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Optical solitons in birefringent fibers with four-wave mixing by extended trial equation method

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Abstract. This paper obtains optical soliton solutions in birefringent fibers that are studied in the presence of four-wave mixing. The extended trial function scheme is the integration algorithm, which is applied. Both Kerr law and parabolic laws are taken into account. The soliton solutions are presented with relevant integrability criteria.

1. Introduction

Birefringence is a very common phenomenon in optical fibers. The aspect of pulse polarization, which leads to birefringence, is unavoidable and unwanted. Therefore, it is important to study these split pulses and retrieve their soliton solutions through a variety of integration schemes that are at disposal in present times [1–30]. An additional factor is that this birefringence comes with four-Wave Mixing (4WM) that serves as an additional unwanted feature. This is true for both Kerr law and parabolic law fibers. It is well known that Kerr law nonlinearity, also known as cubic nonlinearity, arises when refractive index is dependent on the intensity of light [2]. The second form of nonlinearity that is studied in this paper is parabolic law, which is alternatively referred to as the cubic-quintic nonlinear form. This law appears in crystals; in particular, for p -toluene sulfonate crystals, the quintic nonlinearity dominates [2]. This paper retrieves soliton solutions to birefringent fibers in the presence of 4 WM for both of these waveguides by the aid of a rich and powerful integration scheme.

There are a variety of mathematical methods that have been implemented to study the phenomena of birefringence in optical fibers. These are the familiar method of undetermined coefficients labeled in the papers as ansatz approach [1,3,11,23–26], Darboux transform [29], modified simple equation method [4], trial function method [4], and the extended trial function scheme [5,8–10,13,30]. The last approach, namely the extended trial function scheme, was first reported during 2013 [13]. Later, this method has gained popularity and has been successfully applied to a variety of nonlinear phenomena. These include the study of shallow-water waves [8], Dense Wavelength Division Multiplexing (DWDM) topology [9], magneto-optic waveguides [10], and optical soliton dynamics with Biswas-Milovic equation [30]. Each of these algorithms has its relative merits and demerits, as indicated in the respective works. This paper will apply the extended trial function method in detail to retrieve soliton solutions to birefringent fibers studied

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in the presence of Four-Wave Mixing (FWM) with two nonlinear forms, and they are Kerr law and parabolic law. Notably, one needs to apply the phase-matching condition for integrability purposes because of the presence of 4 WM.

While the governing model is considered with Kerr and parabolic laws of nonlinearity, there are a few features that are deliberately not considered. They are the effects of higher order dispersions such as third-Order Dispersion (3OD) and fourth-Order Dispersion (4OD). These terms when treated with additional strong perturbative effects such as self-steepening and nonlinear dispersion will be studied in the future with this integration scheme. An additional integration scheme to retrieve soliton solutions with 3OD and 4OD is the application of Lie transform, which is beyond the scope of the current work. The current research focuses on the unperturbed model with the effect of 4WM effects only. It is also well known that these 3OD and 4OD introduce the effect of dispersive wave emission, also commonly known as soliton radiation. These are non-trivial aspects that are also omitted from this paper. The effects of soliton radiation cannot be handled by the aid of extended trial function method. Instead, additional integration schemes are to be implemented in this case. They stem from the method beyond all-order asymptotics, variational principle, or the inverse scattering transform. Such studies are reserved for future endeavors. The rest of the paper gears up details of extended trial function scheme applied to birefringent fibers with 4WM that are enumerated in the subsequent sections.

2. Kerr law nonlinearity

The simplified version of the governing equation for soliton dynamics in birefringent fibers that follow Kerr law nonlinearity is given by the coupled Nonlinear Schrödinger's Equation (NLSE) in its dimensionless form [4,24,25]:

$$iq_t+aq_{xx}+bq_{xt}+\left(\xi_1|q|^2+\eta_1|r|^2\right)q+\sigma_1q^*r^2=0, \quad (1)$$

$$ir_t+ar_{xx}+br_{xt}+\left(\xi_2|q|^2+\eta_2|r|^2\right)r+\sigma_2r^*q^2=0. \quad (2)$$

This coupled system of NLSEs given by Eqs. (1) and (2) governs soliton propagation through nonlinear optical fibers with Kerr law nonlinearity. Here, a represents the coefficients of Group Velocity Dispersion (GVD), while b gives coefficients of spatio-temporal dispersion. Then, ξ_l and η_l for $l=1,2$ are the coefficients of Self-Phase Modulation (SPM) and Cross-Phase Modulation (XPM), respectively. Finally, σ_l gives the 4 WM terms.

