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Bayesian analysis of heterogeneous doubly censored lifetime data using the 3-component mixture of Rayleigh distributions: A Monte Carlo simulation study

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Abstract. This study considers Bayesian estimation of parameters of a heterogeneous 3-Component Mixture of Rayleigh Distributions (3-CMRD) generating a mixture of data. Being the most popular and reasonable sampling scheme in reliability and survival analyses, the doubly censored sampling scheme is considered in this research. The Bayes estimators and their posterior risks were derived under various situations. In addition, hyperparameters were elicited, and algebraic expressions for posterior predictive distribution and Bayesian predictive intervals were derived. Assuming the informative and the non-informative priors, a comprehensive Monte Carlo simulation was conducted to examine the performance of the Bayes estimators under symmetric and asymmetric loss functions. Finally, to highlight its practical importance, the proposed 3-component mixture model was applied to doubly censored lifetime data from a real-life situation. It was observed that in the analysis of doubly censored data in Bayesian framework, the SRIGP paired with SELF (DLF) was a suitable choice for estimating mixing proportion (component) parameters.

1. Introduction

Most of the lifetime applications in survival analyses involve making inference on the basis of censored data. These data may be doubly, right, or left censored. Censoring is an asset of datasets, not of parameters, as an unavoidable aspect of the real-life applications. In daily life, many kinds of censored data are used, including doubly censored, right censored, and left censored. In survival analysis, data are always subject to censoring. When the survival time is larger (smaller) than the observed left (right) censoring time, the sampling scheme is called left (right) censoring scheme. It is interesting to note that in left censoring sampling scheme, one can only have the information that the survival time is larger than or equal to the observed left censoring time. When both the final and initial times are interval-censored, it is a doubly censoring sampling scheme and the data obtained are thus called doubly censored data. Valuable accounts of doubly censoring sampling scheme for simple and mixture distributions have been given by Fernandez [1], Khan et al. [2], Kim and Song [3], Khan et al. [4], Pak et al. [5], Feroze and Aslam [6], and Sindhu et al. [7].

The Rayleigh distribution has been successfully

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used as a lifetime distribution, especially when lifetime of an item depends on its age. It is commonly used as a suitable lifetime model in reliability engineering and physics. For example, it has been considered in the modeling of wave heights [8], light and sound energy [9], wind power and radio motions [10], ultrasound image [11], etc. In such studies, it is reasonable to assume that lifetime of a given object depends upon its age. Besides the applications in physics, the Rayleigh distribution has received reasonable attention in reliability analysis and probability theory. It seems acceptable to state that in modeling the lifetimes of items, the Rayleigh distribution is a better choice than many others.

Mixture models play an active role in different real-life studies. Using mixture models when the data are assumed only from mixture models is called mixture distributions with direct application. They have been used fruitfully in many areas like industrial engineering [12], biology [13], social sciences [14], economics [15], life testing [16], reliability analysis [17], etc. Even when available data are considered to be generated from a mixture of two or more distributions, mixture models are useful. This motivated us to mix two or more statistical models to get a new mixture model and make Bayesian inference. For a successful Bayesian inference, we sought help from Santos [18], Al-Hussaini and Hussein [19], Mohammadi and Salehi-Rad [20], Ahmad and Al-Zaydi [21], Mohammadi et al. [22], Ali [23], Ateya [24], Mohamed et al. [25], and Zhang and Huang [26]. Specifically, we plan to develop a 3-CMRD under doubly censored sampling scheme for efficient modeling of the given lifetime data. The Bayesian inference is made using the Uniform Prior (UP) and the Jeffreys' Prior (JP) as Non-Informative Priors (NIPs), and Inverted Chi-square Prior (ICP) and Square Root Inverted Gamma Prior (SRIGP) as Informative Priors (IPs) under three loss functions, namely, PLF (precautionary loss function), SELF (Squared Error Loss Function), and DLF (DeGroot Loss Function). To accomplish the task, the direct application of mixture models is considered.

The remainder of this study is arranged as follows. The 3-CMRD is given in Section 2. Section 3 is about developing the likelihood for censored data. Sections 4 and 5 are devoted to derivation of joint and marginal posterior distributions, respectively. Elicitation of hyper parameters is considered in Section 6. Bayes Estimators (BEs) and associated Posterior Risks (PRs) are derived in Section 7. Section 8 studies the use of posterior predictive distributions and Bayesian predictive intervals. For illustrative purposes, a Monte Carlo simulation study is performed in Section 9. Sections 10 and 11 consist in a real-life example and concluding remarks, respectively.

2. The 3-component mixture model

The probability density function (pdf) of a finite 3-CMRD with unknown component parameters λ_j $(j = 1, 2, 3)$ and mixing proportions p_k $(k = 1, 2)$ is:

$$f(y;\boldsymbol{\Omega}) = p_1 \frac{y}{\lambda_1^2} \exp\left(-\frac{y^2}{2\lambda_1^2}\right) + p_2 \frac{y}{\lambda_2^2} \exp\left(-\frac{y^2}{2\lambda_2^2}\right) + (1-p_1-p_2) \frac{y}{\lambda_3^2} \exp\left(-\frac{y^2}{2\lambda_3^2}\right), \tag{1}$$

where $\lambda_j > 0, p_k \geq 0,\ \sum_{k=1}^{2} p_k \leq 1$ and $\boldsymbol{\Omega} = (\lambda_1, \lambda_2, \lambda_3, p_1, p_2)$.

The cumulative distribution function (cdf) of a finite 3-CMRD with unknown component parameters λ_j $(j = 1, 2, 3)$ and mixing proportions p_k $(k = 1, 2)$ is:

$$F(y;\boldsymbol{\Omega}) = 1 - p_1 \exp\left(-\frac{y^2}{2\lambda_1^2}\right) - p_2 \exp\left(-\frac{y^2}{2\lambda_2^2}\right) - (1-p_1-p_2) \exp\left(-\frac{y^2}{2\lambda_3^2}\right). \tag{2}$$

3. The doubly censored sampling scheme

Assume n units are used in a real-life experiment from a 3-CMRD. Let $y_r, y_{r+1}, ..., y_w$ be the only ordered values that can be pointed out. The remaining $r-1$ smaller values and $n-w$ larger values are assumed to be censored. The failed items can be observed as a subset of either the 1st or 2nd or 3rd subpopulation. Let $y_{1r_1}, ..., y_{1w_1},\ y_{2r_2}, ..., y_{2w_2}$, and $y_{3r_3}, ..., y_{3w_3}$ be failed values belonging to the 1st, 2nd, and 3rd subpopulations, respectively. The remaining values, which are smaller than y_r and larger than y_w, are assumed to be censored from each component, where $y_r = \min(y_{1r_1}, y_{2r_2}, y_{3r_3})$ and $y_w = \max(y_{1w_1}, y_{2w_2}, y_{3w_3})$. Also, the numbers of failed values, $s_1 = w_1 - r_1 + 1$, $s_2 = w_2 - r_2 + 1$, and $s_3 = w_3 - r_3 + 1$, can be observed from the 1st, 2nd, and 3rd subpopulations, respectively. The remaining $n - (w - r + 3)$ values are taken as censored and $w-r+3$ are considered as uncensored values. Also, by setting, we have $r = r_1 + r_2 + r_3$, $w = w_1 + w_2 + w_3$, and $s = s_1 + s_2 + s_3$. Now, using the above notations, the likelihood function for doubly censored data $y = \{(y_{1r_1}, ..., y_{1w_1}), (y_{2r_2}, ..., y_{2w_2}), (y_{3r_3}, ..., y_{3w_3})\}$, coming from a 3-component mixture model, is given by:

$$L(\boldsymbol{\Omega}\,|\mathbf{y}) \propto \left\{\prod_{i=r_1}^{w_1} p_1 f_1(y_{1i})\right\} \left\{\prod_{i=r_2}^{w_2} p_2 f_2(y_{2i})\right\} \left\{\prod_{i=r_3}^{w_3} (1-p_1-p_2) f_3(y_{3i})\right\} \times$$

$$\times \left\{F_1\left(y_{1r_1}\right)\right\}^{r_1-1}\left\{F_2\left(y_{2r_2}\right)\right\}^{r_2-1}$$
$$\left\{F_3\left(y_{3r_3}\right)\right\}^{r_3-1}\left\{1-F\left(y_w\right)\right\}^{n-w}. \quad (3)$$

