}$, where the real part is $\mathrm{Re}(s) = \sigma \in \mathbb{R}$ and the imaginary part is $Im(s) = b \in \mathbb{R}$ with the real number $\mathbb{R}$, is $\mathrm{Re}(s) = \frac{1}{2}$, which is the well-known critical line.

1. Introduction

The Riemann Hypothesis (RH) is one of the seven millennium prize problems that was pointed out by the Clay Mathematics Institute in 2000 (see [1]) and considered as the Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems (see [2-5] for more details). It was formulated in 1859 by Bernhard Riemann as part of his attempt to illustrate how the prime numbers can be distributed on the number line (see [6]). As the key connection of the RH, the Riemanns Zeta Function (RZF) plays an important role in the study of the prime numbers (see [3,4]). In fact, the work of the RZF in the set of real numbers was proposed in 1749 by Euler for the first time (see [7]) and developed by Chebyshev (see [8]; also see [9]). As one of the important works developed in the set of complex numbers, this is well known as the RH, which is applied in the fields of physics, probability theory, and statistics. Furthermore, there is an important relation between the RH and the classical Hamiltonian operator (see [10]; also see [11,12]).

Let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ denote the sets of complex, real, and natural numbers, respectively. The real part of $s \in \mathbb{C}$ is represented by $\mathrm{Re}(s)$ and the imaginary part by $Im(s)$. We now start with the RZF defined as follows [6]:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1}$$

where $s \in \mathbb{C}$, $s = \mathrm{Re}(s) + iIm(s) = \sigma + ib \in \mathbb{C}$, with $i = \sqrt{-1}$, $\mathrm{Re}(s) = \sigma \in \mathbb{R}$, and $Im(s) = b \in \mathbb{R}$. It follows from Eq. (1) that for $\mathrm{Re}(s) > 1$ (see [13]; also see [14], p. 420):

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} \frac{1}{1 - p^{-s}}, \tag{2}$$

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which implies that (see [5], p. 28):

$$\log\zeta(s)=\sum_{p}\log\frac{1}{1-p^{-s}}=\sum_{p,n}\log\frac{1}{np^{ns}}, \tag{3}$$

where p and n represent all primes and all positive integrals, respectively.

On the one hand, Eq. (1) satisfies (see [15], p. 14; also see [5], p. 426):

$$\zeta(1-s)=2(2\pi)^{-\frac{s}{2}}\cos\left(\frac{1-s}{2}\right)\Gamma(s)\zeta(s), \tag{4}$$

where $s\in\mathbb{C}$ and $s\neq 1$, yielding the alternative form of Eq. (4) as follows (see [16], p. 13):

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \tag{5}$$

where $s\in\mathbb{C}$, $s\neq 1$, and $\Gamma(\cdot)$ is the well-known Euler Gamma function [17]. The important works demonstrate that Eq. (4) has a holomorphic continuation to all values of s except $s=1$, proposed by Neukirch (see [5], p. 426), and that Eq. (5) has a holomorphic continuation to all values of s except $s=1$, proposed by Titchmarsh (see [16], p. 13).

The symmetrical form of Eq. (4) and Eq. (5), proved in 1859 by Riemann, is given as follows (see [6]; also see [5], p. 425):

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s), \tag{6}$$

which is the well-known Riemann's Functional Equation (RFE) for all complex numbers $s\in\mathbb{C}$ and $\mathrm{Re}(s)>1$. It is proved that Eq. (6) is valid when any one of two sides in Eq. (6) has a holomorphic continuation to all values of s except $s=0$ and $s=1$ (see [5], p. 425), e.g., the complex domain of Eq. (6) can be extended from $\mathrm{Re}(s)>1$ (see [6]) to all values of s except $s=0$ and $s=1$ (see [5], p. 425).