After picking the starting hypothesis to be:

$$q(x,t)=P_1(\zeta)\exp[i\phi(x,t)], \quad (3)$$

and:

$$r(x,t)=P_2(\zeta)\exp[i\phi(x,t)], \quad (4)$$

where:

$$\zeta=k(x-vt), \quad (5)$$

and the phase component ϕ is:

$$\phi(x,t)=-\kappa x+\omega t+\theta, \quad (6)$$

so that the real-part equation changes to:

$$k^2(a-bv)P_1''+\left(b\omega\kappa-\omega-a\kappa^2\right)P_1+\xi_1P_1^3+\left(\eta_1+\sigma_1\right)P_1P_2^2=0, \quad (7)$$

and:

$$k^2(a-bv)P_2''+\left(b\omega\kappa-\omega-a\kappa^2\right)P_2+\xi_2P_2^3+\left(\eta_2+\sigma_2\right)P_2P_1^2=0. \quad (8)$$

The imaginary equation yields the speed of the soliton, as indicated earlier [25]. From the phase component, κ is the soliton frequency, while ω gives the soliton wave number and, finally, θ is the soliton phase constant. This coupled system given by Eqs. (7) and (8) will be now analyzed further, in the next subsection, to seek bright, dark, and singular solitons.

2.1. Extended trial function approach

To start with the extraction of solutions to Eqs. (7) and (8), the following assumption for the soliton structure is made:

$$P_1=\sum_{i=0}^{\varsigma}\tau_i\Psi^i, \quad (9)$$

$$P_2=\sum_{i=0}^{\tilde{\varsigma}}\tilde{\tau}_i\Psi^i, \quad (10)$$

where:

$$(\Psi')^2=\Lambda(\Psi)=\frac{\Phi(\Psi)}{\Upsilon(\Psi)}=\frac{\mu_\varrho\Psi^\varrho+\cdots+\mu_1\Psi+\mu_0}{\chi_\rho\Psi^\rho+\cdots+\chi_1\Psi+\chi_0}. \quad (11)$$

Here, $\tau_0,\cdots,\tau_\varsigma$, $\tilde{\tau}_0,\cdots,\tilde{\tau}_{\tilde{\varsigma}}$, $\mu_0,\cdots,\mu_\varrho$, and $\chi_0,\cdots,\chi_\rho$ are constants to be determined later. Then, Eq. (11) can be reduced to the elementary integral form as follows:

$$\pm(\zeta-\zeta_0)=\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}}=\int\sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}}d\Psi. \quad (12)$$

According to the balancing principle [28], one determines a relation of ϱ, ρ, ς, and $\tilde{\varsigma}$ is given by:

$$\varsigma=\tilde{\varsigma}=\frac{\varrho-\rho-2}{2}. \quad (13)$$

In order to make $\Phi(\Psi)$ and $\Upsilon(\Psi)$ polynomials in Eq. (11) have the possible least degree, that is, to have the simplest form of the integral given in Eq. (12), let us choose $\varrho = 4$, $\rho = 0$, and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (13). This means that the extended trial function approach suggests using the finite expansions:

$$P_1 = \tau_0 + \tau_1 \Psi, \tag{14}$$

$$P_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \tag{15}$$

where τ_i, $\tilde{\tau}_i$ $(i = 0, 1)$ are constants to be determined later such that $\tau_1 \neq 0$ and $\tilde{\tau}_1 \neq 0$, and Ψ satisfies Eq. (11). Substituting these expansions into Eqs. (7) and (8) and solving the resulting system of algebraic equations, we recover by Eqs . (16) as shown in Box I, where $\mathcal{H}$ is given by:

$$\mathcal{H} = (\eta_1 + \sigma_1)(\eta_2 + \sigma_2) - \xi_1 \xi_2. \tag{17}$$

Substituting values of Eqs. (16) into Eqs. (11) and (12) leads to:

$$\pm(\zeta - \zeta_0) = \Omega \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{18}$$

where:

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4},$$

$$\Omega = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{19}$$

As a consequence, one recovers the traveling wave solutions to the model governed by Eqs. (1) and (2) in the forms:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, Eqs. (20) and (21) are obtained as shown in Box II;

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, Eqs. (22) and (23) are obtained as shown in Box III;

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, Eqs. (24)–(27) are obtained as shown in Box IV;

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, Eqs. (28) and (29) are obtained as shown in Box V;

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ Eqs. (30) and (31) are obtained as shown in Box VI, where modulus m is given by:

$$m^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \tag{32}$$

It is noted that λ_j for $j = 1, \cdots, 4$ is the root of the following equation:

$$\Lambda(\Psi) = 0. \tag{33}$$

Under the conditions $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$, and $\zeta_0 = 0$, Solutions (20)–(29) can be reduced to exact solutions in the following forms:

Plane wave solutions are obtained by Eqs. (34)–(37) as shown in Box VII.

Singular optical soliton solutions are obtained by Eqs. (38) and (39) as shown in Box VIII.

