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Research Note

Steady-state thermal and mechanical stresses in Two-Dimensional Functionally Graded Piezoelectric Materials (2D-FGPMs) for a hollow infinite cylinder

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Abstract. In this paper, the general analysis of two-dimensional steady-state thermal and mechanical stresses for a hollow thick infinite cylinder made of Functionally Graded Piezoelectric Materials (2D-FGPMs) is performed and developed. The general form of thermal, mechanical, and electrical boundary conditions is considered on the inside and outside surfaces. A direct method is used to solve the heat conduction equation and the non-homogenous system of partial differential Navier equations, using the complex Fourier series and the power law functions method. The material properties are assumed dependent on the radial and circumferential variables and are expressed as power law functions along the radial and circumferential directions.

1. Introduction

Functionally Graded Piezoelectric Materials (FGPMs) represent a kind of piezoelectric materials with material composition and properties varying continuously along certain directions. The piezoelectric devices can be entirely made of FGPM or use FGPM as a transit interlayer among different piezoelectric materials. FGPM is the composite material intentionally designed so that desirable properties for some specific applications can be obtained. The advantages of the new kind of material can improve the reliability of lifespan of piezoelectric devices. Shelley et al. [1] presented functionally graded piezoelectric ceramics. Chen et al. [2] solved the stability problem of piezoelectric FGM rectangular plates subjected to non-uniformly distributed load heat and voltage. Dai et al. [3] obtained the analytical solutions of stresses in functionally graded piezoelectric hollow structures. Akbari Alashti et al. [4] presented the thermo-elastic analysis of a functionally graded spherical shell with piezoelectric layers by a differential quadrature method.

Nemat-Alla [5] presented a reduction of thermal stresses by developing two-dimensional functionally graded materials. Asgari and Akhlaghi [6] presented thermo-mechanical analysis of 2D-FGM thick hollow cylinder using graded-finite elements. The effects of variation of materials distribution in two radial and axial directions on the temperature, displacements, and stress distributions were studied. Darabseh and Alshear [7] presented thermoelastic analysis of a 2D-FGM hollow circular cylinder with a finite length by the finite-element method. Jabbari et al. [8,9] presented mechanical and thermal stresses in FGPPM hollow cylinders. Meshkini et al. [10] presented asymmetric

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mechanical and thermal stresses in a 2D-FGPPMs hollow cylinder. In addition, Meshkini et al. [11] performed an analytical investigation of a 2D-PPMS hollow infinite cylinder under Thermo-Electro-Mechanical (TEM) Loadings. They applied the separation of variables and the complex Fourier series to solve the heat conduction and Navier equations.

In this study, an analytical method is presented with respect to mechanical and thermal stress analyses for a hollow infinite cylinder made of Functionally Graded Piezoelectric Materials (2D-FGPMs). In the present study, the material properties are assumed to be expressed by power functions in radial and circumferential directions. The effects of electric potential coefficient on displacement, electric potential, and stresses are studied. Temperature distribution is considered in the steady-state asymmetric case, and mechanical and thermal boundary conditions are considered in the general form. The Navier equations in terms of displacements are derived and solved analytically by the direct method; therefore, any boundary conditions for stresses and displacements can be satisfied.

2. Governing equations

2.1. Heat conduction problem

The first law of thermodynamics for the energy equation in the steady-state condition of two-dimensional cylinder is as follows (see Appendix B):

$$\frac{1}{r}\left(krT_{,r}\right)_{,r}+\frac{1}{r^2}\left(kT_{,\theta}\right)_{,\theta}=0,$$

$$a\leq r\leq b,\qquad -\pi\leq\theta\leq+\pi, \tag{1}$$

where $T(r,\theta)$ is temperature distribution, and $k(r,\theta)$ is the thermal conduction coefficient.

The thermal boundary conditions are assumed to be as follows:

$$S_{11}T(a,\theta)+S_{12}T_{,r}(a,\theta)=f_1(\theta), \tag{2}$$

$$S_{21}T(b,\theta)+S_{22}T_{,r}(b,\theta)=f_2(\theta), \tag{3}$$

where S_{ij} represents the constant thermal parameters related to conduction and convection coefficients. Functions $f_1(\theta)$ and $f_2(\theta)$ are known as the inner and outer radii, respectively. It is assumed that nonhomogeneous thermal conduction coefficient $k(r,\theta)$ is the power function of radial and circumferential (r,θ) as in $k(r,\theta)=k_0(\frac{r}{a})^{m_3}e^{n_3}\theta$.

Using the definition for the material properties, the temperature equation becomes:

$$T_{,rr}+(m_3+1)\frac{1}{r}T_{,r}+\frac{1}{r^2}\left(n_3T_{,\theta}+T_{,\theta\theta}\right)=0. \tag{4}$$

The solution of Eq. (3) is written in the form of complex Fourier series as follows:

$$T(r,\theta)=\sum_{q=-\infty}^{\infty}T_q(r)e^{iq\theta}. \tag{5}$$

By substituting Eq. (5) into Eq. (4), the following equation is obtained:

$$T_q''(r)+(m_3+1)\frac{1}{r}T_q'(r)+\frac{1}{r^2}(iqn_3-q^2)T_q(r)=0. \tag{6}$$

Eq. (6) is the Euler equation and has solutions in the following form:

$$T_q(r)=A_qr^{\beta}. \tag{7}$$

By substituting Eq. (7) into Eq. (6), the following characteristic equation is obtained:

$$\beta^2+m_3\beta+(iqn_3-q^2)=0. \tag{8}$$

The roots of Eq. (8) are:

$$\beta_{q_{1,2}}=\frac{-m_3}{2}\mp\left[\frac{m_3^2}{4}+(q^2-iqn_3)\right]^{1/2}. \tag{9}$$

Thus:

$$T_q(r)=A_{q_1}r^{\beta_{q_1}}+A_{q_2}r^{\beta_{q_2}}. \tag{10}$$

Substituting Eq. (10) into Eq. (5) gives:

$$T(r,\theta)=\sum_{q=-\infty}^{\infty}\left(A_{q_1}r^{\beta_{q_1}}+A_{q_2}r^{\beta_{q_2}}\right)e^{iq\theta}. \tag{11}$$

Constants A_{q_1} and A_{q_2} are presented in Appendix A.

2.2. Stress analysis

The governing two-dimensional strain-displacement and electric field intensity relations in cylindrical coordinates are [11]:

$$\varepsilon_{rr}=\frac{\partial u}{\partial r},\qquad \varepsilon_{\theta\theta}=\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r},$$

$$\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right),$$

$$E_r=\frac{\partial\psi}{\partial r},\qquad E_\theta=\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \tag{12}$$

where u, v, and ψ are the displacement components, electric potential, and the radial and circumferential directions, respectively.

The asymmetric stress-strain and electric displacement relations of FGPMs are as follows [11]:

$$\sigma_{rr}=C_{11}\varepsilon_{rr}+C_{12}\varepsilon_{\theta\theta}+e_{21}E_r-C_1^TT(r,\theta),$$

$$\sigma_{\theta\theta}=C_{12}\varepsilon_{rr}+C_{22}\varepsilon_{\theta\theta}+e_{22}E_r-C_2^TT(r,\theta),$$

$$\sigma_{zz}=C_{12}\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}\right)+e_{23}E_r-C_3^TT(r,\theta),$$

$$\sigma_{r\theta}=2C_{44}\varepsilon_{r\theta}+e_{24}E_\theta,$$

$$D_{rr}=e_{21}\varepsilon_{rr}+e_{22}\varepsilon_{\theta\theta}-\varepsilon_{22}E_r+g_{21}T(r,\theta),$$

$$D_{\theta\theta}=2e_{24}\varepsilon_{r\theta}-\varepsilon_{21}E_\theta+g_{22}T(r,\theta), \tag{13}$$

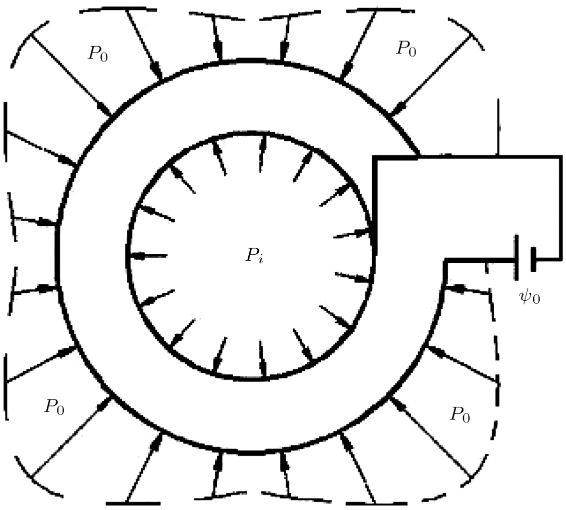

Figure 1. Geometric model of a 2D-FGPM hollow infinite cylinder under 2D-TEM.

where C_i^T represents thermal moduli which can be expressed by elastic constants and linear thermal expansion coefficients α_i and $C_1^T = C_{11}\alpha_r + 2C_{12}\alpha_\theta$, $C_2^T = 2C_{12}\alpha_r + C_{22}\alpha_\theta$.

Under this consideration, $\alpha_r = \alpha_\theta = \alpha$ [8]. Therefore:

$$C_1^T = (C_{11} + 2C_{12})\alpha, \qquad C_2^T = (2C_{12} + C_{22})\alpha,$$
$$C_3^T = C_1^T. \tag{14}$$

The stress and electric displacement equilibrium equations are written as follows:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}.\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0,$$
$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}.\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r}(\sigma_{r\theta}) = 0,$$
$$\frac{\partial D_{rr}}{\partial r} + \frac{1}{r}.\frac{\partial D_{\theta\theta}}{\partial \theta} + \frac{1}{r}(D_{rr}) = 0. \tag{15}$$

Figure 1 shows the geometric model of a 2D-FGPM hollow infinite cylinder under two-dimensional electro thermo mechanical loads. To obtain the equilibrium equations in terms of the displacement components for the 2D-FGPM cylinder, the functional relationship of the material properties must be known. Because the cylinder material is assumed to be graded along r, θ-direction, the coefficient of thermal expansion and elastic and electric constants are assumed to be described with the power laws as follows:

$$\alpha = \alpha_0 \tilde{r}^{m_1} e^{n_1\theta}, \qquad C_{ij} = \bar{C}_{ij}\tilde{r}^{m_2} e^{n_2\theta},$$
$$K = k_0 \tilde{r}^{m_3} e^{n_3\theta}, \qquad e_{2i} = \bar{e}_{2i}\tilde{r}^{m_4} e^{n_4\theta},$$
$$\varepsilon_{2i} = \bar{\varepsilon}_{2i}\tilde{r}^{m_5} e^{n_5\theta}, \qquad g_{2i} = \bar{g}_{2i}\tilde{r}^{m_6} e^{n_6\theta}, \tag{16}$$

where:

$$\tilde{r} = \frac{r}{a}. \tag{17}$$

In addition, a is the inner radius.

