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Exact mathematical solution for nonlinear free transverse vibrations of beams

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Abstract. In the present paper, an exact mathematical solution is obtained for the nonlinear free transverse vibration of beams for the first time. The governing nonlinear partial differential equation in un-deformed coordinates system is converted in two coupled partial differential equations in deformed coordinates system. Then, a mathematical explanation is obtained for the nonlinear mode shapes as well as natural frequencies versus geometrical and material properties of the beam. It is shown that as the sth mode of transverse vibration is excited, the $2s$th mode of the in-plane vibration will be developed. The results of the present work is compared with those obtained by the Galerkin method and the observed agreement will confirm the exact mathematical solution. It is shown that the governing equation is linear in the time domain. As a parameter, amplitude to length ratio (Λ/l) is proposed to show when the nonlinear terms become dominant in the behavior of structure.

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1. Introduction

Analysis of mechanical structures such as beams, plates, and shells has been the subject of numerous researches due to their wide applications in industry. These researches are arranged based on theories and methods long evolved. To explain and prove the problems, beam is a good suggestion due to its simple equations. However, the governing equations of all mechanical structures follow common principles arising from theorizing. Stress of beam under poor bending was a challenging problem at the beginning. The famous mathematician, Euler offered the proper formulation for the first time, based on which Bernoulli [1], who was Euler's assistant, developed his analogy for the vibration of beams only with bending rigidity. The error of results in Euler-Bernoulli theory was increased in higher modes and thicker structures, since no transverse shear stress was included. For this reason, this theory forecasted natural frequency higher and deflection of the beam lower than the real values. To consider the effect of shear deformation, Timoshenko [2,3] formulated a beam equation in which transverse shear strain was constant along thickness. Shear stresses in Timoshenko theory were also constant across beam thickness and their values were equal to those for mid-plane while in real cases, shear stress was a second-order function of thickness and its value at the bottom and top surfaces was zero. Therefore, this theory demanded utilizing a shear correction coefficient for modification. Reddy [4] proposed a Third-order Shear Deformation Theory (TSDT) in which shear stress was the second-order function of thickness and on the top and bottom surfaces, its value was zero. Therefore, there was no need to apply shear correction coefficient to TSDT.

When the lateral deflection of beam is large, in-plane forces are important in transverse vibration. In this case, bending stiffness and stretching stiffness have

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interaction with each other [5]. Pai and Nayfeh [6] showed that Von-Karman strains could not be used to derive a fully nonlinear beam model. To fully account for geometry nonlinearity, the second Piola-Kirchhoff stresses should be used instead of Cauchy stresses. However, in this case, the evaluated stretching effect is not consistent yet by boundary condition. Since stiffness is dependent upon properties of the material in fiber reinforced beams, if stress and strain are defined with respect to un-deformed coordinates, geometrical nonlinearity can be mistaken as material nonlinearity. Consequently, Pai et al. [7] showed that local stress and strain were required. A number of researchers focused on finding the exact relation of deformed and un-deformed coordinates to precisely model geometrical nonlinearities [8–19]. The main problem in these researches was that finite rotations in Euler coordinates transformation were neither independent from each other nor along a three-orthogonal axis. Alkire [12] showed that different sequences of Euler rotations would result in different equations of motion. Ho et al. [20], using Green strain in longitudinal direction, investigated large-amplitude motion of simply supported beam. Heyliger and Reddy [21] and Sheinman and Adan [22], by applying Von-Karman strains, investigated large deflections of beams. Bolotin [23] and Moody [24] showed that nonlinear inertia effects were not as significant as nonlinear elasticity effects. However, Crespo da Silva and Glynn [11,25] showed that the generally ignored nonlinear terms, deduced by curvature, were of the same order of inertia nonlinearity terms and they might have a remarkable effect on the behavior of a structure. Nayfeh and Pai [26] showed that the nonlinear terms in mechanical structures were of hardening type and dominated lower modes. On the other hand, they showed that the nonlinear inertia terms were of softening type and became more effective at higher modes. Many other valuable researches have been conducted by other scientists to address the influence of nonlinear terms on the behavior of a beam [27].

Some other researches have recently been performed on the nonlinear vibration behavior of beams. Ahmed and Rhali [28] established a theoretical framework for the nonlinear transverse vibration of Euler-Bernoulli beams in which a finite number of masses were placed on arbitrary points of the length of a beam. Wang et al. [29] investigated the principal parametric resonance of axially accelerating hyperplastic beam. Seddighi and Eipakchi [30] adopted the multiple scales method to study the dynamics response of an axially moving viscoelastic beam with time-dependent speed. Casalotti et al. [31] studied multi-mode vibration absorption capability of a nonlinear Euler-Bernoulli beam. Flexural vibration superposition of Euler-Bernoulli beam was investigated in [32]. Asghari et al. [33] reviewed a nonlinear size-dependent Timoshenko beam model based on the modified couple stress theory. Lewandowski and Wielentejczyk [34] studied the problem of nonlinear steady-state vibration of beams harmonically excited by harmonic forces. The problem of geometrically nonlinear steady-state vibrations of beams excited by harmonic forces was investigated in [35]. Roozbahani et al. [36] studied the nonlinear vibrations of beams by considering Von-Karman's nonlinear strains and shear deformable theory of Timoshenko. In this research, the beam was excited by applying a suddenly electrostatic force. Alipour et al. [37] presented an analytical solution for nonlinear vibration of beams having been actuated electrostatically. Stojanpvoc [38] investigated the nonlinear vibrations of Timoshenko beams on nonlinear elastic foundation by considering geometrical nonlinearities.

In the literature, no specific way has been presented to determine the exact mathematical solution for geometrically nonlinear vibration of beams. In this paper, the relation between un-deformed and deformed coordinates of beam is determined, since, as it is shown, geometrically nonlinear terms are projections of axial stresses in vertical direction. The nonlinear PDE in Lagrange view is shown to be convertible in two linear PDEs in Euler view. These equations have interaction with each other and, as a result, are solved together. The mode shapes as well as natural frequency of nonlinear transverse vibration of beam will be obtained in this study for the first time. It is shown that the Ordinary Differential Equation (ODE) of beam is linear in time. The beam is considered to be 2D under nonlinear elasticity and Euler-Bernoulli hypothesis in which large deflections stretch its length. The results for the case of a pinned-pinned beam are compared with those obtained by Galerkin method and good agreement will be observed. The effect of amplitude on mode shapes and natural frequency is reviewed and the results are discussed.