Specifically, the likelihood function from a 3-CMRD for the doubly censored data is given by:

$$L\left(\mathbf{\Omega}\,|\mathbf{y}\right) \propto \left\{\prod_{i=r_1}^{w_1} p_1 \frac{y_{1i}}{\lambda_1^2} \exp\left(-\frac{y_{1i}^2}{2\lambda_1^2}\right)\right\}$$
$$\left\{\prod_{i=r_2}^{w_2} p_2 \frac{y_{2i}}{\lambda_2^2} \exp\left(-\frac{y_{2i}^2}{2\lambda_2^2}\right)\right\}$$
$$\left\{\prod_{i=r_3}^{w_3} \left(1-p_1-p_2\right) \frac{y_{3i}}{\lambda_3^2} \exp\left(-\frac{y_{3i}^2}{2\lambda_3^2}\right)\right\}$$
$$\times \left\{1-\exp\left(-\frac{y_{1r_1}^2}{2\lambda_1^2}\right)\right\}^{r_1-1}$$
$$\left\{1-\exp\left(-\frac{y_{2r_2}^2}{2\lambda_2^2}\right)\right\}^{r_2-1}$$
$$\left\{1-\exp\left(-\frac{y_{3r_3}^2}{2\lambda_3^2}\right)\right\}^{r_3-1}$$
$$\times \left\{p_1 \exp\left(-\frac{y_w^2}{2\lambda_1^2}\right) + p_2 \exp\left(-\frac{y_w^2}{2\lambda_2^2}\right)\right.$$
$$\left. + \left(1-p_1-p_2\right) \exp\left(-\frac{y_w^2}{2\lambda_3^2}\right)\right\}^{n-w}.$$

After simplification, the above likelihood function may be written as:

$$L\left(\mathbf{\Omega}\,|\mathbf{y}\right) \propto \sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}\left(-1\right)^{u_k}$$
$$\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}$$
$$\exp\left[-\frac{1}{\lambda_1^2}\left\{\frac{1}{2}\sum_{i=r_1}^{w_1} y_{1i}^2 + u_1 \frac{y_{1r_1}^2}{2}\right.\right.$$
$$\left.\left. + \left(n-w-u_4\right)\frac{y_w^2}{2}\right\}\right]$$
$$\times \exp\left[-\frac{1}{\lambda_3^2}\left\{\frac{1}{2}\sum_{i=r_2}^{w_2} y_{2i}^2 + u_2 \frac{y_{2r_2}^2}{2} + \left(u_4-u_5\right)\frac{y_w^2}{2}\right\}\right]$$
$$\exp\left[-\frac{1}{\lambda_3^2}\left\{\frac{1}{2}\sum_{i=r_3}^{w_3} y_{3i}^2 + u_3 \frac{y_{3r_3}^2}{2} + u_5 \frac{y_w^2}{2}\right\}\right]$$
$$\times \lambda_1^{-2s_1}\lambda_2^{-2s_2}\lambda_3^{-2s_3} p_1^{s_1+n-w-u_4} p_2^{s_2+u_4-u_5}$$
$$\left(1-p_1-p_2\right)^{s_3+u_5}. \quad (4)$$

4. The joint and marginal posterior distributions

The joint posterior distributions giving the doubly censored data $\mathbf{y}$ are presented using two NIPs (UP and JP) and two IPs (ICP and SRIGP).

4.1. The Uniform Prior (UP)

There are situations in which little prior knowledge on the parameter(s) of interest is available. In these situations, the UP and the JP are used as suitable non-informative priors. We take the improper UP for the unknown component parameters λ_j, i.e., $\lambda_j \sim uniform\left(0,\infty\right)$, $j=1,2,3$, and the UP for the unknown proportion parameters p_k, i.e., $p_k \sim uniform\left(0,1\right)$, $k=1,2$. Thus, the joint prior distribution of parameters λ_j and p_k is:

$$\pi_1\left(\mathbf{\Omega}\right) \propto 1; \quad \lambda_j > 0, \quad p_k \geq 0, \quad \sum_{k=1}^{2} p_k \leq 1. \quad (5)$$

Now, given $\mathbf{y}$, using the UP, the joint posterior distribution of parameters λ_j and p_k is defined as shown in Box I. Substituting the relative expressions in Eq. (6), we obtain Eq. (7) shown in Box II, where:

$$A_{11} = s_1 - \frac{1}{2}, \quad A_{21} = s_2 - \frac{1}{2}, \quad A_{31} = s_3 - \frac{1}{2},$$
$$B_{11} = \frac{1}{2}\sum_{i=r_1}^{w_1} y_{1i}^2 + u_1 \frac{y_{1r_1}^2}{2} + \left(n-w-u_4\right)\frac{y_w^2}{2},$$
$$B_{21} = \frac{1}{2}\sum_{i=r_2}^{w_2} y_{2i}^2 + u_2 \frac{y_{2r_2}^2}{2} + \left(u_4-u_5\right)\frac{y_w^2}{2},$$
$$B_{31} = \frac{1}{2}\sum_{i=r_3}^{w_3} y_{3i}^2 + u_3 \frac{y_{3r_3}^2}{2} + u_5 \frac{y_w^2}{2},$$
$$A_{01} = s_1 + n - w - u_4 + 1, \quad B_{01} = s_2 + u_4 - u_5 + 1,$$

$$g_1\left(\mathbf{\Omega}\,|\mathbf{y}\right) = \frac{L\left(\mathbf{\Omega}\,|\mathbf{y}\right)\pi_1\left(\mathbf{\Omega}\right)}{\int_0^1\int_0^{1-p_2}\int_0^\infty\int_0^\infty\int_0^\infty L\left(\mathbf{\Omega}\,|\mathbf{y}\right)\pi_1\left(\mathbf{\Omega}\right) d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2}. \quad (6)$$

Box I

$$g_1\left(\boldsymbol{\Omega}\left|\mathbf{y}\right.\right)=$$

$$\frac{1}{\Psi_1}\left\{\begin{array}{l}\sum\limits_{u_1=0}^{r_1-1}\sum\limits_{u_2=0}^{r_2-1}\sum\limits_{u_3=0}^{r_3-1}\sum\limits_{u_4=0}^{n-w}\sum\limits_{u_5=0}^{u_4}\prod\limits_{k=1}^{3}(-1)^{u_k}\left(\begin{array}{c}r_k-1\\u_k\end{array}\right)\left(\begin{array}{c}n-w\\u_4\end{array}\right)\left(\begin{array}{c}u_4\\u_5\end{array}\right)\lambda_1^{-(2A_{11}+1)}\lambda_2^{-(2A_{21}+1)}\lambda_3^{-(2A_{31}+1)}\\ \exp\left(-\frac{B_{11}}{\lambda_1^2}\right)\exp\left(-\frac{B_{21}}{\lambda_2^2}\right)\exp\left(-\frac{B_{31}}{\lambda_3^2}\right)p_1^{A_{01}-1}p_2^{B_{01}-1}(1-p_1-p_2)^{C_{01}-1}\end{array}\right\}. \quad (7)$$

Box II

$$C_{01}=s_3+u_5+1,$$

$$\Psi_1=\frac{1}{8}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}$$

$$\left(\begin{array}{c}r_k-1\\u_k\end{array}\right)\left(\begin{array}{c}n-w\\u_4\end{array}\right)\left(\begin{array}{c}u_4\\u_5\end{array}\right)\Gamma\left(A_{11}\right)$$

$$\Gamma\left(A_{21}\right)\Gamma\left(A_{31}\right)B_{11}^{-A_{11}}B_{21}^{-A_{21}}B_{31}^{-A_{31}}$$

$$B\left(A_{01},B_{01},C_{01}\right).$$

4.2. The Jeffreys' Prior (JP)

The JP for the unknown component parameters λ_j $(j=1,2,3)$ is obtained as $p(\lambda_j)\propto\left(\left|I\left(\lambda_j\right)\right|\right)^{1/2}$, where $I\left(\lambda_j\right)=-E\left[\frac{\partial^2 f(y;\lambda_j)}{\partial\lambda_j^2}\right]$ is the Fisher's information. Also, the UP is assumed for p_k, i.e., $p_k\sim uniform\,(0,1)$, $k=1,2$. The joint prior distribution of parameters λ_j and p_k is:

$$\pi_2\left(\boldsymbol{\Omega}\right)\propto\frac{1}{\lambda_1\lambda_2\lambda_3},\quad \lambda_j>0, p_k\geq 0,\quad \sum_{k=1}^{2}p_k\leq 1. \quad (8)$$