On the other hand, the functional equation given as:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s), \tag{7}$$

on the complex domain $\mathrm{Re}(s)>1$, proposed in 1859 by Riemann, was reported in [6,18], rediscovered in 1992 by Neukirch (see [5], p. 422) to show that it has a holomorphic continuation to the entire complex plane for all values of s except $s=0$ and $s=1$, and also developed in 2004 by Gelbart and Miller (see, e.g. [19]). The properties of Eq. (7) are given as follows:

(A1) Eq. (7) has a holomorphic continuation to the entire complex plane $s\in\mathbb{C}$, with simple poles at $s=0$ and $s=1$ (see [5], p. 425; also see [19]);

(A2) Eq. (6) is valid for the entire complex plane $s\in\mathbb{C}$ except $s=0$ and $s=1$ (see [5], p. 425; also see [19]);

(A3) $\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)+\frac{1}{s}+\frac{1}{1-s}$ is bounded in the vertical strip ($\infty>\mathrm{Re}(s)>-\infty$) (see [19]).

Moreover, the functional equation defined as:

$$\frac{1}{s(s-1)}+\int_{1}^{\infty}\phi(t)\left(t^{\frac{s}{2}-1}+t^{-\frac{1+s}{2}}\right)dt, \tag{8}$$

in the complex domain $\mathrm{Re}(s)>1$ was presented in [6] and investigated in 1987 by Titchmarsh (see [16], p. 22). Moreover, Titchmarsh proposed that Eq. (8) has an analytic continuation to the entire complex plane (see [16], p. 22). As is well known, Riemann showed that Eq. (7) was equal to Eq. (8) due to Eq. (6) (see [6]; also see [5], p. 425), e.g.:

$$\begin{aligned}&\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)\\&\quad=\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s)\\&\quad=\frac{1}{s(s-1)}+\int_{1}^{\infty}\phi(t)\left(t^{\frac{s}{2}-1}+t^{-\frac{1+s}{2}}\right)dt.\end{aligned} \tag{9}$$

However, the Riemann and Titchmarsh's works of the characteristics of Eq. (8) were not noticed by Neukirch (see [5], p. 425) and Gelbart and Miller (see [19]). In the Neukirch's work, Eqs. (7)-(9) are called the Completed Zeta Function (CZF) (see [5], p. 422); also see [19]). Meanwhile, Tate also proposed the CZF within the Fourier analysis in number fields (see [19]; also see [20] for more details). It is a key topic to show the proof of the RH based on the properties and theorems involving the CZF.

Moreover, in order to consider the proof of the RH, some problems of Eq. (1) were investigated in [15,16,21-28]. In particular, the zeros of Eq. (1) exist in two different types as follows (see, e.g. [15,16,21,22]):

(B1) Eq. (1) has the trivial zeros at $s=-2n$, where $n\in\mathbb{N}$ due to the poles of $\Gamma(s/2)$ (see [23]; also see [5], p. 432);

(B2) Eq. (1) has the non-trivial zeros at the complex numbers $s=\frac{1}{2}+ib$, where $b\in\mathbb{R}$, $\mathrm{Re}(s)=\frac{1}{2}$ is called the critical line (see [24-26]) and the open strip $\{s\in\mathbb{C}:0<\mathrm{Re}(s)<1\}$ is called the critical strip (see, e.g. [27,28]).

As a matter of fact, the critical strip in (B2) was discussed and solved (for example, see [15,16,21,22]). However, the critical line in (B2) is one of the most famous unsolved problems in mathematics. The critical strip was solved in 1895 by Riemann with the use of Eq. (6) (see [6]; also see [6], p. 432). As is well known, the RH has stated that Eq. (1) has the non-trivial zeros at the complex numbers $s = \frac{1}{2} + ib$ (see [1], p. 107).

It is evident that the RH remains unproved to this today (see, e.g. [12]). In this case, we have to continue to face the formidable challenge. The main aim of the present article is to provide the proof of the RH based on the properties and theorems of the CZF and the relations between the RZF and the CZF. The structure of the paper is designed as follows. In Section 2, the definitions and properties of the CZF are introduced and investigated in detail. The proof of the RH is addressed in Section 3. The discussion and remarking comments are given in Section 4. Finally, the conclusion is drawn in Section 5.

2. Preliminaries

In this section, the CZF, which will be used to structure the proof of the RH, is introduced in detail. The properties and theorems for the CZF are also investigated.

Definition 1. *Let $\Theta : X \to Y$, be the CZF defined by (see [6,18,19]; also see [5], p. 422):*

$$\Theta(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s), \tag{10}$$

where $X \in \mathbb{C}$ and $Y \in \mathbb{C}$.