2. Governing equation

2.1. Lagrange coordinates system

Consider the beam shown in Figure 1, whose geometrical and material properties are: width b, thickness h, length l, density ρ, and elasticity modulus E. The deformation field of beam under Euler-Bernoulli hypothesis is as follows [39]:

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_0(x,t)}{\partial x}, \tag{1}$$

$$w(x,z,t) = w_0(x,t). \tag{2}$$

In Eqs. (1) and (2), u_0 and w_0 are deformations of mid-

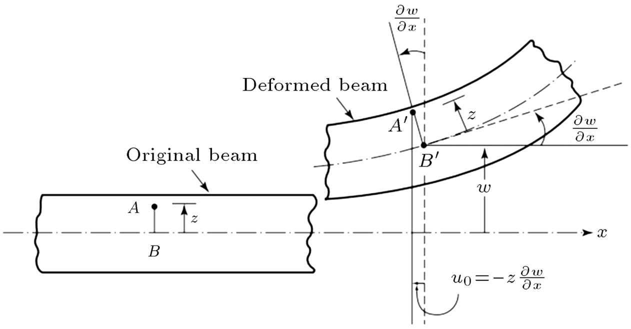

Figure 1. Geometry of a beam before and after deformation [44].

plane; also, u and w are deformations of any arbitrary point along x and z axes, respectively.

Now, consider the Lagrangian x, y, z and Eulerian x^*, y^*, z^* coordinates systems with a common origin before deformation (Figure 2). Equations of motion are derived in Lagrangian coordinates system, because it has a fixed origin. In the linear case, on each face of the infinitesimal element, there is only one component of stress, which is in the direction of the coordinate axes. However, in the nonlinear case, there are three effective stress components, which are in the direction of the coordinate axes. As shown in Figure 2, σ_{xz} is the only stress component in the z direction in the linear case, whereas for the nonlinear case, $\sigma_{xx}\partial w/\partial x + \sigma_{xy}\partial w/\partial y + \sigma_{xz}$ are applied in this direction. As a consequence, it can be noted that the nonlinear terms in equations of motion are the projects of in-plane stresses, which are achieved due to the large slopes. Thus, the governing nonlinear equations in lagerangian coordinate system can be expressed as [40]:

$$\frac{\partial}{\partial x}\left(\sigma_{xx} + \sigma_{xy}\frac{\partial u}{\partial y} + \sigma_{xz}\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\sigma_{yx} + \sigma_{yy}\frac{\partial u}{\partial y} + \sigma_{yz}\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(\sigma_{zx} + \sigma_{zy}\frac{\partial u}{\partial y} + \sigma_{zz}\frac{\partial u}{\partial z}\right) + f_x = \rho u_{,tt}, \quad (3)$$

$$\frac{\partial}{\partial x}\left(\sigma_{xx}\frac{\partial v}{\partial x} + \sigma_{xy} + \sigma_{xz}\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial y}\left(\sigma_{yx}\frac{\partial v}{\partial x} + \sigma_{yy} + \sigma_{yz}\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial z}\left(\sigma_{zx}\frac{\partial v}{\partial x} + \sigma_{zy} + \sigma_{zz}\frac{\partial v}{\partial z}\right) + f_y = \rho v_{,tt}, \quad (4)$$

$$\frac{\partial}{\partial x}\left(\sigma_{xx}\frac{\partial w}{\partial x} + \sigma_{xy}\frac{\partial w}{\partial y} + \sigma_{xz}\right) + \frac{\partial}{\partial y}\left(\sigma_{yx}\frac{\partial w}{\partial x} + \sigma_{yy}\frac{\partial w}{\partial y} + \sigma_{yz}\right) + \frac{\partial}{\partial z}\left(\sigma_{zx}\frac{\partial w}{\partial x} + \sigma_{zy}\frac{\partial w}{\partial y} + \sigma_{zz}\right) + f_z = \rho w_{,tt}. \quad (5)$$

The above equations are well-known Lagrangian ones using Kirchhoff stress components in nonlinear elasticity and no Euler-Bernoulli beam assumption has been applied yet. Density ρ is considered to be constant. To obtain Euler-Bernoulli beam equations, we should consider that:

1. All of the stress components can be neglected compared with σ_{xx} and σ_{xz};
2. The existing variables in the governing equations, which are the results of the beam deformation field (Eqs. (1) and (2)), are only the unknown functions of x and determined with respect to beam area;
3. The initially perpendicular straight lines to mid-plane remain straight and perpendicular after deformation.

In addition to the mentioned assumptions, in studying the transverse vibrations of mechanical structures, the