Finally, bright optical soliton solutions are obtained by Eqs. (40) and (41) as shown in Box IX, where:

$$M = \frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}, \tag{42}$$

$$\tau_1 = \pm\frac{i\tilde{\tau}_1\sqrt{\eta_1 - \xi_2 + \sigma_1}}{\sqrt{-\eta_2 + \xi_1 - \sigma_2}},$$

$$\tilde{\tau}_0 = \mp\frac{i\tau_0\sqrt{-\eta_2 + \xi_1 - \sigma_2}}{\sqrt{\eta_1 - \xi_2 + \sigma_1}},$$

$$\chi_0 = -\frac{2k^2\mu_4(a - bv)(\eta_2 - \xi_1 + \sigma_2)}{\tilde{\tau}_1^2\mathcal{H}},$$

$$\mu_3 = \mp\frac{4i\mu_4\tau_0\sqrt{-\eta_2 + \xi_1 - \sigma_2}}{\tilde{\tau}_1\sqrt{\eta_1 - \xi_2 + \sigma_1}},$$

$$\mu_1 = \pm\frac{2i\tau_0\sqrt{-\eta_2 + \xi_1 - \sigma_2}\left[4\mu_4\tau_0^2(\eta_2 - \xi_1 + \sigma_2) - \mu_2\tilde{\tau}_1^2(\eta_1 - \xi_2 + \sigma_1)\right]}{\tilde{\tau}_1^3(\eta_1 - \xi_2 + \sigma_1)^{3/2}},$$

$$\omega = \frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1 - \xi_2 + \sigma_1) + 2\mu_4(\eta_2 - \xi_1 + \sigma_2)\left(a\kappa^2(\eta_1 - \xi_2 + \sigma_1) - 3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa - 1)(\eta_1 - \xi_2 + \sigma_1)(\eta_2 - \xi_1 + \sigma_2)},$$

$$\mu_0 = \mu_0, \qquad \mu_2 = \mu_2, \qquad \mu_4 = \mu_4, \qquad \tau_0 = \tau_0, \qquad \tilde{\tau}_1 = \tilde{\tau}_1. \tag{16}$$

Box I

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1\pm\frac{\tau_1\Omega}{k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right], \quad (20)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1\pm\frac{\tilde{\tau}_1\Omega}{k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right]. \quad (21)$$

Box II

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{4\tau_1\Omega^2(\lambda_2-\lambda_1)}{4\Omega^2-\left[(\lambda_1-\lambda_2)\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right)\right]^2}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right], \quad (22)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{4\tilde{\tau}_1\Omega^2(\lambda_2-\lambda_1)}{4\Omega^2-\left[(\lambda_1-\lambda_2)\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right)\right]^2}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right]. \quad (23)$$

Box III

$$\widetilde{M}=\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{\lambda_3-\lambda_2}, \quad (43)$$

$$Q=\frac{k\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega}, \quad (44)$$

$$R=\frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}. \quad (45)$$

Note that the amplitudes of the solitons are given by Eqs. (42) and (43), while the inverse width of the soliton is given by Eq. (44). These solitons are valid for $\tau_1 < 0$ and $\tilde{\tau}_1 < 0$. Moreover, under the conditions $\tau_0 = -\tau_1\lambda_2$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_2$, and $\zeta_0 = 0$, Jacobi elliptic function solutions (30) and (31) are reduced to Eqs. (46) and (47) as shown in Box X, where:

$$M_1=\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}, \quad (48)$$

$$\widetilde{M}_1=\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}, \quad (49)$$

$$R_1=\frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4}, \quad (50)$$

$$Q_j=\frac{(-1)^j k\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega} \quad \text{for} \quad j=1,2. \quad (51)$$

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right)\right]-1}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right], \quad (24)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right)\right]-1}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right], \quad (25)$$

and:

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{\tau_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right)\right]-1}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right], \quad (26)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right)\right]-1}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right]. \quad (27)$$

Box IV

Remark 1: When the modulus $m \to 1$, singular optical soliton solutions are recovered by Eqs. (52) and (53) as shown in Box XI, where $\lambda_3 = \lambda_4$.

Remark 2: However, if $m \to 0$, the following periodic singular solutions are obtained by Eqs. (54) and (55) as shown in Box XII, where $\lambda_2 = \lambda_3$.