By substituting Eq. (15) into Eq. (17), the Navier equations in terms of the displacement components are obtained as follows:

$$u_{,rr} + \left((m_2+1) + (m_2-1)\frac{\bar{C}_{12}}{\bar{C}_{11}}\right)\frac{1}{r}u_{,r}$$
$$+ \left(\frac{m_2\bar{C}_{12} - \bar{C}_{22}}{\bar{C}_{11}}\right)\frac{1}{r^2}u + \left(\frac{n_2\bar{C}_{44}}{\bar{C}_{11}}\right)\frac{1}{r}v,r$$
$$- \left(\frac{n_2\bar{C}_{44}}{\bar{C}_{11}}\right)\frac{1}{r^2}v + \left(\frac{\bar{C}_{44}}{\bar{C}_{11}}\right)\frac{1}{r^2}u_{,\theta\theta}$$
$$+ \left(\frac{n_2\bar{C}_{44}}{\bar{C}_{11}}\right)\frac{1}{r^2}u_{,\theta} + \left(\frac{\bar{C}_{12} + \bar{C}_{44}}{\bar{C}_{11}}\right)\frac{1}{r}v_{,r\theta}$$
$$+ \left(\frac{m_2\bar{C}_{12} - \bar{C}_{22} - \bar{C}_{44}}{\bar{C}_{11}}\right)\frac{1}{r^2}v_{,\theta} + \left[\left(\frac{\bar{e}_{21}}{\bar{C}_{11}}\right)\psi_{,rr}\right.$$
$$+ \left(\frac{(m_4+1)\bar{e}_{21} - \bar{e}_{22}}{\bar{C}_{11}}\right)\frac{1}{r}\psi_{,r} + \left(\frac{\bar{e}_{24}}{\bar{C}_{11}}\right)\frac{1}{r^2}\psi_{,\theta\theta}$$
$$\left. + \left(\frac{n_4\bar{e}_{24}}{\bar{C}_{11}}\right)\frac{1}{r^2}\psi_{,\theta}\right]\tilde{r}^{(m_4-m_2)}e^{(n_4-n_2)\theta}$$
$$= \left[\left((m_1 + m_2 + 1)\right.\right.$$
$$\left. + \frac{2(m_1+m_2)\bar{C}_{12} - \bar{C}_{22}}{\bar{C}_{11}}\right)\frac{1}{r}T$$
$$\left. + \left(1 + \frac{2\bar{C}_{12}}{\bar{C}_{11}}\right)T_{,r}\right]\alpha_0\tilde{r}^{m_1}e^{n_1\theta},$$

$$v_{,rr} + (m_2+1)\frac{1}{r}v_{,r} - (m_2+1)\frac{1}{r^2}v$$
$$+ \left(n_2\frac{\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{1}{r^2}v_{,\theta} + \left(\frac{\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{1}{r^2}v_{,\theta\theta}$$
$$+ \left((m_2+1) + \frac{\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{1}{r^2}u_{,\theta} + \left(n_2\frac{\bar{C}_{12}}{\bar{C}_{44}}\right)\frac{1}{r}u_{,r}$$
$$+ \left(1 + \frac{\bar{C}_{12}}{\bar{C}_{44}}\right)\frac{1}{r}u_{,r\theta} + \left(n_2\frac{\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{1}{r^2}u$$
$$+ \left[\left(n_4\frac{\bar{e}_{22}}{\bar{C}_{44}}\right)\frac{1}{r}\psi_{,r} + \left(\frac{\bar{e}_{22} + \bar{e}_{24}}{\bar{C}_{44}}\right)\frac{1}{r}\psi_{,r\theta}\right.$$

$$+\left((m_4+2)\frac{\bar{e}_{24}}{\bar{C}_{44}}\right)\frac{1}{r^2}\psi_{,\theta}\Bigg]\tilde{r}^{(m_4-m_2)}e^{(n_4-n_2)\theta}$$

$$=\Bigg[\left((n_1+n_2)\frac{2\bar{C}_{12}+\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{1}{r}T$$

$$+\left(\frac{2\bar{C}_{12}+\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{1}{r}T_{,\theta}\Bigg]\alpha_0\tilde{r}^{m_1}e^{n_1\theta},$$

$$\psi_{,rr}+(m_5+1)\frac{1}{r}\psi_{,r}+\left(n_5\frac{\bar{\varepsilon}_{21}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r^2}\psi_{,\theta}$$

$$+\left(\frac{\bar{\varepsilon}_{21}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r^2}\psi_{,\theta\theta}-\Bigg[\left(\frac{\bar{e}_{21}}{\bar{\varepsilon}_{22}}\right)u_{,rr}$$

$$+\left(\frac{(m_4+1)\bar{e}_{21}+\bar{e}_{22}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r}u_{,r}+\left(\frac{m_4\bar{e}_{22}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r^2}u$$

$$+\left(\frac{n_4\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r}v_{,r}-\left(\frac{n_4\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r^2}v$$

$$+\left(\frac{(m_4+1)\bar{e}_{22}-\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r^2}v_{,\theta}$$

$$+\left(\frac{\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r}v_{,r\theta}\Bigg]\tilde{r}^{(m_4-m_5)}e^{(n_4-n_5)\theta}$$

$$=\Bigg[\left(\frac{(m_6+1)\bar{g}_{21}+n_6\bar{g}_{22}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r}T+\left(\frac{\bar{g}_{22}}{\bar{\varepsilon}_{22}}\right)T_{,r}$$

$$+\left(\frac{\bar{g}_{22}}{\bar{\varepsilon}_{22}}\right)\frac{1}{r}T_{,\theta}\Bigg]\tilde{r}^{(m_6-m_5)}e^{(n_6-n_5)\theta}. \tag{18}$$

The Navier equations (18) represent a non-homogeneous system of partial differential equations with non-constant coefficients. For simplicity, without the loss of generality, the power law of material properties is considered to be:

$$m_2=m_4, \qquad m_4=m_5,$$

$$m_6-m_5=m_1, \qquad n_2=n_4,$$

$$n_4=n_5, \qquad n_6-n_5=n_1.$$

3. Solution of the Navier equation

$$u(r,\theta)=\sum_{q=-\infty}^{\infty}u_q(r)e^{(iq+n_1)\theta},$$

$$v(r,\theta)=\sum_{q=-\infty}^{\infty}v_q(r)e^{(iq+n_1)\theta},$$

$$\psi(r,\theta)=\sum_{q=-\infty}^{\infty}\psi_q(r)e^{(iq+n_1)\theta}. \tag{19}$$

Substituting Eqs. (11) and (19) into Eq. (18) yields:

$$u_q''+(\lambda_1)\frac{1}{r}u_q'+(\lambda_2+i\lambda_3)\frac{1}{r^2}u_q$$

$$+(\lambda_4+i\lambda_5)\frac{1}{r}v_q'+(\lambda_6+i\lambda_7)\frac{1}{r^2}v_q$$

$$+(\lambda_8)\psi_q''+(\lambda_9)\frac{1}{r}\psi_q'+(\lambda_{10}+i\lambda_{11})\frac{1}{r^2}\psi_q$$

$$=\left(\frac{1}{a^{m_1}}\right)((\lambda_{12}+\beta_{q_1}\lambda_{13})A_{q_1}r^{m_1+\beta_{q_1}-1}$$

$$+(\lambda_{12}+\beta_{q_2}\lambda_{13})A_{q_2}r^{m_1+\beta_{q_2}-1}),$$

$$v_q''+(\lambda_{14})\frac{1}{r}v_q'-(\lambda_{15}-i\lambda_{16})\frac{1}{r^2}v_q$$

$$+(\lambda_{17}+i\lambda_{18})\frac{1}{r}u_q'+(\lambda_{19}+i\lambda_{20})\frac{1}{r^2}u_q$$

$$+(\lambda_{21}+i\lambda_{22})\frac{1}{r}\psi_q'+(\lambda_{23}+i\lambda_{24})\frac{1}{r^2}\psi_q$$

$$=\left(\frac{1}{a^{m_1}}\right)(\lambda_{25}+i\lambda_{26})(A_{q_1}r^{\beta_{q_1}+m_1-1}$$

$$+A_{q_2}r^{\beta_{q_2}+m_1-1}),$$

$$\psi_q''+(\lambda_{27})\frac{1}{r}\psi_q'+(\lambda_{28}+i\lambda_{29})\frac{1}{r^2}\psi_q-(\lambda_{30})u_q''$$

$$-(\lambda_{31})\frac{1}{r}u_q'-(\lambda_{32})\frac{1}{r^2}u_q+(\lambda_{33}+i\lambda_{34})\frac{1}{r}v_q'$$

$$+(\lambda_{35}+i\lambda_{36})\frac{1}{r^2}v_q$$

$$=\left(\frac{1}{a^{m_1}}\right)((\lambda_{37}+i\lambda_{38}+\beta_{q_1}\lambda_{39})A_{q_1}r^{\beta_{q_1}+m_1-1}$$

$$+(\lambda_{37}+i\lambda_{38}+\beta_{q_2}\lambda_{39})A_{q_2}r^{\beta_{q_2}+m_1-1}). \tag{20}$$

Eqs. (20) represent a system of ordinary differential equations with general and particular solutions.

The general solutions are assumed as follows:

$$u_q^g(r)=Dr^\eta,$$

$$v_q^g(r)=Er^\eta,$$

$$\psi_q^g(r)=Fr^\eta. \tag{21}$$

Substituting Eqs. (24) into Eqs. (23) yields:

$$[\eta(\eta-1)+\lambda_1\eta+\lambda_2+i\lambda_3]D + [\lambda_4\eta+\lambda_5+i(\lambda_6\eta+\lambda_7)]E + [\eta(\eta-1)\lambda_8+\lambda_9\eta+\lambda_{10}+i\lambda_{11}]F=0,$$
$$[\lambda_{19}+\lambda_{17}\eta+i(\lambda_{18}\eta+\lambda_{20})]D + [\eta(\eta-1)+\lambda_{14}\eta-\lambda_{15}+i\lambda_{16}]E + [\lambda_{21}\eta+\lambda_{23}+i\lambda_{22}\eta+\lambda_{24})]F=0,$$
$$[\eta(\eta-1)\lambda_{30}-\lambda_{31}\eta-\lambda_{32}]D + [\lambda_{33}\eta+\lambda_{35}+i(\lambda_{34}\eta+\lambda_{36})]E + [\eta(\eta-1)+\lambda_{27}\eta+\lambda_{28}+i\lambda_{29}]F=0. \tag{22}$$

Constant λ_i is presented in Appendix A.

A nontrivial solution is obtained by setting the determinant of the coefficients of Eqs. (22) equal to zero, where six order polynomial characteristic equations are obtained, presenting six eigen values η_{q_1} to η_{q_6}.

Thus, the general solutions are follows:

$$u_q^g(r)=\sum_{j=1}^{6} D_{q_j} r^{\eta_{q_j}} \Rightarrow u_q^g(r)=\sum_{j=1}^{6} D_{q_j} r^{\eta_{q_j}},$$
$$v_q^g(r)=\sum_{j=1}^{6} E_{q_j} r^{\eta_{q_j}} \Rightarrow v_q^g(r)=\sum_{j=1}^{6} X_{q_j} D_{q_j} r^{\eta_{q_j}},$$
$$\psi_q^g(r)=\sum_{j=1}^{6} F_{q_j} r^{\eta_{q_j}} \Rightarrow \psi_q^g(r)=\sum_{j=1}^{6} Y_{q_j} D_{q_j} r^{\eta_{q_j}}, \tag{23}$$

where X_{q_j} is the relation between constants D_{q_j} and E_{q_j}, and Y_{q_j} is the relation between constants D_{q_j} and F_{q_j} obtained from Eqs. (22); they are presented in Appendix A.

The particular solutions $u_q^p(r)$ and $v_q^p(r)$ are assumed as follows:

$$u_q^p(r)=I_{q_1} r^{\beta_{q_1}+m_1+1}+I_{q_2} r^{\beta_{q_2}+m_1+1},$$
$$v_q^p(r)=I_{q_3} r^{\beta_{q_1}+m_1+1}+I_{q_4} r^{\beta_{q_2}+m_1+1},$$
$$\psi_q^p(r)=I_{q_5} r^{\beta_{q_1}+m_1+1}+I_{q_6} r^{\beta_{q_2}+m_1+1}. \tag{24}$$

Substituting Eqs. (24) into the non-homogeneous form of Eq. (20) gives I_{q_1} to I_{q_6} as they are presented in Appendix A.