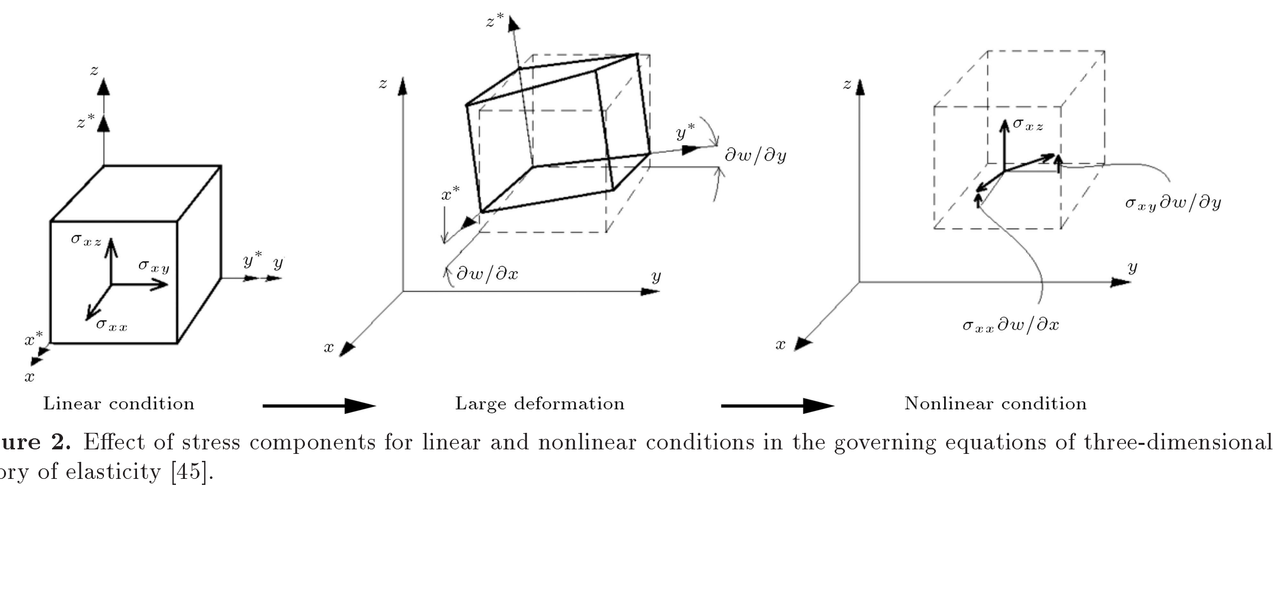

Figure 2. Effect of stress components for linear and nonlinear conditions in the governing equations of three-dimensional theory of elasticity [45].

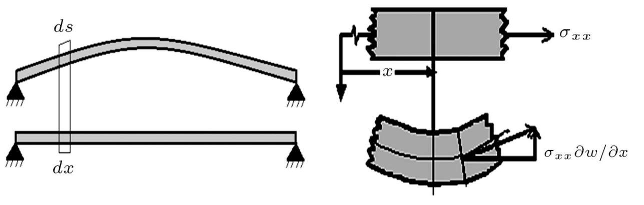

Figure 3. Stretch of beam develops in-plane stress, which appears in transverse vibration.

terms $\partial u/\partial x_i$ and $\partial v/\partial x_i$ are negligible compared with $\partial w/\partial x_i$. As it is observed in Figure 3(b), transverse deflection of the beam only can induce the nonlinear terms involving $\partial w/\partial x$ and the other two slopes ($\partial u/\partial x_i$ and $\partial v/\partial x_i$) become remarkable when the beam undergoes large in-plane deformations.

Using the third assumption, we have:

$$Q_x = \frac{\partial M_x}{\partial x}. \tag{6}$$

The variables in Eq. (6) are defined as $Q_x = \int \sigma_{xz} dA$ and $M_x = \int z.\sigma_{xx} dA$ (these stress components can be seen in the first term of Eq. (5)). In the governing Eqs. (3)–(5), the differentials are with respect to x, y, and z, while the variables are only unknown functions of x. Therefore, by integration in the beam area, the governing equations would be only in terms of differentials with respect to x. Applying beam assumptions, neglecting body forces f_i, and replacing Eq. (6) into (5), we achieve the governing equations in the following form:

$$\frac{\partial N_x}{\partial x} = m\frac{\partial^2 u_0}{\partial t^2}, \tag{7}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x} + N_x\frac{\partial w_0}{\partial x}\right) = m\frac{\partial^2 w_0}{\partial t^2}. \tag{8}$$

One can see that Eq. (2) is used in Eq. (8) (i.e. $\partial w/\partial x = \partial w_0/\partial x$). In these equations, the new variables are $N_x = \int \sigma_{xx} dA$ and $m = \int \rho dA$. To resolve the governing equations exactly, we should have a better understanding of the existing terms in Eqs. (7) and (8). Each term in the above equations has been appeared due to a particular type of deformation in the structure. The strain field in nonlinear vibration is $\varepsilon_x = \partial u/\partial x + (\partial w/\partial x)^2/2$ [39], where u is defined by Eq. (1). In Eq. (7), N_x arises from u_0 and plays no role in transverse vibrations. Therefore, Eq. (7) can be neglected when $u_0 = 0$. However, M_x in Eq. (8) arises from bending effect and is deduced by the second part of Eq. (1), which plays the main role in the transverse vibration. The most important term in the nonlinear transverse vibration is $N_x\partial w_0/\partial x$. Although it is deduced by an in-plane deformation, it only appears in transvers vibration and plays no role in the in-plane vibration. The geometrical justification for this term is given in Figure 3. In Figure 3(a), as can be observed, large deflection leads to increase in the length of the beam and the resulting strain can be considered as $(dS-dx)/dx = (\partial w/\partial x)^2/2$, which has already been shown in the strain field. As a result, although $u_0 = 0$, the stress component σ_{xx} is developed. Since $\partial w/\partial x$ is big, as shown in Figure 3(b), in this case, σ_{xx} is effective in transverse vibrations.

2.2. Euler coordinates system

Consider Figure 2 in which the components of stress field constitute the linear governing equations in the x^*, y^*, z^* coordinates system. It is obvious that the governing equations in this coordinates system are very simple. In three-dimensional theory of elasticity, the relation between two coordinates is unknown. However, here, as shown in Figure 3(b), the normal stress is along beam length appearing in z direction due to large deflection. It is clear in Figure 4 that the rotation value of the Euler configuration with respect to Lagrange configuration is equal to $\partial w/\partial x$. As a consequence, the problem is linear in $x^* - z^*$ coordinates system. This coordinates system has been placed on the mid-plane. The deformation field variables in Eulerian coordinates system are u_0^* and w_0^* for the mid-plane; also, u^* and w^* are the deformations of any arbitrary point of the beam along x^* and z^* directions, respectively. Although it can be neglected when $u_0 = 0$, Eq. (7) should be included in the Eulerian coordinates system, since $u_0^* \neq 0$. Because in Eulerian coordinates system, the term deduced by increasing length of the beam N_{x^*} appears in x^* direction, not z^*. The governing equations in Eulerian coordinates system are in the following form:

$$\frac{\partial N_{x^*}}{\partial x^*} = m\frac{\partial^2 u_0^*}{\partial t^2}, \tag{9}$$

$$\frac{\partial^2 M_{x^*}}{\partial x^{*2}} = m\frac{\partial^2 w_0^*}{\partial t^2}. \tag{10}$$

As a proof, these equations can be obtained simply using a well-known infinitesimal element in which, based on Figure 4, transverse shear stress is along z^* and normal stress is along x^*.