In other words, the CZF is defined as follows (see [5], p. 422; also see [18,19]):

$$\Theta(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s), \tag{11}$$

which has a holomorphic continuation to all values of s except $s \neq 0$ and $s \neq 1$ (see [5], p. 425).

Note that Neukirch proposed that the CZF had an analytic continuation to the entire complex plane $s \in \mathbb{C}$, with simple poles at $s = 0$ and $s = 1$ (see [5], p. 425; also see [19] for more details).

We now discuss different representations of the CZF as follows.

2.1. The CZF of first type

According to the Riemann's formula in the complex domain (see [6]; also see [21], p. 15):

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty \phi(t) t^{\frac{s}{2}-1} dt, \tag{12}$$

we may rewrite Eq. (11) as follows (see [16], p. 22):

$$\Theta(s) = \int_0^\infty \phi(t) t^{\frac{s}{2}-1} dt, \tag{13}$$

where the Jacobi's theta function is defined as follows (see, e.g. [29]):

$$\phi(t) = \sum_{n=1}^{\infty} e^{-n^2\pi t}, \tag{14}$$

where $t > 0$ and $n \in \mathbb{N}$. Here, Eq. (13) is called the CZF of the first type, which has a holomorphic continuation to the entire complex plane s with the exceptions of 0 and 1.

As an alternative form of Eq. (13), with the help of the expression (see [5], p. 422):

$$\phi(t) = \frac{1}{2}(\lambda(it) - 1),$$

where (see [5], p. 422):

$$\lambda(t) = 1 + \sum_{n=1}^{\infty} e^{-n^2\pi i t},$$

we have (see [5], p. 422):

$$\Theta(s) = \frac{1}{2}\int_0^\infty (\lambda(it) - 1) t^{\frac{s}{2}-1} dt, \tag{15}$$

which is a holomorphic continuation to the entire complex plane s except $s = 0$ and $s = 1$ (see [5], p. 425).

Similarly, we note that Edwards considered Eq. (13) with a holomorphic continuation to the complex plane $\mathrm{Re}(s) > 1$ (see [21], p. 15). However, Riemann may consider Eq. (13) as the entire complex plane $s \in \mathbb{C}$, with simple poles at $s = 0$ and $s = 1$ due to Eq. (6) (see [6]). Actually, it is clear that both Eqs. (13) and (15) have holomorphic continuations to the entire complex plane with simple poles at $s = 0$ and $s = 1$ (see [5], p. 422; also see [19]).

2.2. The CZF of second type

With the use of the Riemann's formula (see [6] for more details; also see [16], p. 22):

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{-\frac{1+s}{2}}\right) dt, \tag{16}$$

which can be, with the aid of Eq. (6), rewritten as follows (for example, see [16], p. 22):

$$\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s) = \frac{1}{s(s-1)} + \int_1^\infty \phi(t)\left(t^{-\frac{1+s}{2}} + t^{\frac{s}{2}-1}\right) dt, \tag{17}$$

we define the CZF that has a holomorphic continuation to the entire complex plane with simple poles at $s = 0$ and $s = 1$ as follows:

$$\Theta(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$
$$= \int_0^1 \phi(t)t^{\frac{s}{2}-1}dt + \int_1^\infty \phi(t)t^{\frac{s}{2}-1}dt$$
$$= \frac{1}{s(s-1)} + \int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{-\frac{1+s}{2}}\right)dt, \quad (18)$$

since Eq. (16) has a holomorphic continuation for all values of s except $s = 0$ and $s = 1$ (see [16], p. 22). Here, Eq. (18) is called the CZF of second type, which has a holomorphic continuation to the entire complex plane $s \in \mathbb{C}$, with simple poles at $s = 0$ and $s = 1$. In fact, this has been observed by Titchmarsh (although he did not point out the poles $s = 0$ and $s = 1$) (see [16], p. 22). It is evident that Eq. (18) is a holomorphic continuation to the entire complex plane $s \in \mathbb{C}$, with simple poles at $s = 0$ and $s = 1$ due to Eq. (6) (see [5], p. 425; also see [19]).