The complete solutions for $u_q(r)$, $v_q(r)$, and $\psi_q(r)$ are the sum of the general and particular solutions which are as follows:

$$u_q(r)=\sum_{j=1}^{6} D_{qj} r^{\eta_{qj}}+I_{q_1} r^{\beta_{q_1}+m_1+1} + I_{q_2} r^{\beta_{q_2}+m_1+1},$$
$$v_q(r)=\sum_{j=1}^{6} X_{qj} D_{qj} r^{\eta_{qj}}+I_{q_3} r^{\beta_{q_1}+m_1+1} + I_{q_4} r^{\beta_{q_2}+m_1+1},$$
$$\psi_q(r)=\sum_{j=1}^{6} Y_{qj} D_{qj} r^{\eta_{qj}}+I_{q_5} r^{\beta_{q_1}+m_1+1} + I_{q_6} r^{\beta_{q_2}+m_1+1}. \tag{25}$$

Substituting Eqs. (22) into Eqs. (16) gives:

$$u(r,\theta)=\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\left[\sum_{j=1}^{6} D_{qj} r^{\eta_{qj}}+I_{q_1} r^{\beta_{q_1}+m_1+1} + I_{q_2} r^{\beta_{q_2}+m_1+1}\right] e^{(iq+n_1)\theta},$$
$$v(r,\theta)=\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\left[\sum_{j=1}^{6} X_{qj} D_{qj} r^{\eta_{qj}}+I_{q_3} r^{\beta_{q_1}+m_1+1} + I_{q_4} r^{\beta_{q_2}+m_1+1}\right] e^{(iq+n_1)\theta},$$
$$\psi(r,\theta)=\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\left[\sum_{j=1}^{6} Y_{qj} D_{qj} r^{\eta_{qj}}+I_{q_5} r^{\beta_{q_1}+m_1+1} + I_{q_6} r^{\beta_{q_2}+m_1+1}\right] e^{(iq+n_1)\theta}. \tag{26}$$

By substituting Eqs. (26) into Eqs. (12), the strains and electric intensity are obtained as follows:

$$\varepsilon_{rr}=\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\left[\sum_{j=1}^{6}(\eta_{qj}) D_{qj} r^{\eta_{qj}-1} + (\beta_{q_1}+m_1+1) I_{q_1} r^{\beta_{q_1}+m_1} + (\beta_{q_2}+m_1+1) I_{q_2} r^{\beta_{q_2}+m_1}\right] e^{(iq+n_1)\theta},$$
$$\varepsilon_{\theta\theta}=\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\left[\sum_{j=1}^{6}(iq+n_1)(X_{qj}+1) D_{qj} r^{\eta_{qj}-1} + \left((iq+n_1) I_{q_3}+I_{q_1}\right) r^{\beta_{q_1}+m_1} + \left((iq+n_1) I_{q_4}+I_{q_2}\right) r^{\beta_{q_2}+m_1}\right] e^{(iq+n_1)\theta},$$

$$\varepsilon_{r\theta}=\frac{1}{2}\Bigg(\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\Bigg[\sum_{j=1}^{6}\Big((iq+n_1)+(\eta_{qj}-1)X_{qj})\Big)D_{qj}r^{\eta_{qj}-1}+\big((iq+n_1)I_{q_1}+(\beta_{q_1}+m_1)I_{q_3}\big)r^{\beta_{q_1}+m_1}+\big((iq+n_1)I_{q_2}+(\beta_{q_2}+m_1)I_{q_4}\big)r^{\beta_{q_2}+m_1}\Bigg]\Bigg)e^{(iq+n_1)\theta},$$

$$E_r=\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\Bigg[\sum_{j=1}^{6}(\eta_{qj})Y_{qj}D_{qj}r^{\eta_{qj}-1}+(\beta_{q_1}+m_1+1)I_{q_5}r^{\beta_{q_1}+m_1}+(\beta_{q_2}+m_1+1)I_{q_6}r^{\beta_{q_2}+m_1}\Bigg]e^{(iq+n_1)\theta},$$

$$E_\theta=\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\Bigg[\sum_{j=1}^{6}(iq+n_1)Y_{qj}D_{qj}r^{\eta_{qj}-1}+(iq+n_1)I_{q_5}r^{\beta_{q_1}+m_1}+(iq+n_1)I_{q_6}r^{\beta_{q_2}+m_1}\Bigg]e^{(iq+n_1)\theta}. \tag{27}$$

By substituting Eqs. (27) into Eqs. (13), the stress and electric displacement are obtained as follows:

$$\sigma_{rr}=\frac{1}{a^{m_2}}\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\Bigg[\sum_{j=1}^{6}\Big(\bar{C}_{11}\Big(\eta_{qj}D_{qj}r^{\eta_{qj}+m_2-1}+(\beta_{q_1}+m_1+1)I_{q_1}r^{\beta_{q_1}+m_1+m_2}+(\beta_{q_2}+m_1+1)I_{q_2}r^{\beta_{q_2}+m_1+m_2}-\frac{\alpha_0}{a^{m_1}}\left(A_{q1}r^{\beta_{q_1}+m_1+m_2}+A_{q2}r^{\beta_{q_2}+m_1+m_2}\right)\Big)+\bar{C}_{12}\big((iq+n_1)(X_{qj}+1)D_{qj}r^{\eta_{qj}+m_2-1}+((iq+n_1)I_{q_3}+I_{q_1})r^{\beta_{q_1}+m_1+m_2}+((iq+n_1)I_{q_4}+I_{q_2})r^{\beta_{q_2}+m_1+m_2}-\frac{2\alpha_0}{a^{m_1}}(A_{q1}r^{\beta_{q_1}+m_1+m_2}+A_{q2}r^{\beta_{q_2}+m_1+m_2})\big)\Big)e^{n_2\theta}+\bar{e}_{21}((\eta_{qj})Y_{qj}D_{qj}r^{\eta_{qj}+m_2-1}+(\beta_{q_1}+m_1+1)I_{q_5}r^{\beta_{q_1}+m_1+m_2}+(\beta_{q_2}+m_1+1)I_{q_6}r^{\beta_{q_2}+m_1+m_2})e^{n_2\theta}\Bigg]e^{(iq+n_1)\theta},$$

$$\sigma_{\theta\theta}=\frac{1}{a^{m_2}}\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\Bigg[\sum_{j=1}^{6}\Big(\bar{C}_{12}\Big(\eta_{qj}D_{qj}r^{\eta_{qj}+m_2-1}+(\beta_{q_1}+m_1+1)I_{q_1}r^{\beta_{q_1}+m_1+m_2}+(\beta_{q_2}+m_1+1)I_{q_2}r^{\beta_{q_2}+m_1+m_2}-\frac{\alpha_0}{a^{m_1}}(A_{q1}r^{\beta_{q_1}+m_1+m_2}+A_{q2}r^{\beta_{q_2}+m_1+m_2})\Big)+\bar{C}_{22}\Big((iq+n_1)(X_{qj}+1)D_{qj}r^{\eta_{qj}+m_2-1}+((iq+n_1)I_{q_3}+I_{q_1})r^{\beta_{q_1}+m_1+m_2}+((iq+n_1)I_{q_4}+I_{q_2})r^{\beta_{q_2}+m_1+m_2}-\frac{2\alpha_0}{a^{m_1}}(A_{q1}r^{\beta_{q_1}+m_1+m_2}+A_{q2}r^{\beta_{q_2}+m_1+m_2})\Big)\Big)e^{n_2\theta}+\bar{e}_{22}((\eta_{qj})Y_{qj}D_{qj}r^{\eta_{qj}+m_2-1}+(\beta_{q_1}+m_1+1)I_{q_5}r^{\beta_{q_1}+m_1+m_2}+(\beta_{q_2}+m_1+1)I_{q_6}r^{\beta_{q_2}+m_1+m_2})e^{n_2\theta}\Bigg]e^{(iq+n_1)\theta},$$

$$\sigma_{r\theta}=\frac{1}{a^{m_2}}\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\Bigg[\sum_{j=1}^{6}\bar{C}_{44}((iq+n_1)+((\eta_{qj}-1)X_{qj})D_{qj}r^{\eta_{qj}+m_2-1}+((iq+n_1)I_{q_1}+(\beta_{q_1}+m_1)I_{q_3})r^{\beta_{q_1}+m_1+m_2}+((iq+n_1)I_{q_2}+(\beta_{q_2}+m_1)I_{q_4})r^{\beta_{q_2}+m_1+m_2})e^{n_2\theta}-\bar{e}_{24}((iq+n_1)Y_{qj}D_{qj}r^{\eta_{qj}+m_2-1}+(iq+n_1)I_{q_5}r^{\beta_{q_1}+m_1+m_2}+(iq+n_1)I_{q_6}r^{\beta_{q_2}+m_1+m_2})e^{n_2\theta}\Bigg]e^{(iq+n_1)\theta},$$

$$\sigma_{zz} = \frac{1}{a^{m_2}} \sum_{\substack{q=-\infty \\ q\neq 0}}^{\infty} \left[\sum_{j=1}^{6} \left(\bar{C}_{12} \left(\eta_{qj} D_{qj} r^{\eta_{qj}+m_2-1} \right.\right.\right.$$

$$+ (\beta_{q_1} + m_1 + 1) I_{q_1} r^{\beta_{q_1}+m_1+m_2}$$

$$+ (\beta_{q_2} + m_1 + 1) I_{q_2} r^{\beta_{q_2}+m_1+m_2}$$

$$+ ((iq + n_1) I_{q_3} + I_{q_1}) r^{\beta_{q_1}+m_1+m_2}$$

$$+ ((iq + n_1) I_{q_4} + I_{q_2}) r^{\beta_{q_2}+m_1+m_2}$$

$$- \frac{3\alpha_0}{a^{m_1}} (A_{q1} r^{\beta_{q_1}+m_1+m_2}$$

$$\left.\left. + A_{q2} r^{\beta_{q_2}+m_1+m_2}) \right) \right) e^{n_2\theta}$$

$$+ \bar{e}_{23}((\eta_{qj}) Y_{qj} D_{qj} r^{\eta_{qj}+m_2-1}$$

$$+ (\beta_{q_1} + m_1 + 1) I_{q_5} r^{\beta_{q_1}+m_1+m_2}$$

$$\left. + (\beta_{q_2} + m_1 + 1) I_{q_6} r^{\beta_{q_2}+m_1+m_2}) e^{n_2\theta} \right] e^{(iq+n_1)\theta},$$

$$D_{rr} = \frac{1}{a^{m_2}} \sum_{\substack{q=-\infty \\ q\neq 0}}^{\infty} \left[\sum_{j=1}^{6} (\bar{e}_{21}(\eta_{qj} D_{qj} r^{\eta_{qj}+m_2-1} \right.$$

$$+ (\beta_{q_1} + m_1 + 1) I_{q_1} r^{\beta_{q_1}+m_1+m_2}$$

$$+ (\beta_{q_2} + m_1 + 1) I_{q_2} r^{\beta_{q_2}+m_1+m_2})$$

$$+ \bar{e}_{22}((iq + n_1)(X_{qj} + 1) D_{qj} r^{\eta_{qj}+m_2-1}$$

$$+ ((iq + n_1) I_{q_3} + I_{q_1}) r^{\beta_{q_1}+m_1+m_2}$$

$$+ ((iq + n_1) I_{q_4} + I_{q_2}) r^{\beta_{q_2}+m_1+m_2})) e^{n_2\theta}$$

$$- \bar{\varepsilon}_{22}((\eta_{qj}) Y_{qj} D_{qj} r^{\eta_{qj}+m_2-1}$$

$$+ (\beta_{q_1} + m_1 + 1) I_{q_5} r^{\beta_{q_1}+m_1+m_2}$$

$$+ (\beta_{q_2} + m_1 + 1) I_{q_6} r^{\beta_{q_2}+m_1+m_2}) e^{n_2\theta}$$

$$+ \frac{\bar{g}_{21}}{a^{m_1}} (A_{q_1} r^{\beta_{q_1}+m_1+m_2}$$

$$\left. + A_{q_2} r^{\beta_{q_2}+m_1+m_2}) e^{(n_1+n_2)\theta} \right] e^{(iq+n_1)\theta},$$