3. Parabolic law nonlinearity

Optical solitons in birefringent fibers with parabolic law nonlinearity are governed by the following coupled NLSE [25]:

$$iq_t + aq_{xx} + \left(k_1|q|^2 + 2k_1|r|^2\right)q$$

$$+\left(k_2|q|^4 + 3k_2|r|^4 + 6k_2|q|^2|r|^2\right)q$$

$$+\left(k_1+3k_2|q|^2+2k_2|r|^2\right)r^2q^* + k_2r^3(q^*)^2 = 0, \quad (56)$$

$$ir_t + ar_{xx} + \left(k_1|r|^2 + 2k_1|q|^2\right)r$$

$$+\left(k_2|r|^4 + 3k_2|q|^4 + 6k_2|q|^2|r|^2\right)r$$

$$+\left(k_1+3k_2|r|^2+2k_2|q|^2\right)q^2r^* + k_2q^3(r^*)^2 = 0. \quad (57)$$

Eqs. (56) and (57) represent the model for the propagation of optical solitons through birefringent fibers that maintain parabolic law nonlinearity. In Eqs. (56) and (57), a is the coefficient of GVD, while k_l for $l = 1, 2$ are the coefficients of SPM and XPM terms, respectively. The last two terms in Eqs. (56) and (57) account for 4 WM. Here, 4 WM is a nonlinear effect that stems from third-order nonlinearity. It occurs when at least two different frequency components co-propagate in

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2\tau_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{k\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega}\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right], \quad (28)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1-\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{k\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega}\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right]. \quad (29)$$

Box V

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega}\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right),m\right]}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right], \quad (30)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega}\left(k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)-\zeta_0\right),m\right]}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right]. \quad (31)$$

Box VI

some nonlinear medium. In addition, x represents the spatial variable, while t represents temporal variable. Finally, $q(x,t)$ and $r(x,t)$ are the complex-valued wave profiles for the polarized solitons [1,2,8–11].

In this case, substituting Eqs. (3)–(6) into Eqs. (56) and (57) and, then, decomposing into real and imaginary parts give:

$$\left(\omega+a\kappa^2\right)P_l-k_1P_l^3-3k_1P_lP_{\bar{l}}^2-k_2P_l^5-5k_2P_lP_{\bar{l}}^4 - 9k_2P_l^3P_{\bar{l}}^2-k_2P_l^2P_{\bar{l}}^3-ak^2P_l''=0, \quad (58)$$

and:

$$-k\left(v+2a\kappa\right)P_l'=0, \quad (59)$$

respectively, where $l=1,2$ and $\bar{l}=3-l$. From the imaginary-part equation, it is possible to obtain the speed of the soliton as follows:

$$v=-2a\kappa. \quad (60)$$

Real-part Eq. (58) is written as the following coupled system of equations:

$$\left(\omega+a\kappa^2\right)P_1-k_1P_1^3-3k_1P_1P_2^2-k_2P_1^5-5k_2P_1P_2^4 - 9k_2P_1^3P_2^2-k_2P_1^2P_2^3-ak^2P_1''=0, \quad (61)$$

$$q(x,t)=\left\{\pm\frac{\tau_1\Omega}{k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right],\quad(34)$$

$$r(x,t)=\left\{\pm\frac{\tilde{\tau}_1\Omega}{k\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right],\quad(35)$$

$$q(x,t)=\left\{\frac{4\tau_1\Omega^2(\lambda_2-\lambda_1)}{4\Omega^2-\left[k(\lambda_1-\lambda_2)\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]^2}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right],\quad(36)$$

$$r(x,t)=\left\{\frac{4\tilde{\tau}_1\Omega^2(\lambda_2-\lambda_1)}{4\Omega^2-\left[k(\lambda_1-\lambda_2)\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]^2}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right].\quad(37)$$

Box VII

$$q(x,t)=\left\{\frac{\tau_1(\lambda_2-\lambda_1)}{2}\left(1\mp\coth\left[\frac{k(\lambda_1-\lambda_2)}{2\Omega}\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]\right)\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right],\quad(38)$$

$$r(x,t)=\left\{\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{2}\left(1\mp\coth\left[\frac{k(\lambda_1-\lambda_2)}{2\Omega}\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]\right)\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}(\eta_1-\xi_2+\sigma_1)+2\mu_4(\eta_2-\xi_1+\sigma_2)\left(a\kappa^2(\eta_1-\xi_2+\sigma_1)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)(\eta_1-\xi_2+\sigma_1)(\eta_2-\xi_1+\sigma_2)}\right)t+\theta\right\}\right].\quad(39)$$

Box VIII

$$(\omega+a\kappa^2)P_2-k_1P_2^3-3k_1P_2P_1^2-k_2P_2^5-5k_2P_2P_1^4$$
$$-9k_2P_2^3P_1^2-k_2P_2^2P_1^3-ak^2P_2''=0.\quad(62)$$

Using the balancing procedure between P_l'' and P_l^5 in Eqs. (61) and (62), we recover that:

$$N=\frac{1}{2}.\quad(63)$$

To obtain an analytic solution, we employ the trans-

$$q(x,t)=\left\{\frac{M}{R+\cosh\left[Q\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right],\tag{40}$$

$$r(x,t)=\left\{\frac{\widetilde{M}}{R+\cosh\left[Q\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right].\tag{41}$$