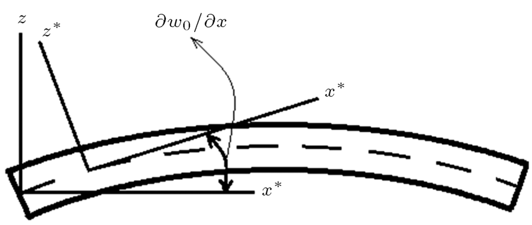

Figure 4. Rotation of Euler configuration with respect to Lagrange coordinates is equal to beam slope curve.

3. Exact mathematical solution

The governing equation of nonlinear transvers vibration of beam in Lagrange coordinates system can be obtained by using the corresponding form of Eq. (8) in terms of the strain field components [39] in the following form:

$$-EI\frac{\partial^4 w_0}{\partial x^4}+\frac{3}{2}AE\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial w_0}{\partial x}\right)^2=m\frac{\partial^{w_0}}{\partial t^2}. \quad (11)$$

The above equation is a nonlinear partial differential equation and has no exact mathematical solution due to the presence of the nonlinear term. In this equation, I and A are surface inertia and area of beam, respectively. Referring to Figure 3, the applied forces in nonlinear vibration are shown in Figure 5. In this figure, the vector R can be representative of the left-hand side of Eq. (8). It is obvious that Eq. (11) can be decomposed into two coupled equations in the following form:

$$EA\frac{\partial^2 u_0^*}{\partial {x^*}^2}=m\frac{\partial^2 u_0^*}{\partial t^2}, \quad (12)$$

$$-EI\frac{\partial^4 w_0^*}{\partial {x^*}^4}=m\frac{\partial^2 w_0^*}{\partial t^2}. \quad (13)$$

In the above equations, u_0^* is the stretch value representing the change in the length and in-plane deformation is neglected ($u_0=0$). The boundary conditions for simply supported beam cause the deformation in Eq. (12) as well as the moment and deflection in Eq. (13) to be equal to zero:

$$u^*\left(x^*,t\right)=w^*\left(x^*,t\right)=\frac{\partial^2 w^*\left(x^*,t\right)}{\partial {x^*}^2}=0,$$

$$x^*=0,l^*. \quad (14)$$

In Eq. (14), we have $l^*=\int_0^l\sqrt{(1+(\partial w/\partial x)^2)}dx$. For the linear transverse vibration, Eq. (13) is solved independently and when the in-plane vibration is considered, Eq. (12) should be re-solved. However, for the present problem, both equations are effective, simultaneously, and they have interaction with each other. Therefore, in the nonlinear free vibrations, in addition to Eqs. (12)–(14), the geometry of deformation seen in Figure 5 should be satisfied. By considering the method of separation of variables, we consider the following functions:

$$u_0^*(x^*,t)=U^*(x^*)\sin(\omega t), \quad (15)$$

$$w_0^*(x^*,t)=W^*(x^*)\sin(\omega t), \quad (16)$$

$$w(x,t)=W(x)\sin(\omega t). \quad (17)$$

The three variables $u_0^*(x^*,t)$, $w_0^*(x^*,t)$, and $w(x,t)$ constitute a deformation field, simultaneously. As a result, the time-dependent function $\sin(\omega t)$ is considered for them. Meanwhile, the geometrical constraint presented in Figure 5, ($\vec{W}^*+\vec{U}_i=\vec{W}$) induces the following equations:

$$U_i=\left|W^*\frac{dW^*}{dx^*}\right|, \qquad \left(\frac{\partial w^*}{\partial x^*}\cong\frac{\partial w}{\partial x}\right), \quad (18)$$

$$W(x)=\sqrt{\left(U_i(x^*)\right)^2+(W^*(x^*))^2}. \quad (19)$$

In Eq. (18), the variable U_i is considered to be an only positive number, because for the immovable boundary conditions considered here, only the beam is allowed to stretch. Since Eqs. (12) and (13) are to be solved simultaneously, the mode shapes of U^* may not satisfy the geometry constraint in Figure 5. Hence, U_i in Eq. (18) is considered by combining several modes of U^*. In fact, the shown constraint in Figure 5 determines U^*; accordingly, the resultant vector in Eq. (19) is vertical. Replacing Eqs. (15) and (16) into Eqs. (12) and (13), respectively, one can separately find solutions to them for boundary conditions in Eq. (14) in the following form [41]:

$$U_r^*(x^*)=A_r\sin\left(\frac{r\pi}{l^*}x^*\right), \qquad \omega_r^2=\frac{EA}{m}\left(\frac{r\pi}{l^*}\right)^2, \quad (20)$$

$$W_s^*(x^*)=A_s\sin\left(\frac{s\pi}{l^*}x^*\right), \qquad \omega_s^2=\frac{EI}{m}\left(\frac{s\pi}{l^*}\right)^4. \quad (21)$$

While independent solutions have been obtained, the interaction between them should be included. Inserting

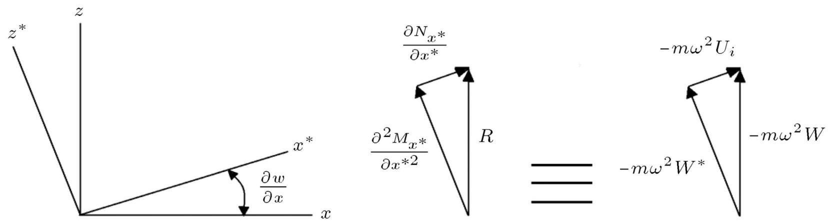

Figure 5. The nonlinear governing equation in Lagrange coordinates can be decomposed into two linear coupled equations in Euler coordinates.