$$D_{\theta\theta} = \frac{1}{a^{m_2}} \sum_{\substack{q=-\infty \\ q\neq 0}}^{\infty} \left[\sum_{j=1}^{6} \bar{e}_{24}((iq + n_1) \right.$$

$$+ ((\eta_{qj} - 1) X_{qj}) D_{qj} r^{\eta_{qj}+m_4-1} + ((iq + n_1) I_{q_1}$$

$$+ (\beta_{q_1} + m_1) I_{q_3}) r^{\beta_{q_1}+m_1+m_4} + (iq + n_1) I_{q_2}$$

$$+ (\beta_{q_2} + m_1) I_{q_4}) r^{\beta_{q_2}+m_1+m_4}) e^{n_4\theta}$$

$$- \bar{\varepsilon}_{21}((iq + n_1) Y_{qj} D_{qj} r^{\eta_{qj}+m_5-1}$$

$$+ (iq + n_1) I_{q_5} r^{\beta_{q_1}+m_1+m_5}$$

$$+ (iq + n_1) I_{q_6} r^{\beta_{q_2}+m_1+m_5}) e^{n_5\theta}$$

$$+ \frac{\bar{g}_{22}}{a^{m_1}} (A_{q_1} r^{\beta_{q_1}+m_1+m_6}$$

$$\left. + A_{q_2} r^{\beta_{q_2}+m_1+m_6}) e^{n_6\theta} \right] e^{(iq+n_1)\theta}. \tag{28}$$

Assume that the six boundary conditions are expressed as follows:

$$u(a,\theta) = w_1(\theta), \qquad \sigma_{rr}(a,\theta) = w_7(\theta),$$

$$u(b,\theta) = w_2(\theta), \qquad \sigma_{rr}(b,\theta) = w_8(\theta),$$

$$v(a,\theta) = w_3(\theta), \qquad \sigma_{r\theta}(a,\theta) = w_9(\theta),$$

$$v(b,\theta) = w_4(\theta), \qquad \sigma_{r\theta}(b,\theta) = w_{10}(\theta),$$

$$\psi(a,\theta) = w_5(\theta), \qquad D_{rr}(a,\theta) = w_{11}(\theta),$$

$$\psi(b,\theta) = w_6(\theta), \qquad D_{rr}(b,\theta) = w_{12}(\theta). \tag{29}$$

Expanding the given boundary conditions in complex Fourier series gives:

$$w_j(\theta) = \sum_{n=-\infty}^{\infty} W_j(q) e^{(iq+n_1)\theta}, \qquad j = 1, \cdots, 6, \tag{30}$$

where:

$$W_j(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w_j(q) e^{-(iq+n_1)\theta} d\theta, \quad j = 1, \cdots, 6. \tag{31}$$

Using the selected six boundary conditions of Eqs. (29), by means of Eqs. (33) and (31), six unknown coefficients, D_{q_1} to D_{q_6}, are calculated.

4. Results and discussion

The proposed analytical solution programmed into MATLAB (2008 $\sim$ 2016) is solved. Table 1 shows a thick hollow infinite cylinder of inner radius $a = 1$ (m) and outer radius $b = 1.2$ (m) of a PZT-4 material with properties.

Table 1. Material properties PZT-4 for 2D-FGPM.

Parameters	Value	Parameters	Value
α_0	0.000012 $\left(\frac{1}{^\circ C}\right)$	$\bar{e}_{23}$	15.1 $\left(\frac{\mathrm{C}}{\mathrm{m}^2}\right)$
$\bar{r}$	1.1 m	$\bar{e}_{24}$	12.7 $\left(\frac{\mathrm{C}}{\mathrm{m}^2}\right)$
$\bar{C}_{11}$	139 GPa	$\bar{\varepsilon}_{21}$	6.5×10^{-9} $\left(\frac{\mathrm{C}^2}{\mathrm{Nm}^2}\right)$
$\bar{C}_{12}$	78 GPa	$\bar{\varepsilon}_{22}$	6.5×10^{-9} $\left(\frac{\mathrm{C}^2}{\mathrm{Nm}^2}\right)$
$\bar{C}_{22}$	139 GPa	$\bar{g}_{21}$	5.4×10^{-5} $\left(\frac{\mathrm{C}}{\mathrm{m}^2\mathrm{k}}\right)$
$\bar{C}_{44}$	30.5 GPa	$\bar{g}_{22}$	5.4×10^{-5} $\left(\frac{\mathrm{C}}{\mathrm{m}^2\mathrm{k}}\right)$
$\bar{e}_{21}$	-5.2 $\left(\frac{\mathrm{C}}{\mathrm{m}^2}\right)$	$m_1, m_2, \cdots, m_6$	m
$\bar{e}_{22}$	-5.2 $\left(\frac{\mathrm{C}}{\mathrm{m}^2}\right)$	$n_1, n_2, \cdots, n_6$	n

Table 2. Thermal and electrical boundary conditions for the first example of FGPM.

Boundary conditions						
$T(a,\theta)$	$\sigma_{rr}(a,\theta)$	$\sigma_{r\theta}(a,\theta)$	$u(b,\theta)$	$\nu(b,\theta)$	$\psi(a,\theta)$	$\psi(b,\theta)$
$60\sin(2\lvert\theta\rvert)$ (°c)	0	0	0	0	0	$\psi_0\theta^2\cos(2\theta)$ (W/A)

Table 3. Mechanical and electrical boundary conditions for the second example of FGPM.

Boundary conditions					
T	$\sigma_{rr}(a,\theta)$	$\sigma_{r\theta}(a,\theta)$	$v(b,\theta)$	$\psi(a,\theta)$	$\psi(b,\theta)$
0	$600\sec h\left(\frac{\theta}{6}^2+\theta\right)$ (MPa)	$30\ \theta^2\cosh\ (\theta)$ (MPa)	0	$\psi_0\theta^2\cos\ (2\theta)$ (W/A)	0

The thermal boundary conditions are substituted into Eqs. (2) and (3) to obtain the temperature distribution, where the constants of integration are obtained from equations in Appendix A. In general, the displacement and stress boundary conditions are substituted into Eqs. (29); in addition, with proper function expansions (Eq. (30)), the constant coefficients of the series expansions are obtained from Eqs. (31).

To examine the proposed method for the solution, two example problems are considered. The first example may have a certain physical interpretation, while the second example is chosen to show the mathematical effectiveness of the proposed method.

As the first example, consider a thick hollow infinite cylinder where the inside boundary is traction-free with a given temperature distribution as in Table 2. The outside boundary is assumed to be radially fixed with zero temperature. Therefore, the assumed boundary conditions yield the function given in Table 2.

In the second example, a thick-walled cylinder may be assumed with zero temperature distribution, yet exposed to mechanical boundary conditions. The stress and displacement boundary conditions are assumed to be selected, such that the mathematical strength of the proposed method is examined. These types of boundary conditions may not be handled with the potential function method. Using Eqs. (29) to (31), the boundary conditions given in terms of the radial and shear stresses as well as electric potential appear in Table 3. These boundary conditions are expanded by the integral series, and unknown coefficients, D_{q_j}, are determined.

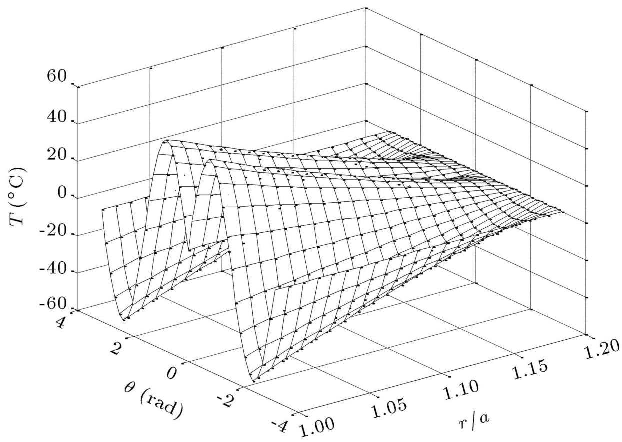

Figure 2. Temperature distribution in the cross-section of cylinder.

Figure 2 shows the temperature distribution in the wall thickness along the radial and circumferential directions. The effect of the power-law index on the temperature distribution is also shown in Figures 3 and 4.

Figure 5 shows the hoop stress in the cross-section of the cylinder. The effect of the power-law index on the distribution of the hoop thermal stress is shown in

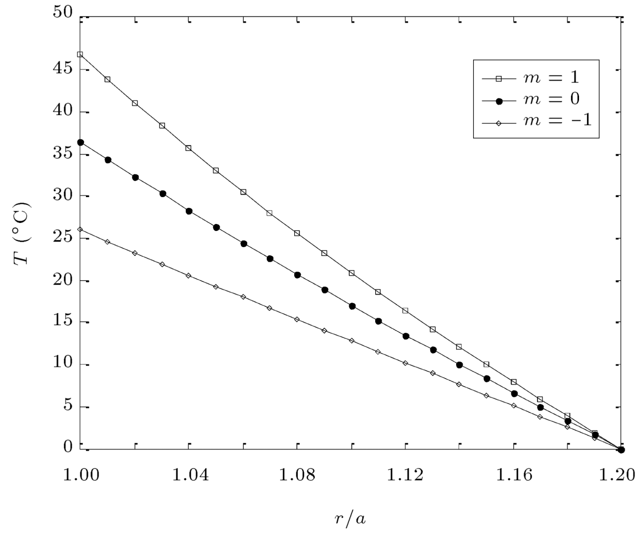

Figure 3. Circumferential temperature distribution at $\theta = \pi/3$.

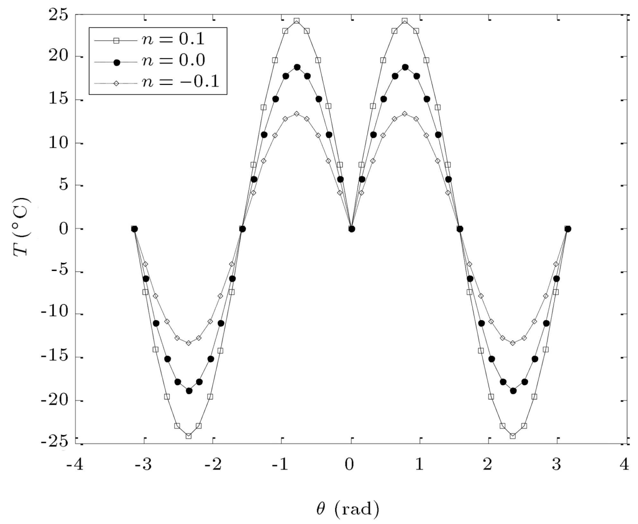

Figure 4. Circumferential temperature distribution at $r = \bar{r}$.