2.3. The relation between two representations of the CZF

With the aid of the expression in all complex domain (see [6]; also see [16], p. 22):

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$
$$= \frac{1}{s(s-1)} + \int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{-\frac{1+s}{2}}\right)dt$$
$$= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{-\frac{1+s}{2}}\right)dt$$
$$= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 \phi\left(\frac{1}{t}\right)t^{\frac{s-3}{2}}dt + \int_1^\infty \phi(t)t^{\frac{s}{2}-1}dt$$
$$= \int_0^1 t^{\frac{s}{2}-1}\left[\frac{1}{t}\phi\left(\frac{1}{t}\right) + \frac{1}{2\sqrt{t}} - \frac{1}{2}\right]t^{\frac{s-3}{2}}dt$$
$$+ \int_1^\infty \phi(t)t^{\frac{s}{2}-1}dt = \int_0^1 \phi(t)t^{\frac{s}{2}-1}dt$$
$$+ \int_1^\infty \phi(t)t^{\frac{s}{2}-1}dt = \int_0^\infty \phi(t)t^{\frac{s}{2}-1}dt, \quad (19)$$

and the property of the Jacobi's theta function (see [29]; also see [6,23]):

$$2\phi(t) + 1 = t^{\frac{1}{2}}\left(2\phi\left(\frac{1}{t}\right) + 1\right), \quad (20)$$

which implies (see [16], p. 22):

$$\sum_{n=\infty}^{\infty} e^{-n^2\pi t} = \frac{1}{\sqrt{t}}\sum_{n=\infty}^{\infty} e^{-n^2\pi/t}, \quad (21)$$

we have (see [16], p. 22):

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$
$$= \frac{1}{s(s-1)} + \int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{-\frac{1+s}{2}}\right)dt$$
$$= \int_0^\infty \phi(t)t^{\frac{s}{2}-1}dt, \quad (22)$$

which can be extended to be a holomorphic continuation for all values of s except $s = 0$ and $s = 1$ (see [16,19] for more details).

Thus, with the aid of Eq. (22), we note that Eqs. (13) (see [19] for more details) and (18) (see [16], p. 22) can be extended to be a holomorphic continuation for all values of s except $s = 0$ and $s = 1$. Meanwhile, it is very clear that both $s = 0$ and $s = 1$ are the poles of $\Theta(s)$ (see [19] for more details). Since Eq. (13) may be considered as an alternative form of Eq. (18) in some cases and they can be expressed in Eq. (22), there exist for $t > 0$ (see [6] for more details; also see [16], p. 22):

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty \phi(t)t^{\frac{s}{2}-1}dt,$$

and for $t > 0$ and $t \neq 1$ (see [16], p. 22; also see [5], p. 425):

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$
$$= \frac{1}{s(s-1)} + \int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{-\frac{1+s}{2}}\right)dt$$
$$= \int_0^1 \phi(t)t^{\frac{s}{2}-1}dt + \int_1^\infty \phi(t)t^{\frac{s}{2}-1}dt,$$

which are holomorphic continuations for all values of s except $s = 0$ and $s = 1$ (see [5], p. 422) since Eq. (6) has a holomorphic continuation for all values of s except $s = 0$ and $s = 1$ (see [5], p. 425; also see [19]).

2.4. The properties and theorems involving the CZF

In this section, the properties and theorems of the CZF are presented in detail.

Property 1 (see [5], p. 425; also see [16], p. 12, and [19]). *The CZF has a holomorphic continuation to the entire complex plane $s \in \mathbb{C}$, with simple poles at $s = 0$ and $s = 1$.*

Proof. *Making use of Eqs. (6), (13), and (18), we directly obtain the result (For more details, see [5,16,19]).*

Property 2 (see [5], p. 432). *Let the CZF and the RZF be holomorphic continuations to the entire complex plane $s \in \mathbb{C}$ except $s = 0$ and $s = 1$, respectively. If $\zeta(s) = 0$ and $s = \sigma + ib$ is a non-trivial zero, then we have:*

$$\Theta(s) = 0. \tag{23}$$

Proof. *Substituting $\zeta(s) = 0$ into Eq. (11), we have the result.*

Property 3 (see [5], p. 432). *Let the CZF and the RZF be holomorphic continuations to the entire complex plane $s \in \mathbb{C}$ except $s = 0$ and $s = 1$, respectively. If $\zeta(s)$ has the trivial zeros, then $\Theta(s) \neq 0$.*