Box IX

$$q(x,t)=\left\{\frac{M_1}{R_1+\mathrm{sn}^2\left[Q_j\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right),\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right],\tag{46}$$

$$r(x,t)=\left\{\frac{\widetilde{M}_1}{R_1+\mathrm{sn}^2\left[Q_j\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right),\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right].\tag{47}$$

Box X

$$q(x,t)=\left\{\frac{M_1}{R_1+\tanh^2\left[Q_j\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right],\tag{52}$$

$$r(x,t)=\left\{\frac{\widetilde{M}_1}{R_1+\tanh^2\left[Q_j\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right].\tag{53}$$

Box XI

$$q(x,t)=\left\{\frac{M_1}{R_1+\sin^2\left[Q_j\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right], \quad (54)$$

$$r(x,t)=\left\{\frac{\widetilde{M}_1}{R_1+\sin^2\left[Q_j\left(x-\left\{\frac{2a\kappa-b\omega}{b\kappa-1}\right\}t\right)\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{\mu_2\tilde{\tau}_1^2\mathcal{H}\left(\eta_1-\xi_2+\sigma_1\right)+2\mu_4\left(\eta_2-\xi_1+\sigma_2\right)\left(a\kappa^2\left(\eta_1-\xi_2+\sigma_1\right)-3\tau_0^2\mathcal{H}\right)}{2\mu_4(b\kappa-1)\left(\eta_1-\xi_2+\sigma_1\right)\left(\eta_2-\xi_1+\sigma_2\right)}\right)t+\theta\right\}\right]. \quad (55)$$

Box XII

formations:

$$P_1=U_1^{\frac{1}{2}}=U_2^{\frac{1}{2}}=P_2, \quad (64)$$

in Eqs. (61) and (62) to find:

$$4\left(\omega+a\kappa^2\right)U_1^2-16k_1U_1^3-64k_2U_1^4$$
$$-ak^2\left\{2U_1U_1''-\left(U_1'\right)^2\right\}=0, \quad (65)$$

$$4\left(\omega+a\kappa^2\right)U_2^2-16k_1U_2^3-64k_2U_2^4$$
$$-ak^2\left\{2U_2U_2''-\left(U_2'\right)^2\right\}=0. \quad (66)$$

This coupled system given by Eqs. (65) and (66) will be now analyzed further in the next subsection.

3.1. Extended trial function approach

This subsection will retrieve optical soliton solutions to the model governed by Eqs. (56) and (57) by the extended trial function approach. By balancing the order of U_lU_l'' and U_l^4 in Eqs. (65) and (66), a relation of ϱ, ρ, ς, and $\tilde{\varsigma}$ is determined as follows:

$$\varsigma=\tilde{\varsigma}=\frac{\varrho-\rho-2}{2}. \quad (67)$$

When $\varrho=4$, $\rho=0$, and $\varsigma=\tilde{\varsigma}=1$ in Eq. (67), the solutions of Eqs. (65) and (66) can be written in the forms:

$$U_1=\tau_0+\tau_1\Psi, \quad (68)$$

$$U_2=\tilde{\tau}_0+\tilde{\tau}_1\Psi, \quad (69)$$

where τ_i, $\tilde{\tau}_i$ $(i=0,1)$ are constants to be determined later such that $\tau_1\neq0$ and $\tilde{\tau}_1\neq0$, and Ψ satisfies Eq. (11). Substituting these solutions into Eqs. (65) and (66) and solving the resulting system of algebraic equations, we recover:

Set 1:

$$\mu_0=\mu_0,\qquad \mu_2=\mu_2,\qquad \chi_0=\chi_0,$$

$$\tau_0=\tilde{\tau}_0,\qquad \tau_1=\tilde{\tau}_1,$$

$$\mu_1=\frac{\mu_0\tilde{\tau}_1}{\tilde{\tau}_0}+\frac{\tilde{\tau}_0\left[ak^2\mu_2+8\chi_0\tilde{\tau}_0\left(k_1+8k_2\tilde{\tau}_0\right)\right]}{ak^2\tilde{\tau}_1},$$

$$\mu_3=-\frac{8\chi_0\tilde{\tau}_1\left(3k_1+32k_2\tilde{\tau}_0\right)}{3ak^2},$$

$$\mu_4=-\frac{64k_2\chi_0\tilde{\tau}_1^2}{3ak^2},$$

$$\omega=-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}. \quad (70)$$

Set 2:

Eqs. (71) are shown in Box XIII. Substituting the solution set (70) into Eqs. (11) and (12) leads to:

$$\pm(\zeta-\zeta_0)=\Omega_1\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \quad (72)$$

where:

$$\Lambda(\Psi)=\Psi^4+\frac{\mu_3}{\mu_4}\Psi^3+\frac{\mu_2}{\mu_4}\Psi^2+\frac{\mu_1}{\mu_4}\Psi+\frac{\mu_0}{\mu_4},$$

$$\Omega_1=\sqrt{\frac{\chi_0}{\mu_4}}. \quad (73)$$

As a consequence, one obtains the traveling wave solutions to the model governed by Eqs. (56) and (57) in the following forms:

$$\mu_0=\mu_0, \qquad \mu_2=\mu_2, \qquad \tau_0=\tilde{\tau}_0, \qquad \tau_1=\tilde{\tau}_1,$$

$$\mu_1=\frac{\mu_0\tilde{\tau}_1^2\left(3k_1+32k_2\tilde{\tau}_0\right)^3+\mu_2\tilde{\tau}_0^2\left(3k_1+16k_2\tilde{\tau}_0\right)^2\left(3k_1+32k_2\tilde{\tau}_0\right)}{\tilde{\tau}_0\tilde{\tau}_1\left(3k_1+16k_2\tilde{\tau}_0\right)\left(9k_1^2+240k_1k_2\tilde{\tau}_0+1280k_2^2\tilde{\tau}_0^2\right)},$$

$$\mu_3=\frac{32k_2\tilde{\tau}_1\left(3k_1+32k_2\tilde{\tau}_0\right)\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left(9k_1^2+240k_1k_2\tilde{\tau}_0+1280k_2^2\tilde{\tau}_0^2\right)},$$

$$\mu_4=\frac{256k_2^2\tilde{\tau}_1^2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left(9k_1^2+240k_1k_2\tilde{\tau}_0+1280k_2^2\tilde{\tau}_0^2\right)},$$

$$\chi_0=-\frac{12ak^2k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left(9k_1^2+240k_1k_2\tilde{\tau}_0+1280k_2^2\tilde{\tau}_0^2\right)},$$

$$\omega=-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}. \tag{71}$$

Box XIII

When $\Lambda(\Psi)=(\Psi-\lambda_1)^4$:

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1\pm\frac{\tilde{\tau}_1\Omega_1}{k\left(x+2a\kappa t\right)-\zeta_0}\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right]; \tag{74}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$:

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{4\tilde{\tau}_1\Omega_1^2(\lambda_2-\lambda_1)}{4\Omega_1^2-\left[(\lambda_1-\lambda_2)\left(k\left(x+2a\kappa t\right)-\zeta_0\right)\right]^2}\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right]; \tag{75}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$:

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega_1}\left(k\left(x+2a\kappa t\right)-\zeta_0\right)\right]-1}\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right], \tag{76}$$

and:

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega_1}\left(k\left(x+2a\kappa t\right)-\zeta_0\right)\right]-1}\right\}^{\frac{1}{2}}\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right]. \tag{77}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$, Eq. (78) is obtained as shown in Box XIV;

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$, Eqs. (79) is obtained as shown in Box XV, where:

$$m^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{80}$$

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1-\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{k\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega_1}(x+2a\kappa t)\right]}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right]. \quad (78)$$

Box XIV

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_1}\left(k\left(x+2a\kappa t\right)-\zeta_0\right),m\right]}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right]. \quad (79)$$

Box XV

Note that λ_j for $j=1,\cdots,4$ is the root of the following equation:

$$\Lambda(\Psi)=0. \quad (81)$$

Under the conditions $\tilde{\tau}_0=-\tilde{\tau}_1\lambda_1$ and $\zeta_0=0$, Solutions (74)–(78) can be reduced to exact solutions in the following forms:

Rational function solutions are:

$$q(x,t)=r(x,t)=\left\{\pm\frac{\tilde{\tau}_1\Omega_1}{k(x+2a\kappa t)}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right], \quad (82)$$

$$q(x,t)=r(x,t)=\left\{\frac{4\tilde{\tau}_1\Omega_1^2(\lambda_2-\lambda_1)}{4\Omega_1^2-[k(\lambda_1-\lambda_2)(x+2a\kappa t)]^2}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right]. \quad (83)$$

Traveling wave solutions are:

$$q(x,t)=r(x,t)=\left\{\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{2}\left(1\mp\coth\left[\frac{k(\lambda_1-\lambda_2)}{2\Omega_1}(x+2a\kappa t)\right]\right)\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right]. \quad (84)$$

Finally, bright soliton solutions are:

$$q(x,t)=r(x,t)=\left\{\frac{M_2}{(R_2+\cosh\left[Q_3\left(x+2a\kappa t\right)\right])^{\frac{1}{2}}}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+6k_1\tilde{\tau}_0+32k_2\tilde{\tau}_0^2+\frac{ak^2\mu_2}{4\chi_0}\right)t+\theta\right\}\right], \quad (85)$$

where:

$$M_2=\left(\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{\lambda_3-\lambda_2}\right)^{\frac{1}{2}}, \quad (86)$$

$$Q_3=\frac{k\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega_1}, \quad (87)$$