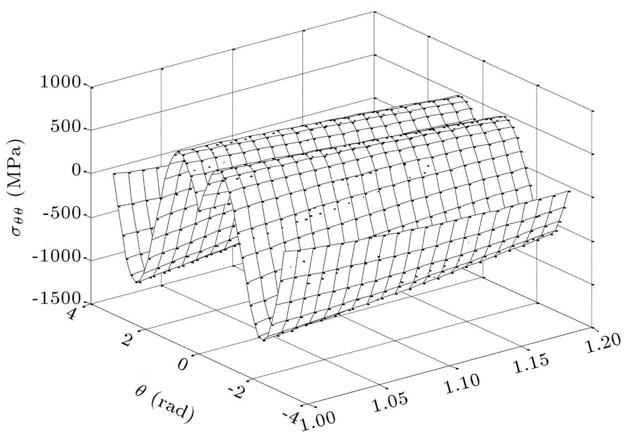

Figure 5. Hoop thermal stress in the cross-section of cylinder.

Figures 6 and 7. These figures are the plot of hoop stresses versus r/a at $\theta = \pi/3$ and θ at $r = \bar{r}$.

Figures 8 and 9 show the hoop thermal stress and radial displacements in the cross-section of cylinder, respectively, where the electric potential coefficient (ψ_0) changes and the other parameters remain fixed.

The radial mechanical stress distributions are shown in Figure 10. Stress patterns on the inside and outside surfaces follow harmonic patterns. The given harmonic boundary conditions for σ_{rr} at $r = a$ have a general influence on the pattern of the stress distributions in the cylinders' cross-section. The effect of the power-law index on the distribution of the radial mechanical stresses is shown in Figures 11 and 12. These figures show the plot of stresses versus r/a at $\theta = \pi/3$ and θ at $r = \bar{r}$. It is shown that as m and n increase, the radial mechanical stresses increase, too.

Figures 13 and 14 show the axial mechanical

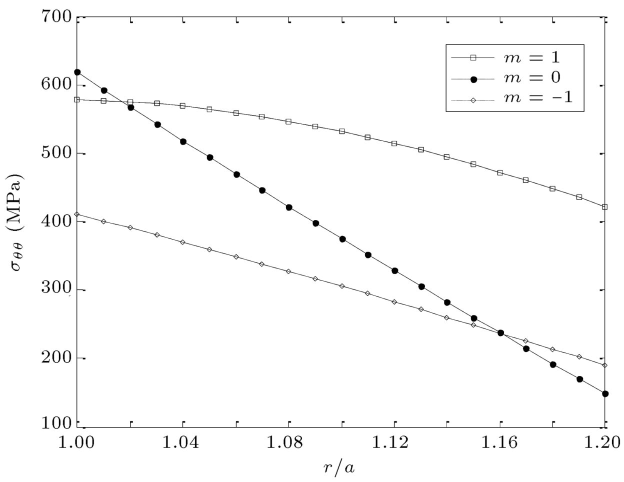

Figure 6. Hoop distribution of circumferential thermal stress, $\sigma_{\theta\theta}$, at $\theta = \pi/3$.

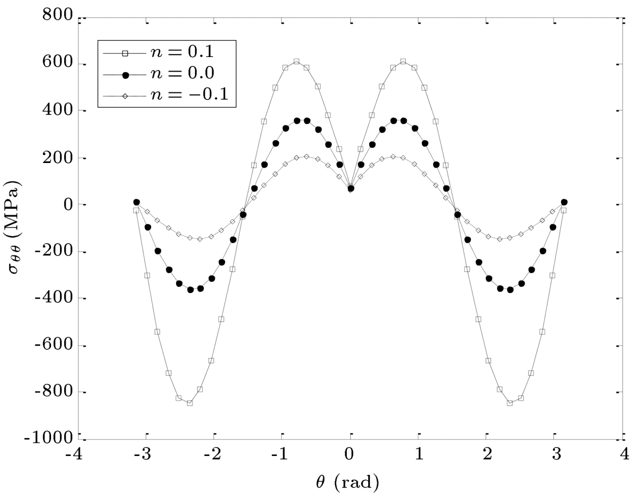

Figure 7. Hoop distribution of circumferential thermal stress, $\sigma_{\theta\theta}$, at $r = \bar{r}$.

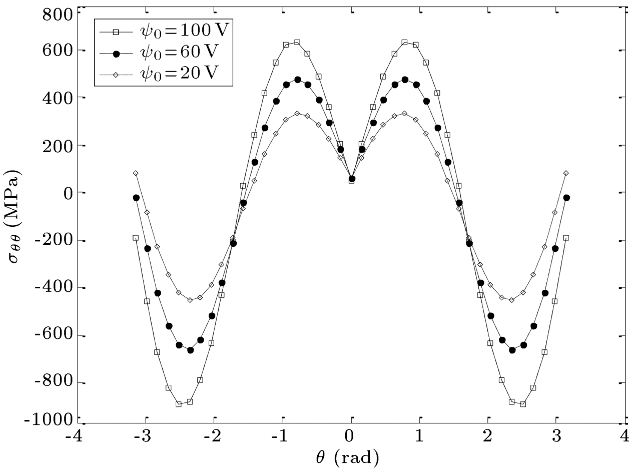

Figure 8. Hoop thermal stress in the cross-section of cylinder in different electric potential coefficients.

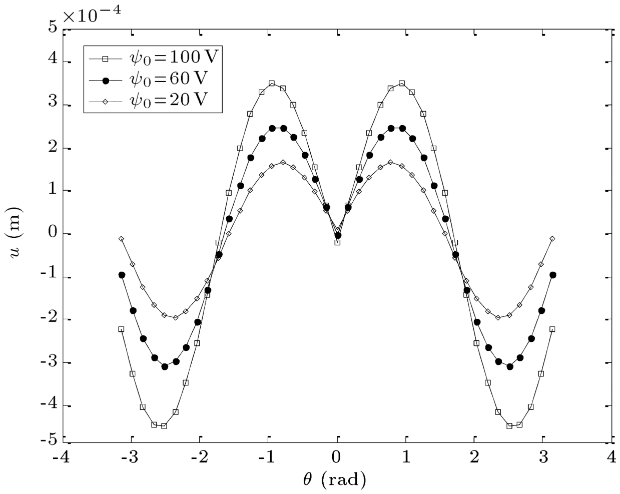

Figure 9. Radial displacement in the cross-section of cylinder in different electric potential coefficients.

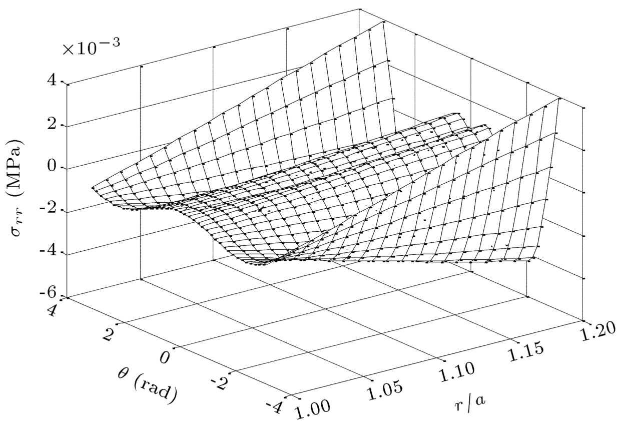

Figure 10. Radial mechanical stress in the cross-section of cylinder.

stresses and circumferential displacements in the cross-section of cylinder, respectively, where electric potential coefficient (ψ_0) is changing.

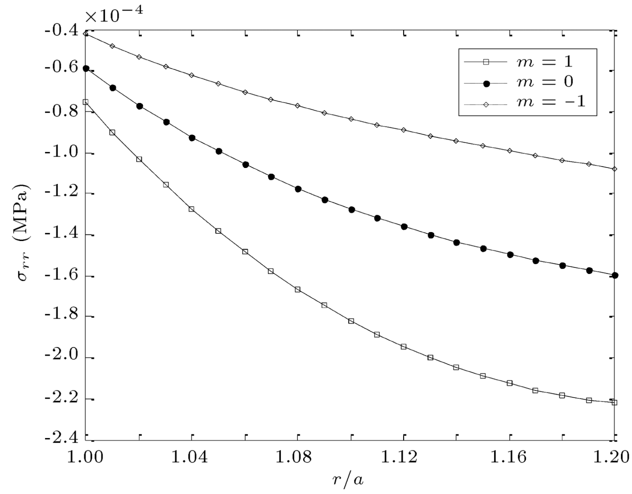

Figure 11. Radial distribution of circumferential mechanical stress, σ_{rr}, at $\theta = \pi/3$.

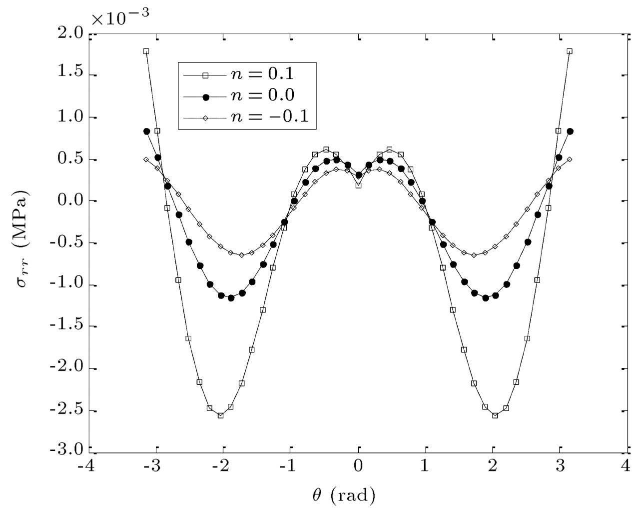

Figure 12. Radial distribution of circumferential mechanical stress, σ_{rr}, at $r = \bar{r}$.

5. Conclusions

In the present work, an attempt was made to study the problem of a general solution for the thermal and mechanical stresses in thick two-dimensional Functionally Graded Piezoelectric Materials (2D-FGPMs) hollow infinite cylinder where the two-dimensional asymmetric steady-state loads are taken into account. The method of solution is based on the direct method and uses the power series, rather than the potential function method. The advantage of this method is its mathematical power to handle both simple and complicated mathematical functions for the thermal and mechanical stress boundary conditions. The potential function method is capable of handling the complicated mathematical functions as in boundary condition. The proposed method does not have the mathematical limitations to handle the general types

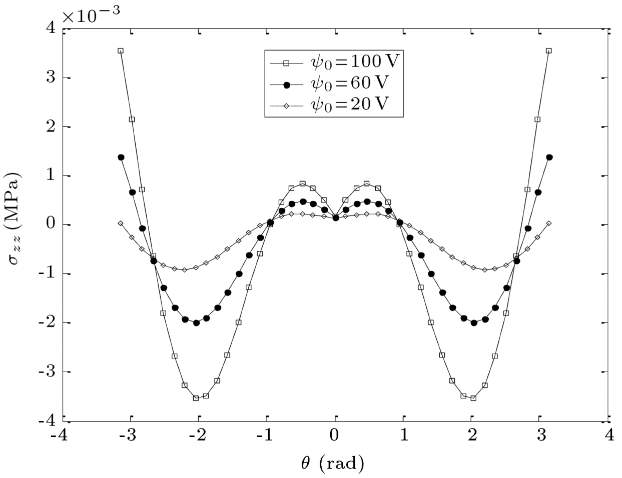

Figure 13. Axial mechanical stress in the cross-section of cylinder in different electric potential coefficients.