Proof. *Since Eq. (1) has the trivial zeros at $s = -2n$, where $n \in \mathbb{N}$, there are the poles of $\Gamma(\frac{s}{2})$ (see [5], p. 432).*

(C1) *In view of $\pi^{-\frac{s}{2}} \neq 0$ at the trivial zeros at $s = -2n$ and by using Eqs. (12) and (13), we have:*

$$\Theta(s)=\int_0^\infty \phi(t)t^{\frac{s}{2}-1}dt=\int_0^\infty \phi(t)t^{-n-1}dt>0, \tag{24}$$

where:

$$\phi(t)t^{-n-1} > 0, \tag{25}$$

with $t > 0$.

Thus, we have the result.

(C2) *In a similar way, by means of Eq. (18), we have (see [6]; also see [21]):*

$$\int_0^1 \phi(t)t^{\frac{s}{2}-1}dt + \int_1^\infty \phi(t)t^{\frac{s}{2}-1}dt$$

$$=\frac{1}{s(s-1)}+\int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{-\frac{1+s}{2}}\right) dt, \tag{26}$$

such that:

$$\Theta(s) =\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-\frac{s}{2}}$$

$$=\frac{1}{s(1-s)}+\int_1^\infty \phi(t)\left(t^{\frac{1-s}{2}-1} + t^{\frac{s}{2}-1}\right) dt$$

$$=\int_0^1 \phi(t)t^{\frac{s}{2}-1}dt+\int_1^\infty \phi(t)t^{\frac{s}{2}-1}dt \neq 0, \tag{27}$$

due to Eq. (25), where $n \in \mathbb{R}$. Thus, we give the result. To sum up, Property 3 holds.

Property 4 (see [5], p. 425; also see [6,18,19,21]). *Let the CZF and the RZF be holomorphic continuations to the entire complex plane $s \in \mathbb{C}$ except $s = 0$ and $s = 1$, respectively. If $s \in \mathbb{C}$, $s \neq 0$, and $s \neq 1$, then we have:*

$$\Theta(s) = \Theta(1-s). \tag{28}$$

Proof. *With the aid of Eq. (6), we have:*

$$\Theta(1-s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s), \tag{29}$$

such that:

$$\Theta(s) = \Theta(1-s), \tag{30}$$

due to the conditions $s \in \mathbb{C}$, $s \neq 0$ and $s \neq 1$.

For more details of Property 4, see [5,6,18,19,21].

We also notice that with the use of Eq. (18), Eq. (30) holds, where:

$$\Theta(s) = \frac{1}{s(1-s)} + \int_1^\infty \phi(t)\left(t^{\frac{s}{2}-1} + t^{\frac{1-s}{2}-1}\right) dt, \tag{31}$$

and:

$$\Theta(1-s) = \frac{1}{s(1-s)} + \int_1^\infty \phi(t)\left(t^{\frac{1-s}{2}-1} + t^{\frac{s}{2}-1}\right) dt. \tag{32}$$

In conclusion, Property 4 holds.

Theorem 1. *Let the CZF and the RZF be holomorphic continuations to the entire complex plane $s \in \mathbb{C}$ except $s = 0$ and $s = 1$, respectively. If $\Theta(s) = 0$, then $\zeta(s) = 0$ and s is the nontrivial zero.*

Proof. *In order to solve the problem, we consider that there might exist $\Theta(s) = 0$ in the following cases: the trivial zeros and the non-trivial zeros of the RZF.*

(D1) *At the trivial zeros of the RZF, by Property 3, we always have:*

$$\Theta(s) > 0, \tag{33}$$

which has the conflict with the condition $\Theta(s) = 0$.

(D2) *At the nontrivial zeros of the RZF, there are the following formulas:*

$$\Gamma(\frac{s}{2}) \neq 0, \tag{34}$$

and:

$$\pi^{-\frac{s}{2}} \neq 0. \tag{35}$$

In this case, we always have:

$$\Theta(s) = \ell\zeta(s), \tag{36}$$

where $\ell = \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}} \neq 0$ *is a constant for any complex number* s *because the nontrivial zeros of the RZF and the complex zeros of the CZF are the same complex number* $s \in \mathbb{C}$.