$$R_2 = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}. \tag{88}$$

Note that the amplitude of the solitons is given by Eq. (86), while the inverse width of the solitons is given by Eq. (87). These solitons are valid for $\tilde{\tau}_1 < 0$. On the other hand, under the conditions $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_2$ and $\zeta_0 = 0$, Jacobi elliptic function solutions (79) are reduced to:

$$q(x,t) = r(x,t)$$

$$= \left\{ \frac{M_3}{\left(R_3 + \mathrm{sn}^2\left[Q_j\left(x+2a\kappa t\right), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{\frac{1}{2}}} \right\}$$

$$\times \exp\left[i\left\{-\kappa x + \left(-a\kappa^2 + 6k_1\tilde{\tau}_0 + 32k_2\tilde{\tau}_0^2 + \frac{ak^2\mu_2}{4\chi_0}\right)t + \theta\right\}\right], \tag{89}$$

where:

$$M_3 = \left(\frac{\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}\right)^{\frac{1}{2}}, \tag{90}$$

$$R_3 = \frac{\lambda_4 - \lambda_2}{\lambda_1 - \lambda_4}, \tag{91}$$

$$Q_j = \frac{(-1)^j k\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_1} \quad \text{for} \quad j = 4, 5. \tag{92}$$

Remark 3: When the modulus $m \to 1$, the second form of singular optical soliton solutions is obtained:

$$q(x,t) = r(x,t) = \left\{ \frac{M_3}{\left(R_3 + \tanh^2\left[Q_j\left(x+2a\kappa t\right)\right]\right)^{\frac{1}{2}}} \right\}$$

$$\times \exp\left[i\left\{-\kappa x + \left(-a\kappa^2 + 6k_1\tilde{\tau}_0 + 32k_2\tilde{\tau}_0^2 + \frac{ak^2\mu_2}{4\chi_0}\right)t + \theta\right\}\right], \tag{93}$$

where $\lambda_3 = \lambda_4$.

Remark 4: However, if $m \to 0$, the following periodic singular solutions emerge:

$$q(x,t) = r(x,t) = \left\{ \frac{M_3}{\left(R_3 + \sin^2\left[Q_j\left(x+2a\kappa t\right)\right]\right)^{\frac{1}{2}}} \right\}$$

$$\times \exp\left[i\left\{-\kappa x + \left(-a\kappa^2 + 6k_1\tilde{\tau}_0 + 32k_2\tilde{\tau}_0^2 + \frac{ak^2\mu_2}{4\chi_0}\right)t + \theta\right\}\right], \tag{94}$$

where $\lambda_2 = \lambda_3$.

Similarly, substituting values of Eqs. (71) into Eqs. (11) and (12) leads to:

$$\pm(\zeta - \zeta_0) = \Omega_2 \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{95}$$

where:

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4},$$

$$\Omega_2 = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{96}$$

As a consequence, one gets the traveling wave solutions to the model governed by Eqs. (56) and (57) in the following forms:

- When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, Eq. (97) is obtained as shown in Box XVI;
- When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, Eq. (98) is obtained as shown in Box XVII;
- When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, Eqs. (99) and (100) are obtained as shown in Box XVIII;
- When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, Eq. (101) is obtained as shown in Box XIX;
- When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, Eq. (102) is obtained as shown in Box XX, where:

$$q(x,t) = r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 \pm \frac{\tilde{\tau}_1\Omega_2}{k\left(x + 2a\kappa t\right) - \zeta_0}\right\}^{\frac{1}{2}}$$

$$\times \exp\left[i\left\{-\kappa x + \left(-a\kappa^2 + \frac{\tilde{\tau}_0\left(3k_1 + 16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2 + 48k_1k_2\mu_2\tilde{\tau}_0 + 256k_2^2\left(\mu_2\tilde{\tau}_0^2 - 6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1 + 16k_2\tilde{\tau}_0\right) - 16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t + \theta\right\}\right]. \tag{97}$$

Box XVI

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{4\tilde{\tau}_1\Omega_2^2(\lambda_2-\lambda_1)}{4\Omega_2^2-\left[(\lambda_1-\lambda_2)\left(k\left(x+2a\kappa t\right)-\zeta_0\right)\right]^2}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \tag{98}$$

Box XVII

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega_2}\left(k\left(x+2a\kappa t\right)-\zeta_0\right)\right]-1}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right], \tag{99}$$

and:

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega_2}\left(k\left(x+2a\kappa t\right)-\zeta_0\right)\right]-1}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \tag{100}$$

Box XVIII

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1-\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{k\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega_2}\left(x+2a\kappa t\right)\right]}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \tag{101}$$

Box XIX

$$q(x,t)=r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2}\left(k\left(x+2a\kappa t\right)-\zeta_0\right),m\right]}\right\}^{\frac{1}{2}}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \tag{102}$$

Box XX

$$q(x,t)=r(x,t)=\left\{\pm\frac{\tilde{\tau}_1\Omega_2}{k\left(x+2a\kappa t\right)}\right\}^{\frac{1}{2}}$$