Given $\mathbf{y}$, the joint posterior distribution of parameters λ_j and p_k is defined by Eq. (9) as shown in Box III. On substituting the relevant likelihood and prior distributions in (9), the joint posterior distribution is given by Eq. (10) as shown in Box IV, where:

$$A_{12}=s_1,\quad A_{22}=s_2,\quad A_{32}=s_3,$$

$$B_{12}=\frac{1}{2}\sum_{i=r_1}^{w_1}y_{1i}^2+u_1\frac{y_{1r_1}^2}{2}+(n-w-u_4)\frac{y_w^2}{2},$$

$$B_{22}=\frac{1}{2}\sum_{i=r_2}^{w_2}y_{2i}^2+u_2\frac{y_{2r_2}^2}{2}+(u_4-u_5)\frac{y_w^2}{2},$$

$$B_{32}=\frac{1}{2}\sum_{i=r_3}^{w_3}y_{3i}^2+u_3\frac{y_{3r_3}^2}{2}+u_5\frac{y_w^2}{2},$$

$$A_{02}=s_1+n-w-u_4+1,\quad B_{02}=s_2+u_4-u_5+1,$$

$$C_{02}=s_3+u_5+1,$$

$$\Psi_2=\frac{1}{8}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}$$

$$\left(\begin{array}{c}r_k-1\\u_k\end{array}\right)\left(\begin{array}{c}n-w\\u_4\end{array}\right)\left(\begin{array}{c}u_4\\u_5\end{array}\right)\Gamma\left(A_{12}\right)$$

$$\Gamma\left(A_{22}\right)\Gamma\left(A_{32}\right)B_{12}^{-A_{12}}B_{22}^{-A_{22}}B_{32}^{-A_{32}}B$$

$$\left(A_{02},B_{02},C_{02}\right).$$

4.3. The Inverted Chi-square Prior (ICP)

When definite prior knowledge is given, it is quantified into an IP distribution. The ICP is assumed, as an IP, for the unknown component parameters λ_j, i.e., $\lambda_j\sim IC\left(a_j,b_j\right)$, $j=1,2,3$. The bivariate beta prior is considered for the unknown proportion parameters, i.e., $(p_1,p_2)\sim Biv\ Beta\,(a,b,c)$. The joint prior distribution of λ_j and p_k is obtained by Eq. (11) as shown in Box V. Given $\mathbf{y}$, the joint posterior distribution of parameters using the ICP is defined by Eq. (12) as shown in Box VI. The specific form of the joint posterior distribution is obtained by Eq. (13) as shown in Box VII, where:

$$A_{13}=s_1+\frac{a_1}{2},\quad A_{23}=s_2+\frac{a_2}{2},$$

$$A_{33}=s_3+\frac{a_3}{2},$$

$$g_2\left(\boldsymbol{\Omega}\left|\mathbf{y}\right.\right)=\frac{L\left(\boldsymbol{\Omega}\left|\mathbf{y}\right.\right)\pi_2\left(\boldsymbol{\Omega}\right)}{\int_0^1\int_0^{1-p_2}\int_0^\infty\int_0^\infty\int_0^\infty L\left(\boldsymbol{\Omega}\left|\mathbf{y}\right.\right)\pi_2\left(\boldsymbol{\Omega}\right)d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2}. \quad (9)$$

Box III

$$\pi_3(\boldsymbol{\Omega})=\frac{2b_1^{\frac{a_1}{2}}\lambda_1^{-(a_1+1)}\exp\left(-\frac{b_1}{2\lambda_1^2}\right)}{2^{\frac{a_1}{2}}\Gamma\left(\frac{a_1}{2}\right)}\frac{2b_2^{\frac{a_2}{2}}\lambda_2^{-(a_2+1)}\exp\left(-\frac{b_2}{2\lambda_2^2}\right)}{2^{\frac{a_2}{2}}\Gamma\left(\frac{a_2}{2}\right)}\frac{2b_3^{\frac{a_3}{2}}\lambda_3^{-(a_3+1)}\exp\left(-\frac{b_3}{2\lambda_3^2}\right)}{2^{\frac{a_3}{2}}\Gamma\left(\frac{a_3}{2}\right)}\frac{p_1^{a-1}p_2^{b-1}(1-p_1-p_2)^{c-1}}{B(a,b,c)}$$

$$\lambda_j>0,\qquad p_k\geq 0,\qquad \sum_{k=1}^{2}p_k\leq 1. \tag{11}$$

Box IV

$$g_2(\boldsymbol{\Omega}|\mathbf{y})=$$

$$\frac{1}{\Psi_2}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\lambda_1^{-(2A_{12}+1)}\lambda_2^{-(2A_{22}+1)}\lambda_3^{-(2A_{32}+1)}\\ \exp\left(-\frac{B_{12}}{\lambda_1^2}\right)\exp\left(-\frac{B_{22}}{\lambda_2^2}\right)\exp\left(-\frac{B_{32}}{\lambda_3^2}\right)p_1^{A_{02}-1}p_2^{B_{02}-1}(1-p_1-p_2)^{C_{02}-1}\end{array}\right\}. \tag{10}$$

Box V

$$B_{13}=\frac{1}{2}\sum_{i=r_1}^{w_1}y_{1i}^2+u_1\frac{y_{1r_1}^2}{2}+(n-w-u_4)\frac{y_w^2}{2}+\frac{b_1}{2},$$

$$B_{23}=\frac{1}{2}\sum_{i=r_2}^{w_2}y_{2i}^2+u_2\frac{y_{2r_2}^2}{2}+(u_4-u_5)\frac{y_w^2}{2}+\frac{b_2}{2},$$

$$B_{33}=\frac{1}{2}\sum_{i=r_3}^{w_3}y_{3i}^2+u_3\frac{y_{3r_3}^2}{2}+u_5\frac{y_w^2}{2}+\frac{b_3}{2},$$

$$A_{03}=s_1+n-w-u_4+a,$$

$$B_{03}=s_2+u_4-u_5+b,$$

$$C_{03}=s_3+u_5+c,$$

$$\Psi_3=\frac{1}{8}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}$$

$$\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{13})$$

$$\Gamma(A_{23})\Gamma(A_{33})B_{13}^{-A_{13}}B_{23}^{-A_{23}}B_{33}^{-A_{33}}$$

$$B(A_{03},B_{03},C_{03}).$$

$$g_3(\boldsymbol{\Omega}|\mathbf{y})=\frac{L(\boldsymbol{\Omega}|\mathbf{y})\pi_3(\boldsymbol{\Omega})}{\int_0^1\int_0^{1-p_2}\int_0^\infty\int_0^\infty\int_0^\infty L(\boldsymbol{\Omega}|\mathbf{y})\pi_3(\boldsymbol{\Omega})\,d\lambda_1d\lambda_2d\lambda_3dp_1dp_2}. \tag{12}$$

Box VI

$$g_3(\boldsymbol{\Omega}|\mathbf{y})=$$

$$\frac{1}{\Psi_3}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\lambda_1^{-(2A_{13}+1)}\lambda_2^{-(2A_{23}+1)}\lambda_3^{-(2A_{33}+1)}\\ \exp\left(-\frac{B_{13}}{\lambda_1^2}\right)\exp\left(-\frac{B_{23}}{\lambda_2^2}\right)\exp\left(-\frac{B_{33}}{\lambda_3^2}\right)p_1^{A_{03}-1}p_2^{B_{03}-1}(1-p_1-p_2)^{C_{03}-1}\end{array}\right\}. \tag{13}$$

Box VII

$$\pi_4(\boldsymbol{\Omega}) = \frac{2b_1^{a_1}\lambda_1^{-(2a_1+1)}\exp\left(-\frac{b_1}{\lambda_1^2}\right)}{\Gamma(a_1)}\frac{2b_2^{a_2}\lambda_2^{-(2a_2+1)}\exp\left(-\frac{b_2}{\lambda_2^2}\right)}{\Gamma(a_2)}\frac{2b_3^{a_3}\lambda_3^{-(2a_3+1)}\exp\left(-\frac{b_3}{\lambda_3^2}\right)}{\Gamma(a_3)}\frac{p_1^{a-1}p_2^{b-1}(1-p_1-p_2)^{c-1}}{B(a,b,c)},$$

$$\lambda_j > 0, p_k \geq 0, \sum_{k=1}^{2} p_k \leq 1. \tag{14}$$

Box VIII

$$g_4(\boldsymbol{\Omega}|\mathbf{y}) = \frac{L(\boldsymbol{\Omega}|\mathbf{y})\,\pi_4(\boldsymbol{\Omega})}{\int_0^1\int_0^{1-p_2}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} L(\boldsymbol{\Omega}|\mathbf{y})\,\pi_4(\boldsymbol{\Omega})\,d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2}. \tag{15}$$