Eqs. (21) into Eq. (18), we find:

$$U_i = \left| A_s^2 \frac{s\pi}{2l^*} \sin\left(\frac{2s\pi}{l^*} x^*\right) \right|. \tag{22}$$

We realize from Eq. (22) that as the sth mode of transverse vibration is excited (W_s^*), the $r = 2s$th mode of in-plane vibration (U_r^*) simultaneously appears. Their combination gives the sth mode of nonlinear transverse vibrations ($W(x)$). Therefore, in the nonlinear transverse vibration of beam for $r = 2s$, the governing equation and boundary conditions as well as the geometrical constraint in Figure 5 are satisfied. If we consider the weight coefficient of contribution of in-plane modes U_r^* to the nonlinear transverse vibration as c_r ($U_i = \sum_{r=1}^{\infty} c_r U_r^*$), Eq. (22) implies that:

$$c_r = 1 \quad (r = 2s), \qquad \text{and} \qquad c_r = 0 \quad (r \neq 2s). \tag{23}$$

Using Eqs. (21) and (22) in Eqs. (15)–(17) results in the following equations:

$$u_0^*(x^*, t) = \left| \Lambda^2 A_s^2 \frac{s\pi}{2l^*} \sin\left(\frac{2s\pi}{l^*} x^*\right) \sin(\omega t) \right|, \tag{24}$$

$$w_0^*(x^*, t) = \Lambda A_s \sin\left(\frac{s\pi}{l^*} x^*\right) \sin(\omega t), \tag{25}$$

$$w(x,t) = \Lambda A_s \sin\left(\frac{s\pi}{l} x\right) \sqrt{1 + \left(\Lambda A_s \frac{s\pi}{l} \cos\left(\frac{s\pi}{l} x\right)\right)^2} \sin(\omega t). \tag{26}$$

In Eqs. (24)–(26), the variable Λ is inserted to consider the effect of amplitude on nonlinear transverse vibration. In fact, the result for ΛA_s as the maximum deflection of beam along z^* represents the amplitude, because normalized mode shapes are used. It is observed that although Eqs. (24) and (25) are versus x^*, Eq. (26) is a function of x because w is perpendicular to x. Eqs. (20) and (22) indicate that the beam vibrates with natural frequency of ω_r as a rod and its natural frequency will be $\omega_r (r = 2s)$, due to stretching stiffness of the beam, if the mode shape deduced by Eq. (18) is imposed on the structure,. On the one hand, according to Eq. (21), the beam vibrates with the natural frequency of ω_s due to its bending stiffness in transverse vibration. However, in the nonlinear free vibrations, two equations are coupled together and both stretching and bending stiffness appear in natural frequency. Let us assume the stretching stiffness of beam is k_r in a way that $\omega_r^2 = k_r/m$ and bending stiffness is k_s so that $\omega_s^2 = k_s/m$. Here, it should be emphasized that k_r and k_s are both linear stiffness. A schematic model of the system is given in Figure 6 to more clarify the concept. It can be concluded from the physics of the problem, as shown in Figure 6, that the springs are parallel to each other, but their amplitudes are unequal. In addition, from the figure, we find out that stiffness after nonlinear vibrations is $k = k_s + \lambda^2 k_r$ and natural frequency will be $\omega^2 = \omega_s^2 + \lambda^2 \omega_r^2$. In this equation, λ is the ratio of amplitude of in-plane vibration to the amplitude of transverse vibrations (i.e. ΛA_s). One can compare Eqs. (24) and (25), which shows $\lambda = A_s \Lambda (s\pi/2l^*)$. Thus, the nonlinear natural frequency of beam ω can be obtained in the following form:

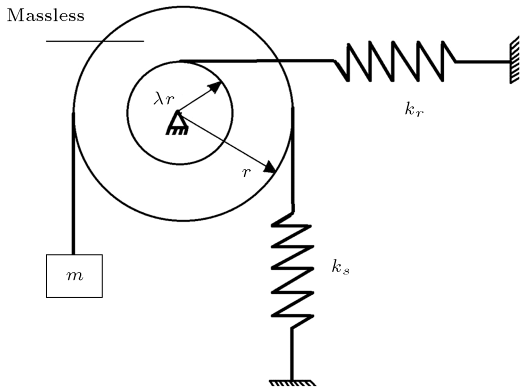

Figure 6. A schematic model of beam showing that its nonlinear stiffness is equal to linear combination of two linear springs.

$$\omega = \left(\frac{s\pi}{l^*}\right)^2 \sqrt{\frac{E}{\rho}\left((A_s\Lambda)^2 + \frac{h^2}{12}\right)}, \quad (s = 1, 2, 3, \cdots). \tag{27}$$

As will be discussed soon, the above natural frequency belongs to a hardening nonlinear system. For the present problem, nonlinear inertia has been neglected as it is not excited. However, for movable boundary conditions, such as a clamped-free beam, in-plane inertia becomes effective in transverse vibration and the problem in this case will be of the softening nonlinear type.

4. Results and discussion

It should be noticed that Eq. (26) does not satisfy Eq. (11). The reason lies in the simplifications made during the extraction process for governing equations of Lagrange coordinates, including the one considered in the Von-Karman strain field:

$$\frac{(dS - dx)}{dx} = \frac{\sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2} dx - dx}{dx} \cong \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \tag{28}$$

The approximation used in Eq. (28) incorporates only two terms in Taylor expansion. The reason for this restriction is that we investigate the problem in Lagrange coordinates system. Hence, even the exact solution to Eq. (11) will have lower accuracy than that to Eq. (26).

4.1. Mode shape

According to the studies in the literature and to the best knowledge of the author, the governing equation in Lagrange coordinates system has no exact mathematical explanation for the mode shape of beam. In the approximate methods such as Galerkin and Rayleigh-Ritz, the linear mode shapes, satisfying boundary conditions, can be used [41]. Therefore, to verify Eq. (26), we consider a fundamental origin in scientific theorizing. Consider two theories; one is general and covers an extensive spectrum of problems and the other is specific to a number of particular problems. Thus, the general theory covers the problems of the other theory. In this case, the limit in the results obtained by the general theory approaches that by the other theory when the particular problems are reviewed. As a consequence, the mode shapes shown in Eq. (26) should be the same as those of the linear problem when $\Lambda \to 0$. This is shown in Figure 7. It is seen that as Λ decreases, the mode shape tends to the shapes for linear vibration. The effect of nonlinear terms in several modes of vibration is depicted in Figure 8. We notice in this figure that the nonlinear terms become more evident in higher modes than in lower modes.