In this case, there is:

$$\Theta(s) = 0, \tag{37}$$

if and only if:

$$\zeta(s) = 0, \tag{38}$$

where s *is the nontrivial zero of* $\zeta(s)$. *Since* $s = 1$ *is the simple pole of the RZF and* $\zeta(0) \neq 0$ *(see [28], p. 18), we obtain the result.*

Corollary 1. *Let the CZF and the RZF be holomorphic continuations to the entire complex plane* $s \in \mathbb{C}$ *except* $s = 0$ *and* $s = 1$, *respectively. If* $\zeta(s) = 0$ *and* s *is the nontrivial zero, then* $\Theta(s) = 0$ *and* s *is the complex zero of* $\Theta(s)$.

Proof. *In a similar way, with the use of Eqs. (34) and (35), we have the result since, by using Eq. (36), we always see that the nontrivial zeros of the RZF and the zeros of the RZF are the same complex number* $s \in \mathbb{C}$, *where* $s \in \mathbb{C}$ *except* $s = 0$ *and* $s = 1$.

Corollary 2. *Let the CZF and the RZF be holomorphic continuations to the entire complex plane* $s \in \mathbb{C}$ *except* $s = 0$ *and* $s = 1$, *respectively. We have* $\zeta(s) = 0$ *and* s *is the nontrivial zero if and only if* $\Theta(s) = 0$ *and* s *is the complex zero of* $\Theta(s)$.

Proof. *With the use of Corollary 1 and Theorem 1, we have the result.*

Corollary 3. *Let the CZF and the RZF be holomorphic continuations to the entire complex plane* $s \in \mathbb{C}$ *except* $s = 0$ *and* $s = 1$, *respectively. If the number of the nontrivial zeros of the RZF is* $\aleph_\zeta(\zeta(s)t)$ *and the number of the zeros of the RZF is* $\aleph_\Theta(\zeta(s))$, *then there is:*

$$\aleph_\zeta(\zeta(s)) = \aleph_\Theta(\zeta(s)). \tag{39}$$

Proof. *According to Corollary 1 and Theorem 1,* s *will be the nontrivial zero of the RZF and the zero of* $\Theta(s)$ *such that there exists Eq. (36) for any* s *with* $s \in \mathbb{C}$, $s \neq 0$, *and* $s \neq 1$. *That is to say, we observe that the zero of* $\Theta(s)$ *has a one-to-one map of the RZF for the same complex number* s, *where* $s \in \mathbb{C}$, $s \neq 0$, *and* $s \neq 1$ *and* $\zeta(s)$ *is one-valued and finite for all finite values* $s \in \mathbb{C}$ *except* $s \neq 0$ *and* $s \neq 1$ *since the special values are valid (see [5], p. 432). Thus, Eq. (39) holds.*

Note that Corollary 3 is the special case of the Siegel work (see [30], p. 179).

Corollary 4. *If* $\Theta(s) = 0$, *then:*

$$1 - s = 1 - \sigma - ib \tag{40}$$

is the zero of $\Theta(s)$, *where* $s = \sigma + ib$, $\sigma \in \mathbb{R}$, *and* $b \in \mathbb{R}$.

Proof. *We now structure:*

$$1 - s = 1 - \sigma - ib \in \mathbb{C}, \tag{41}$$

which leads:

$$1 - s = 1 - \sigma - ib = 1 - (\sigma + ib) \in \mathbb{C}. \tag{42}$$

With the use of Property 4, we have:

$$\Theta(s) = 0, \tag{43}$$

which leads to:

$$\Theta(1 - s) = 0. \tag{44}$$

Thus, it follows from Eq. (44) that $1 - s = 1 - \sigma + ib$ *is the complex zero of* $\Theta(s)$.

In this case, we have the following result.

Corollary 5. *If* $\zeta(s) = 0$, *where* s *is the nontrivial zero for* $s \in \mathbb{C}$, $s \neq 0$ *and* $s \neq 1$, *then:*

$$1 - s = 1 - \sigma - ib \tag{45}$$

is the nontrivial zero of $\zeta(s)$, *where* $s = \sigma + ib$ *with* $\sigma \in \mathbb{R}$ *and* $b \in \mathbb{R}$.