Box IX

4.4. The posterior distribution using the square root inverted gamma prior

The Square Root Inverted Gamma Prior (SRIGP) is taken as an IP for the unknown component parameters λ_j, i.e., $\lambda_j \sim SRIG(a_j, b_j)$, $j = 1,2,3$, and the bivariate beta prior is assumed for the unknown proportion parameters p_k $(k = 1,2)$, i.e., $(p_1, p_2) \sim Biv\ Beta(a,b,c)$. The joint prior distribution of parameters λ_j and p_k is obtained by Eq. (14) as shown in Box VIII. Given $\mathbf{y}$, the joint posterior distribution of parameters λ_j and p_k is now defined by Eq. (15) as shown in Box IX. The final expression for the joint posterior distribution is now derived by Eq. (16) as shown in Box X, where:

$$A_{14} = s_1 + a_1, \quad A_{24} = s_2 + a_2, \quad A_{34} = s_3 + a_3,$$

$$B_{14} = \frac{1}{2}\sum_{i=r_1}^{w_1} y_{1i}^2 + u_1\frac{y_{1r_1}^2}{2} + (n-w-v_4)\frac{y_w^2}{2} + b_1,$$

$$B_{24} = \frac{1}{2}\sum_{i=r_2}^{w_2} y_{2i}^2 + u_2\frac{y_{2r_2}^2}{2} + (u_4-u_5)\frac{y_w^2}{2} + b_2,$$

$$B_{34} = \frac{1}{2}\sum_{i=r_3}^{w_3} y_{3i}^2 + u_3\frac{y_{3r_3}^2}{2} + u_5\frac{y_w^2}{2} + b_3,$$

$$A_{04} = s_1 + n - w - u_4 + a, \quad B_{04} = s_2 + u_4 - u_5 + b,$$

$$C_{04} = s_3 + u_5 + c,$$

$$\Psi_4 = \frac{1}{8}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k} \binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{14}) \Gamma(A_{24})\Gamma(A_{34})B_{14}^{-A_{14}}B_{24}^{-A_{24}}B_{34}^{-A_{34}} B(A_{04}, B_{04}, C_{04}).$$

5. The marginal posterior distributions

Given $\mathbf{y}$, the marginal posterior distributions of parameters λ_j and p_k assuming the NIPs and the IPs are derived by Eqs. (17) and (18) as shown in Box XI, where ϖ, π and η are given different values of:

(i) $\varpi = 1, \pi = 2,$ and $\eta = 3$;

(ii) $\pi = 1, \varpi = 2,$ and $\eta = 3$;

(iii) $\pi = 1, \eta = 2,$ and $\varpi = 3$.

$$g_4(\boldsymbol{\Omega}|\mathbf{y}) = \frac{1}{\Psi_4}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\lambda_1^{-(2A_{14}+1)}\lambda_2^{-(2A_{24}+1)}\lambda_3^{-(2A_{34}+1)} \\ \exp\left(-\frac{B_{14}}{\lambda_1^2}\right)\exp\left(-\frac{B_{24}}{\lambda_2^2}\right)\exp\left(-\frac{B_{34}}{\lambda_3^2}\right)p_1^{A_{04}-1}p_2^{B_{04}-1}(1-p_1-p_2)^{C_{04}-1}\end{array}\right\}. \tag{16}$$

Box X

$$h_\xi\left(\lambda_\varpi\left|\mathbf{y}\right.\right)=\int_0^1\int_0^{1-p_2}\int_0^\infty\int_0^\infty g_\xi\left(\boldsymbol{\Omega}\left|\mathbf{y}\right.\right)$$

$$d\lambda_\pi d\lambda_\eta dp_1 dp_2, \tag{17}$$

$$h_\xi\left(\lambda_\varpi\left|y\right.\right)=$$

$$\frac{1}{4\Psi_\xi}\left\{\begin{array}{l}\sum\limits_{u_1=0}^{r_1-1}\sum\limits_{u_2=0}^{r_2-1}\sum\limits_{u_3=0}^{r_3-1}\sum\limits_{u_4=0}^{n-w}\sum\limits_{u_5=0}^{u_4}\prod\limits_{k=1}^{3}(-1)^{u_k}\left(\begin{array}{c}r_k-1\\u_k\end{array}\right)\left(\begin{array}{c}n-w\\u_4\end{array}\right)\left(\begin{array}{c}u_4\\u_5\end{array}\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\lambda_\varpi^{-(A_{\varpi\xi}+1)}\exp\left(-\frac{B_{\varpi\xi}}{\lambda_\varpi^2}\right)\end{array}\right\}. \tag{18}$$

Box XI

$$h_\xi\left(p_\alpha\left|\mathbf{y}\right.\right)=\int_0^{1-p_\alpha}\int_0^\infty\int_0^\infty\int_0^\infty g_\xi\left(\boldsymbol{\Omega}\left|\mathbf{y}\right.\right)$$

$$d\lambda_1 d\lambda_2 d\lambda_3 dp_\beta, \tag{19}$$

Eq. (20) is shown in Box XII, where α and β take different values as:

(i) $\alpha=1,\beta=2,\Upsilon=B,$ and $\Delta=A$;

(ii) $\beta=1,\alpha=2,\Upsilon=A,$ and $\Delta=B$.

Also, $\xi=1$ for the UP, $\xi=2$ for the JP, $\xi=3$ for the ICP, and $\xi=4$ for the SRIGP.

6. Elicitation

Elicitation is used to enumerate prior professional information of a person about some unknown quantity of interest and it can be utilized to improve any numerical data that we may have. Under Bayesian paradigm, elicitation and specification of the prior distribution is a very complicated and common problem. In Bayesian framework, elicitation mostly rises as a tool of identifying the prior distribution for a parameter, which is random. In various sampling models, different methods for specification of thoughts to determine the hyperparameters have been developed. For elicitation of hyperparameters, there are many methods given in the literature.

To elicit (determine) a prior distribution, Aslam [27] developed some criteria dependent on the Prior Predictive Distribution (PPD). He used confidence level, predictive mode, and predictive probabilities for eliciting hyperparameters. The general criterion of judgment is to associate the PPD with the assessment of the expert and select hyperparameters that make the judgment agree strictly with a member of the family. Then, following the laws of probability, the professional should be consistent with the determining probabilities. Certain contradictions may occur, which are not important. The function $\Phi\left(\omega_1,\omega_2\right)$ can be used for elicitation of hyperparameters ω_1 and ω_2 as $\Phi\left(\omega_1,\omega_2\right)=\min\limits_{\omega_1,\omega_2}\sum\limits_z\left\{\frac{p(z)-p_0(z)}{p(z)}\right\}^2$, where $p(z)$ represents the predictive probabilities considered by ω_1 and ω_2, which are hyperparameters, and $p_0(z)$ indicates the determined predictive probabilities. Now, for eliciting the hyperparameters, the above equations are solved simultaneously through Mathematica software. Thus, in this article, a method based on predictive probabilities is used. The PPDs given in Eqs. (22) and (24) are used for eliciting the hyperparameters of the ICP and SRIGP, respectively.

6.1. Elicitation for the ICP

The PPD using the ICP is derived as:

$$h_\xi\left(p_\alpha\left|y\right.\right)=$$

$$\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum\limits_{u_1=0}^{r_1-1}\sum\limits_{u_2=0}^{r_2-1}\sum\limits_{u_3=0}^{r_3-1}\sum\limits_{u_4=0}^{n-w}\sum\limits_{u_5=0}^{u_4}\prod\limits_{k=1}^{3}(-1)^{u_k}\left(\begin{array}{c}r_k-1\\u_k\end{array}\right)\left(\begin{array}{c}n-w\\u_4\end{array}\right)\left(\begin{array}{c}u_4\\u_5\end{array}\right)\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Upsilon_{0\xi},C_{0\xi}\right)p_\alpha^{\Delta_{0\xi}-1}(1-p_\alpha)^{\Upsilon_{0\xi}+C_{0\xi}-1}\end{array}\right\}. \tag{20}$$

Box XII

$$p(y) = \int_0^1 \int_0^{1-p_2} \int_0^\infty \int_0^\infty \int_0^\infty f(y|\mathbf{\Omega})\, \pi`_3(\mathbf{\Omega})\, d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2, \quad (21)$$

$$p(y) = \frac{1}{(a+b+c)} \left\{ \frac{a a_1 b_1^{\frac{a_1}{2}} y}{(b_1+y^2)^{\frac{a_1}{2}+1}} + \frac{b a_2 b_2^{\frac{a_2}{2}} y}{(b_2+y^2)^{\frac{a_2}{2}+1}} + \frac{c a_3 b_3^{\frac{a_3}{2}} y}{(b_3+y^2)^{\frac{a_3}{2}+1}} \right\}. \quad (22)$$