4.2. Natural frequency

According to the literature on the problem [27], the ODE of the system is nonlinear in the time domain.

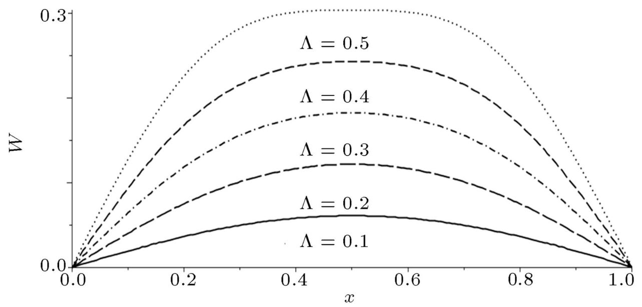

Figure 7. Nonlinear mode shapes tend to the shape of linear vibration when amplitude is small.

For a nonlinear ODE, natural frequency is a function of time ($\omega = \omega(t)$) [42]. However, Eq. (27) is constant in time domain and only varies as the amplitude of vibration changes by Λ. In addition, Eqs. (12) and (13) are linear functions of time. In fact, the nonlinear effects in the Euler coordinates system are included in governing equations by applying Eq. (18). In Eq. (18), a nonlinear mode shape is imposed on the in-plane vibration of beam and, based on fundamentals of vibration [41], this does not make the problem of longitudinal vibration nonlinear. To find the reason for this paradox, we review the effect of nonlinear terms on the behavior of structure and natural frequency, qualitatively.

4.2.1. Effect of natural frequency on linear resonance

Consider a one degree of freedom system with the natural frequency of $\omega_n = 1/\pi$. In this case, the direction of motion changes each second. Because natural frequency is constant and period of the system is only a function of natural frequency, the period is constant, too. The work done by excitation force can be calculated by $\Phi = \int_0^t F\dot{w}dt$. If the excitation force and the motion are in the same direction, acceleration of the system is positive; otherwise, it will be negative. Consider the case in which a harmonic force $F = F_0 g(\Omega, t)$ is applied to the system. $g(\Omega, t)$ can be any arbitrary harmonic function. Excitation frequency (Ω) shows when the direction of the applied force changes. Hence, if $\Omega = \omega_n$, the applied force and the motion of the system have the same direction and energy of the system will continuously increase ($t \to \infty \Rightarrow \Phi \to \infty$). This phenomenon is known as resonance. In the nonlinear resonance, the same thing occurs, but natural frequency changes during the motion of the beam, which will be discussed in the next section.

4.2.2. Effect of natural frequency on nonlinear resonance

Linear natural frequency is only dependent on the material properties of the system. As a general definition, natural frequency of a mechanical system is the ratio of the internal driving forces to the internal inertia forces of that system. For example, in a one

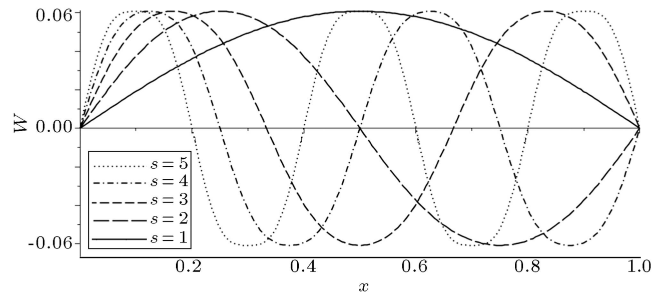

Figure 8. Nonlinear terms become more effective on mode shapes in higher modes of vibration.

degree of freedom mass-spring system, $\omega_n = \sqrt{K/M}$, where spring stiffness (K) represents the driving forces and mass (M) denotes the inertia forces, which resists acceleration.

Natural frequency of linear systems is constant and occurs because the driving and inertia forces do not vary during vibration. However, in the nonlinear vibration, driving forces change with a change in the amplitude of motion. For example, consider a nonlinear vibrating system with the following equation of motion:

$$M\ddot{f} + Kf + \alpha f^3 = 0 \quad (\text{i.e.. } \ k_1 = K, \ k_2 = \alpha f^2).$$

Since the driving forces are dependent on displacement, the effect of α in this equation is to change the driving forces. If α is positive, increasing amplitude of motion increases the driving forces and if α is negative, large amplitude decreases these forces. This is why in the hardening nonlinear systems, natural frequency increases by amplitude of the motion whereas natural frequency of softening systems has an inverse relation with amplitude of motion. Nayfeh and Mook [42] expressed that the initial condition was an effective parameter on the nonlinear frequency. The proposed amplitude in this study can include initial conditions, too. The reason is that the sum of initial kinetic and potential energy determines the amplitude for maximum potential energy of the system at initial conditions. Moreover, the behavior of nonlinear frequency in terms of amplitude of vibration can be described in a better way, which will be shown soon.

In free vibration, energy of the system is constant and equal to the initial energy. However, in the forced vibration, it changes over time. Therefore, in the nonlinear forced vibration, the initial conditions give no information about the nonlinear frequency, except at the initial time (i.e., $t = 0$). However, if the amplitude of motion is considered, it will be possible to analyze the nonlinear frequency at any moment of the motion.

Now, the resonance phenomenon of nonlinear systems can be simply explained. In the linear resonance (see Section 4.2.1), the excitation force and the motion of system are in the same direction, because the excitation frequency is equal to the natural frequency. Regarding the fact that the natural frequency of nonlinear oscillation is dependent on the amplitude of motion, resonance phenomenon increases the amplitude of motion and, consequently, the natural frequency will be changed. Thus, the system is either softening ($\omega_n < \Omega$) or hardening ($\omega_n > \Omega$).