Proof. *In the similar way of Corollary 2, we directly obtain the result.*

To sum up, by using Eqs. (13) and (18), Theorem 1, Corollary 2, Corollary 3, and Corollary 4 are valid.

3. The proof of the RH

In this section, we now consider the proof of the RH. As a direct result of Theorem 1 and Corollary 1 (as an alternative, Corollary 2 is valid), we see that Theorem 2 is sufficient and necessary for the RH. We have the identity theorem of the RH as follows.

Theorem 2. *Let the CZF and the RZF be holomorphic continuations to the entire complex plane* $s \in \mathbb{C}$ *except* $s = 0$ *and* $s = 1$, *respectively. Let* $\Theta(s) = 0$ *and* *let $s = \sigma + ib$ be the zero, where $\sigma \in \mathbb{R}$ and $b \in \mathbb{R}$. Then, we have $Re(s) = \sigma = \frac{1}{2}$.*

Proof. *Taking $s = \sigma + ib$, we have:*

$$\Theta(\sigma + ib) = \int_0^\infty \phi(t) t^{\frac{\sigma+ib}{2}-1} dt = 0, \tag{46}$$

and:

$$\Theta(1-(\sigma + ib)) = \int_0^\infty \phi(t) t^{\frac{1-(\sigma+ib)}{2}-1} dt = 0. \tag{47}$$

Making use of the complex exponents with positive real bases (for example, see [31]):

$$t^\gamma = e^{\gamma \ln(t)}, \tag{48}$$

where $t > 0$, $\gamma \in \mathbb{C}$, and $x = \ln(t)$ is represented as the unique real solution to the functional equation $t = e^x$ (for example, see [31]), we have:

$$t^{\frac{\sigma+ib}{2}-1} = e^{\left(\frac{\sigma+ib}{2}-1\right)\ln t} = e^{\frac{(\sigma-2)\ln t}{2}} e^{\frac{ib\ln t}{2}}, \tag{49}$$

and:

$$t^{\frac{1-(\sigma+ib)}{2}-1} = e^{\left(-\frac{\sigma+1+ib}{2}\right)\ln t} = e^{-\frac{(\sigma+1)\ln t}{2}} e^{-\frac{ib\ln t}{2}}, \tag{50}$$

such that, from Eqs. (46) and (47), there are:

$$\begin{aligned}
&\Theta(\sigma + ib) \\
&= \int_0^\infty \phi(t) e^{\frac{(\sigma-2)\ln t}{2}} e^{\frac{ib\ln t}{2}} dt \\
&= \int_0^\infty \phi(t) e^{\frac{(\sigma-2)\ln t}{2}} \left(\cos\left(\frac{b\ln t}{2}\right) + i\sin\left(\frac{b\ln t}{2}\right)\right) dt \\
&= 0,
\end{aligned} \tag{51}$$

and:

$$\begin{aligned}
&\Theta(1-(\sigma + ib)) \\
&= \int_0^\infty \phi(t) e^{-\frac{(\sigma+1)\ln t}{2}} e^{-\frac{ib\ln t}{2}} dt \\
&= \int_0^\infty \phi(t) e^{-\frac{(\sigma+1)\ln t}{2}} \left(\cos\left(\frac{b\ln t}{2}\right) - i\sin\left(\frac{b\ln t}{2}\right)\right) dt \\
&= 0,
\end{aligned} \tag{52}$$

respectively.