Using the above PPD Eq. (22), 9 integrals for the limits of values of random variable Y, i.e., $1/2 \le y \le 1$, $1 \le y \le 3/2$, $3/2 \le y \le 2$, $2 \le y \le 5/2$, $5/2 \le y \le 3$, $3 \le y \le 7/2$, $7/2 \le y \le 4$, $4 \le y \le 9/2$, and $9/2 \le y \le 5$ are considered with probabilities 0.26, 0.24, 0.15, 0.10, 0.05, 0.03, 0.02, 0.01, and 0.005, respectively. It is worth mentioning that these probabilities might have been taken from the opinion of expert(s) about the likelihood of the considered intervals. Also, various intervals could be considered. These 9 derived equations are simultaneously solved using Mathematica software. Using the methodology defined above, the values of hyperparameters are $a_1 = 4.87245$, $b_1 = 4.61091$, $a_2 = 4.42412$, $b_2 = 4.18569$, $a_3 = 3.89641$, $b_3 = 3.76739$, $a = 2.16005$, $b = 3.88109$, and $c = 3.09061$.

6.2. Elicitation for the SRIGP

The PPD for a random variable, Y, using the SRIGP is obtained as:

$$p(y) = \int_0^1 \int_0^{1-p_2} \int_0^\infty \int_0^\infty \int_0^\infty f(y|\mathbf{\Omega})\, \pi_4(\mathbf{\Omega})\, d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2, \quad (23)$$

$$p(y) = \frac{1}{(a+b+c)} \left\{ \frac{a a_1 b_1^{a_1} y}{\left(b_1+\frac{y^2}{2}\right)^{a_1+1}} + \frac{b a_2 b_2^{a_2} y}{\left(b_2+\frac{y^2}{2}\right)^{a_2+1}} + \frac{c a_3 b_3^{a_3} y}{\left(b_3+\frac{y^2}{2}\right)^{a_3+1}} \right\}. \quad (24)$$

Using the procedure defined above, we obtain the values of hyperparameters as $a_1 = 4.54009$, $b_1 = 4.12418$, $a_2 = 4.75611$, $b_2 = 4.42104$, $a_3 = 3.97389$, $b_3 = 3.830018$, $a = 9.09837$, $b = 5.91833$, and $c = 6.84910$.

7. Bayes estimators and posterior risks

In this section, the PLF, SELF, and DLF are used to acquire BEs and their PRs assuming the different prior distributions. Legendre [28] defined the SELF as $L(\lambda, \hat{\psi}) = (\lambda - \hat{\psi})^2$, which is a symmetric loss function. Norstrom [29] defined PLF as an asymmetric loss function by $L(\lambda, \hat{\psi}) = \hat{\psi}^{-1}(\lambda - \hat{\psi})^2$. DeGroot [30] defined a loss function of $L(\lambda, \hat{\psi}) = \hat{\psi}^{-2}(\lambda - \hat{\psi})^2$. Table 1, given below, provides the general expressions of the BEs and their PRs.

7.1. The BEs and PRs under SELF assuming the UP, JP, ICP, and SRIGP

The respective marginal posterior distribution yields the algebraic expressions for BEs and PRs for parameters λ_j and p_k under SELF as shown in Box XIII.

7.2. The BEs and PRs under PLF assuming the UP, JP, ICP, and SRIGP

The algebraic expressions of BEs and PRs for parameters λ_j and p_k under PLF are derived with the corresponding marginal posterior distribution as shown in Box XIV.

7.3. The BEs and PRs under DLF assuming the UP, JP, ICP, and SRIGP

The respective marginal posterior distribution yields the algebraic expressions for BEs and PRs for parameters λ_j and p_k under DLF as shown in Box XV.

8. Posterior predictive distribution and Bayesian predictive interval

The posterior predictive distribution (ppd) is the distribution for a future value, say Y_{n+1}, based on the already observed sample vector $\mathbf{y}$. Given doubly censored data, say $\mathbf{y}$, the ppd for Y_{n+1} is obtained by Eqs. (37) and (38) as shown in Box XVI.

For constructing a Bayesian Predictive Interval (BPI), suppose that L and U are the two endpoints of BPI. A $100(1-\gamma)\%$ BPI can be constructed through

Table 1. BEs and PRs under SELF, PLF, and DLF.

Loss function	Bayes estimators	Posterior risks
$SELF = L(\lambda, \hat{\psi}) = (\lambda - \hat{\psi})^2$	$\hat{\psi} = E_{\lambda\vert\mathbf{y}}(\lambda)$	$\rho(\hat{\psi}) = E_{\lambda\vert\mathbf{y}}(\lambda^2) - [E_{\lambda\vert\mathbf{y}}(\lambda)]^2$
$PLF = L(\lambda, \hat{\psi}) = \hat{\psi}^{-1}(\lambda - \hat{\psi})^2$	$\hat{\psi} = [E_{\lambda\vert\mathbf{y}}(\lambda^2)]^{1/2}$	$\rho(\hat{\psi}) = 2[E_{\lambda\vert\mathbf{y}}(\lambda^2)]^{1/2} - 2E_{\lambda\vert\mathbf{y}}(\lambda)$
$DLF = L(\lambda, \hat{\psi}) = \hat{\psi}^{-2}(\lambda - \hat{\psi})^2$	$\hat{\psi} = E_{\lambda\vert\mathbf{y}}(\lambda^2)/E_{\lambda\vert\mathbf{y}}(\lambda)$	$\rho(\hat{\psi}) = 1 - \{E_{\lambda\vert\mathbf{y}}(\lambda)\}^2 / E_{\lambda\vert\mathbf{y}}(\lambda^2)$

$$\hat{\lambda}_{\varpi(SELF)} = \frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-\frac{1}{2}\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}-\frac{1}{2}\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}, \tag{25}$$

$$\hat{p}_{\alpha(SELF)} = \frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Upsilon_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+1,\Upsilon_{0\xi}+C_{0\xi}\right)\end{array}\right\}, \tag{26}$$

$$\begin{aligned}\rho\left(\hat{\lambda}_{\varpi(SELF)}\right) = {} & \frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-1\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-(A_{\varpi\xi}-1)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}\\ & -\left[\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-\frac{1}{2}\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}-\frac{1}{2}\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}\right]^2,\end{aligned} \tag{27}$$

$$\begin{aligned}\rho\left(\hat{p}_{\alpha(SELF)}\right) = {} & \frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Upsilon_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\Upsilon_{0\xi}+C_{0\xi}\right)\end{array}\right\}\\ & -\left[\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Upsilon_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+1,\Upsilon_{0\xi}+C_{0\xi}\right)\end{array}\right\}\right]^2.\end{aligned} \tag{28}$$

Box XIII

the ppd defined in Eq. (38) as:

$$\int_0^L f\left(y_{n+1}\left|\mathbf{y}\right.\right)dy_{n+1} = \frac{\gamma}{2}, \quad \text{and}$$

$$\int_U^\infty f\left(y_{n+1}\left|\mathbf{y}\right.\right)dy_{n+1} = \frac{\gamma}{2}.$$

After simplification, the BPI (L, U) can be acquired by using Eqs. (39) and (40) as shown in Box XVII.