As a result, resonance phenomenon will be finished ($\Omega \neq \omega_n$). This is why in the nonlinear case, the excitation frequency is considered as $\Omega = \omega_n + \epsilon\sigma$ [42]. Using this scheme, the detuning parameter can vary in a way that $\Omega = \omega_{nonlinear}$ is satisfied for the whole process. This is shown in Figure 9. At the origin ($\sigma = 0$), excitation frequency and linear natural frequency are equal, resulting in the resonance. In a hardening system, increase in the amplitude due to the resonance results in $\omega_n > \Omega$. Therefore, to remain in the resonance region, σ should increase according to the variation of ω_n. By re-occurrence of the resonance, the amplitude and subsequently, the natural frequency increase again. Hence, σ should increase again and this process is repeated frequently. Thus, in the hardening case (Figure 9(b)), the curve bends away from the vertical axis to the right side. On the other hand, in the softening nonlinear systems, increasing amplitude leads to $\omega_n < \Omega$. Therefore, in these systems, σ should decrease with the change of ω_n and, as a result, frequency response curve will be bent to the left hand (Figure 9(c)).

4.2.3. Nonlinear frequency of beam

As a well-known phenomenon, the reason of which was described in the previous section, natural frequency of a nonlinear system increases as the amplitude is enhanced (hardening). For a nonlinear PDE (beam equation), either the stiffness is a function of time or it is independent of time. In the first case, the governing ODE is nonlinear ($k_2 = \alpha f^2$) and natural frequency changes, although the amplitude is constant ($\omega = \omega(t)$). However, in the second case, natural frequency is constant when amplitude is constant. Therefore, according to the literature, ODE for nonlinear vibration of beam is nonlinear, which results in $\omega = \omega(t)$, and according to present exact mathematical solution, the governing ODE is linear. Now, we review the problem

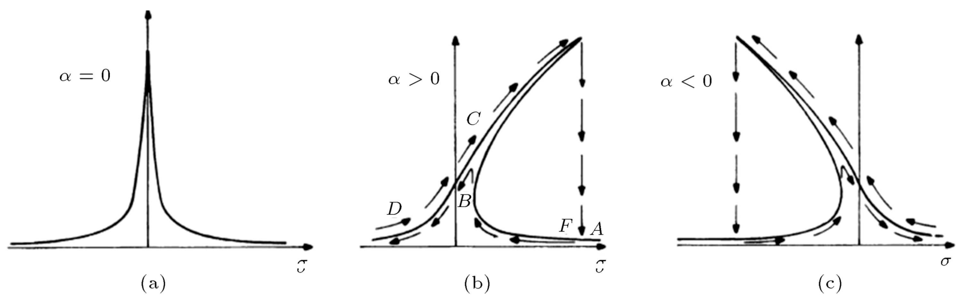

Figure 9. Frequency response of (a) linear, (b) hardening nonlinear, and (c) softening nonlinear systems [31].

to find the fact first from Euler's point of view and then, from Lagrange's point of view.

From Euler's point of view, regarding Eq. (27), the stiffness of beam is obtained from $k = k_s + \lambda^2 k_r$. Both bending stiffness and stretching stiffness are linear (i.e. $k_s \neq k_s(w, \Lambda), k_r \neq k_r(u, \Lambda)$). The other parameter is a function of amplitude $\lambda = \lambda(\Lambda)$ and therefore, $\omega = \omega(\Lambda)$. This is due to the fact that a nonlinear PDE is linear over time, but nonlinear in space.

From Lagrange's point of view, applying approximate methods such as Galerkin to Eq. (11) gives the same ODE discussed in Section 4.2.2, i.e. $M\ddot{f} + Kf + \alpha f^3 = 0$. This equation is a nonlinear ODE whose nonlinearity order is three. The number 3 appears in this equation from $N_x \partial w_0/\partial x$ in Eq. (8). The stress component here is N_x arising from the stretching effect $(E(\partial w_0/\partial x)^2/2)$. By multiplying it by the slope, the resulting equation is a third-order function of slope and, as a result, a third-order function of $f(t)$.

Let us investigate N_x, which is a second order function of $f(t)$, since in the Taylor expansion of Eq. (28), two terms are included. Therefore, by considering more terms in Taylor expansion, the order of the resulting ODE will be, for example, five or seven. As we know, the governing equation of any arbitrary system represents its inherent properties and these properties are not dependent upon the number of terms in Taylor expansion considered by the researcher. On the other hand, the coefficient $\partial w_0/\partial x$ has been multiplied by N_x in Eq. (8) to give its projection in the vertical direction, while in Euler coordinates, N_{x^*} has not been multiplied by the slope since it is along x^*. As a result, if we consider two terms in Taylor expansion, the resulting ODE will be of the second-order nonlinearity in Euler coordinates, while it is of the third-order nonlinearity in Lagrange coordinates system. Based on the fundamentals in deriving the equations, the governing equations should be independent of the considered coordinates system based on which they are extracted. However, we observe that the problem in the Lagrange coordinates does not satisfy this basic concept and depends on the number of terms in Taylor expansion.

To overcome the problem, here, the author suggests that, to derive the nonlinear strain field, first, the time-dependent function should be separated:

$$\frac{d(S-x)}{dx} = \frac{\sqrt{1+\left(\frac{dW}{dx}\right)^2}dx - dx}{dx} \cong \frac{1}{2}\left(\frac{dW}{dx}\right)^2. \tag{29}$$

The above equation is, in fact, another form of Eq. (18), which is independent of time. We conclude that Eq. (28) violates the shown geometrical constraint in Figure 5. Now, considering more terms in Taylor expansion has no effect on the inherent properties of the system. Using Eq. (29) in the governing equation and separating the time-dependent function, one can obtain $((\partial w/\partial x)^3 = (dW/dx)^3.f(t))$:

$$-\frac{EI}{m}\frac{d^4W}{dx^4} + \frac{3}{2}\frac{AE}{m}\left(\frac{d^2W}{dx^2}\right)\left(\frac{dW}{dx}\right)^2 + \omega^2 W = 0. \tag{30}$$

The first term in Eq. (30) involves bending stiffness k_s and the second term involves stretching stiffness $\lambda^2 k_r$, the sum of which gives total stiffness k. Therefore, for verification of the results of the present work, one can apply Galerkin method to Eq. (30) and calculate the bending and stretching stiffness. The comparison is shown in Table 1. The considered geometrical and material properties in this table are: (E = 200 GPa, h = 0.05 m, b = 0.02 m, l = 2 m, ρ = 2700 kg/m^3, Λ = 0.1 m). It is clear that there is a good agreement between the results. The Von-Karman strain field

Table 1. Comparison of nonlinear natural frequency in Eq. (27) with those obtained by the Galerkin method.