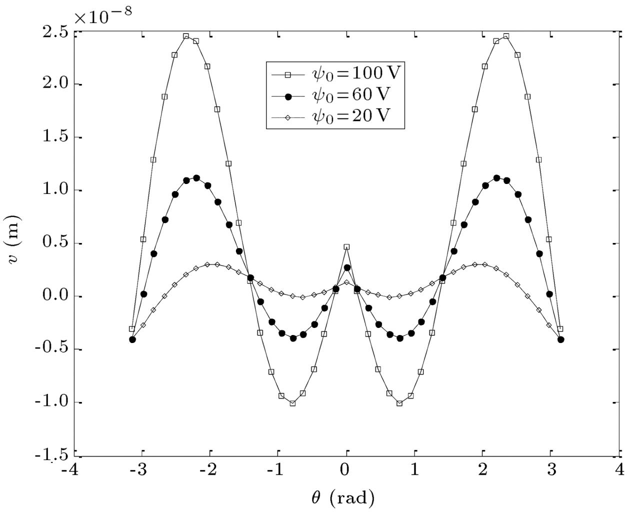

Figure 14. Circumferential displacement in the cross-section of cylinder in different electric potential coefficients.

of boundary conditions, which usually occur in the potential function method.

Nomenclature

A_{q_1}, A_{q_2}	Thermal constant
a	Inner radius
b	Outer radius
S_{ij}	Constant temperature parameters
C_{ij}	Elastic constant
e_{2i}	Piezoelectric constant
ε_{2i}	Dielectric constant
g_{2j}	Pyroelectric coefficient
d_i	Mechanical and thermal constants
I_{ij}	Constant mechanical parameters
f_1, f_2	Inner and outer temperature boundary conditions
$w_1, \cdots, w_{12}$	Inner and outer mechanical boundary conditions
m_i, n_i	Material properties parameters $i = 1 \cdots 6$
k	Thermal conduction coefficient
k_0	Material parameter
r, θ, z	Cylinder coordinate
T	Cylinder temperature
T_q	Coefficient of sine Fourier series
u, v	Displacement components
α	Thermal expansion coefficient
α_0	Material constant
ε_{ij}	Strain tensor $(i,j) = (r,\theta)$
$\in$	Volumetric strain ($\in = \varepsilon_{rr} + \varepsilon_{\theta\theta}$)
σ_{ij}	Stress tensor $(i,j) = (r,\theta)$
D_{ij}	Electric displacement $(i,j) = (r,\theta)$
$\psi(r,\theta)$	Electric potential
E_r, E_θ	Electric field intensity

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Appendix A

d_1 to d_{24} are coefficients I_{q_i} in Appendix A (see Eq. (A.4)):

$$\begin{aligned} d_1 =&(\beta_{q_1}+m_1+1)(\beta_{q_1}+m_1)\\ &+\left((m_2+1)+(m_2-1)\frac{\bar{C}_{12}}{\bar{C}_{11}}\right)(\beta_{q_1}+m_1+1)\\ &+\left(\frac{m_2\bar{C}_{12}-\bar{C}_{22}}{\bar{C}_{11}}+((n_1+n_2)n_1\right.\\ &\left.+iq(2n_1+n_2)-q^2)\frac{\bar{C}_{44}}{\bar{C}_{11}}\right), \end{aligned}$$

$$\begin{aligned} d_2 =&(\beta_{q_2}+m_1+1)(\beta_{q_2}+m_1)\\ &+\left((m_2+1)+(m_2-1)\frac{\bar{C}_{12}}{\bar{C}_{11}}\right)(\beta_{q_2}+m_1+1)\\ &+\left(\frac{m_2\bar{C}_{12}-\bar{C}_{22}}{\bar{C}_{11}}+[(n_1+n_2)n_1\right.\\ &\left.+iq(2n_1+n_2)-q^2]\frac{\bar{C}_{44}}{\bar{C}_{11}}\right), \end{aligned}$$

$$\begin{aligned} d_3 =&\left((iq+n_1)\frac{\bar{C}_{12}}{\bar{C}_{11}}\right.\\ &\left.+(iq+n_1+n_2)\frac{\bar{C}_{44}}{\bar{C}_{11}}\right)(\beta_{q_1}+m_1+1)\\ &+\left((iq+n_1)\frac{m_2\bar{C}_{12}-\bar{C}_{22}}{\bar{C}_{11}}-(iq+n_1+n_2)\frac{\bar{C}_{44}}{\bar{C}_{11}}\right), \end{aligned}$$

$$\begin{aligned} d_4 =&\left((iq+n_1)\frac{\bar{C}_{12}}{\bar{C}_{11}}\right.\\ &\left.+(iq+n_1+n_2)\frac{\bar{C}_{44}}{\bar{C}_{11}}\right)(\beta_{q_2}+m_1+1)\\ &+\left((iq+n_1)\frac{m_2\bar{C}_{12}-\bar{C}_{22}}{\bar{C}_{11}}-(iq+n_1+n_2)\frac{\bar{C}_{44}}{\bar{C}_{11}}\right), \end{aligned}$$

$$\begin{aligned} d_5 =&\left(\frac{\bar{e}_{21}}{\bar{C}_{11}}\right)(\beta_{q_1}+m_1+1)(\beta_{q_1}+m_1)\\ &+\left(\frac{(m_4+1)\bar{e}_{21}-\bar{e}_{22}}{\bar{C}_{11}}\right)(\beta_{q_1}+m_1+1)\\ &+\left([(n_1+n_4)n_1+iq(2n_1+n_4)-q^2]\frac{\bar{e}_{24}}{\bar{C}_{11}}\right), \end{aligned}$$

$$\begin{aligned} d_6 =&\left(\frac{\bar{e}_{21}}{\bar{C}_{11}}\right)(\beta_{q_2}+m_1+1)(\beta_{q_2}+m_1)\\ &+\left(\frac{(m_4+1)\bar{e}_{21}-\bar{e}_{22}}{\bar{C}_{11}}\right)(\beta_{q_2}+m_1+1)\\ &+\left([(n_1+n_4)n_1+iq(2n_1+n_4)-q^2]\frac{\bar{e}_{24}}{\bar{C}_{11}}\right), \end{aligned}$$

$$\begin{aligned} d_7 =&\left(\left((m_1+m_2+1)+\frac{2(m_1+m_2)\bar{C}_{12}-\bar{C}_{22}}{\bar{C}_{11}}\right)\right.\\ &\left.+\left(\frac{\bar{C}_{11}+2\bar{C}_{12}}{\bar{C}_{11}}\right)\beta_{q_1}\right)\frac{\alpha_0}{a^{m_1}}A_{q_1}, \end{aligned}$$

$$\begin{aligned} d_8 =&\left(\left((m_1+m_2+1)+\frac{2(m_1+m_2)\bar{C}_{12}-\bar{C}_{22}}{\bar{C}_{11}}\right)\right.\\ &\left.+\left(\frac{\bar{C}_{11}+2\bar{C}_{12}}{\bar{C}_{11}}\right)\beta_{q_2}\right)\frac{\alpha_0}{a^{m_1}}A_{q_2}, \end{aligned}$$

$$\begin{aligned} d_9 =&(\beta_{q_1}+m_1+1)(\beta_{q_1}+m_1)+(m_2+1)(\beta_{q_1}+m_1+1)\\ &-\left((m_2+1)-[(n_1+n_2)+iq(n_2+2)-q^2]\frac{\bar{C}_{22}}{\bar{C}_{44}}\right), \end{aligned}$$

$$\begin{aligned} d_{10} =&(\beta_{q_2}+m_1+1)(\beta_{q_2}+m_1)+(m_2+1)(\beta_{q_2}+m_1+1)\\ &-\left((m_2+1)-[(n_1+n_2)+iq(n_2+2)-q^2]\frac{\bar{C}_{22}}{\bar{C}_{44}}\right), \end{aligned}$$

$$\begin{aligned} d_{11} =&\left((iq+n_1)+(iq+n_1+n_2)\frac{\bar{C}_{12}}{\bar{C}_{44}}\right)(\beta_{q_1}+m_1+1)\\ &+\left((iq+n_1)(m_2+1)+(iq+n_1+n_2)\frac{\bar{C}_{22}}{\bar{C}_{44}}\right), \end{aligned}$$

$$d_{12}=\left((iq+n_1)+(iq+n_1+n_2)\frac{\bar{C}_{12}}{\bar{C}_{44}}\right)(\beta_{q_2}+m_1+1)$$

$$+\left((iq+n_1)(m_2+1)+(iq+n_1+n_2)\frac{\bar{C}_{22}}{\bar{C}_{44}}\right),$$

$$d_{13}=\left((iq+n_1)\frac{\bar{e}_{24}}{\bar{C}_{44}}\right.$$

$$\left.+(iq+n_1+n_4)\frac{\bar{e}_{22}}{\bar{C}_{44}}\right)(\beta_{q_1}+m_1+1)$$

$$+\left((iq+n_1)(m_4+2)\frac{\bar{e}_{24}}{\bar{C}_{44}}\right),$$

$$d_{14}=\left((iq+n_1)\frac{\bar{e}_{24}}{\bar{C}_{44}}\right.$$

$$\left.+(iq+n_1+n_4)\frac{\bar{e}_{22}}{\bar{C}_{44}}\right)(\beta_{q_2}+m_1+1)$$

$$+\left((iq+n_1)(m_4+2)\frac{\bar{e}_{24}}{\bar{C}_{44}}\right),$$

$$d_{15}=(n_1+n_2+iq)\left(\frac{2\bar{C}_{12}+\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{\alpha_0}{a^{m_1}}A_{q_1},$$

$$d_{16}=(n_1+n_2+iq)\left(\frac{2\bar{C}_{12}+\bar{C}_{22}}{\bar{C}_{44}}\right)\frac{\alpha_0}{a^{m_1}}A_{q_2},$$

$$d_{17}=(\beta_{q_1}+m_1+1)(\beta_{q_1}+m_1)$$

$$+(\beta_{q_1}+m_1+1)(m_5+1)$$

$$+\left(((n_1+n_5)n_1+iq(n_5+2n_1)-q^2)\frac{\bar{\varepsilon}_{21}}{\bar{\varepsilon}_{22}}\right),$$

$$d_{18}=(\beta_{q_2}+m_1+1)(\beta_{q_2}+m_1)$$

$$+(\beta_{q_2}+m_1+1)(m_5+1)$$

$$+\left(((n_1+n_5)n_1+iq(n_5+2n_1)-q^2)\frac{\bar{\varepsilon}_{21}}{\bar{\varepsilon}_{22}}\right),$$

$$d_{19}=-\left(\frac{\bar{e}_{21}}{\bar{\varepsilon}_{22}}\right)(\beta_{q_1}+m_1+1)(\beta_{q_1}+m_1)$$

$$-\left(\frac{(m_4+1)\bar{e}_{21}+\bar{e}_{22}}{\bar{\varepsilon}_{22}}\right)(\beta_{q_1}+m_1+1)$$

$$-\left(\frac{m_4\bar{e}_{22}}{\bar{\varepsilon}_{22}}\right),$$

$$d_{20}=-\left(\frac{\bar{e}_{21}}{\bar{\varepsilon}_{22}}\right)(\beta_{q_2}+m_1+1)(\beta_{q_2}+m_1)$$

$$-\left(\frac{(m_4+1)\bar{e}_{21}+\bar{e}_{22}}{\bar{\varepsilon}_{22}}\right)(\beta_{q_2}+m_1+1)$$

$$-\left(\frac{m_4\bar{e}_{22}}{\bar{\varepsilon}_{22}}\right),$$

$$d_{21}=\left((iq+n_1+n_4)\frac{\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right)(\beta_{q_1}+m_1+1)$$

$$+\left((iq+n_1)(m_4+1)\frac{\bar{e}_{22}}{\bar{\varepsilon}_{22}}-(iq+n_1+n_4)\frac{\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right),$$

$$d_{22}=\left((iq+n_1+n_4)\frac{\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right)(\beta_{q_2}+m_1+1)$$