9. Monte Carlo simulation

It is well clear from the algebraic expressions of Bayes estimator and posterior risks defined in Section 7 that it is difficult to analytically compare different Bayes

$$\hat{\lambda}_{\varpi(PLF)}=$$
$$\sqrt{\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-1\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-(A_{\varpi\xi}-1)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}}, \tag{29}$$

$$\hat{p}_{\alpha(PLF)}=$$
$$\sqrt{\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Upsilon_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\Upsilon_{0\xi}+C_{0\xi}\right)\end{array}\right\}}, \tag{30}$$

$$\rho\left(\hat{\lambda}_{\varpi(PLF)}\right)=$$
$$2\sqrt{\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-1\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-(A_{\varpi\xi}-1)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}}$$
$$-\frac{1}{4\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-\tfrac{1}{2}\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}-\frac{1}{2}\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}, \tag{31}$$

$$\rho\left(\hat{p}_{\alpha(PLF)}\right)=$$
$$2\sqrt{\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Upsilon_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\Upsilon_{0\xi}+C_{0\xi}\right)\end{array}\right\}}$$
$$-\frac{1}{4\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Upsilon_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+1,\Upsilon_{0\xi}+C_{0\xi}\right)\end{array}\right\}. \tag{32}$$

Box XIV

estimators. The Bayes estimators considered in this study may be compared numerically by using Monte Carlo simulations. Through a Monte Carlo simulation study, the Bayes estimators are compared under various loss functions, priors, sample sizes, left and right test termination times, and parametric values. For each of the five unknown parameters λ_j and p_k in a 3-CMRD, simulated BEs and PRs are reported in Tables 2-4 using the following simulation technique:

1. Consider that sample size n is fixed. Using the Mathematica software, a sample from the mixture

$$\hat{\lambda}_{\varpi(DLF)} = \frac{\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{\varpi\xi}-1)\Gamma(A_{\pi\xi})\Gamma(A_{\eta\xi})\\ B_{\varpi\xi}^{-(A_{\varpi\xi}-1)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B(A_{0\xi},C_{0\xi})B(B_{0\xi},A_{0\xi}+C_{0\xi})\end{array}\right\}}{\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-\frac{1}{2}\right)\Gamma(A_{\pi\xi})\Gamma(A_{\eta\xi})\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}-\frac{1}{2}\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B(A_{0\xi},C_{0\xi})B(B_{0\xi},A_{0\xi}+C_{0\xi})\end{array}\right\}}, \quad (33)$$

$$\hat{p}_{\alpha(DLF)} = \frac{\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi})\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B(\Upsilon_{0\xi},C_{0\xi})B(\Delta_{0\xi}+2,\Upsilon_{0\xi}+C_{0\xi})\end{array}\right\}}{\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi})\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B(\Upsilon_{0\xi},C_{0\xi})B(\Delta_{0\xi}+1,\Upsilon_{0\xi}+C_{0\xi})\end{array}\right\}}, \quad (34)$$

$$\rho\left(\hat{\lambda}_{\varpi(DLF)}\right) = 1-\frac{\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{\varpi\xi}-\frac{1}{2}\right)\Gamma(A_{\pi\xi})\Gamma(A_{\eta\xi})\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}-\frac{1}{2}\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B(A_{0\xi},C_{0\xi})B(B_{0\xi},A_{0\xi}+C_{0\xi})\end{array}\right\}^2}{8\Psi_\xi\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{\varpi\xi}-1)\Gamma(A_{\pi\xi})\Gamma(A_{\eta\xi})\\ B_{\varpi\xi}^{-(A_{\varpi\xi}-1)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B(A_{0\xi},C_{0\xi})B(B_{0\xi},A_{0\xi}+C_{0\xi})\end{array}\right\}}, \quad (35)$$

$$\rho\left(\hat{p}_{\alpha(DLF)}\right) = 1-\frac{\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi})\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B(\Upsilon_{0\xi},C_{0\xi})B(\Delta_{0\xi}+1,\Upsilon_{0\xi}+C_{0\xi})\end{array}\right\}^2}{8\Psi_\xi\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi})\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B(\Upsilon_{0\xi},C_{0\xi})B(\Delta_{0\xi}+2,\Upsilon_{0\xi}+C_{0\xi})\end{array}\right\}}. \quad (36)$$

Box XV

$$f\left(y_{n+1}\left|\mathbf{y}\right.\right)=\int_0^1\int_0^{1-p_2}\int_0^\infty\int_0^\infty\int_0^\infty p\left(y_{n+1}\left|\mathbf{\Omega}\right.\right)g_\xi\left(\mathbf{\Omega}\left|\mathbf{y}\right.\right)d\lambda_1 d\lambda_2 d\lambda_3 dp_1 dp_2, \quad (37)$$

$$f\left(y_{n+1}\left|\mathbf{y}\right.\right)=$$

$$\frac{y_{n+1}}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}+1\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ \left(B_{1\xi}+\frac{y_{n+1}^2}{2}\right)^{-\left(A_{1\xi}+1\right)}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi}+1,C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}$$

$$+\frac{y_{n+1}}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}+1\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}\left(B_{2\xi}+\frac{y_{n+1}^2}{2}\right)^{-\left(A_{2\xi}+1\right)}B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi}+1,A_{0\xi}+C_{0\xi}\right)\end{array}\right\} \quad (38)$$

$$+\frac{y_{n+1}}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}+1\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}\left(B_{3\xi}+\frac{y_{n+1}^2}{2}\right)^{-\left(A_{3\xi}+1\right)}B\left(A_{0\xi},C_{0\xi}+1\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}.$$

Box XVI

distribution is generated as follows:

(i) Assuming $f_1\left(y;\lambda_1\right)=\frac{y}{\lambda_1^2}\exp\left(-\frac{y^2}{2\lambda_1^2}\right)$, generate np_1 random values;

(ii) Assuming $f_2\left(y;\lambda_2\right)=\frac{y}{\lambda_2^2}\exp\left(-\frac{y^2}{2\lambda_2^2}\right)$, generate np_2 random values;

(iii) Assuming $f_3\left(y;\lambda_3\right)=\frac{y}{\lambda_3^2}\exp\left(-\frac{y^2}{2\lambda_3^2}\right)$, generate the remaining $n\left(1-p_1-p_2\right)$ random values.

2. Take a sample censored at fixed test termination times on left test termination time y_r and right test termination time y_w;
3. Select values which are less than y_r and greater than y_w as censored ones;
4. Using the censored sample selected through the steps 1-3, calculate BE $\hat{\delta}_i$ and PR $\rho\left(\hat{\delta}_i\right)$ by solving Eqs. (25)-(36);
5. Repeat steps 1-4, 500 times;
6. Calculate the simulated BEs and their simulated PRs as $\hat{\delta}=\frac{1}{500}\sum_{i=1}^{500}\left(\hat{\delta}_i\right)$ and $\rho\left(\hat{\delta}\right)=\frac{1}{500}\sum_{i=1}^{500}\rho\left(\hat{\delta}_i\right)$, respectively;
7. Repeat steps 1-6 for sample size $n=40,80,140$ with parameters $\left(\lambda_1,\lambda_2,\lambda_3,p_1,p_2\right)\in(13,11,9,0.4,0.4)$ and left and right test termination times $\left(y_r,y_w\right)\in(2,30)$.

From the simulated results arranged in Tables 2-4, it is noticed that the amount of under-estimation (and/or over-estimation) of parameters λ_j and p_k assuming different NIPs and IPs under symmetric loss function (SELF) and asymmetric loss functions (PLF and DLF) is smaller for larger sample sizes with fixed y_r and y_w. Also, the degree of over-estimation (and/or under-estimation) of parameters λ_j and p_k is grater for larger y_r and smaller y_w values. The differences of the BEs of parameters λ_j and p_k from their assumed values decrease to zero by increasing the sample size.

The PR of the BE is a notable measure for the assessment of the performance of the BEs. It is observed that the amounts of PRs of the BEs of parameters λ_j and p_k using different prior loss functions considered in this study are inversely proportional to sample size for fixed left test termination time y_r and right test termination time y_w.

When selecting a suitable prior, it is observed that the IP (ICP and SRIGP) is a more efficient prior than the NIP under the considered loss functions. Also, it is evident that the SRIGP (JP) materializes as the preeminent prior compared to the ICP (UP) amongst the different IPs (NIPs) due to smaller associated PR. On the other hand, in estimating the component

$$\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\frac{\Gamma(A_{1\xi}+1)\Gamma(A_{2\xi})\Gamma(A_{3\xi})}{A_{1\xi}}\\ \left({B_{1\xi}}^{-A_{1\xi}}-\left(B_{1\xi}+\frac{L^2}{2}\right)^{-A_{1\xi}}\right)B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi}+1,C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}$$

$$+\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi}+1)\Gamma(A_{3\xi})}{A_{2\xi}}\\ B_{1\xi}^{-A_{1\xi}}\left({B_{2\xi}}^{-A_{2\xi}}-\left(B_{2\xi}+\frac{L^2}{2}\right)^{-A_{2\xi}}\right)B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi}+1,A_{0\xi}+C_{0\xi}\right)\end{array}\right\} \quad (39)$$

$$+\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi}+1)}{A_{3\xi}}\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}\left({B_{3\xi}}^{-A_{3\xi}}-\left(B_{3\xi}+\frac{L^2}{2}\right)^{-A_{3\xi}}\right)B\left(A_{0\xi},C_{0\xi}+1\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}=\frac{\gamma}{2},$$

and:

$$\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\frac{\Gamma(A_{1\xi}+1)\Gamma(A_{2\xi})\Gamma(A_{3\xi})}{A_{1\xi}}\\ \left(B_{1\xi}+\frac{U^2}{2}\right)^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi}+1,C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}$$

$$+\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi}+1)\Gamma(A_{3\xi})}{A_{2\xi}}\\ B_{1\xi}^{-A_{1\xi}}\left(B_{2\xi}+\frac{U^2}{2}\right)^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi}+1,A_{0\xi}+C_{0\xi}\right)\end{array}\right\} \quad (40)$$

$$+\frac{1}{8\Psi_\xi}\left\{\begin{array}{l}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-1}\sum_{u_3=0}^{r_3-1}\sum_{u_4=0}^{n-w}\sum_{u_5=0}^{u_4}\prod_{k=1}^{3}(-1)^{u_k}\binom{r_k-1}{u_k}\binom{n-w}{u_4}\binom{u_4}{u_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi}+1)}{A_{3\xi}}\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}\left(B_{3\xi}+\frac{U^2}{2}\right)^{-A_{3\xi}}B\left(A_{0\xi},C_{0\xi}+1\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}=\frac{\gamma}{2}$$