$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right], \tag{105}$$

$$q(x,t)=r(x,t)=\left\{\frac{4\tilde{\tau}_1\Omega_2^2(\lambda_2-\lambda_1)}{4\Omega_2^2-\left[k(\lambda_1-\lambda_2)\left(x+2a\kappa t\right)\right]^2}\right\}^{\frac{1}{2}}$$

$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \tag{106}$$

Box XXI

$$q(x,t)=r(x,t)=\left\{\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{2}\left(1\mp\coth\left[\frac{k(\lambda_1-\lambda_2)}{2\Omega_2}\left(x+2a\kappa t\right)\right]\right)\right\}^{\frac{1}{2}}$$

$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \tag{107}$$

Box XXII

$$m^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{103}$$

Note that λ_j for $j=1,\cdots,4$ is the root of the following equation:

$$\Lambda(\Psi)=0. \tag{104}$$

Under the conditions $\tilde{\tau}_0=-\tilde{\tau}_1\lambda_1$ and $\zeta_0=0$, Eqs. (97)–(101) can be reduced to exact solutions in the following forms:

Plane wave solutions are obtained by Eqs. (105) and (106) as shown in Box XXI;

Traveling wave solutions (singular optical solitons) are obtained by Eq. (107) as shown in Box XXII;

Finally, bright soliton solutions are obtained by Eq. (108) as shown in Box XXIII, where:

$$M_4=\left(\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{\lambda_3-\lambda_2}\right)^{\frac{1}{2}}, \tag{109}$$

$$Q_6=\frac{k\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega_2}, \tag{110}$$

$$R_4=\frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}. \tag{111}$$

$$q(x,t)=r(x,t)=\left\{\frac{M_4}{\left(R_4+\cosh\left[Q_6\left(x+2a\kappa t\right)\right]\right)^{\frac{1}{2}}}\right\}$$

$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \tag{108}$$

Box XXIII

$$q(x,t)=r(x,t)=\left\{\frac{M_5}{\left(R_5+\text{sn}^2\left[Q_j\left(x+2a\kappa t\right),\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{\frac{1}{2}}}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \quad (112)$$

Box XXIV

$$q(x,t)=r(x,t)=\left\{\frac{M_5}{\left(R_5+\tanh^2\left[Q_j\left(x+2a\kappa t\right)\right]\right)^{\frac{1}{2}}}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \quad (116)$$

Box XXV

Note that the amplitude of the solitons is given by Eq. (109), while the inverse width of the solitons is given by Eq. (110). These solitons are valid for $\tilde{\tau}_1 < 0$. Furthermore, under the conditions $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_2$ and $\zeta_0 = 0$, Jacobi elliptic function solutions (102) are reduced to Eq. (112) as shown in Box XXIV, where:

$$M_5=\left(\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{\frac{1}{2}}, \quad (113)$$

$$R_5=\frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4}, \quad (114)$$

$$Q_j=\frac{(-1)^j k\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2} \quad \text{for} \quad j=7,8. \quad (115)$$

Remark 5: When the modulus $m \to 1$, the second form of singular optical soliton solutions is obtained by Eq. (116) as shown in Box XXV, where $\lambda_3 = \lambda_4$.

Remark 6: However, if $m \to 0$, the periodic singular solutions is obtained by Eq. (117) as shown in Box XXVI, where $\lambda_2 = \lambda_3$.

4. Conclusions

This paper analyzed optical solitons in birefringent fibers with Four-Ware Mixing (4WM) for Kerr and parabolic laws of nonlinearity. The extended trial function approach retrieved bright and singular soliton solutions along with several other forms of waves that include periodic singular waves and other solutions. These solutions appeared with different constraint conditions that guarantee the existence of a variety of waves. The phase-matching condition enables the extraction of these waves for birefringent fibers with 4WM. The results of this paper are very important

$$q(x,t)=r(x,t)=\left\{\frac{M_5}{\left(R_5+\sin^2\left[Q_j\left(x+2a\kappa t\right)\right]\right)^{\frac{1}{2}}}\right\}$$
$$\times\exp\left[i\left\{-\kappa x+\left(-a\kappa^2+\frac{\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)\left[-9k_1^2\mu_2+48k_1k_2\mu_2\tilde{\tau}_0+256k_2^2\left(\mu_2\tilde{\tau}_0^2-6\mu_0\tilde{\tau}_1^2\right)\right]}{48k_2\left[\mu_2\tilde{\tau}_0\left(3k_1+16k_2\tilde{\tau}_0\right)-16k_2\mu_0\tilde{\tau}_1^2\right]}\right)t+\theta\right\}\right]. \quad (117)$$

Box XXVI

in the field of nonlinear fiber optics since these exact soliton solutions will be necessary to implement in the telecommunications industry.

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