$$+\left((iq+n_1)(m_4+1)\frac{\bar{e}_{22}}{\bar{\varepsilon}_{22}}-(iq+n_1+n_4)\frac{\bar{e}_{24}}{\bar{\varepsilon}_{22}}\right),$$

$$d_{23}=\left((m_6+1)\frac{\bar{g}_{21}}{\bar{\varepsilon}_{22}}+(iq+n_6+\beta_{q_1})\frac{\bar{g}_{22}}{\bar{\varepsilon}_{22}}\right)\frac{1}{a^{m_1}}A_{q_1},$$

$$d_{24}=\left((m_6+1)\frac{\bar{g}_{21}}{\bar{\varepsilon}_{22}}+(iq+n_6+\beta_{q_2})\frac{\bar{g}_{22}}{\bar{\varepsilon}_{22}}\right)\frac{1}{a^{m_1}}A_{q_2}, \tag{A.1}$$

$$\widehat{N}_{1q_j}=\eta_{q_j}(\eta_{q_j}-1)+\lambda_1\eta+\lambda_2+i\lambda_3,$$

$$\widehat{N}_{2q_j}=\lambda_4\eta_{q_j}+\lambda_5+i(\lambda_6\eta_{q_j}+\lambda_7),$$

$$\widehat{N}_{3q_j}=\eta_{q_j}(\eta_{q_j}-1)\lambda_8+\lambda_9\eta_{q_j}+\lambda_{10}+i\lambda_{11},$$

$$\widehat{N}_{4q_j}=\lambda_{19}+\lambda_{17}\eta_{q_j}+i(\lambda_{18}\eta_{q_j}+\lambda_{20}),$$

$$\widehat{N}_{5q_j}=\eta_{q_j}(\eta_{q_j}-1)+\lambda_{14}\eta_{q_j}-\lambda_{15}+i\lambda_{16},$$

$$\widehat{N}_{6q_j}=\lambda_{21}\eta_{q_j}+\lambda_{23}+i(\lambda_{22}\eta_{q_j}+\lambda_{24}),$$

$$\widehat{N}_{7q_j}=\eta_{q_j}(\eta_{q_j}-1)\lambda_{30}-\lambda_{31}\eta_{q_j}-\lambda_{32},$$

$$\widehat{N}_{8q_j}=\lambda_{33}\eta_{q_j}+\lambda_{35}+i(\lambda_{34}\eta_{q_j}+\lambda_{36}),$$

$$\widehat{N}_{9q_j}=\eta_{q_j}(\eta_{q_j}-1)+\lambda_{27}\eta_{q_j}+\lambda_{28}+i\lambda_{29}, \tag{A.2}$$

$$\begin{bmatrix}\widehat{N}_{1q_j} & \widehat{N}_{2q_j} & \widehat{N}_{3q_j}\\ \widehat{N}_{4q_j} & \widehat{N}_{5q_j} & \widehat{N}_{6q_j}\\ \widehat{N}_{7q_j} & \widehat{N}_{8q_j} & \widehat{N}_{9q_j}\end{bmatrix}\begin{bmatrix}D_{q_j}\\ E_{q_j}\\ F_{q_j}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Longrightarrow\begin{vmatrix}\widehat{N}_{1q_j} & \widehat{N}_{2q_j} & \widehat{N}_{3q_j}\\ \widehat{N}_{4q_j} & \widehat{N}_{5q_j} & \widehat{N}_{6q_j}\\ \widehat{N}_{7q_j} & \widehat{N}_{8q_j} & \widehat{N}_{9q_j}\end{vmatrix}=0, \tag{A.3}$$

$$X_{qj}=\frac{E_{q_j}}{D_{q_j}}=\frac{\widehat{N}_{1q_j}\widehat{N}_{6q_j}-\widehat{N}_{3q_j}\widehat{N}_{4q_j}}{\widehat{N}_{3q_j}\widehat{N}_{5q_j}-\widehat{N}_{2q_j}\widehat{N}_{6q_j}},$$

$$Y_{qj}=\frac{F_{q_j}}{D_{q_j}}=\frac{\widehat{N}_{4q_j}\widehat{N}_{8q_j}-\widehat{N}_{5q_j}\widehat{N}_{7q_j}}{\widehat{N}_{5q_j}\widehat{N}_{9q_j}-\widehat{N}_{6q_j}\widehat{N}_{8q_j}}\qquad j=1,\cdots,6,$$

Eq. (A.4) is shown in Box A.I. where:

$$\lambda_1=(m_2+1)+(m_2-1)\frac{\bar{C}_{12}}{\bar{C}_{11}},$$

$$\lambda_2=\frac{(m_2\bar{C}_{12}-\bar{C}_{22})+[(n_1^2+n_1n_2)-q^2]\bar{C}_{44}}{\bar{C}_{11}},$$

$$\lambda_3=\frac{(2n_1+n_2)\bar{C}_{44}}{\bar{C}_{11}}q,$$

$$\lambda_4=\frac{n_1\bar{C}_{12}+(n_1+n_2)\bar{C}_{44}}{\bar{C}_{11}},$$

$$\lambda_5=\left(\frac{\bar{C}_{12}+\bar{C}_{44}}{\bar{C}_{11}}\right)q,$$

$$\lambda_6=\frac{(m_2\bar{C}_{12}-\bar{C}_{22})n_1-(n_1+n_2)\bar{C}_{44}}{\bar{C}_{11}},$$

$$\lambda_7=\left(\frac{m_2\bar{C}_{12}-\bar{C}_{22}-\bar{C}_{44}}{\bar{C}_{11}}\right)q,$$

$$\lambda_8=\frac{\bar{e}_{21}}{\bar{C}_{11}},\qquad \lambda_9=\frac{(m_4+1)\bar{e}_{21}-\bar{e}_{22}}{\bar{C}_{11}},$$

$$\lambda_{10}=[(n_1^2+n_1n_4)-q^2]\frac{\bar{e}_{24}}{\bar{C}_{11}},$$

$$\lambda_{11}=\left((2n_1+n_4)\frac{\bar{e}_{24}}{\bar{C}_{11}}\right)q,$$

$$\lambda_{12}=\alpha_0\left((m_1+m_2+1)+\frac{2(m_1+m_2)\bar{C}_{12}-\bar{C}_{22}}{\bar{C}_{11}}\right),$$

$$\lambda_{13}=\alpha_0\left(1+\frac{2\bar{C}_{12}}{\bar{C}_{11}}\right),\qquad \lambda_{14}=m_2+1,$$

$$\lambda_{15}=(m_2+1)-[(n_1+n_2)-q^2]\frac{\bar{C}_{22}}{\bar{C}_{44}},$$

$$\lambda_{16}=\left((n_2+2)\frac{\bar{C}_{22}}{\bar{C}_{44}}\right)q,$$

$$\lambda_{17}=n_1+(n_1+n_2)\frac{\bar{C}_{12}}{\bar{C}_{44}},\qquad \lambda_{18}=\left(1+\frac{\bar{C}_{12}}{\bar{C}_{44}}\right)q,$$

$$\lambda_{19}=(m_2+1)n_1+(n_1+n_2)\frac{\bar{C}_{22}}{\bar{C}_{44}},$$

$$\lambda_{20}=\left((m_2+1)+\frac{\bar{C}_{22}}{\bar{C}_{44}}\right)q,$$

$$\lambda_{21}=\frac{(n_1+n_4)\bar{e}_{22}+n_1\bar{e}_{24}}{\bar{C}_{44}},$$

$$\lambda_{22}=\left(\frac{\bar{e}_{22}+\bar{e}_{24}}{\bar{C}_{44}}\right)q,$$

$$\lambda_{23}=(m_4+2)n_1\frac{\bar{e}_{24}}{\bar{C}_{44}},\qquad \lambda_{24}=(m_4+2)q\frac{\bar{e}_{24}}{\bar{C}_{44}},$$

$$\lambda_{25}=\alpha_0(n_1+n_2)\left(\frac{2\bar{C}_{12}+\bar{C}_{22}}{\bar{C}_{44}}\right),$$

$$I_{q_1}=\frac{d_7d_9d_{17}-d_3d_{15}d_{17}-d_5d_9d_{23}-d_7d_{13}d_{19}+d_3d_{13}d_{23}+d_5d_{15}d_{19}}{d_1d_9d_{17}-d_3d_{11}d_{17}-d_1d_{13}d_{19}-d_5d_9d_{21}+d_5d_{11}d_{19}+d_3d_{13}d_{21}},$$

$$I_{q_2}=\frac{d_8d_{10}d_{18}-d_4d_{16}d_{18}-d_6d_{10}d_{24}-d_8d_{14}d_{20}+d_4d_{14}d_{24}+d_6d_{16}d_{20}}{d_2d_{10}d_{18}-d_4d_{12}d_{18}-d_2d_{14}d_{20}-d_6d_{10}d_{22}+d_6d_{12}d_{20}+d_4d_{14}d_{22}},$$

$$I_{q_3}=\frac{d_1d_{15}d_{17}-d_7d_{11}d_{17}-d_1d_{13}d_{23}-d_5d_{15}d_{21}+d_5d_{11}d_{23}+d_7d_{13}d_{21}}{d_1d_9d_{17}-d_3d_{11}d_{17}-d_1d_{13}d_{19}-d_5d_9d_{21}+d_5d_{11}d_{19}+d_3d_{13}d_{21}},$$

$$I_{q_4}=\frac{d_2d_{16}d_{18}-d_8d_{12}d_{18}-d_2d_{14}d_{24}-d_6d_{16}d_{22}+d_6d_{12}d_{24}+d_8d_{14}d_{22}}{d_2d_{10}d_{18}-d_4d_{12}d_{18}-d_2d_{14}d_{20}-d_6d_{10}d_{22}+d_6d_{12}d_{20}+d_4d_{14}d_{22}},$$

$$I_{q_5}=\frac{d_1d_9d_{23}-d_1d_{15}d_{19}-d_3d_{11}d_{23}-d_7d_9d_{21}+d_7d_{11}d_{19}+d_3d_{15}d_{21}}{d_1d_9d_{17}-d_3d_{11}d_{17}-d_1d_{13}d_{19}-d_5d_9d_{21}+d_5d_{11}d_{19}+d_3d_{13}d_{21}},$$

$$I_{q_6}=\frac{d_2d_{10}d_{24}-d_2d_{16}d_{20}-d_4d_{12}d_{24}-d_8d_{10}d_{22}+d_8d_{12}d_{20}+d_4d_{16}d_{22}}{d_2d_{10}d_{18}-d_4d_{12}d_{18}-d_2d_{14}d_{20}-d_6d_{10}d_{22}+d_6d_{12}d_{20}+d_4d_{14}d_{22}},\tag{A.4}$$