Box XVII

parameters λ_j, the DLF shows superior performance to SELF and PLF, whereas SELF shows better performance then the other two loss functions in estimating the proportion parameters p_k. Selecting the best loss function and prior has no dependency on sample sizes y_r and y_w. However, it is noteworthy that choosing the loss function (prior) with a prior (loss function) is dependent on the amount of PRs associated with it.

10. Example of real data

Gómez et al. [31] reported real data on the life of weak crack of Kevlar 373/epoxy, which was subject to fixed force at the 90% stress level until it thoroughly failed. Gómez et al. [31] revealed that the mixture data $\mathbf{z}$ could be shown by exponential distribution. For exponential random mixture data ($\mathbf{z}$), the transformation $y = \sqrt{2z}$ provides the Rayleigh random mixture data ($\mathbf{y}$). Therefore, as this transformation agrees well with our findings, we can apply the mixture data of Gómez et al. [31] to the proposed Bayesian methodology. To illustrate the proposed methodology, the data are randomly grouped into three sets of values with 26 values belonging to the 1st subpopulation, 25 values belonging to the 2nd subpopulation, and 25 values belonging to 3rd subpopulation. Now, we have the situation in which the mixture data are doubly censored. To implement censored sampling, $z_{1r_1}, ..., z_{1w_1}$,

Table 2. BEs and PRs of 3-CMED under SELF with parameters $\lambda_1 = 13, \lambda_2 = 11, \lambda_3 = 9, p_1 = 0.4$, and $p_2 = 0.4$.

y_r, y_w	n	Prior distribution		$\hat{\lambda}_{1(SELF)}$	$\hat{\lambda}_{2(SELF)}$	$\hat{\lambda}_{3(SELF)}$	$\hat{p}_{1(SELF)}$	$\hat{p}_{2(SELF)}$
2, 30	40	UP	BE	13.33840	11.46070	9.979170	0.390525	0.396327
			PR	4.369720	3.121660	5.685300	0.005761	0.005754
		JP	BE	12.86390	11.27800	9.783080	0.391943	0.397957
			PR	3.821840	2.873360	4.990160	0.005721	0.005720
		ICP	BE	11.80070	10.49960	8.257880	0.366211	0.397800
			PR	2.793170	2.200540	2.716760	0.004559	0.004692
		SRIGP	BE	11.13290	9.501270	7.243980	0.427406	0.342300
			PR	2.067660	1.513920	1.464150	0.003814	0.003505
	80	UP	BE	13.12030	11.30900	9.510120	0.394990	0.398038
			PR	1.991010	1.486130	2.178200	0.003046	0.003040
		JP	BE	12.92660	11.20420	9.323220	0.395260	0.398459
			PR	1.861470	1.417360	1.984710	0.003027	0.003031
		ICP	BE	12.38550	10.68930	8.626310	0.381692	0.398773
			PR	1.578580	1.209790	1.458730	0.002678	0.002708
		SRIGP	BE	11.82570	10.26080	8.099090	0.416901	0.365138
			PR	1.316480	1.019100	1.062860	0.002391	0.002278
	140	UP	BE	13.07193	11.18586	9.392980	0.396800	0.399156
			PR	1.263000	0.954000	1.324000	0.002077	0.002071
		JP	BE	12.95280	11.09090	9.208700	0.397699	0.399242
			PR	1.246000	0.944000	1.229000	0.002073	0.002065
		ICP	BE	12.52868	10.89802	8.762496	0.386731	0.399186
			PR	1.106822	0.870031	0.974588	0.001896	0.001911
		SRIGP	BE	12.33457	10.41274	8.334605	0.413209	0.373871
			PR	0.982727	0.736358	0.772555	0.001744	0.001681

Table 3. BEs and PRs of 3-CMED under PLF with parameters $\lambda_1 = 13, \lambda_2 = 11, \lambda_3 = 9, p_1 = 0.4$, and $p_2 = 0.4$.

y_r, y_w	n	Prior distribution		$\hat{\lambda}_{1(PLF)}$	$\hat{\lambda}_{2(PLF)}$	$\hat{\lambda}_{3(PLF)}$	$\hat{p}_{1(PLF)}$	$\hat{p}_{2(PLF)}$
2, 30	40	UP	BE	13.39110	11.45110	10.35580	0.398138	0.404467
			PR	0.303573	0.253295	0.520760	0.014644	0.014390
		JP	BE	13.25400	11.39070	9.934260	0.398041	0.403746
			PR	0.300589	0.250244	0.476557	0.014512	0.014216
		ICP	BE	11.89710	10.53390	8.447830	0.373705	0.402624
			PR	0.231132	0.205001	0.299366	0.012359	0.011751
		SRIGP	BE	11.00500	9.726970	7.428160	0.430554	0.348558
			PR	0.181545	0.158089	0.195688	0.008908	0.010163
	80	UP	BE	13.14930	11.34840	9.674050	0.398764	0.403144
			PR	0.142221	0.124301	0.215089	0.007606	0.007528
		JP	BE	13.05410	11.18350	9.474640	0.399227	0.403078
			PR	0.142092	0.123109	0.210275	0.007580	0.007505
		ICP	BE	12.37780	10.81110	8.698990	0.384788	0.402108
			PR	0.126007	0.111468	0.160471	0.006991	0.006754
		SRIGP	BE	11.99510	10.20380	8.095940	0.420207	0.366766
			PR	0.111184	0.097178	0.125259	0.005720	0.006231
	140	UP	BE	13.07060	11.31620	9.385300	0.399170	0.403120
			PR	0.097000	0.086000	0.133000	0.005210	0.005140
		JP	BE	12.98190	11.18812	9.236380	0.399596	0.403001
			PR	0.095540	0.084440	0.127500	0.005205	0.005128
		ICP	BE	12.53180	10.92642	8.863021	0.389835	0.401886
			PR	0.087255	0.078797	0.108817	0.004884	0.004767
		SRIGP	BE	12.22169	10.49760	8.367542	0.414654	0.376752
			PR	0.078975	0.070002	0.088693	0.004224	0.004486

Table 4. BEs and PRs of 3-CMED under DLF with parameters $\lambda_1 = 13, \lambda_2 = 11, \lambda_3 = 9, p_1 = 0.4$, and $p_2 = 0.4$.

y_r, y_w	n	Prior distribution		$\hat{\lambda}_{1(DLF)}$	$\hat{\lambda}_{2(DLF)}$	$\hat{\lambda}_{3(DLF)}$	$\hat{p}_{1(DLF)}$	$\hat{p}_{2(DLF)}$
2, 30	40	UP	BE	13.44550	11.73720	10.69460	0.403827	0.412936
			PR	0.023118	0.022182	0.050456	0.036722	0.035164
		JP	BE	13.30780	11.51810	10.08720	0.403184	0.411814
			PR	0.022265	0.021672	0.045749	0.036399	0.035092
		ICP	BE	12.18270	10.45980	8.567050	0.379911	0.408243
			PR	0.018709	0.018640	0.033179	0.032853	0.029963
		SRIGP	BE	11.17940	9.697940	7.428130	0.435719	0.352969
			PR	0.015748	0.015472	0.024486	0.020509	0.028010
	80	UP	BE	13.15920	11.42870	9.783750	0.402739	0.406934
			PR	0.011135	0.011192	0.022317	0.019267	0.018736
		JP	BE	13.05000	11.31450	9.569450	0.402929	0.406882
			PR	0.011112	0.011120	0.021650	0.019104	0.018716
		ICP	BE	12.48150	10.82950	8.738990	0.388961	0.405646
			PR	0.010194	0.010335	0.017825	0.018099	0.016851
		SRIGP	BE	12.09400	10.28450	8.133410	0.422726	0.371004
			PR	0.009248	0.009396	0.014753	0.013615	0.016796
	140	UP	BE	13.10719	11.27572	9.436629	0.402564	0.405187
			PR	0.007248	0.007399	0.013894	0.012938	0.012736
		JP	BE	13.10140	11.15356	9.357930	0.402339	0.404606
			PR	0.007240	0.007360	0.013670	0.012920	0.012680
		ICP	BE	12.67745	10.83887	8.955196	0.392153	0.404140
			PR	0.006939	0.007182	0.012190	0.012526	0.011917
		SRIGP	BE	12.37931	10.49015	8.399147	0.417437	0.378596
			PR	0.006451	0.006651	0.010473	0.010140	0.011871