From Eq. (51), we may give:

$$\int_0^\infty \phi(t) e^{\frac{(\sigma-2)\ln t}{2}} \cos\left(\frac{b\ln t}{2}\right) dt = 0, \tag{53}$$

and:

$$\int_0^\infty \phi(t) e^{\frac{(\sigma-2)\ln t}{2}} \sin\left(\frac{b\ln t}{2}\right) dt = 0. \tag{54}$$

Similarly, from Eq. (52), we obtain:

$$\int_0^\infty \phi(t) e^{-\frac{(\sigma+1)\ln t}{2}} \cos\left(\frac{b\ln t}{2}\right) dt = 0, \tag{55}$$

and:

$$\int_0^\infty \phi(t) e^{-\frac{(\sigma+1)\ln t}{2}} \sin\left(\frac{b\ln t}{2}\right) dt = 0. \tag{56}$$

Thus, it is clear that there exists [32,33]:

$$\sin\left(\frac{b\ln t}{2}\right) = 0, \tag{57}$$

and:

$$b\ln t = \pm 2\pi\rho, \tag{58}$$

where $t > 0$ and $\rho \in \mathbb{N}_0$, such that:

$$\int_0^\infty \phi(t) e^{\frac{(\sigma-2)\ln t}{2}} dt = 0, \tag{59}$$

and:

$$\int_0^\infty \phi(t) e^{-\frac{(\sigma+1)\ln t}{2}} dt = 0, \tag{60}$$

hold for $t > 0$.

With the use of Eqs. (59) and (60), there is for any $t > 0$:

$$\int_0^\infty \phi(t) e^{\frac{(\sigma-2)\ln t}{2}} dt = \int_0^\infty \phi(t) e^{-\frac{(\sigma+1)\ln t}{2}} dt, \tag{61}$$

which yields:

$$e^{\frac{(\sigma-2)\ln t}{2}} = e^{-\frac{(\sigma+1)\ln t}{2}}. \tag{62}$$

It follows from Eq. (62) that:

$$\frac{\sigma-2}{2} = -\frac{\sigma+1}{2}, \tag{63}$$

which deduces:

$$\sigma = \frac{1}{2}. \tag{64}$$

Thus, the proof of Theorem 2 is finished.

To sum up, the proof of the RH is finished since Theorem 2 is necessary and sufficient for the RH.

4. Discussion and remarking comments

In this work, we present the special cases and new results for the RH as follows:

(E1) From Eqs. (11) and (39), for $b = 0$, we have $s = \frac{1}{2}$ and $\Theta(\frac{1}{2}) = \Theta(1 - \frac{1}{2}) \neq 0$;

(E2) For any $b \in \mathbb{R}$ and $b \neq \mathbb{R}$, there is $\Theta(\frac{1}{2} + ib) = \Theta(\frac{1}{2} - ib) = 0$;

(E3) For any $b \in \mathbb{R}$ and $b \neq \mathbb{R}$, there is $\zeta(\frac{1}{2} + ib) = \zeta(\frac{1}{2} - ib) = 0$, which has infinitely many real zeros (see [27]);

(E4) The Riemann's ξ-function, denoted as $\xi(s)$, is defined as (see [6]):

$$\xi(s) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s), \tag{65}$$

which can be represented in the form of:

$$\xi(s) = \frac{s(s-1)}{2}\Theta(s). \tag{66}$$

(E5) The Landau Ξ-function is defined as (see [34]):

$$\Xi(b) = \xi(\frac{1}{2} + ib), \tag{67}$$

which can be rewritten as follows:

$$\Xi(b) = -\left(\frac{1}{4} + b^2\right)\Theta\left(\frac{1}{2} + ib\right). \tag{68}$$

In view of Eqs. (11) and (39), the functional equation is defined as follows:

$$\mathbb{M}(b) = \Theta\left(\frac{1}{2} + ib\right). \tag{69}$$

With the use of the symmetrical relation (see [34]):

$$\Xi(-b) = \Xi(b), \tag{70}$$

we have:

$$\mathbb{M}(b) = \mathbb{M}(-b). \tag{71}$$

We observe that the zeros of $\Theta(s)$ are presented as the symmetrical distribution in $s = \frac{1}{2} + ib$ due to:

$$\mathbb{M}(b) = -\frac{4}{1+b^2} + 2\int_1^\infty \phi(t)e^{-\frac{3\ln t}{4}}\cos\frac{b\ln t}{2}dt, \tag{72}$$

which is the distribution of the non-trivial zeros of the RZF.

5. Conclusion

In the present work, the proof of the RH was proposed based on the relation between the CZF and the RZF. The theorems and corollaries containing the CZF and RZF were proposed. In addition, a new function like the Landau Ξ-function via the CZF was suggested. It is important for us to research the prime numbers.

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