Mode (s)	Method	$\lambda\omega_r$ ($r = 2s$)	ω_s	ω
1	Exact	1292.3822	306.51536	1328.2332
	Galerkin	1300.4345	306.51536	1332.0695
2	Exact	5169.5290	1226.0614	5312.9330
	Galerkin	5201.7382	1226.0614	5344.2780
3	Exact	11631.440	2758.6382	11954.099
	Galerkin	11703.911	2758.6382	12024.625
4	Exact	20678.611	4904.2458	21251.112
	Galerkin	20806.953	4904.2458	21377.112
5	Exact	32309.356	7662.8841	33205.831
	Galerkin	32510.864	7662.8841	33401.737

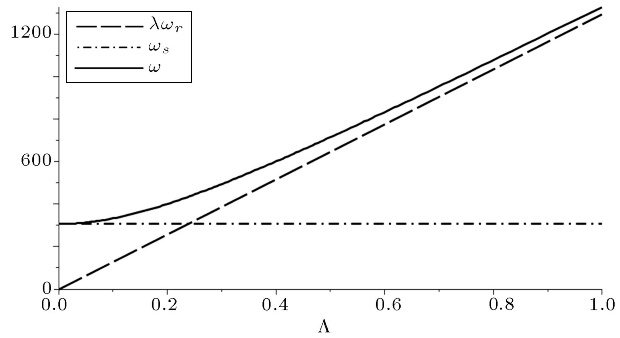

Figure 10. Effect of amplitude on nonlinear frequency.

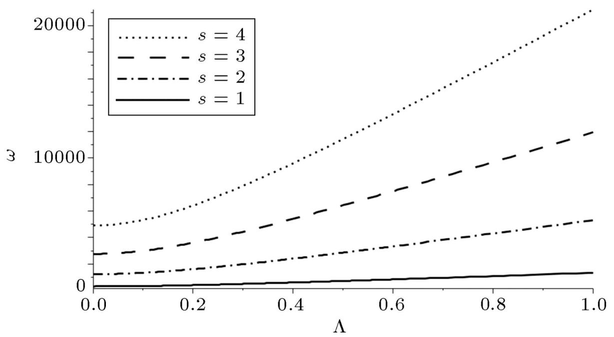

Figure 11. Effect of nonlinear terms on natural frequency increases as the number of modes increases.

satisfies neither Eq. (12) and boundary condition of Eq. (14) nor the geometrical constraint in Eq. (18). For this reason, it has a certain error in Table 1.

The effect of amplitude Λ on ω_s, $\lambda\omega_r$, and ω is depicted in Figure 10. As expected, bending stiffness is not affected by amplitude. For small deflection, the bending effect is dominant; but, as can be observed, when amplitude increases, the stretching effects become dominant. The effect of nonlinearity on higher modes is shown in Figure 11. It is obviously observed that the stretching effect on higher modes is more significant. Material properties of the beam in these figures are the same used in Table 1.

Based on the literature, it is considered that when the ratio of amplitude to thickness is $\Lambda/h > 0.5$, the nonlinear terms should be considered in the governing equations [43]. However, in Eq. (27), thickness of structure is not seen in stretching stiffness. According to the same equation, the nonlinear terms are proportional to Λ/l. The same conclusion can be drawn from Eq. (26). Thus, thickness appears to have no direct effect on stretching stiffness. This is shown in Figure 12. It is obvious that for a specified thickness and amplitude, the effect of nonlinear terms becomes more sensible as the length of beam is reduced. Nevertheless, the effect of thickness is not necessarily neutral. Since bending stiffness is proportional to the second order of thickness, when the beam is thin, thickness is very effective with respect to amplitude enhancement. However, this effectiveness is reduced as the beam becomes thicker. This issue is represented in Figure 13.

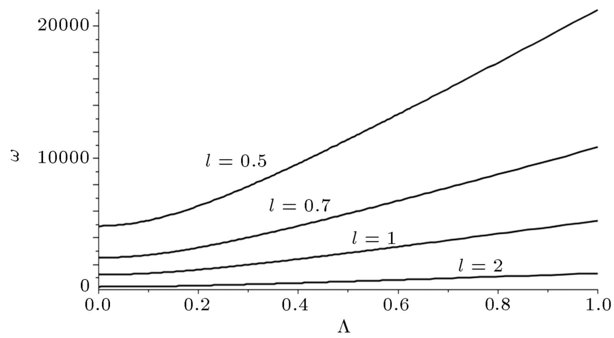

Figure 12. Effect of amplitude to length ratio on the nonlinear natural frequency.

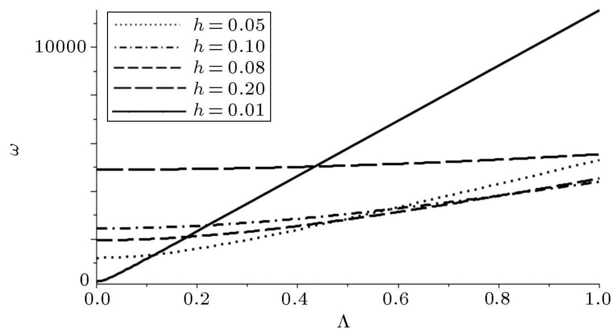

Figure 13. Effect of thickness on nonlinear natural frequency.

5. Conclusion

The nonlinear partial differential equation of beam in Lagrange coordinates system, as the vector of the resultant forces, was decomposed into its constituting vectors along and perpendicular to the beam. The result was two linear partial differential equations, which were solved together since they had interaction with each other. Mode shapes and natural frequency of nonlinear vibration of the beam were achieved for the first time. Nonlinear mode shapes were shown to tend to the shapes for the linear vibration for small amplitude. Natural frequency, arising from bending and stretching stiffness, was compared with those given by Galerkin method and good agreement was observed. It was shown that to the sth mode of nonlinear vibration, the sth mode of bending stiffness and the $2s$th mode of in-plane stiffness contributed, simultaneously. The governing Ordinary Differential Equation (ODE) of the system was shown to be linear in the time domain. It was shown that the effect of nonlinear terms in the governing equation was proportional to amplitude to length ratio (Λ/l).

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Biography

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