$z_{2r_2}, ..., z_{2w_2}$, and $z_{3r_3}, ..., z_{3w_3}$ failed values belonging to the 1st, 2nd, and 3rd subpopulations, respectively, are considered. The remaining values greater than 3.4 and less than 0.5 are taken to be censored from each population such that $z_r = \min(z_{1r_1}, z_{2r_2}, z_{3r_3}) = 0.5$ and $z_w = \max(z_{1w_1}, z_{2w_2}, z_{3w_3}) = 3.4$, whereas the numbers of failed values $s_1 = w_1 - r_1 + 1 = 19$, $s_2 = w_2 - r_2 + 1 = 20$, and $s_3 = w_3 - r_3 + 1 = 19$ can be observed in subpopulations 1, 2, and 3, respectively. The remaining $n-(w-r+3) = 18$ values are censored values and $w - r + 3 = 58$ are the uncensored values such that $r = r_1 + r_2 + r_3$, $w = w_1 + w_2 + w_3$, and $s = s_1 + s_2 + s_3$. The total number of tests conducted is 76, i.e., $n = 76$. The data are summarized as follows:

$$n_1 = 26, \quad r_1 = 4, \ w_1 = 22,$$

$$s_1 = w_1 - r_1 + 1 = 19,$$

$$y^2_{1r_1} = 2z_{1r_1} = 1, \ \ y^2_{1w_1} = 2z_{1w_1} = 6.8,$$

$$\sum_{i=r_1}^{w_1} y^2_{1i} = 2\sum_{i=r_1}^{w_1} z_{1i} = 61.0512,$$

$$n_2 = 25, \quad r_2 = 3, \quad w_2 = 22,$$

$$s_2 = w_2 - r_2 + 1 = 20,$$

$$y^2_{1r_1} = 2z_{1r_1} = 1, \ \ y^2_{1w_1} = 2z_{1w_1} = 6.8,$$

$$\sum_{i=r_2}^{w_2} y^2_{2i} = 2\sum_{i=r_2}^{w_2} z_{2i} = 63.9028,$$

$$n_3 = 25, \quad r_3 = 3, \quad w_1 = 21,$$

$$s_3 = w_3 - r_3 + 1 = 19,$$

$$y^2_{1r_1} = 2z_{1r_1} = 1, \ \ y^2_{1w_1} = 2z_{1w_1} = 6.8,$$

$$\sum_{i=r_3}^{w_3} y^2_{3i} = 2\sum_{i=r_3}^{w_3} z_{3i} = 59.4332,$$

$$n = n_1 + n_2 + n_3 = 76,$$

$$r = r_1 + r_2 + r_3 = 10,$$

$$w = w_1 + w_2 + w_3 = 65, \text{ and}$$

$$s = s_1 + s_2 + s_3 = 58.$$

Since $n - (w - r + 3) = 18$, we have 23.68% doubly censored sample.

It can be seen that the results for real data , given in Table 5, coincide with the simulated results. Thus, the performance of the SRIGP is the best among all the considered NIPs (UP and JP) and IPs (ICP and SRIGP) by having the minimum amounts of PRs for the BEs. Also, it is noticed that the results are comparatively more precise for the JP (SRIGP) than for the UP (ICP) among the NIPs (IPs) under SELF, PLF, and DLF. Moreover, it can be seen that SELF (DLF) performs better than DLF and PLF (SELF and PLF) in estimating mixing proportion parameters p_k (component parameters λ_j).

A BPI is an interval related to a variable yet to be detected, with a quantified probability of the variable lying in the interval. Using the above data, 90% BPI for weak crack of Kevlar 373/epoxy subject to fixed force at 90% stress level in the future, assuming NIP and IP, is presented in Table 6.

When an NIP is to be used, the BPIs for the JP are narrower than for the UP. Similarly, when IPs are available, the BPIs for the SRIGP are narrower than for the ICP. The Bayesian prediction intervals are narrower with IPs (ICP or SRIGP) than with NIPs (UP or JP).

11. Concluding remarks

Under doubly censoring sampling scheme, we considered the Bayesian analysis of 3-CMRD to model lifetime data. The Monte Carlo simulation study and real life application led to the following conclusions.

From the simulated results given in Tables 2-4, increase in sample size resulted in improved Bayes estimators of parameters λ_j and p_k. Although Bayes estimators either overestimated or underestimated the parameters, the amounts of over-estimation and/or under-estimation for parameters λ_j and p_k were quite higher (lower) with relatively smaller (larger) sample sizes for fixed left and right test termination times. Similarly, sample size and test termination times affected the PRs. Specifically, as sample size increased (decreased), the amounts of posterior risks of Bayes estimators of parameters λ_j and p_k decreased (increased) for fixed test termination times. This observation held for each loss function considered in this study, no matter which prior was used. However, the SELF (DLF) was proven to be the preferable choice for estimating mixing proportion (component) parameters.

As an overall conclusive statement, we can say that in the Bayesian estimation of parameters under doubly censoring sampling scheme, the SRIGP paired with DLF was the preferable option to estimate λ_j and the SRIGP paired with SELF was the suitable choice to estimate p_k. Also, the results given in Tables 5 and 6, which were achieved for real-life mixture data, were compatible with the results of the simulation study, showing the correctness of the simulation scheme.

Table 5. BEs and PRs of 3-CMED under SELF, PLF, and DLF with real-life data.

Prior	Loss function		$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{p}_1$	$\hat{p}_2$
SELF	UP	BE	1.88287	1.93224	1.91006	0.32644	0.34565
		PR	0.04975	0.05210	0.05392	0.00364	0.00376
	JP	BE	1.86053	1.90956	1.88617	0.32643	0.34571
		PR	0.04758	0.04984	0.05145	0.00364	0.00376
	ICP	BE	1.77120	1.83070	1.81269	0.31700	0.35193
		PR	0.03793	0.04073	0.04255	0.00310	0.00329
	SRIGP	BE	1.72387	1.74247	1.74344	0.36969	0.31738
		PR	0.03257	0.03286	0.03546	0.00283	0.00263
PLF	UP	BE	1.89603	1.94567	1.92413	0.33197	0.35104
		PR	0.02633	0.02687	0.02813	0.01105	0.01080
	JP	BE	1.87327	1.92256	1.89976	0.33196	0.35111
		PR	0.02549	0.02601	0.02718	0.01105	0.01080
	ICP	BE	1.78188	1.84179	1.82439	0.32185	0.35657
		PR	0.02135	0.02218	0.02340	0.00970	0.00928
	SRIGP	BE	1.73329	1.75187	1.75358	0.37350	0.32149
		PR	0.01884	0.01881	0.02028	0.00762	0.00824
DLF	UP	BE	1.90929	1.95920	1.93829	0.33759	0.35653
		PR	0.01384	0.01376	0.01456	0.03302	0.03052
	JP	BE	1.88610	1.93566	1.91344	0.33758	0.35660
		PR	0.01356	0.01348	0.01426	0.03304	0.03054
	ICP	BE	1.79262	1.85295	1.83616	0.32678	0.36127
		PR	0.01195	0.01201	0.01278	0.02992	0.02585
	SRIGP	BE	1.74276	1.76133	1.76378	0.37734	0.32566
		PR	0.01084	0.01071	0.01153	0.02029	0.02545

Table 6. BPI (L, U) of the 3-CMRD with real life data.

UP		**JP**		**ICP**		**SRIGP**	
L	U	L	U	L	U		
0.60114	4.78425	0.59415	4.72487	0.56980	4.51303	0.54853	4.32760

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