Box A.I

$$\lambda_{26} = \alpha_0 \left(\frac{2\bar{C}_{12} + \bar{C}_{22}}{\bar{C}_{44}}\right) q, \qquad \lambda_{27} = (m_5 + 1),$$

$$\lambda_{28} = [(n_1^2 + n_1 n_5) - q^2]\frac{\bar{\varepsilon}_{21}}{\bar{\varepsilon}_{22}},$$

$$\lambda_{29} = (2n_1 + n_5)q\frac{\bar{\varepsilon}_{21}}{\bar{\varepsilon}_{22}}, \qquad \lambda_{30} = -\frac{\bar{e}_{21}}{\bar{\varepsilon}_{22}},$$

$$\lambda_{31} = \frac{(m_4 + 1)\bar{e}_{21} + \bar{e}_{22}}{\bar{\varepsilon}_{22}}, \qquad \lambda_{32} = \frac{m_4 \bar{e}_{22}}{\bar{\varepsilon}_{22}},$$

$$\lambda_{33} = (n_1 + n_4)\frac{\bar{e}_{24}}{\bar{\varepsilon}_{22}}, \qquad \lambda_{34} = \frac{\bar{e}_{24}}{\bar{\varepsilon}_{22}}q,$$

$$\lambda_{35} = \frac{(m_4 + 1)n_1\bar{e}_{22} - (n_1 + n_4)\bar{e}_{24}}{\bar{\varepsilon}_{22}},$$

$$\lambda_{36} = \left(\frac{(m_4 + 1)\bar{e}_{22} - \bar{e}_{24}}{\bar{\varepsilon}_{22}}\right) q,$$

$$\lambda_{37} = \frac{(m_6 + 1)\bar{g}_{21} + (n_6)\bar{g}_{22}}{\bar{\varepsilon}_{22}},$$

$$\lambda_{38} = \frac{\bar{g}_{22}}{\bar{\varepsilon}_{22}}q, \qquad \lambda_{39} = \frac{\bar{g}_{22}}{\bar{\varepsilon}_{22}}. \tag{A.5}$$

Also:

$$\sum_{q=-\infty}^{\infty} \left[\left(S_{11}a^{\beta_{q_1}} + S_{12}\beta_{q_1}a^{\beta_{q_1}-1}\right) A_{q_1} + \left(S_{11}a^{\beta_{q_2}} + S_{12}\beta_{q_2}a^{\beta_{q_2}-1}\right) A_{q_2}\right] e^{iq\theta} = f_1(\theta),$$

$$\sum_{q=-\infty}^{\infty} \left[\left(S_{21}b^{\beta_{q_1}} + S_{22}\beta_{q_1}b^{\beta_{q_1}-1}\right) A_{q_1} + \left(S_{21}b^{\beta_{q_2}} + S_{22}\beta_{q_2}b^{\beta_{q_2}-1}\right) A_{q_2}\right] e^{iq\theta} = f_2(\theta),$$

$$\left(S_{11}a^{\beta_{q_1}} + S_{12}\beta_{q_1}a^{\beta_{q_1}-1}\right) A_{q_1} + \left(S_{11}a^{\beta_{q_2}} + S_{12}\beta_{q_2}a^{\beta_{q_2}-1}\right) A_{q_2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\theta)e^{-iq\theta}d\theta,$$

$$\left(S_{21}b^{\beta_{q_1}} + S_{22}\beta_{q_1}b^{\beta_{q_1}-1}\right) A_{q_1} + \left(S_{21}b^{\beta_{q_2}} + S_{22}\beta_{q_2}b^{\beta_{q_2}-1}\right) A_{q_2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\theta)e^{-iq\theta}d\theta,$$

$$A_{q_1} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\widehat{S}_1 - \widehat{S}_2}\left[\left(S_{21}b^{\beta_{q_2}} + S_{22}\beta_{q_1}b^{\beta_{q_2}-1}\right) f_1(\theta) - \left(S_{11}a^{\beta_{q_2}} + S_{12}\beta_{q_2}a^{\beta_{q_2}-1}\right) f_2(\theta)\right] e^{-iq\theta}d\theta\right),$$

$$A_{q_2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\widehat{S}_1 - \widehat{S}_2}\left[\left(S_{11}a^{\beta_{q_1}} + S_{12}\beta_{q_1}a^{\beta_{q_1}-1}\right) f_2(\theta) - \left(S_{21}b^{\beta_{q_1}} + S_{22}\beta_{q_1}b^{\beta_{q_1}-1}\right) f_1(\theta)\right] e^{-iq\theta}d\theta\right),$$

$$\widehat{S}_1 = \left(S_{11}a^{\beta_{q_1}} + S_{12}\beta_{q_1}a^{\beta_{q_1}-1}\right)\left(S_{21}b^{\beta_{q_2}} + S_{22}\beta_{q_2}b^{\beta_{q_2}-1}\right),$$

$$\widehat{S}_2 = \left(S_{11}a^{\beta_{q_2}} + S_{12}\beta_{q_2}a^{\beta_{q_2}-1}\right)\left(S_{21}b^{\beta_{q_1}} + S_{22}\beta_{q_1}b^{\beta_{q_1}-1}\right). \tag{A.6}$$

Appendix B

Heat conduction equations are as follows:

$$(k_{ij}T_{,j})_{,i} = \rho c\dot{T} + R = 0, \qquad k = k(r,\theta),$$

$$(k_{ij}T_{,i})_{,i} = 0 \qquad i = j, \quad (kT_{,i})_{,i} = 0,$$

$$kT_{,i} = C \qquad (k_{,i}T_{,i}) + (kT_{,ii}) = 0,$$

$$\left(\frac{\partial k}{\partial x}\frac{\partial T}{\partial x}\right) + \left(\frac{\partial k}{\partial y}\frac{\partial T}{\partial y}\right) + k(\nabla^2 T) = 0. \tag{B.1}$$

Eq. (1) in Cartesian coordinates is as follows:

$$\left(\frac{\partial k}{\partial x}\frac{\partial T}{\partial x}\right) + \left(\frac{\partial k}{\partial y}\frac{\partial T}{\partial y}\right) + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0,$$

$$x = r\cos\theta \to r = \frac{x}{\cos\theta},$$

$$y = r\sin\theta \to r = \frac{y}{\sin\theta}. \tag{B.2}$$

By simplifying the above equations:

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x}, \qquad \frac{\partial k}{\partial x} = \frac{\partial k}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial x},$$

$$x = r\cos\theta,$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y}, \qquad \frac{\partial k}{\partial y} = \frac{\partial k}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial y},$$

$$y = r\sin\theta,$$

$$\frac{\partial r}{\partial x} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \sin\theta,$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin\theta, \qquad \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta,$$

$$\left[\begin{array}{l} \left(\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\cos\theta + \sin\theta\frac{\partial T}{\partial y}\right) r\sin\theta \\ \left(\frac{\partial T}{\partial \theta} = -r\sin\theta\frac{\partial T}{\partial x} + r\cos\theta\frac{\partial T}{\partial y}\right)\cos\theta \end{array}\right]$$

$$+\left[\begin{array}{l} \left(r\sin\theta\frac{\partial T}{\partial r} = \cancel{r\sin\theta\cos\theta\frac{\partial T}{\partial x}} + \sin^2\theta\frac{\partial T}{\partial y}\right) \\ \left(\cos\theta\frac{\partial T}{\partial \theta} = \cancel{-r\sin\theta\cos\theta\frac{\partial T}{\partial x}} + r\cos^2\theta\frac{\partial T}{\partial y}\right) \end{array}\right]$$

$$+ r\sin\theta\frac{\partial T}{\partial r} + \cos\theta\frac{\partial T}{\partial \theta} = r(\sin^2\theta + \cos^2\theta)\frac{\partial T}{\partial y},$$

$$\frac{\partial T}{\partial y} = \frac{1}{r}\left[r\sin\theta\frac{\partial T}{\partial r} + \cos\theta\frac{\partial T}{\partial \theta}\right],$$

$$\frac{\partial T}{\partial y} = \sin\theta\frac{\partial T}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial T}{\partial \theta},$$

$$\left(\frac{\partial T}{\partial r} = \cos\theta\frac{\partial T}{\partial x} + \sin\theta\frac{\partial T}{\partial y}\right),$$

$$\frac{\partial T}{\partial x} = \frac{1}{\cos\theta}\left[\frac{\partial T}{\partial r} - \sin\theta\frac{\partial T}{\partial y}\right]$$

$$= \frac{1}{\cos\theta}\left[\frac{\partial T}{\partial r} - \sin\theta\left(\sin\theta\frac{\partial T}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial T}{\partial \theta}\right)\right],$$

$$\frac{\partial T}{\partial x} = \frac{1}{\cos\theta}\left[\frac{\partial T}{\partial r} - \sin^2\theta\frac{\partial T}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial T}{\partial \theta}\right],$$

$$\frac{\partial T}{\partial y} = \frac{1}{\cos\theta}\left[\cancel{(\sin^2\theta + \cos^2\theta)}\frac{\partial T}{\partial r} - \cancel{\sin^2\theta\frac{\partial T}{\partial r}} - \frac{\sin\theta\cos\theta}{r}\frac{\partial T}{\partial \theta}\right],$$

$$\frac{\partial T}{\partial x} = \frac{1}{\cos\theta}\left[\cos^2\theta\frac{\partial T}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial T}{\partial \theta}\right],$$

$$\frac{\partial T}{\partial x} = \cos\theta\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\frac{\partial T}{\partial \theta}. \tag{B.3}$$

Similarly:

$$\frac{\partial k}{\partial y} = \sin\theta\frac{\partial k}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial k}{\partial \theta},$$

$$\frac{\partial k}{\partial x} = \cos\theta\frac{\partial k}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial k}{\partial \theta},$$

$$\frac{\partial k}{\partial x}\frac{\partial T}{\partial x} = \left(\cos\theta\frac{\partial k}{\partial r} - \frac{\sin\theta}{r}\frac{\partial k}{\partial \theta}\right)\left(\cos\theta\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\frac{\partial T}{\partial \theta}\right)$$

$$= \left(\cos^2\theta\frac{\partial k}{\partial r}\frac{\partial T}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial r}\frac{\partial T}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}\right),$$

$$\frac{\partial k}{\partial y}\frac{\partial T}{\partial y} = \left(\sin\theta\frac{\partial k}{\partial r} + \frac{\cos\theta}{r}\frac{\partial k}{\partial \theta}\right)\left(\sin\theta\frac{\partial T}{\partial r} + \frac{\cos\theta}{r}\frac{\partial T}{\partial \theta}\right)$$

$$= \left(\sin^2\theta\frac{\partial k}{\partial r}\frac{\partial T}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial r}\frac{\partial T}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}\right),$$

$$\left(\frac{\partial k}{\partial x}\frac{\partial T}{\partial x}\right) + \left(\frac{\partial k}{\partial y}\frac{\partial T}{\partial y}\right)$$

$$= \left\{\begin{array}{l} \left(\cos^2\theta\frac{\partial k}{\partial r}\frac{\partial T}{\partial r} - \cancel{\frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial r}\frac{\partial T}{\partial \theta}}\right. \\ \left.- \cancel{\frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial r}} + \frac{\sin^2\theta}{r^2}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}\right) \\ + \left(\sin^2\theta\frac{\partial k}{\partial r}\frac{\partial T}{\partial r} + \cancel{\frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial r}\frac{\partial T}{\partial \theta}}\right. \\ \left.+ \cancel{\frac{\sin\theta\cos\theta}{r}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial r}} + \frac{\cos^2\theta}{r^2}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}\right) \end{array}\right.$$

$$\frac{\partial k}{\partial r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta} = \frac{\partial k}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial T}{\partial y},$$

$$(kT_{,i})_{,i} = \left(\frac{\partial k}{\partial r}\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}\right) + k(\nabla^2 T) = 0,$$

$$\left(\frac{\partial k}{\partial r}\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}\right) + k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right) = 0,$$

$$k\,\frac{\partial^2 T}{\partial r^2} + \cancel{\frac{\partial k}{\partial r}\frac{\partial T}{\partial r}} + k\,\frac{1}{r}\frac{\partial T}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}} + \frac{k}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \tag{A.4}$$

Finally:

$$\left(\frac{\partial k}{\partial r}\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial k}{\partial \theta}\frac{\partial T}{\partial \theta}\right)$$
$$+k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right)=0$$
$$\rightarrow \frac{1}{r}\left(krT_{,r}\right)_{,r}+\frac{1}{r^2}\left(kT_{,\theta}\right)_{,\theta}=0.$$

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